Optimal Stopping with Random Intervention Times*

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Abstract

We consider a class of optimal stopping problems where the ability to stop depends on exogenous Poisson signal process – one can only stop at the Poisson jump times. Even though the time variable in these problems has a discrete aspect, a variational inequality can be obtained by considering an underlying continuous time structure. Depending on whether stopping is allowed at $t = 0$, the value function exhibits different properties across the optimal exercise boundary. Indeed, the value function is only $\mathcal{C}^0$ across the optimal boundary when stopping is allowed at $t = 0$ and $\mathcal{C}^2$ otherwise, both contradicting the usual $\mathcal{C}^1$ smoothness that is necessary and sufficient for the application of the principle of smooth fit. Also discussed is an equivalent stochastic control formulation for these stopping problems. Finally, we derive the asymptotic behavior of the value functions and optimal exercise boundaries as the intensity of the Poisson process goes to infinity, or roughly speaking, as the problems converge to the classical continuous-time optimal stopping problems.

Key words: Optimal stopping, variational inequality, stochastic control, Poisson process.
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1 Introduction

Optimal stopping problems have many applications in engineering, economics and finance; see e.g. [1]-[4], [6], [8]-[11]. The usual setting is either in continuous time where stopping times can take any value in a certain time interval, or in discrete time where stopping times can only take values in a pre-specified grid. Explicit solutions to these problems, which are valuable for subsequent analysis, are rarely available except in some simple continuous time problems with infinite time horizon. To fix ideas, let us consider a classical irreversible investment problem, which is also equivalent to an perpetual American option pricing problem; see e.g. [6, 9].

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Suppose a firm is considering a certain investment opportunity. At any time t, the firm has an option to pay a fixed cost K to install an investment project, whose market value S_t is stochastic and assumed to be a geometric Brownian motion:

$$\frac{dS_t}{S_t} = b\,dt + \sigma\,dW_t, \qquad S_0 = x; \quad t \geq 0.$$

Here $W = (W_t, \mathcal{F}_t)$ is a standard Brownian motion, and $\sigma > 0$ and b are constants that stand for volatility and drift, respectively. The payoff from this option at time t is therefore $(S_t - K)^+ \doteq \max\{S_t - K, 0\}$, and the firm wants to maximize its expected present value by judiciously choosing an investment time. This is equivalent to an optimal stopping problem

$$v(x) \doteq \sup_{\tau \in \mathcal{S}} \mathsf{E}^x \left[e^{-r\tau}(S_\tau - K)^+ \right],$$

where r is the discount rate ($r > b$), $\mathcal{S}$ is the set of all admissible stopping times (investment times), and E^x denotes expected value given $S_0 = x$.

Usually $\mathcal{S}$ is the set of all stopping times taking values in $[0, \infty]$, which means that the firm can pay the cost K and install the project at *any* time. A closed-form solution has been obtained under these assumptions. A discrete time version of this optimal stopping problem would restrict, for example, $\mathcal{S}$ to be the set of all stopping times taking values in the time grid $\{nh : n \geq 0\}$, where $h > 0$ is a constant. However, to the best knowledge of the authors a closed form solution has not been obtained for this problem.

Even though the continuous-time model is more likely to yield an explicit solution, the lack of any restriction on the possible stopping times is sometimes unrealistic. In this paper we investigate an optimal stopping problem that is intermediate between the continuous and discrete time problems just discussed. Suppose that there exists an exogenous, *uncontrolled* Poisson process (say N) serving as a signal process, and that the controller can stop only at the times when the Poisson process N makes a jump. In other words, the investment can be only be made at the *random* times $T_1 < T_2 < \cdots < T_n < \cdots$, where $\{T_1, T_2 - T_1, T_3 - T_2, \cdots\}$ are independent and identically distributed (iid) exponential random variables. In this framework, the investor does not have total freedom over the possible investment times – he has to rely on the Poisson process to give him a certain signal (in this case, the jumps), in order to trigger the investment. For example, it may be that an investment is possible only at those times when certain assets are made available, and not otherwise.

This formulation can be easily extended to problems with more complicated payoff. Although the set of times when one can stop is discrete, explicit solutions are possible for some simple models with an infinite-time horizon. To the best of our knowledge, this type of constraint on the stopping times has not been studied before. Its counterpart in stochastic control was first studied by [12], where an exogenous Poisson process is used to model liquidity effects; see also [13] for an application to problems in singular control.

The paper is organized as follows. In Section 2 we introduce two optimal stopping problems and their associated discrete-time versions. In Section 3 we derive the associated variational inequality by exploiting an underlying continuous time structure. The rest of Section 3 is devoted to solving the variational inequality and proving a verification theorem. An equivalent stochastic control formulation is discussed in Section 4. An asymptotic analysis is carried out in Section 5 as λ (the intensity of the exogenous Poisson process) goes to infinity, in order to relate this formulation to the usual continuous-time model. For the sake of completeness, a very brief account of the analogous continuous-time optimal stopping problem is also included.

Remark 1. In the standard continuous time formulation the so-called principle of smooth fit is used to identify the boundary of the continuation region. The application of this principle usually produces a solution that is continuously differentiable but not twice continuously differentiable across this boundary. An interesting feature of the problem we study is that this qualitative property is never observed. In fact, the value function of the optimal stopping problem turns out to be either $\mathcal{C}^0$ or $\mathcal{C}^2$, depending on whether stopping is allowed at $t = 0$ or not.

2 The optimal stopping problem

Consider a probability space $(\Omega, \mathcal{F}, \mathsf{P}; \mathsf{F})$ with filtration $\mathsf{F} = (\mathcal{F}_t)$ satisfying the usual conditions: right-continuity and completion by P-negligible sets. The *state process* $S = (S_t, \mathcal{F}_t)$ is assumed to be a geometric Brownian motion with

$$\frac{dS_t}{S_t} = b\,dt + \sigma\,dW_t, \qquad S_0 = x. \tag{2.1}$$

Here $W = (W_t, \mathcal{F}_t)$ is a standard F-Brownian motion and b and $\sigma > 0$ are constants. We also assume the probability space is rich enough to carry a Poisson process (the signal process) $N = (N_t, \mathcal{F}_t)$ with intensity λ. *Throughout this paper we assume that the Brownian motion W and the Poisson process N are independent.* The n-th jump time of the Poisson process is denoted by T_n, with the conventions $T_0 \equiv 0$, $T_\infty \equiv \infty$.

We study the following optimal stopping problem. Let $\mathcal{S}$ be the set of admissible stopping times. The objective is to maximize the discounted payoff

$$v(x) \doteq \sup_{\tau \in \mathcal{S}} \mathsf{E}^x \left[e^{-r\tau}(S_\tau - K)^+ \right]. \tag{2.2}$$

Here r and K are both positive constants, and we will assume that $r > b$. Note that the case where $r \le b$ is trivial — the process $\left\{ e^{-rt}(S_t - K)^+;\ t \ge 0 \right\}$ becomes a submartingale, the value function is infinity, and an optimal strategy does not exist.

Suppose that the decision maker can only stop the process at the jump times of the Poisson processes. In other words,

$$\mathcal{S} = \{\mathsf{F}\text{-stopping time } \tau : \text{ for every } \omega \in \Omega,\ \tau(\omega) = T_n(\omega) \text{ for some } n = 1, 2, \cdots, \infty\} \tag{2.3}$$

We have the following preliminary result, whose proof is elementary and thus omitted.

Lemma 1. *For any random variable $N : \Omega \to \{0, 1, 2, \cdots, \infty\}$, the following two statements are equivalent.*

1. *The random variable N is a G-stopping time and $N \ge 1$; here $\mathsf{G} = (\mathcal{G}_n) \doteq (\mathcal{F}_{T_n})$.*
2. *The random variable $\tau \doteq T_N \in \mathcal{S}$.*

This lemma ensures the one-to-one correspondence between the admissible stopping time set $\mathcal{S}$ and the set

$$\mathcal{N} \doteq \{N \geq 1 : \ N \text{ is a } \mathsf{G}\text{-stopping time}\}. \tag{2.4}$$

Define the G-adapted process

$$Z_n \doteq (T_n, S_{T_n}), \qquad n \geq 0. \tag{2.5}$$

It follows that

$$v(x) = \sup_{N \in \mathcal{N}} \mathsf{E}^x \left[\phi(Z_N) \,|\, Z_0 = (0, x) \right], \qquad \text{where} \ \ \phi(Z_n) \doteq e^{-rT_n}(S_{T_n} - K)^+. \tag{2.6}$$

Note that in the preceding setup, the decision maker is not allowed to stop at $t = 0$ (or, $n = 0$). If we remove this restriction, we have the following optimal stopping problem. The set of admissible stopping times becomes

$$\mathcal{S}_0 = \{\mathsf{F}\text{-stopping time } \tau : \ \text{for every } \omega \in \Omega, \, \tau(\omega) = T_n(\omega) \text{ for some } n = 0, 1, 2, \cdots, \infty\} \tag{2.7}$$

and the value function is defined as

$$v_0(x) \doteq \sup_{\tau \in \mathcal{S}_0} \mathsf{E}^x \left[e^{-r\tau}(S_\tau - K)^+ \right]. \tag{2.8}$$

It follows similarly that

$$v_0(x) = \sup_{N \in \mathcal{N}_0} \mathsf{E} \left[\phi(Z_N) \,|\, Z_0 = (0, x) \right], \qquad \text{where} \ \ \mathcal{N}_0 \doteq \{N \geq 0; \ N \text{ is a } \mathsf{G}\text{-stopping time}\}. \tag{2.9}$$

There are several reasons for the simultaneous introduction of problems (2.2), (2.6), (2.8) and (2.9). One is that the variational inequality is very easy to derive for the continuous-time problem (2.2), from which the value function for problem (2.8) can be obtained. However, the verification theorem for (2.2) is achieved through problem (2.8), and it depends heavily on the discrete-time formulations (2.6), (2.9).

Remark 2. The optimal stopping problem (2.2) is time-homogeneous, in the sense that at any time t, the next opportunity of stopping will appear after an exponential time of rate λ (due to the memoryless property). However, it is interesting to note that the discrete-time version (2.6) is *not* time-homogeneous since one can stop at any n except $n = 0$. An analogous observation applies to the stopping problem (2.8). The discrete-time version (2.9) is time-homogeneous, while the continuous-time version (2.8) is not, since at any t *except* $t = 0$ one has to wait an exponential time for the stopping opportunity to come.

Remark 3. One would naturally expect that the following two equalities hold.

$$v_0(x) = \max\left\{(x - K)^+, \ v(x)\right\} \tag{2.10}$$

$$v(x) = \mathsf{E}^x \int_0^\infty e^{-rt} v_0(S_t) \cdot \lambda e^{-\lambda t} \, dt. \tag{2.11}$$

The intuition behind the two equalities is clear. The first equality says that at $t = 0$, one can either stop to get payoff $(x - K)^+$ or continue to get payoff $v(x)$. Hence the value v_0 is the maximum of the two. As for the second equality, in the problem defining V one has to wait an exponential time, and is then allowed to stop at any of the following Poisson jump times (including the first one). Since the Poisson process is independent of the driving Brownian motion, this gives the integral representation for v in terms of v_0. A detailed proof is given in Theorem 2.

3 Variational Inequality and Verification Theorem

Here we consider the optimization problem (2.2), that is,

$$v(x) \doteq \sup_{\tau \in \mathcal{S}} \mathsf{E}\left[e^{-r\tau}(S_\tau - K)^+ \,\middle|\, S_0 = x\right].$$

Let us proceed heuristically for a while. From now on, we will use $\mathcal{L}$ to denote the infinitesimal generator of the geometric Brownian motion $S = (S_t, \mathcal{F}_t)$. Thus

$$\big(\mathcal{L}u\big)(x) = \frac{1}{2}x^2\sigma^2 u''(x) + bxu'(x), \qquad \forall\, x \geq 0. \tag{3.1}$$

It is natural to guess that the optimal strategy should take the following form: *try* stopping whenever the state process S exceeds some threshold $x^* > K$, and continue otherwise. In other words, the optimal strategy should be

$$\tau^* \doteq \inf\{T_n : \; n \geq 1, \; S_{T_n} \geq x^*\}.$$

If it is optimal to continue when the state process is below x^*, then we would expect

$$-rv + \mathcal{L}v = 0, \qquad \forall\, x \in (0, x^*). \tag{3.2}$$

When $x > x^*$, however, one cannot stop unless the Poisson process N makes a jump. Note that in a small time interval of length dt, the Poisson process has probability $\lambda\, dt$ to make a jump, i.e., with probability $\lambda\, dt$ the process will be stopped with a payoff $x - K$, and the process will continue with probability $1 - \lambda\, dt$. This formally suggests that for $x > x^*$,

$$\begin{aligned} v(x) &= \lambda\, dt \cdot (x - K) + (1 - \lambda\, dt) \cdot \mathsf{E}\left[e^{-r \cdot dt} v(S_{dt}) \,|\, S_0 = x\right] \\ &= \lambda(x - K)\, dt + (1 - \lambda\, dt)\left[v(x) + (-rv + \mathcal{L}v)\, dt\right] \\ &= v(x) + (-rv + \mathcal{L}v)\, dt + \lambda \cdot [(x - K) - v]\, dt, \end{aligned}$$

which yields

$$-rv + \mathcal{L}v + \lambda \cdot [(x - K) - v] = 0, \qquad \forall\, x > x^*. \tag{3.3}$$

Many optimal stopping problems in continuous time yield a value function which is $\mathcal{C}^1$ across the optimal exercise boundary (cf. Section 5). An interesting observation here is that v is likely to be $\mathcal{C}^2$ across x^*. Indeed, we expect that $v(x) > (x - K)^+$ for all $x < x^*$ (otherwise, one should try to stop instead of continuing), and similarly $v(x) < (x - K)^+$ for all $x > x^*$ (here the strict inequality follows from the fact that the decision maker does not have total freedom and cannot stop arbitrarily). If one believes that the smooth-fit principle still holds in this case (i.e. the value function v is $\mathcal{C}^1$ across the boundary $x = x^*$), then $v(x^*) = x^* - K$, which in turn implies that v is $\mathcal{C}^2$ across the optimal exercise boundary x^* from equations (3.2) and (3.3) for v.

We obtain the following variational inequality from the preceding heuristic argument.

Variational Inequality. Find a non-negative, twice continuously differentiable function $V: \mathsf{R}^+ \to \mathsf{R}^+$, and a constant $x^* > K$, such that

$$
\begin{array}{lrcll}
(3.4) & V(0+) & = & 0; & \\
(3.5) & V(x^*) - (x^* - K)^+ & = & 0; & \\
(3.6) & -rV + \mathcal{L}V & = & 0; & 0 < x < x^* \\
(3.7) & -rV + \mathcal{L}V + \lambda\left[(x-K) - V\right] & = & 0; & x \ge x^* \\
(3.8) & V(x) - (x-K)^+ & > & 0; & 0 < x < x^* \\
(3.9) & V(x) - (x-K)^+ & < & 0; & x > x^*.
\end{array}
$$

This variational inequality admits a unique solution that can be calculated explicitly. It turns out that the solution is indeed the value function and x^* is indeed the optimal exercise boundary.

We have the following theorem.

Theorem 1. *Let $\big(V(x), x^*\big)$ be the unique solution to the variation inequality* (3.4)–(3.9)*, and let*

$$v(x) \doteq \sup_{\tau \in \mathcal{S}} \mathsf{E}\left[e^{-r\tau}(S_\tau - K)^+ \,\big|\, S_0 = x\right].$$

Then $V(x) = v(x)$ for all $x \in (0, \infty)$. Furthermore, the following stopping time is optimal:

$$\tau^* \doteq \inf\{T_n: \ n \ge 1, \ S_{T_n} \ge x^*\}.$$

The rest of the section is devoted to solving the variational inequality (3.4)–(3.9) and to proving Theorem 1. We have to show that the solution $\big(V(x), x^*\big)$ is unique, and that the conjectured strategy is indeed optimal. The proof also implies the following result.

Theorem 2. *The value function v_0 of the optimal stopping problem* (2.8) *equals*

$$V_0(x) \doteq \max\left\{(x-K)^+, \ V(x)\right\} = \begin{cases} (x-K)^+, & x \ge x^* \\ V(x), & 0 \le x < x^*, \end{cases}$$

and the optimal stopping policy is

$$\tau_0^* \doteq \inf\{T_n: \ n \ge 0, \ S_{T_n} \ge x^*\}.$$

Furthermore,

$$v(x) = \mathsf{E}^x \int_0^\infty e^{-rt} v_0(S_t) \cdot \lambda e^{-\lambda t}\, dt.$$

3.1 Solution to the variational inequality

To solve the variational inequality (3.4)–(3.9), we first observe that equation (3.6) implies

$$V(x) = A_+ x^{\alpha_+} + A_- x^{\alpha_-}, \qquad 0 \le x < x^*.$$

Here α_+ and α_- are the two roots of the quadratic equation

$$f(\alpha) \doteq \frac{1}{2}\alpha^2 + \left(\frac{b}{\sigma^2} - \frac{1}{2}\right)\alpha - \frac{r}{\sigma^2} = 0 \qquad \text{or} \qquad \alpha_\pm = \left(\frac{1}{2} - \frac{b}{\sigma^2}\right) \pm \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}.$$

Since $\alpha_- < 0$, it follows from equation (3.4) that $A_- = 0$. Hence, with $A \doteq A_+$, $\alpha \doteq \alpha_+$, we have

$$(3.10) \qquad V(x) = Ax^{\alpha} \quad \text{where} \quad \alpha = \left(\frac{1}{2} - \frac{b}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}, \qquad \forall\, 0 \le x < x^*.$$

For $x > x^*$, it is not difficult to verify that equation (3.7) implies that

$$V(x) = B_+ x^{\beta_+} + B_- x^{\beta_-} + \frac{\lambda}{\lambda + r - b} x - \frac{\lambda K}{\lambda + r}, \qquad \forall\, x > x^*.$$

Here β_+ and β_- are the two roots of quadratic equation

$$g(\beta) \doteq \frac{1}{2}\beta^2 + \left(\frac{b}{\sigma^2} - \frac{1}{2}\right)\beta - \frac{\lambda + r}{\sigma^2} = 0 \ \text{ or } \ \beta_{\pm} = \left(\frac{1}{2} - \frac{b}{\sigma^2}\right) \pm \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + \frac{2(\lambda + r)}{\sigma^2}}.$$

Moreover, it is very easy to show that $\beta_+ > 1$ since $r > b$. Therefore, we must have $B_+ = 0$ from equation (3.9) and the non-negativity of V, which in turn implies that, with $B \doteq B_-$ and $\beta \doteq \beta_-$,
(3.11)

$$V(x) = Bx^{\beta} + \frac{\lambda}{\lambda + r - b} x - \frac{\lambda K}{\lambda + r} \ \text{ with } \ \beta = \left(\frac{1}{2} - \frac{b}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + \frac{2(\lambda + r)}{\sigma^2}}, \ \ \forall\, x > x^*.$$

There are three unknowns (A, B, x^*). However, the continuity of $V(x)$ and $V'(x)$ across the optimal exercise boundary x^*, as well as equation (3.5), gives that

$$V(x^*+) = V(x^*-) = (x^* - K)^+ = x^* - K, \qquad V'(x^*+) = V'(x^*-).$$

Here we have used the assumption $x^* > K$. Therefore

$$\begin{aligned} A(x^*)^{\alpha} &= x^* - K, \\ B(x^*)^{\beta} + \frac{\lambda}{\lambda + r - b} x^* - \frac{\lambda K}{\lambda + r} &= x^* - K, \\ \beta B(x^*)^{\beta - 1} + \frac{\lambda}{\lambda + r - b} &= \alpha A(x^*)^{\alpha - 1}. \end{aligned}$$

One can directly calculate the unique solution

$$(3.12) \qquad x^* = \frac{\alpha - \frac{r}{\lambda + r}\beta}{\alpha - \frac{r - b}{\lambda + r - b}\beta - \frac{\lambda}{\lambda + r - b}} \cdot K,$$

and that the pair of coefficients (A, B) are determined by

$$(3.13) \qquad A = \frac{x^* - K}{(x^*)^{\alpha}}, \qquad B = \frac{\frac{r - b}{\lambda + r - b} x^* - \frac{r}{\lambda + r} K}{(x^*)^{\beta}}.$$

We have the following proposition.

Proposition 1. *The pair* $\big(V(x), x^*\big)$ *determined by equations* (3.10), (3.11), (3.12) *and* (3.13) *is the unique solution to the variational inequality* (3.4)-(3.9).

Proof: We first show that the optimal exercise boundary satisfies $x^* > K$, and $A > 0, B > 0$. It is very easy to see that $\alpha > 1$ and $\beta < 0$. Hence, the numerator and denominator in equation (3.12) are both positive since $r > b$. In particular, x^* is positive. We study the following two cases.

1. *Case $b < 0$:* It suffices to show that

$$x^* > K \qquad \left(\text{hence, } \; x^* > \tfrac{r}{\lambda+r}\tfrac{\lambda+r-b}{r-b}K\right).$$

   A bit of algebra shows that this is equivalent to the inequality $\beta > \frac{\lambda+r}{b}$. However, since β solves the quadratic equation $g(\beta) = 0$, with

$$g\left(\frac{\lambda+r}{b}\right) = \frac{(\lambda+r)(\lambda+r-b)}{2b^2} > 0, \qquad g(0) = -\frac{\lambda+r}{\sigma^2} < 0,$$

   the claim follows.

2. *Case $b \geq 0$:* It suffices to show that

$$x^* > \frac{r}{\lambda+r}\frac{\lambda+r-b}{r-b}K \qquad (\text{hence, } \; x^* > K)\,.$$

   Elementary manipulations show that this is equivalent to the inequality $\alpha < \frac{r}{b}$. However, α solves the quadratic equation $f(\alpha) = 0$, with

$$f\left(\frac{r}{\sigma}\right) = \frac{r(r-b)}{2b^2} > 0, \qquad f(0) = -\frac{r}{\sigma^2} < 0,$$

   and the claim follows readily.

It remains to show that V is non-negative and the inequalities (3.8), (3.9) hold. However, since $A > 0, \alpha > 1$ and $B > 0, \beta < 0$, it is easy to see that V is convex on the intervals $[0, x^*)$ and $[x^*, \infty)$, respectively. Since V is twice continuously differentiable, it must be convex on the whole interval $[0, \infty)$, and thus non-negative. It is easy to check that

$$\lim_{x\to\infty} V'(x) = \lim_{x\to\infty}\left(\beta B x^{\beta-1} + \frac{\lambda}{\lambda+r-b}\right) = \frac{\lambda}{\lambda+r-b} < 1.$$

It follows that $V'(x) < 1$ for all $x \in [0, \infty)$, which in turn implies inequalities (3.8) and (3.9) with the aid of equality (3.5). □

Remark 4. We have shown that V satisfies the following dynamic programming equation

$$-rV + \mathcal{L}V + \lambda \cdot \max\big\{(x-K)^+ - V, 0\big\} = 0, \qquad x \in (0, \infty).$$

3.2 Verification theorem

In this subsection, we will simultaneously prove Theorems 1 and 2. Let $\big(V(x), x^*\big)$ be the unique solution to the variational inequality (3.4) - (3.9), and define

$$V_0(x) = \max\big\{(x-K)^+,\ V(x)\big\} = \begin{cases} (x-K)^+, & x \geq x^* \\ V(x)\ , & 0 \leq x < x^* \end{cases},$$

We divide the proof into several steps.

Step 1: We first show the equality

$$V(x) = \mathsf{E}^x \int_0^\infty e^{-rt} V_0(S_t) \cdot \lambda e^{-\lambda t}\, dt \tag{3.14}$$

holds. Indeed, it follows from Remark 4 that

$$-rV + \mathcal{L}V + \lambda \cdot (V_0 - V) = -(r+\lambda)V + \mathcal{L}V + \lambda V_0 \equiv 0.$$

Consider the process $M = (M_t, \mathcal{F}_t)$ where

$$M_T \doteq e^{-(r+\lambda)T} V(S_T) + \int_0^T \lambda e^{-(r+\lambda)t} V_0(S_t)\, dt, \qquad T \geq 0.$$

It follows from Itô's rule that

$$\begin{aligned} M_T &= V(x) + \int_0^T e^{-(r+\lambda)t} \left[-(r+\lambda)V + \mathcal{L}V + \lambda V_0\right](S_t)\, dt + \int_0^T e^{-(r+\lambda)t} V'(S_t) \cdot \sigma S_t\, dW_t \\ &= V(x) + \int_0^T e^{-(r+\lambda)t} V'(S_t) \cdot \sigma S_t\, dW_t. \end{aligned}$$

Since $V' < 1$ (see the proof of Proposition 1), it is not difficult to verify that the stochastic integral defines a martingale, and therefore

$$V(x) = \mathsf{E}^x M_0 = \mathsf{E}^x M_T = \mathsf{E}^x e^{-(r+\lambda)T} V(S_T) + \mathsf{E}^x \int_0^T \lambda e^{-(r+\lambda)t} V_0(S_t)\, dt, \quad \forall\, T \geq 0.$$

However, since there exists a constant c such that $V(x) \leq x + c$ for all $x \in [0, \infty)$ and $r > b$,

$$\limsup_{T\to\infty} \mathsf{E}^x e^{-(r+\lambda)T} V(S_T) \leq \limsup_{T\to\infty} \mathsf{E}^x \left[e^{-(r+\lambda)T}(S_T + c)\right] = \limsup_{T\to\infty} e^{-(r+\lambda)T}(e^{bT} + c) = 0.$$

Letting $T \to \infty$, (3.14) follows from the monotone convergence theorem.

Step 2: Here we show that $v_0(x) = V_0(x)$ is the value function and τ_0^* defines an optimal stopping strategy for the optimization problem (2.8). First, it follows from Step 1 that

$$V_0(x) \geq V(x) = \int_0^\infty e^{-rt} V_0(S_t) \cdot \lambda e^{-\lambda t}\, dt = \mathsf{E}^x e^{-rU} V_0(S_U)$$

where U is an independent exponential random variable with rate λ. This implies that the process $\left(e^{-rT_n} V_0(S_{T_n}), \mathcal{G}_n\right)$ is indeed a non-negative supermartingale. It follows from the optional sampling theorem that

$$V_0(x) \geq \mathsf{E}^x \left[e^{-rT_N} V_0(S_{T_N})\right] \geq \mathsf{E}^x \left[e^{-rT_N}(S_{T_N} - K)^+\right]$$

for all G-stopping times N. Hence, we have $V_0(x) \geq v_0(x)$ after taking supremum over N on the right-hand side. It remains to show that

$$V_0(x) = \mathsf{E}^x \left[e^{-r\tau_0^*}(S_{\tau_0^*} - K)^+\right],$$

where by definition

$$\tau_0^* = T_{N_0^*}, \qquad \text{with} \;\; N_0^* \doteq \inf\{n \geq 0: \quad S_{T_n} \geq x^*\}.$$

To this end, it suffices to show that the process

$$Q \doteq \left(e^{-rT_{N_0^* \wedge n}} V_0(S_{T_{N_0^* \wedge n}}); \mathcal{G}_n\right)$$

is a uniformly integrable martingale. Indeed, if this is true, then the optional sampling theorem yields

$$V_0(x) = \mathsf{E}^x Q_{N_0^*} = \mathsf{E}^x \left[e^{-r\tau_0^*} V_0(S_{\tau_0^*})\right] = \mathsf{E}^x \left[e^{-r\tau_0^*}(S_{\tau_0^*} - K)^+\right].$$

The martingale property is fairly easy to prove. For $n \geq 1$, we have

$$\begin{aligned}
\mathsf{E}\left[Q_n \,\middle|\, \mathcal{G}_{n-1}\right] &= \mathsf{E}\left[e^{-rT_n} V_0(S_{T_n}) \cdot 1_{\{N_0^* \geq n\}} \,\middle|\, \mathcal{G}_{n-1}\right] + \sum_{i=0}^{n-1} \mathsf{E}\left[e^{-rT_i} V_0(S_{T_i}) \cdot 1_{\{N_0^* = i\}} \,\middle|\, \mathcal{G}_{n-1}\right] \\
&= 1_{\{N_0^* \geq n\}} \mathsf{E}\left[e^{-rT_n} V_0(S_{T_n}) \,\middle|\, \mathcal{G}_{n-1}\right] + \sum_{i=0}^{n-1} e^{-rT_i} V_0(S_{T_i}) \cdot 1_{\{N_0^* = i\}} \\
&= 1_{\{N_0^* \geq n\}} e^{-rT_{n-1}} \mathsf{E}^{S_{T_{n-1}}} e^{-rU} V_0(S_U) + \sum_{i=0}^{n-1} e^{-rT_i} V_0(S_{T_i}) \cdot 1_{\{N_0^* = i\}} \\
&= 1_{\{N_0^* \geq n\}} e^{-rT_{n-1}} V(S_{T_{n-1}}) + \sum_{i=0}^{n-1} e^{-rT_i} V_0(S_{T_i}) \cdot 1_{\{N_0^* = i\}} \\
&= 1_{\{N_0^* \geq n\}} e^{-rT_{n-1}} V_0(S_{T_{n-1}}) + \sum_{i=0}^{n-1} e^{-rT_i} V_0(S_{T_i}) \cdot 1_{\{N_0^* = i\}} \\
&= Q_{n-1}.
\end{aligned}$$

Here the third equality follows from the strong Markov property with U being an independent exponential random variable of rate λ. The fourth equality follows from Step 1 and the fifth equality follows from the fact that on set $\{N_0^* \geq n\}$ we have $S_{T_{n-1}} < x^*$, which implies that $V_0(S_{T_{n-1}}) = V(S_{T_{n-1}})$ on this set by the definition of V_0.

In order to show that Q is uniformly integrable, it suffices to show that $\sup_{t\geq 0} e^{-rt} V_0(S_t)$ is integrable. However, since $V_0(x) \leq x + c$ for some constant c, we only need to show that $\sup_{t \geq 0} e^{-rt} S_t$ is integrable. Note that

$$\sup_{t\geq 0} e^{-rt} S_t = e^Y, \qquad \text{where} \;\; Y \doteq \sup_{t \geq 0}\left(\sigma W_t - \frac{1}{2}\sigma^2 t + (b-r)t\right).$$

The distribution of Y can be found in standard textbooks; e.g. [7]. We have

$$\mathsf{E}\left[\sup_{t\geq 0} e^{-rt} S_t\right] = 1 + \int_0^\infty e^y \mathsf{P}(Y \geq y)\, dy = 1 + \int_0^y e^y e^{\frac{2y}{\sigma}\left(\frac{b-r}{\sigma} - \frac{\sigma}{2}\right)}\, dy = 1 + \frac{\sigma^2}{2(r-b)}.$$

Step 3: Finally, we show that $v(x) = V(x)$ is the value function and τ^* defines an optimal stopping strategy for the optimization problem (2.2). Indeed, since $\left(e^{-rT_n} V_0(S_{T_n}), \mathcal{G}_n\right)$ is a non-negative supermartingale (see Step 2), we have that, for all G-stopping times $N \geq 1$,

$$\begin{aligned} \mathsf{E}^x \left[e^{-rT_N} (S_{T_N} - K)^+ \right] &\leq \mathsf{E}^x \left[e^{-rT_N} V_0(S_{T_N}) \right] \\ &\leq \mathsf{E}^x \left[e^{-rT_1} V_0(S_{T_1}) \right] \\ &= \mathsf{E}^x \int_0^\infty e^{-rt} V_0(S_t) \cdot \lambda e^{-\lambda t} \, dt \\ &= V(x). \end{aligned}$$

Taking the supremum over such N on the left-hand side, we have $v(x) \leq V(x)$ for all x. It remains to show that

$$V(x) = \mathsf{E}^x \left[e^{-r\tau^*} (S_{\tau^*} - K)^+ \right].$$

However, by conditioning on the first jump time T_1, it follows from the strong Markov property that

$$\begin{aligned} \mathsf{E}^x \left[e^{-r\tau^*} (S_{\tau^*} - K)^+ \right] &= \int_0^\infty \mathsf{E}^x \left[e^{-r\tau^*} (S_{\tau^*} - K)^+ \,\middle|\, T_1 = t \right] \cdot \lambda e^{-\lambda t} \, dt \\ &= \mathsf{E}^x \int_0^\infty e^{-rt} \mathsf{E}^{S_t} \left[e^{-r\tau_0^*} (S_{\tau_0^*} - K)^+ \right] \cdot \lambda e^{-\lambda t} \, dt \\ &= \mathsf{E}^x \int_0^\infty e^{-rt} V_0(S_t) \cdot \lambda e^{-\lambda t} \, dt \\ &= V(x). \end{aligned}$$

This completes the proof. □

4 An Equivalent Stochastic Control Formulation

The optimal stopping problem (2.2) exhibits several interesting properties: (i) it is easy to obtain the dynamic programming equation formally from the continuous-time version even though the underlying problems are essentially discrete; (ii) the value function is $\mathcal{C}^2$; (iii) the verification theorem is proved by introducing the auxiliary optimal stopping problem (2.8).

Now let us consider the problem (2.2) from the point of view of stochastic control. At any time t, the decision maker has indeed two control strategies: either continue or *try* stopping. Such an strategy can be represented by a (control) process with binary values. Hence we can heuristically translate the optimal stopping problem into a stochastic control problem: suppose that $u = (u_t, \mathcal{F}_t)$ is a control process, with $u_t = 1$ for trying to stop and $u_t = 0$ for continuing. The objective is to maximize the associated expected payoff by judiciously choosing a control process. The proposed control problem should possess the following property: the value function should equal v, and the optimal control process should take form $u_t^* = 1_{\{S_t \geq x^*\}}$. (Note that for nondegenerate stochastic optimal control problems it is often true that the value function is $\mathcal{C}^2$.)

Such a control problem exists and has a very clear interpretation. Define the set of *admissible controls*

$$\mathcal{A} \doteq \{ u = (u_t, \mathcal{F}_t) : u \text{ is measurable and adapted, with } u_t = 0 \text{ or } 1 \},$$

and the associate payoff

$$J(x; u) \doteq \mathsf{E}^x \int_0^\infty \lambda u_t e^{-rt - \lambda \int_0^t u_s \, ds} (S_t - K)^+ \, dt. \tag{4.1}$$

The objective is to maximize the expected payoff

$$\bar{v}(x) \doteq \sup_{u \in \mathcal{A}} J(x; u). \tag{4.2}$$

Remark 5. The stochastic control formulation (4.1), (4.2) is equivalent to the optimal stopping problem (2.2), thanks to the Theorem 3 below. Here is a heuristic explanation of why we define the associated payoff J as in (4.1). Consider a control process $u \in \mathcal{A}$ which is $\mathcal{F}_t^W$-adapted, where $\mathcal{F}^W$ is the filtration generated by the Brownian motion W (since W and N are independent, the consideration of such control processes should be sufficient, at least intuitively). The control u represents an investment strategy, and the investment (i.e., stopping) will take place the *first* time that both $u = 1$ and the Poisson process makes a jump. Let P_t stands for the conditional probability conditioning on $\mathcal{F}_t^W$, and let m denote Lebesgue measure. Under the proposed investment strategy, we have formally

$$\begin{aligned} &\mathsf{P}_t\big(\text{the investment takes place on time interval } [t, t+dt)\,\big) \\ &\quad= \mathsf{P}_t\big(u_t = 1 \text{ and a Poisson jump occurs at time interval } [t, t+dt)\,\big) \\ &\qquad\qquad \times\ \mathsf{P}_t\big(\text{no investment takes place on time interval } [0, t)\big) \end{aligned}$$

However,

$$\begin{aligned} &\mathsf{P}_t\big(u_t = 1 \text{ and a Poisson jump occurs at time interval } [t, t+dt)\,\big) \\ &\qquad = 1_{\{u_t=1\}} \cdot \mathsf{P}_t\big(\text{a Poisson jump occurs at time interval } [t, t+dt)\,\big) \\ &\qquad = u_t \cdot \lambda\, dt. \end{aligned}$$

and

$$\begin{aligned} &\mathsf{P}_t\big(\text{no investment takes place on time interval } [0, t)\big) \\ &\qquad = \mathsf{P}_t\big(\text{no Poisson jumps at time } s, \text{ for all } s \in [0, t) \text{ such that } u_s = 1\big) \\ &\qquad = e^{-\lambda \cdot m(\{s \in [0,t):\ u_s = 1\})} \\ &\qquad = e^{-\lambda \int_0^t u_s\, ds}. \end{aligned}$$

Therefore, the total expected payoff from the investment policy associated with process u is

$$\begin{aligned} &\int_0^\infty \mathsf{E}^x \left[e^{-rt}(S_t - K)^+ \cdot 1_{\{\text{the investment takes place at time interval } [t,\ t+dt)\}}\right] \\ &\quad= \mathsf{E}^x \int_0^\infty e^{-rt}(S_t - K)^+ \cdot \mathsf{P}_t\big(\text{the investment takes place at time interval } [t, t+dt)\,\big) \\ &\quad= \mathsf{E}^x \int_0^\infty e^{-rt}(S_t - K)^+ \cdot \lambda u_t e^{-\lambda \int_0^t u_s\, ds}\, dt, \end{aligned}$$

which is exactly J as defined in (4.1); here the first equality follows by conditioning on $\mathcal{F}_t^W$.

We have the following result. Let v and x^* denote the value function and optimal exercise boundary for problem (2.2).

Theorem 3. *$\bar{v}(x) = v(x)$ for $x \in (0, \infty)$, and the optimal control process is given by $u^* = (u_t^*, \mathcal{F}_t)$, with*

$$u_t^* \doteq 1_{\{S_t \geq x^*\}}.$$

Proof: We first show that $v(x) \geq \bar{v}(x)$. Consider any admissible control policy $u \in \mathcal{A}$, and the process

$$X_t \doteq e^{-rt-\lambda\int_0^t u_s\,ds} v(S_t), \qquad t \geq 0.$$

It follows from Itô's formula that for every $T \geq 0$

$$X_T = v(x) + \int_0^T e^{-rt-\lambda\int_0^t u_s\,ds}\big(-\lambda u_t v - rv + \mathcal{L}v\big)(S_t)\,dt + M_T,$$

where

$$M_T \doteq \int_0^T e^{-rt-\lambda\int_0^t u_s\,ds} v'(S_t)\cdot\sigma S_t\,dW_t, \qquad T \geq 0.$$

Since $|v'(x)| \leq 1$ for $x \in (0,\infty)$, $M = (M_t, \mathcal{F}_t)$ is a martingale, and hence $E^x M_T = 0$. Also, since $-rv + \mathcal{L}v \leq 0$ (see Remark 4) and $u_t \in \{0,1\}$, we have

$$X_T \leq v(x) + \int_0^T e^{-rt-\lambda\int_0^t u_s\,ds} u_t\big(-\lambda v - rv + \mathcal{L}v\big)(S_t)\,dt + M_T, \qquad \forall\ T \geq 0.$$

Again it follows from Remark 4 that

$$-\lambda v - rv + \mathcal{L}v = -\lambda\max\{(x-K)^+, V\} \leq -\lambda(x-K)^+,$$

which implies that

$$X_T \leq v(x) - \int_0^T e^{-rt-\lambda\int_0^t u_s\,ds}\lambda u_t(S_t-K)^+\,dt + M_T, \qquad \forall\ T \geq 0.$$

Taking expectation on both sides gives

$$v(x) \geq \mathsf{E}^x\int_0^T e^{-rt-\lambda\int_0^t u_s\,ds}\lambda u_t(S_t-K)^+\,dt + \mathsf{E}^x X_T \geq \mathsf{E}^x\int_0^T e^{-rt-\lambda\int_0^t u_s\,ds}\lambda u_t(S_t-K)^+\,dt, \quad \forall\ T \geq 0.$$

Letting $T \to \infty$, the monotone convergence theorem implies $v(x) \geq J(x;u)$. Since u is arbitrary, $v(x) \geq \bar{v}(x)$.

It remains to show that $v(x) \leq J(x;u^*)$, which implies that $v \leq \bar{v}$. If this is true then $v = \bar{v}$, and u^* is the optimal control. Observe that (3.6) and (3.7) imply

$$\big(-\lambda u_t^* v - rv + \mathcal{L}v\big)(S_t) = u_t^*\big(-\lambda v - rv + \mathcal{L}v\big)(S_t) = -\lambda u_t^*(S_t-K)^+.$$

We have similarly

$$X_T^* = v(x) - \int_0^T e^{-rt-\lambda\int_0^t u_s^*\,ds}\lambda u_t^*(S_t-K)^+\,dt + M_T^*, \qquad \forall\ T \geq 0,$$

where X^* and the martingale M^* are defined as before with u^* replacing u. We have shown

$$\mathsf{E}^x X_T^* + \int_0^T e^{-rt-\lambda\int_0^t u_s^*\,ds}\lambda u_t^*(S_t-K)^+\,dt \geq v(x).$$

Now let $T \to \infty$ and observe that for some constant c,

$$\limsup_{T\to\infty}\mathsf{E}^x X_T^* \leq \limsup_{T\to\infty}\mathsf{E}^x e^{-rT}v(S_T) \leq \limsup_{T\to\infty}\mathsf{E}^x e^{-rT}(S_T + c) = 0.$$

The last equality follows since $r > b$. An application of the monotone convergence theorem gives $v(x) \leq J(x;u^*)$. $\square$

Remark 6. It is easy to see from its proof that Theorem 3 still holds if the set of admissible controls is defined as

$$\mathcal{A} \doteq \{u = (u_t, \mathcal{F}_t) : u \text{ is measurable and adapted, with } \ u_t \in [0,1]\}.$$

Remark 7. This equivalent stochastic control formulation has the following advantage: suppose a cost is charged during the times when assets are made available, i.e., there is a charge at all times when the controller is actively "trying" to stop. Although this problem cannot be formulated as an optimal stopping problem, it can be formulated as an extension of the stochastic control problem considered in this section.

5 Asymptotics as $\lambda \to \infty$

A relevant question for this formulation is the following: what is the cost of having such constraints on the stopping times? In other words, how does this problem differ from the classical continuous-time optimal stopping, which can be regarded as the limiting model when $\lambda \to \infty$. For the convenience of the reader, we include a brief account of the corresponding continuous-time optimal stopping problem, whose applications include the irreversible investment and perpetual American option pricing; see, e.g., [6, 9] for more details.

5.1 Review of the irreversible investment problem

Let $\bar{\mathcal{S}}$ denote all F-stopping times taking values in $[0,\infty]$. The optimal stopping problem under consideration is

$$w(x) \doteq \sup_{\tau \in \bar{\mathcal{S}}} \mathsf{E}\left[e^{-r\tau}(S_\tau - K)^+ \,\middle|\, S_0 = x\right]. \tag{5.1}$$

It can be shown that the optimal stopping time is

$$\sigma^* \doteq \inf\{t \geq 0 : \quad S_t \geq x_0^*\},$$

for some $x_0^* > K$. It can further be shown that on the "continuation" region $(0, x_0^*)$, w satisfies the equation

$$-rw + \mathcal{L}w = 0.$$

This implies $w(x) = A_0 x^\alpha$ for some constant A_0, with α defined as in (3.10). On the "stopping" region $[x_0^*, \infty)$, we have

$$w(x) = (x - K)^+ = x - K.$$

The unknown pair of constants (A_0, x_0^*) can be determined through the so-called *smooth-fit principle*, which asserts that the value function is $\mathcal{C}^1$ across the optimal exercise boundary $x = x_0^*$. It follows that

$$w(x) = \begin{cases} A_0 x^\alpha, & 0 < x < x_0^* \\ x - K, & x_0^* \leq x \end{cases}; \qquad \text{where } \ x_0^* = \frac{\alpha}{\alpha - 1}K, \ \ A_0 = \frac{x_0^* - K}{(x_0^*)^\alpha}. \tag{5.2}$$

It is easy to see that w is *only* $\mathcal{C}^1$ across the optimal exercise boundary $x = x_0^*$.

5.2 Asymptotics

Here we study the asymptotics as λ, the intensity of the signaling Poisson process, goes to infinity (or the mean interjump time $h = \lambda^{-1}$ goes to zero). It is natural to expect that the value functions and the optimal exercise boundary for problem (2.2) approach those of the corresponding optimal stopping problem (5.1). In this section, we will denote by $v_0^{(h)}$ (resp., $v^{(h)}$) the value function of problem (2.8) (resp., (2.2)), and the optimal exercise boundaryby x_h^*.

The following result says that the value functions $v^{(h)}$ and $v_0^{(h)}$ converges to w with rate λ^{-1}. In other words, the cost of the constraint on the stopping times is approximately λ^{-1} times a constant when λ is large enough. The optimal exercise boundaries x_h^*, however, converge with rate $\sqrt{\lambda^{-1}}$.

Theorem 4. *Let $h = \lambda^{-1}$ be the mean interjump time. The optimal exercise boundaries x_h^* satisfy*

$$x_h^* = x_0^* - \frac{\sqrt{2}}{2}\sigma x_0^* \cdot \sqrt{h} + o(\sqrt{h}).$$

The value functions $v_0^{(h)}$ and $v^{(h)}$ satisfy

$$v_0^{(h)}(x) = v^{(h)}(x) = w(x) - \frac{1}{4}\alpha(\alpha-1)\sigma^2 w(x)\cdot h + o(h), \qquad \forall\ 0 < x < x_0^*$$

and

$$\begin{aligned} v_0^{(h)}(x) &= w(x) \\ v^{(h)}(x) &= w(x) + \big[Kr - (r-b)x\big]h + o(h), \end{aligned} \qquad \forall\ x > x_0^*,$$

as well as

$$\begin{aligned} v_0^{(h)}(x_0^*) &= w(x_0^*) \\ v^{(h)}(x_0^*) &= w(x_0^*) + (1-e^{-1})\big[Kr - (r-b)x_0^*\big]h + o(h). \end{aligned}$$

Remark 8. The expansion above is not uniform in $x \in (0,\infty)$. An analogous uniform asymptotic expansion can also be obtained, and in fact follows from the same detailed calculations as those given below.

Proof of Theorem 4: The proof is straightforward computation. To ease the notation, let $\iota \doteq \sqrt{h}$. We first establish the asymptotics of the optimal exercise boundaries x_h^*. Let β_h denote the constant β in equation (3.11), and let (A_h, B_h) denote the constants in (3.13) (note that α does not depend on h). It follows from (3.12) that

$$x_h^* = \frac{\alpha - \frac{r\iota}{1+r\iota^2}(\iota\beta_h)}{\alpha - \frac{(r-b)\iota}{1+(r-b)\iota^2}(\iota\beta_h) - \frac{1}{1+(r-b)\iota^2}} \cdot K.$$

Although $\beta_h \to -\infty$ as $h \to 0$, we have

$$\iota\beta_h = \sqrt{h}\beta_h \ \to\ -\frac{\sqrt{2}}{\sigma} \quad \text{as} \ \ h \to 0 \ \ (\text{or } \iota \to 0).$$

It follows that the denominator and numerator in the expression for x_h^* converge to $\alpha - 1$ and α respectively, as $h \to 0$; in particular, $x_h^* \to x_0^*$. Keeping in mind $h = \iota^2$, it is easy to check that

$$\left.\frac{dx_h^*}{d\iota}\right|_{\iota=0} = \left(\frac{r\sqrt{2}}{\sigma(\alpha-1)} - \frac{\alpha\sqrt{2}(r-b)}{\sigma(\alpha-1)^2}\right)\cdot K = \frac{\sqrt{2}(\alpha b - r)K}{\sigma(\alpha-1)^2} = -\frac{\sqrt{2}}{2}\frac{\sigma^2\alpha(\alpha-1)}{\sigma(\alpha-1)^2}K = -\frac{\sqrt{2}}{2}\sigma x_0^*.$$

The third equality follows from $f(\alpha) = 0$, with f defined as in Subsection 3.1, and the last equality follows from (5.2).

It remains to calculate the asymptotics of $v_0^{(h)}$ and $v^{(h)}$. First observe that $x_h^* \le x_0^*$ for all $h > 0$ since $w(x) \ge v_0^{(h)}(x) > (x-K)^+$ for all $x < x_h^*$. We consider the following three cases separately.

The case $0 < x < x_0^*$. Since $x_h^* \to x_0^*$, we can consider h small enough such that $x < x_h^*$. It follows that

$$v_0^{(h)}(x) = v^{(h)}(x) = A_h x^\alpha; \qquad \text{where } \ A_h = \frac{x_h^* - K}{(x_h^*)^\alpha} \ \text{ from (3.13).}$$

It then follows from (5.2) that

$$\left.\frac{dA_h}{d\iota}\right|_{\iota=0} = (1-\alpha)(x_h^*)^{-\alpha-1}\left(x_h^* - \frac{\alpha}{\alpha-1}K\right)\cdot\left.\frac{dx_h^*}{d\iota}\right|_{\iota=0} = 0$$

and

$$\begin{aligned}
\left.\frac{d^2A_h}{d\iota^2}\right|_{\iota=0} &= (1-\alpha)(x_h^*)^{-\alpha-1}\left(x_h^* - \frac{\alpha}{\alpha-1}K\right)\cdot\left.\frac{d^2x_h^*}{d\iota^2}\right|_{\iota=0} \\
&\qquad + \alpha(\alpha-1)(x_h^*)^{-\alpha-2}\left(x_h^* - \frac{\alpha+1}{\alpha-1}K\right)\cdot\left.\left(\frac{dx_h^*}{d\iota}\right)^2\right|_{\iota=0} \\
&= 0 + \alpha(\alpha-1)(x_0^*)^{-\alpha-2}\left(x_0^* - \frac{\alpha+1}{\alpha-1}K\right)\cdot\left(-\frac{\sqrt{2}}{2}\sigma x_0^*\right)^2 \\
&= -\frac{1}{2}\sigma^2(x_0^*)^{-\alpha}\alpha K = -\frac{1}{2}\sigma^2\frac{A_0}{x_0^* - K}\alpha K = -\frac{1}{2}\alpha(\alpha-1)\sigma^2\cdot A_0.
\end{aligned}$$

The case $x > x_0^*$. In this case, we have $v_0^{(h)}(x) = w(x) = (x-K)^+ = x - K$, and

$$v^{(h)}(x) = B_h x^{\beta_h} + \frac{1}{1+(r-b)h}x - \frac{K}{1+rh}$$

thanks to (3.11). It suffices to observe that the term

$$0 \le B_h x^{\beta_h} = \left(\frac{x}{x_h^*}\right)^{\beta_h}\left(\frac{r-b}{\lambda+r-b}x_h^* - \frac{r}{\lambda+r}K\right) \le \left(\frac{x}{x_0^*}\right)^{\beta_h} x_0^*$$

converges to zero exponentially fast since $\sqrt{h}\beta_h \to -\frac{\sqrt{2}}{\sigma}$.

The case $x = x_0^*$**.** In this case, we have $v_0^{(h)}(x_0^*) = w(x_0^*) = (x_0^* - K)^+ = x_0^* - K$, and

$$v^{(h)}(x_0^*) = B_h(x_0^*)^{\beta_h} + \frac{1}{1+(r-b)h} x_0^* - \frac{K}{1+rh}.$$

It remains to show that

$$B_h(x_0^*)^{\beta_h} = e^{-1}\left[(r-b)x_0^* - Kr\right]h + o(h).$$

However, it follows from the asymptotics of x_h^* that

$$\begin{aligned} B_h(x_0^*)^{\beta_h} &= \left(\frac{x_h^*}{x_0^*}\right)^{-\beta_h}\left(\frac{(r-b)h}{1+(r-b)h} - \frac{rh}{1+rh}K\right) \\ &= \left[1 - \frac{\sqrt{2}}{2}\sigma\sqrt{h} + o(\sqrt{h})\right]^{-\beta_h} \cdot \left[\big((r-b)x_0^* - Kr\big)h + o(h)\right] \\ &= e^{-1}\big((r-b)x_0^* - Kr\big)h + o(h); \end{aligned}$$

here the last equality follows from

$$\lim_{h\to 0}\frac{\sqrt{2}}{2}\sigma\sqrt{h}\cdot\beta_h = \frac{\sqrt{2}}{2}\sigma\cdot\left(-\frac{\sqrt{2}}{\sigma}\right) = -1.$$

This completes the proof. □

6 Summary

In this paper, we have considered a class of optimal stopping problems in which the decision maker does not have total freedom in choosing the stopping times. Instead, an uncontrolled exogenous Poisson process is introduced so that the stopping occurs only if the Poisson process gives a jump.

Different value functions are obtained, depending on whether stopping is allowed at time $t = 0$. These two problems are very closely related. One problem (where no stopping is allowed at $t = 0$) makes it possible to obtain explicit solutions for both problems, while the other problem (where one can stop at $t = 0$) helps in the proof of the verification theorem. It is also interesting to notice that the two value functions are $\mathcal{C}^2$ and $\mathcal{C}^0$ across the optimal exercise boundary, contradicting the usual $\mathcal{C}^1$ fit for optimal stopping problems (smooth-fit-principle). An equivalent stochastic control formulation for the first problem is discussed.

Also studied are the asymptotics of the constrained optimal stopping problem as the intensity of the Poisson process goes to infinity. We find that the optimal exercise boundary converges with rate $\sqrt{\lambda^{-1}}$, and the cost of the constraint is of magnitude λ^{-1} for large λ.

References

[1] BATHER, J.A. & CHERNOFF, H. (1966) Sequential decisions in the control of a spaceship. *Proc. Fifth Berkeley Syposium on Mathematical Statistics and Probability* **3**, 181-207.

[2] BENSOUSSAN, A. (1984) On the theory of option pricing. *Acta Appl. Math.* **2**, 139-158.

[3] BRENNAN, M.J. & SCHWARTZ, E.S. (1985) Evaluating natural rescource investments. *Journal of Business* **58**, 135-157.

[4] BROADIE, M. & GLASSERMAN, P. (1997) Pricing American-style securities using simulation. *J. Econ. Dynam. Control* **21**, 1323-1352.

[5] BRÉMAUD, P. (1981) *Point Processes and Queues: Martingale Dynamics.* Springer-Verlag, New York.

[6] DIXIT, A.K. & PINDYCK, R.S. (1994) *Investment under uncertainty.* Princeton University Press.

[7] KARATZAS, I. & SHREVE, S.E. (1991) *Brownian Motion and Stochastic Calculus.* Springer-Verlag, New York.

[8] JACKA, S.D. (1991) Optimal stopping and the American put. *Math. Finance* **1**, 1-14.

[9] MCDONALD, R. & SIEGEL, D. (1986) The value of waiting to invest. *Quarterly J. Econ.* **101**, 707-728.

[10] MERTON, R.C. (1990) *Continuous-Time Finance.* Cambridge-Oxford: Blackwell.

[11] MYNENI, R. (1992) The pricing of the American option. *Ann. Appl. Probab.* **2**, 1-23.

[12] ROGERS, L.C.G. (2000) A simple model of liquidity effects. *Submitted for publication.*

[13] WANG, H. (2001) Some control problems with random intervention times. To appear in *Adv. Appl. Probab.* **6**.