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Optimal Subspace Techniques for DOA Estimation in the Presence of Noise and Model Errors

M. Jansson[1], B. Ottersten[1], M. Viberg[2], A. L. Swindlehurst[3]

1) Department of Signals, Sensors and Systems
KTH – Royal Institute of Technology, Stockholm, Sweden
2) Department of Signals and Systems
Chalmers University of Technology, Göteborg, Sweden
3) Department of Electrical and Computer Engineering
Brigham Young University, Provo, Utah

Signal parameter estimation and specifically direction of arrival (DOA) estimation for sensor array data is encountered in a number of applications ranging from electronic surveillance to wireless communications. Subspace based methods have shown to provide computationally as well as statistically efficient algorithms for DOA estimation. Estimator performance is ultimately limited by model disturbances such as measurement noise and model errors. Herein, we review a recently proposed framework that allows the derivation of optimal subspace methods taking both finite sample effects (noise) and model perturbations into account. We show how this general estimator reduces to well known techniques for cases when one disturbance dominates completely over the other.

1.1 Introduction

Subspace based techniques have been shown to be powerful tools in many signal processing applications where the observed data consist of low rank signals in noise. Some examples include sensor array signal processing, harmonic retrieval, factor analysis, timing estimation, frequency offset estimation, image processing, system identification, and blind channel identification. By appropriate use of the underlying low rank data model and the associated signal/noise characteristics, subspace estimation techniques can often be made computationally and/or statistically efficient.

This chapter focuses on subspace techniques for direction of arrival (DOA) estimation from data collected by a sensor array. This is quite a mature field of research by now and many tutorial papers and books have been presented, some detailing specific aspects and others giving broader views (e.g., [Krim and Viberg, 1996; Van Trees, 2002], and the references therein). We have no ambition whatsoever to give a comprehensive account of the development of

the field of DOA estimation in this chapter. Rather, we will pursue a specific view of the problem and consider the sensitivity of subspace DOA estimation methods to noise and errors in the array model. Although this may seem to be a very narrow view of the field, it will be argued that many existing DOA estimation techniques can be seen as special cases of the estimator presented later in this chapter.

An important aspect of high resolution DOA estimation is algorithm sensitivity to noise and model errors. Performance of DOA detection and estimation algorithms is ultimately limited by noise in the array measurements and errors in the array model. In many cases the array response is not exactly known and the deviations of the true response from that of the model may severely influence the performance. Indeed, many authors have studied the quantitative effects of model errors on both DOA and signal waveform estimation [Zhu and Wang, 1988; Wong et al., 1988; Swindlehurst and Kailath, 1990; Friedlander, 1990a,b; Swindlehurst and Kailath, 1992, 1993; Li and Vaccaro, 1992; Viberg and Swindlehurst, 1994a; Soon and Huang, 1992; Kangas et al., 1994, 1996; Ramsdale and Howerton, 1980; Compton, 1982; Quazi, 1982; Friedlander and Weiss, 1994; Yang and Swindlehurst, 1995].

If the array response is known to be dependent on some unknown factors, a natural approach is to parameterize the array model not only by the DOA parameters but also by some additional calibration-dependent parameters. These parameters can include, for example, sensor element positions, gain and phase offsets, mutual coupling, element directivity, etc.. Given such a model, it is natural to attempt to estimate the unknown (nuisance) model parameters simultaneously with the signal parameters. This approach is often referred to as auto-calibration [Paulraj and Kailath, 1985; Gustafsson et al., 1996; Rockah and Schultheiss, 1987a,b; Weiss and Friedlander, 1989; Wahlberg et al., 1991; Wylie et al., 1994; Viberg and Swindlehurst, 1994b; Flieller et al., 1995; Swindlehurst, 1996]. Using auto-calibration techniques may however be problematic in certain cases. One obvious reason is that the number of unknown parameters that need to be estimated from the data can be quite large, which may lead to difficulties in numerically calculating the optimal solution. An even more critical issue to consider when auto-calibration techniques are employed is whether or not both the DOA and the model error parameters are simultaneously identifiable. For example, it is clearly not possible to estimate both DOAs and sensor positions simultaneously without the use of additional information [Lo and Marple, 1987; Weiss and Friedlander, 1989; Ng and Nehorai, 1993; Koerber and Fuhrmann, 1993; Yip and Zhou, 1995; McArthur and Reilly, 1994].

An alternative to auto-calibration approaches is to use techniques that assume the model error parameters to be realizations of some underlying random vector with a known a priori distribution. With this modification of the problem formulation in a Bayesian framework, difficulties with parameter identifiability may be alleviated. It also allows a systematic approach to estimator design such as, for example, maximum a posteriori approaches [Viberg and Swindlehurst, 1994b; Wahlberg et al., 1991]. Another way to exploit the assumption of random model error parameters is to, at least implicitly, consider the random model error as an additional noise term in the data model. To reduce the sensitivity of the DOA estimator to the model errors, statistically optimal weighting matrices can then be derived. Methods that follow this latter approach were presented in [Swindlehurst and Kailath, 1992, 1993] which studied MUSIC and subspace fitting methods under the assumption that the estimation errors due to model imperfections dominated the effects of additive noise.

The best performance is achieved by appropriately taking into account both noise and model error effects. Optimally weighted methods treating the combined effects of model errors and noise were presented in [Viberg and Swindlehurst, 1994a; Jansson et al., 1998]. This may be viewed as a pragmatic approach even when the model errors cannot be considered random (which very well may be the typical case) and we will pursue this approach herein. The perturbation parameter covariance matrix can be seen as a design variable reflecting the expected level of parameter variability.

The remainder of this chapter is organized as follows. In the next section, the data model and some fundamental facts are introduced. Section 1.3 presents the general array error model which is used herein along with three explicit examples of typical model errors. In Section 1.4, DOA estimation using subspace techniques is formulated in a generalized signal subspace fitting framework. This framework is then utilized in Section 1.5 to present an optimally weighted DOA estimation method which accounts for both finite sample effects due to the noise and small array model errors. Section 1.5 then continues with a discussion around the performance of this general estimator and how it is related to many existing optimal and suboptimal DOA estimators in different special cases. Finally, the presented results are illustrated in a numerical example in Section 1.6 and the chapter ends with some concluding remarks in Section 1.7.

1.2 Data Model

Assume that the output of an array of m sensors is given by the model

$$\mathbf{x}(t) = \mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\rho})\mathbf{s}(t) + \mathbf{n}(t),$$

where $\mathbf{s}(t) \in \mathbb{C}^{d\times 1}$ contains the emitted signal waveforms and $\mathbf{n}(t) \in \mathbb{C}^{m\times 1}$ is an additive noise vector which is independent of the signal term. The array steering matrix is defined as

$$\mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\rho}) = \begin{bmatrix} \mathbf{a}(\boldsymbol{\theta}_1, \boldsymbol{\rho}) & \dots & \mathbf{a}(\boldsymbol{\theta}_d, \boldsymbol{\rho}) \end{bmatrix},$$

where $\mathbf{a}(\boldsymbol{\theta}_i, \boldsymbol{\rho}) \in \mathbb{C}^{m\times 1}$ denotes the array response to a unit waveform associated with the signal parameter $\boldsymbol{\theta}_i \in \mathbb{R}^{p\times 1}$. We will refer to $\boldsymbol{\theta}_i$ as the direction of arrival (DOA) of the ith signal. The vector $\boldsymbol{\rho} \in \mathbb{R}^{n\times 1}$ contains all additional parameters of the array steering matrix that may be unknown. Examples of typical models for the vector $\boldsymbol{\rho}$ will be considered later. When the array response is a function of the signal parameters only, we simply omit the dependency on $\boldsymbol{\rho}$ and write $\mathbf{A}(\boldsymbol{\theta})$. It is assumed that the array is unambiguous so that the columns in $\mathbf{A}(\boldsymbol{\theta})$ are linearly independent as long as $\boldsymbol{\theta}_i \neq \boldsymbol{\theta}_j,\ i \neq j$.

The signal $\mathbf{s}(t)$ and the noise $\mathbf{n}(t)$ are modeled as zero-mean circular (Gaussian) random vectors with covariances

$$\mathrm{E}\{\mathbf{s}(t)\mathbf{s}^H(s)\} = \mathbf{P}\,\delta_{t,s},$$
$$\mathrm{E}\{\mathbf{n}(t)\mathbf{n}^H(s)\} = \sigma^2\mathbf{I}\,\delta_{t,s},$$

where $(\cdot)^H$ denotes complex conjugate transpose and $\delta_{t,s}$ is the Kronecker delta. Let d' denote the rank of the signal covariance matrix $\mathbf{P}$. Note that $d' < d$ when some of the signals are fully correlated or coherent.

Assuming the signals and the noise to be uncorrelated, the array output covariance matrix is given by

$$\mathbf{R} = \mathrm{E}\{\mathbf{x}(t)\mathbf{x}^H(t)\} = \mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\rho})\mathbf{P}\mathbf{A}^H(\boldsymbol{\theta}, \boldsymbol{\rho}) + \sigma^2\mathbf{I}.$$

The eigendecomposition of $\mathbf{R}$ is

$$\mathbf{R} = \sum_{k=1}^{m} \lambda_k \mathbf{e}_k \mathbf{e}_k^H$$

where $\lambda_1 \geq \cdots \geq \lambda_{d'} > \lambda_{d'+1} = \cdots = \lambda_m = \sigma^2$ are the eigenvalues and $\mathbf{e}_k$ the corresponding eigenvectors. Let $\mathbf{E}_s = \begin{bmatrix} \mathbf{e}_1 & \dots & \mathbf{e}_{d'} \end{bmatrix}$ be the matrix of the *signal* eigenvectors and $\mathbf{E}_n = \begin{bmatrix} \mathbf{e}_{d'+1} & \dots & \mathbf{e}_m \end{bmatrix}$ the matrix of *noise* eigenvectors. The range of $\mathbf{E}_s$ is called the *signal subspace* while the range of $\mathbf{E}_n$ is the *noise subspace.*

From the structure of the covariance matrix model it is clear that

$$\mathcal{R}(\mathbf{E}_s) \subseteq \mathcal{R}(\mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\rho})) \tag{1.1}$$

where $\mathcal{R}(\mathbf{A})$ denotes the range of $\mathbf{A}$. For non-coherent cases when $d' = d$, there is equality in (1.1) and we also have

$$\mathbf{E}_n^H \mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\rho}) = \mathbf{0} \tag{1.2}$$

but note that this does *not* hold when $d' < d$.

The two geometrical facts (1.1)-(1.2) form the basis for all subspace estimation techniques.

From (1.1) it is clear that there exists a full rank matrix $\mathbf{T} \in \mathbb{C}^{d \times d'}$ such that

$$\mathbf{E}_s = \mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\rho})\mathbf{T}. \tag{1.3}$$

For the parameters to be identifiable when using the subspace approach, they have to be uniquely determined from the subspace equation (1.3). In particular, identifiability is guaranteed if

$$\mathbf{A}(\boldsymbol{\theta}_1)\mathbf{T}_1 = \mathbf{A}(\boldsymbol{\theta}_2)\mathbf{T}_2$$

for any two full rank matrices $\mathbf{T}_i \in \mathbb{C}^{d \times d'}$, $i = 1, 2$, implies $\boldsymbol{\theta}_1 = \boldsymbol{\theta}_2$ (with some convention for the ordering of the elements in $\boldsymbol{\theta}_i$) [Wax and Ziskind, 1989]. If the parameters in $\boldsymbol{\rho}$ also need to be estimated, they should naturally also be included in the identifiability condition above.

1.3 Array model errors

When discussing array model errors we will follow the approach mentioned in the introduction which assumes $\boldsymbol{\rho}$ to be a random vector drawn from some distribution. More specifically, the perturbation parameter vector $\boldsymbol{\rho}$ is modeled as a random vector with mean $\mathrm{E}\{\boldsymbol{\rho}\} = \boldsymbol{\rho}_0$ and covariance

$$\mathrm{E}\{(\boldsymbol{\rho} - \boldsymbol{\rho}_0)(\boldsymbol{\rho} - \boldsymbol{\rho}_0)^T\} = \boldsymbol{\Omega}. \tag{1.4}$$

It is assumed that both $\boldsymbol{\rho}_0$ and $\boldsymbol{\Omega}$ are known. Similar to [Viberg and Swindlehurst, 1994a,b; Jansson et al., 1998], we assume that the elements in $\boldsymbol{\Omega}$ are "small" and, hence, consider only small perturbations in $\boldsymbol{\rho}$ around $\boldsymbol{\rho}_0$ to allow a first order perturbation analysis.

Some examples of common array perturbation models are outlined below.

1.3.1 Sensor position errors

Assuming an array composed of identical sensors lying in the same two-dimensional plane as the signals of interest, a general model for the ith element of the array response vector is

$$[\mathbf{a}(\boldsymbol{\theta}_k)]_i = g_i(\boldsymbol{\theta}_k) \exp\left(j\frac{2\pi}{\lambda}[x_i \cos(\boldsymbol{\theta}_k) + y_i \sin(\boldsymbol{\theta}_k)] \right) ,$$

where λ is the carrier wavelength, $g_i(\cdot)$ denotes the gain pattern and (x_i, y_i) denote the position coordinates of the ith sensor. If the sensor positions are not precisely known, then we set

$$\boldsymbol{\rho} = [x_1, y_1, \cdots, x_m, y_m]^T ,$$

letting $\boldsymbol{\rho}_0$ represent the nominal (assumed) vector of sensor positions, and $\boldsymbol{\Omega}$ the covariance matrix of the position errors.

1.3.2 Receiver gain and phase variations

The gain and phase of each sensor's RF receiver front end vary due to a variety of factors, including differences in cable lengths, non-identical components, temperature fluctuations, etc. While to some extent these variations can be calibrated out, there always remain some differences from receiver to receiver. The following model is a simple way of representing these effects:

$$\mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\rho}) = \mathrm{diag}(\boldsymbol{\rho})\mathbf{A}(\boldsymbol{\theta}) ,$$

where $\mathbf{A}(\boldsymbol{\theta})$ is the nominal (calibrated) array response, and $\mathrm{diag}(\boldsymbol{\rho})$ represents a diagonal matrix whose non-zero elements are given by $\boldsymbol{\rho}_i \exp(j\boldsymbol{\rho}_{i+m})$, $i = 1, 2, \ldots, m$. Here, the first m elements of $\boldsymbol{\rho}$ model the gain and the remaining m elements model the phase. Again, $\boldsymbol{\rho}_0$ contains the nominal gain and phase values, and $\boldsymbol{\Omega}$ models the expected variation in the receiver gain and phase. Mutual coupling effects can be modeled using a similar approach, where instead of a diagonal matrix, $\boldsymbol{\rho}$ is used to specify a more complicated matrix with off-diagonal elements that capture the element-to-element coupling.

1.3.3 Generic array perturbations

In many cases, the causes of the array model errors may be many and too complex to model using physical reasoning as in the two cases described

above. One approach that can be employed in such cases is to model $\boldsymbol{\rho}$ implicitly using

$$\tilde{\mathbf{A}} = \mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\rho}) - \mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\rho}_0) \ ,$$

with the statistics of $\tilde{\mathbf{A}}$ directly quantifying the array perturbation. For example, in [Swindlehurst and Kailath, 1993; Viberg and Swindlehurst, 1994a], $\tilde{\mathbf{A}}$ is assumed to be zero mean with covariances given by

$$\mathrm{E}\{\mathrm{vec}(\tilde{\mathbf{A}})\,\mathrm{vec}^H(\tilde{\mathbf{A}})\} = \boldsymbol{\Psi} \otimes \boldsymbol{\Gamma} \tag{1.5}$$

$$\mathrm{E}\{\mathrm{vec}(\tilde{\mathbf{A}})\,\mathrm{vec}^T(\tilde{\mathbf{A}})\} = \mathbf{0}. \tag{1.6}$$

Looking at the ijth block in (1.5), $\mathrm{E}\{\tilde{\mathbf{a}}(\boldsymbol{\theta}_i)\tilde{\mathbf{a}}^H(\boldsymbol{\theta}_j)\} = \boldsymbol{\Psi}_{ij}\boldsymbol{\Gamma}$, it can be seen that this error model assumes the same spatial error covariance matrix $\boldsymbol{\Gamma}$ from sensor to sensor, with a possible DOA dependency modeled via $\boldsymbol{\Psi}_{ij}$.

1.4 DOA estimation and subspace fitting

The subspace relations discussed in Section 1.2 can be used in various ways for the estimation of the model parameters and in particular the DOAs. It has been shown that most approaches quite conveniently can be put into a common subspace fitting framework [Viberg and Ottersten, 1991]. The idea of the signal subspace fitting methods is to minimize a suitable norm of the error between a sample estimate $\hat{\mathbf{E}}_s$ of the signal eigenvector matrix $\mathbf{E}_s$ and the model $\mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\rho})\mathbf{T}$. More precisely, the basic signal subspace fitting criterion is [Viberg and Ottersten, 1991; Ottersten et al., 1993]

$$V(\boldsymbol{\theta}, \boldsymbol{\rho}) = \min_{\mathbf{T}} \ \|\hat{\mathbf{E}}_s - \mathbf{A}(\boldsymbol{\theta}, \boldsymbol{\rho})\mathbf{T}\|^2_{\mathbf{W}} \tag{1.7}$$

where $\mathbf{W}$ denotes a positive definite Hermitian weighting matrix and $\|\mathbf{X}\|^2_{\mathbf{W}} = \mathrm{Tr}(\mathbf{X}\mathbf{W}\mathbf{X}^H)$. The estimates of the parameters are obtained as the minimizing argument of the criterion; i.e.,

$$\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\rho}} = \arg\min_{\boldsymbol{\theta}, \boldsymbol{\rho}} V(\boldsymbol{\theta}, \boldsymbol{\rho}).$$

Here, the possibility of including estimation of the array model parameter vector $\boldsymbol{\rho}$ is explicitly indicated (cf. the discussion about auto-calibration in Section 1.1). However, in the following the focus will be the estimation of the DOA parameters only.

For certain error models, the above weighted norm (1.7) needs to be generalized so that each residual element gets its own weight relative to all the

others. This can be achieved with the more general criterion

$$V(\boldsymbol{\theta}) = \min_{\mathbf{T}} \ \bar{\boldsymbol{\varepsilon}}^H(\boldsymbol{\theta})\mathbf{W}\bar{\boldsymbol{\varepsilon}}(\boldsymbol{\theta}) \tag{1.8}$$

$$\bar{\boldsymbol{\varepsilon}} = \begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon}^* \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \mathrm{vec}(\hat{\mathbf{E}}_s - \mathbf{A}(\boldsymbol{\theta})\mathbf{T})$$

where $\mathbf{W}$ is a positive definite weighting matrix (we use $\mathbf{W}$ as a generic notation for weighting matrices). Above, an extended residual vector $\bar{\boldsymbol{\varepsilon}}$ is obtained by combining the residual $\boldsymbol{\varepsilon}$ and its complex conjugate $\boldsymbol{\varepsilon}^*$. Alternatively, the extended residual vector could have been formed from the real and imaginary parts of $\boldsymbol{\varepsilon}$. Clearly, these two representations yield equivalent results (with appropriate $\mathbf{W}$) since there is an invertible transformation between them.

An alternative DOA estimation criterion can be formulated by utilizing the geometrical relation (1.1) as follows. Let $\mathbf{B}(\boldsymbol{\theta}) \in \mathbb{C}^{m\times(m-d)}$ be a full rank matrix whose columns span the null-space of $\mathbf{A}^H(\boldsymbol{\theta})$. This implies that $\mathbf{B}^H(\boldsymbol{\theta})\mathbf{A}(\boldsymbol{\theta}) = \mathbf{0}$ and $\mathbf{B}^H(\boldsymbol{\theta}_0)\mathbf{E}_s = \mathbf{0}$ where $\boldsymbol{\theta}_0$ denotes the true DOA. Hence, assuming parameter identifiability, the equations $\mathbf{B}^H(\boldsymbol{\theta})\mathbf{E}_s = \mathbf{0}$ determine the true DOA uniquely. Given an estimate $\hat{\mathbf{E}}_s$ of $\mathbf{E}_s$, these equations will not be fulfilled exactly for any $\boldsymbol{\theta}$ and it is reasonable to consider the minimization of a suitable norm of $\mathbf{B}^H(\boldsymbol{\theta})\hat{\mathbf{E}}_s$. Similar to the above, consider

$$V(\boldsymbol{\theta}) = \bar{\boldsymbol{\varepsilon}}^H(\boldsymbol{\theta})\mathbf{W}\bar{\boldsymbol{\varepsilon}}(\boldsymbol{\theta}), \tag{1.9}$$

$$\bar{\boldsymbol{\varepsilon}} = \begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon}^* \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \mathrm{vec}(\mathbf{B}^H(\boldsymbol{\theta})\hat{\mathbf{E}}_s).$$

It can be shown that the estimates obtained by the minimization of the above subspace fitting criteria (1.8) and (1.9) are asymptotically equivalent [Stoica et al., 1997] (cf. [Cardoso and Moulines, 2000]). In other words, for each subspace fitting weighting matrix in (1.8) there is a weighting matrix in (1.9) leading to asymptotically equivalent DOA estimates and vice versa. For this reason, when studying asymptotic equivalence, it is sufficient to study one of these formulations.

However, when attempting to minimize these criteria, some parameterizations can lead to efficient optimization algorithms. Also, the weighting matrices are often parameter as well as data dependent and must be estimated. Estimating the weighting matrices is not equivalent for the two formulations.

Henceforth, we will refer to the above as the generalized weighted subspace fitting (GWSF) method and in particular focus on the second formulation (1.9).

1.5 Special cases of GWSF

The GWSF formulation is in fact very general and many existing subspace DOA estimation methods can be related to GWSF by a proper choice of the weighting matrix.

1.5.1 GWSF for combined noise and array errors

It is well known that, within the class of estimators based on $\bar{\boldsymbol{\varepsilon}}$ in (1.9), the optimal choice of the weighting in terms of minimizing the parameter estimation error variance is

$$\mathbf{W} = \mathbf{C}_{\bar{\boldsymbol{\varepsilon}}}^{-1} \tag{1.10}$$

where $\mathbf{C}_{\bar{\boldsymbol{\varepsilon}}}$ is the asymptotic covariance matrix of the residual vector $\bar{\boldsymbol{\varepsilon}}$ at the true DOA $\boldsymbol{\theta}_0$.

For the combined noise and small array perturbation case (1.4), $\mathbf{C}_{\bar{\boldsymbol{\varepsilon}}}$ can be shown to be [Jansson et al., 1998]

$$\mathbf{C}_{\bar{\boldsymbol{\varepsilon}}} = \bar{\mathbf{L}} + \bar{\mathbf{G}}\bar{\mathbf{G}}^H, \tag{1.11}$$

where

$$\begin{aligned} \bar{\mathbf{L}} &= \begin{bmatrix} \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}^* \end{bmatrix}, & \mathbf{L} &= \left(\sigma^2 \widetilde{\boldsymbol{\Lambda}}^{-2} \boldsymbol{\Lambda}_s \otimes \mathbf{B}^H \mathbf{B}\right), \\ \bar{\mathbf{G}} &= \begin{bmatrix} \mathbf{G} \\ \mathbf{G}^* \end{bmatrix}, & \mathbf{G} &= \left(\mathbf{T}^T \otimes \mathbf{B}^H\right) \mathbf{D}_{\boldsymbol{\rho}} \bar{\boldsymbol{\Omega}}^{1/2}. \end{aligned} \tag{1.12}$$

Here, $\boldsymbol{\Lambda}_s$ is a diagonal matrix containing the d' largest signal eigenvalues of $\mathbf{R}$, $\widetilde{\boldsymbol{\Lambda}} = \boldsymbol{\Lambda}_s - \sigma^2\mathbf{I}$, $\bar{\boldsymbol{\Omega}}^{1/2}$ is a (symmetric) square root of $\bar{\boldsymbol{\Omega}} = N\boldsymbol{\Omega}$, $\mathbf{T} = \mathbf{A}^\dagger \mathbf{E}_s$ where $(\cdot)^\dagger$ denotes the Moore-Penrose pseudoinverse and

$$\mathbf{D}_{\boldsymbol{\rho}} = \left[\frac{\partial \operatorname{vec}(\mathbf{A}(\boldsymbol{\theta},\boldsymbol{\rho}))}{\partial \boldsymbol{\rho}_1} \quad \cdots \quad \frac{\partial \operatorname{vec}(\mathbf{A}(\boldsymbol{\theta},\boldsymbol{\rho}))}{\partial \boldsymbol{\rho}_n}\right]\Big|_{\boldsymbol{\theta}_0,\boldsymbol{\rho}_0}.$$

The covariance matrix (1.11) contains two terms, one due to the noise and another which accounts for the array perturbations. Thus, with some knowledge of the relation between errors due to measurement noise and perturbations in the array model, an optimal trade-off is obtained when forming the weighting matrix. Also, note that the weighting matrix depends on the unknown parameters in $\mathbf{B}$ and $\mathbf{D}_{\boldsymbol{\rho}}$. Fortunately, it is possible to replace the weighting matrix with an estimate thereof without affecting the asymptotic properties of the DOA estimate. More details regarding GWSF for the combined effects of noise and model errors including implementation issues can be found in [Jansson et al., 1998].

In [Wahlberg et al., 1991; Viberg and Swindlehurst, 1994b; Zhu and Wang, 1988] an asymptotically valid Cramér Rao lower bound (CRB) is derived for the problem of interest herein. Below we give the lower bound on the signal parameters only. Assuming that $\hat{\boldsymbol{\theta}}$ is an asymptotically unbiased estimate of $\boldsymbol{\theta}_0$ and that $\boldsymbol{\rho}$ is Gaussian, then for large N and small $\boldsymbol{\Omega}$,

$$\mathrm{E}\{(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)^T\} \;\geq\; \mathbf{CRB}_{\boldsymbol{\theta}} \;\triangleq\; \frac{\sigma^2}{2N}\left[\mathbf{C} - \mathbf{F}_{\boldsymbol{\theta}}^T \boldsymbol{\Upsilon}^{-1} \mathbf{F}_{\boldsymbol{\theta}}\right]^{-1}, \quad (1.13)$$

where

$$\begin{aligned}
\mathbf{C} &= \mathrm{Re}\{\mathbf{D}_{\boldsymbol{\theta}}^H \mathbf{M} \mathbf{D}_{\boldsymbol{\theta}}\},\\
\mathbf{M} &= \mathbf{U}^T \otimes \boldsymbol{\Pi}_{\mathbf{A}}^{\perp},\\
\mathbf{U} &= \mathbf{A}^{\dagger} \mathbf{E}_s \widetilde{\boldsymbol{\Lambda}}^2 \boldsymbol{\Lambda}_s^{-1} \mathbf{E}_s^H \mathbf{A}^{\dagger H}\\
\mathbf{D}_{\boldsymbol{\theta}} &= \left[\frac{\partial \,\mathrm{vec}(\mathbf{A}(\boldsymbol{\theta},\boldsymbol{\rho}))}{\partial \boldsymbol{\theta}_1} \quad \cdots \quad \frac{\partial \,\mathrm{vec}(\mathbf{A}(\boldsymbol{\theta},\boldsymbol{\rho}))}{\partial \boldsymbol{\theta}_d}\right],\\
\mathbf{F}_{\boldsymbol{\theta}} &= \mathrm{Re}\{\mathbf{D}_{\boldsymbol{\rho}}^H \mathbf{M} \mathbf{D}_{\boldsymbol{\theta}}\},\\
\boldsymbol{\Upsilon} &= \mathrm{Re}\{\mathbf{D}_{\boldsymbol{\rho}}^H \mathbf{M} \mathbf{D}_{\boldsymbol{\rho}} + \frac{\sigma^2}{2} \bar{\boldsymbol{\Omega}}^{-1}\}
\end{aligned}$$

and $\boldsymbol{\Pi}_{\mathbf{A}}^{\perp} = \mathbf{I} - \boldsymbol{\Pi}_{\mathbf{A}}$ where $\boldsymbol{\Pi}_{\mathbf{A}} = \mathbf{A}\mathbf{A}^{\dagger}$. The above expressions are evaluated at $\boldsymbol{\theta}_0$ and $\boldsymbol{\rho}_0$.

It is interesting to notice that the CRB for the case with only measurement noise and no calibration errors is $\sigma^2 \mathbf{C}^{-1}/2N$ and clearly is a lower bound for the combined CRB in (1.13) since $\mathbf{F}_{\boldsymbol{\theta}}^T \boldsymbol{\Upsilon}^{-1} \mathbf{F}_{\boldsymbol{\theta}}$ is positive semidefinite.

As shown in [Jansson et al., 1998], the optimally weighted GWSF DOA estimator is consistent and has a limiting zero mean Gaussian distribution with a covariance equal to the CRB matrix in (1.13). Hence, GWSF is a statistically efficient estimator for this quite general estimation problem. The MAP-NSF [Viberg and Swindlehurst, 1994b] and MAPprox [Wahlberg et al., 1991; Jansson et al., 1998] estimators also attain the CRB above. However, GWSF has some advantages compared to those methods, especially for scenarios with coherent or highly correlated emitters. GWSF also allows an efficient polynomial rooting based implementation similar to that of IQML [Bresler and Macovski, 1986] and MODE [Stoica and Sharman, 1990b] for the estimation of the DOAs when the nominal array is uniform and linear (see [Jansson et al., 1998] for details).

In the following sections, two special cases will be studied, namely when either the measurement noise or the model errors dominate. It will be shown that this general estimator will reduce to well known methods in these cases.

1.5.2 GWSF for model errors only

In the previous section, it was assumed that the DOA estimation errors are significantly influenced by both noise effects and array model perturbations. Next we study GWSF for the case when the model errors dominate and neglect the noise effects. In other words, we will study the GWSF criterion when $\sigma^2 \to 0$ (or when $N \to \infty$).

In particular we will study GWSF for the generic array perturbation model discussed in Section 1.3. In the notation from Section 1.5.1, (1.5)-(1.6) become

$$\mathbf{D}_{\boldsymbol{\rho}}\bar{\boldsymbol{\Omega}}\mathbf{D}_{\boldsymbol{\rho}}^H = \boldsymbol{\Psi} \otimes \boldsymbol{\Gamma}$$
$$\mathbf{D}_{\boldsymbol{\rho}}\bar{\boldsymbol{\Omega}}\mathbf{D}_{\boldsymbol{\rho}}^T = \mathbf{0}$$

and the GWSF optimal weighting matrix (1.10),(1.11) reduces to

$$\mathbf{W} = \begin{bmatrix} \mathbf{G}\mathbf{G}^H & \mathbf{0} \\ \mathbf{0} & (\mathbf{G}\mathbf{G}^H)^* \end{bmatrix}^{-1}$$

where $\mathbf{G} = \left(\mathbf{T}^T \otimes \mathbf{B}^H\right)\mathbf{D}_{\boldsymbol{\rho}}\bar{\boldsymbol{\Omega}}^{1/2}$. Hence, the GWSF criterion (1.9),(1.10) simplifies to

$$\begin{aligned} V(\boldsymbol{\theta}) &= 2\,\mathrm{vec}^H(\mathbf{B}^H\hat{\mathbf{E}}_s)[\left(\mathbf{T}^T\boldsymbol{\Psi}\mathbf{T}^* \otimes \mathbf{B}^H\boldsymbol{\Gamma}\mathbf{B}\right)]^{-1}\,\mathrm{vec}(\mathbf{B}^H\hat{\mathbf{E}}_s) \\ &= 2\,\mathrm{Tr}\{\hat{\mathbf{E}}_s^H\mathbf{B}(\mathbf{B}^H\boldsymbol{\Gamma}\mathbf{B})^{-1}\mathbf{B}^H\hat{\mathbf{E}}_s(\mathbf{T}^H\boldsymbol{\Psi}^T\mathbf{T})^{-1}\} \\ &= 2\,\mathrm{Tr}\{\boldsymbol{\Pi}^{\perp}_{\boldsymbol{\Gamma}^{-1/2}\mathbf{A}}\boldsymbol{\Gamma}^{-1/2}\hat{\mathbf{E}}_s(\mathbf{T}^H\boldsymbol{\Psi}^T\mathbf{T})^{-1}\hat{\mathbf{E}}_s^H\boldsymbol{\Gamma}^{-1/2}\} \end{aligned}$$

where in the last equality we used the fact that $\mathbf{B}^H\boldsymbol{\Gamma}^{1/2}\boldsymbol{\Gamma}^{-1/2}\mathbf{A} = \mathbf{0}$ and hence that

$$\boldsymbol{\Pi}_{\boldsymbol{\Gamma}^{1/2}\mathbf{B}} = \boldsymbol{\Pi}^{\perp}_{\boldsymbol{\Gamma}^{-1/2}\mathbf{A}}.$$

The above simplified expression of the GWSF criterion function corresponds exactly to the criterion function used by the robust subspace fitting (RSF) method in [Swindlehurst and Kailath, 1993]. As shown in [Swindlehurst and Kailath, 1993], RSF is an optimally weighted subspace fitting method for the studied generic array response error model.

The RSF perturbation model is in a sense "non-parametric," and is probably most suitable for a case where the array response is measured by a calibration procedure. The GWSF formulation on the other hand allows for general parameterizations of the model error, e.g., in terms of physical quantities.

1.5.3 GWSF with no model errors

Here it is shown that the optimally weighted GWSF simplifies to the WSF method [Viberg and Ottersten, 1991] when no model errors are taken into account. For this case, in (1.12) $\bar{\mathbf{\Omega}} = \mathbf{0}$ and the GWSF criterion (1.9),(1.10),(1.11) can be re-written as

$$\begin{aligned} V(\boldsymbol{\theta}) &= \bar{\boldsymbol{\varepsilon}}^H(\boldsymbol{\theta}) \begin{bmatrix} \mathbf{L}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{L}^{-*} \end{bmatrix} \bar{\boldsymbol{\varepsilon}}(\boldsymbol{\theta}) = 2\,\mathrm{vec}^H(\mathbf{B}^H\hat{\mathbf{E}}_s)\mathbf{L}^{-1}\,\mathrm{vec}(\mathbf{B}^H\hat{\mathbf{E}}_s) \\ &= 2\,\mathrm{vec}^H(\mathbf{B}^H\hat{\mathbf{E}}_s)\left(\sigma^{-2}\widetilde{\mathbf{\Lambda}}^2\mathbf{\Lambda}_s^{-1} \otimes (\mathbf{B}^H\mathbf{B})^{-1}\right)\mathrm{vec}(\mathbf{B}^H\hat{\mathbf{E}}_s) \\ &= 2\,\mathrm{vec}^H(\mathbf{B}^H\hat{\mathbf{E}}_s)\,\mathrm{vec}((\mathbf{B}^H\mathbf{B})^{-1}\mathbf{B}^H\hat{\mathbf{E}}_s\sigma^{-2}\widetilde{\mathbf{\Lambda}}^2\mathbf{\Lambda}_s^{-1}) \\ &= 2\,\mathrm{Tr}(\hat{\mathbf{E}}_s^H\mathbf{B}(\mathbf{B}^H\mathbf{B})^{-1}\mathbf{B}^H\hat{\mathbf{E}}_s\sigma^{-2}\widetilde{\mathbf{\Lambda}}^2\mathbf{\Lambda}_s^{-1}) \\ &= \frac{2}{\sigma^2}\,\mathrm{Tr}(\mathbf{\Pi}_{\mathbf{A}}^{\perp}\hat{\mathbf{E}}_s\mathbf{W}_{\mathrm{WSF}}\hat{\mathbf{E}}_s^H) \end{aligned} \tag{1.14}$$

where $\mathbf{W}_{\mathrm{WSF}} = \widetilde{\mathbf{\Lambda}}^2\mathbf{\Lambda}_s^{-1}$. Above, we again used the fact that $\mathbf{B}$ spans the nullspace of $\mathbf{A}$ and, hence, $\mathbf{B}(\mathbf{B}^H\mathbf{B})^{-1}\mathbf{B}^H = \mathbf{\Pi}_{\mathbf{B}} = \mathbf{\Pi}_{\mathbf{A}}^{\perp}$. Equation (1.14) is exactly $2/\sigma^2$ times the WSF criterion as introduced in [Viberg and Ottersten, 1991]. It is known that WSF is asymptotically statistically efficient and, hence, GWSF will also be efficient.

In the WSF paper [Viberg and Ottersten, 1991] it was shown that several known methods can be viewed as special cases of the subspace fitting formulation. In particular, it was shown that DML [Böhme, 1984], MD-MUSIC [Schmidt, 1981; Roy, 1987; Cadzow, 1988], TLS-ESPRIT [Roy et al., 1986; Paulraj et al., 1986; Roy and Kailath, 1989], and ML-ESPRIT [Roy, 1987] all have the same asymptotic performance as certain members of the signal subspace fitting (SSF) family of methods (obtained by choosing different weighting matrices in the SSF criterion (1.7)). Since SSF is a special case of GWSF, weighting matrices can be chosen so that GWSF is asymptotically equivalent to the above mentioned methods as well.

To see the connection between (one-dimensional spectral) MUSIC [Schmidt, 1979, 1981] and subspace fitting, consider the following subspace fitting criterion

$$V(\theta) = \min_{\mathbf{T}} \|\hat{\mathbf{E}}_s - \mathbf{a}(\theta)\mathbf{T}\|_F^2 = \mathrm{Tr}\{(\mathbf{I} - \mathbf{a}(\mathbf{a}^H\mathbf{a})^{-1}\mathbf{a}^H)\hat{\mathbf{E}}_s\hat{\mathbf{E}}_s^H\}$$

where the dependence on θ was omitted for simplicity and where $\|\mathbf{X}\|_F^2 =$

$\mathrm{Tr}\{\mathbf{X}\mathbf{X}^H\}$ is the squared Frobenius norm. Next notice that

$$\begin{aligned}
&\min_{\theta} \mathrm{Tr}\{(\mathbf{I} - \mathbf{a}(\mathbf{a}^H\mathbf{a})^{-1}\mathbf{a}^H)\hat{\mathbf{E}}_s\hat{\mathbf{E}}_s^H\} \\
&\quad = \max_{\theta} \mathrm{Tr}\{\mathbf{a}(\mathbf{a}^H\mathbf{a})^{-1}\mathbf{a}^H\hat{\mathbf{E}}_s\hat{\mathbf{E}}_s^H\} = \min_{\theta} \mathrm{Tr}\{\mathbf{a}(\mathbf{a}^H\mathbf{a})^{-1}\mathbf{a}^H\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^H\} \\
&\quad = \min_{\theta} \frac{\mathbf{a}^H\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^H\mathbf{a}}{\mathbf{a}^H\mathbf{a}} = \max_{\theta} \frac{\mathbf{a}^H\mathbf{a}}{\mathbf{a}^H\hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^H\mathbf{a}}
\end{aligned}$$

since $\hat{\mathbf{E}}_s\hat{\mathbf{E}}_s^H = \mathbf{I} - \hat{\mathbf{E}}_n\hat{\mathbf{E}}_n^H$. The last expression involves the well known MUSIC pseudo-spectrum. Clearly, MUSIC can be viewed as a (un-weighted) one-dimensional subspace fitting method where the different DOAs are found by locating the d smallest local minima.

1.6 Numerical simulation example

In this section, the performance of different DOA estimators are illustrated by means of a simulation example involving DOA estimation errors due to both noise and array model imperfections. Consider a uniform linear array of $m = 6$ sensors separated by a half wavelength. Two signals impinge from the directions 0° and 10° relative to broadside. The signals are uncorrelated and of equal power. The sample size is fixed to $N = 200$ while the signal to noise ratio (SNR) is varied. The nominal unit gain sensors are perturbed by additive Gaussian random variables with a standard deviation of 5%. The nominal phases of the sensors are also perturbed by adding uncorrelated Gaussian random variables with standard deviations 0.05. This phase error corresponds approximately to a direction error of 1° around broadside.

Fig. 1.1 shows the root-mean-square (RMS) errors versus the SNR. Only the RMS errors for $\boldsymbol{\theta}_1$ are displayed; the results corresponding to $\boldsymbol{\theta}_2$ are similar. The empirical RMS values are computed from 1000 independent Monte Carlo trials. The DOA estimators included in the comparison are: Root-MUSIC [Barabell, 1983; Schmidt, 1979], ESPRIT (using maximum overlap sub-arrays) [Roy et al., 1986; Paulraj et al., 1986; Roy and Kailath, 1989], WSF [Viberg and Ottersten, 1991] and GWSF [Jansson et al., 1998] (see Section 1.5.1). The WSF and GWSF methods are implemented in their "rooting versions" [Jansson et al., 1998; Stoica and Sharman, 1990b].

The approximate CRB given by (1.13) that accounts both for array model errors and the noise is denoted MAP-CRB in Fig. 1.1. The CRB for the ideal nominal case without array model errors is denoted NOM-CRB (i.e., the CRB accounting only for the additive noise, see e.g. [Stoica and Nehorai, 1990]). Also included in Fig. 1.1 is a curve resulting from a theoretical

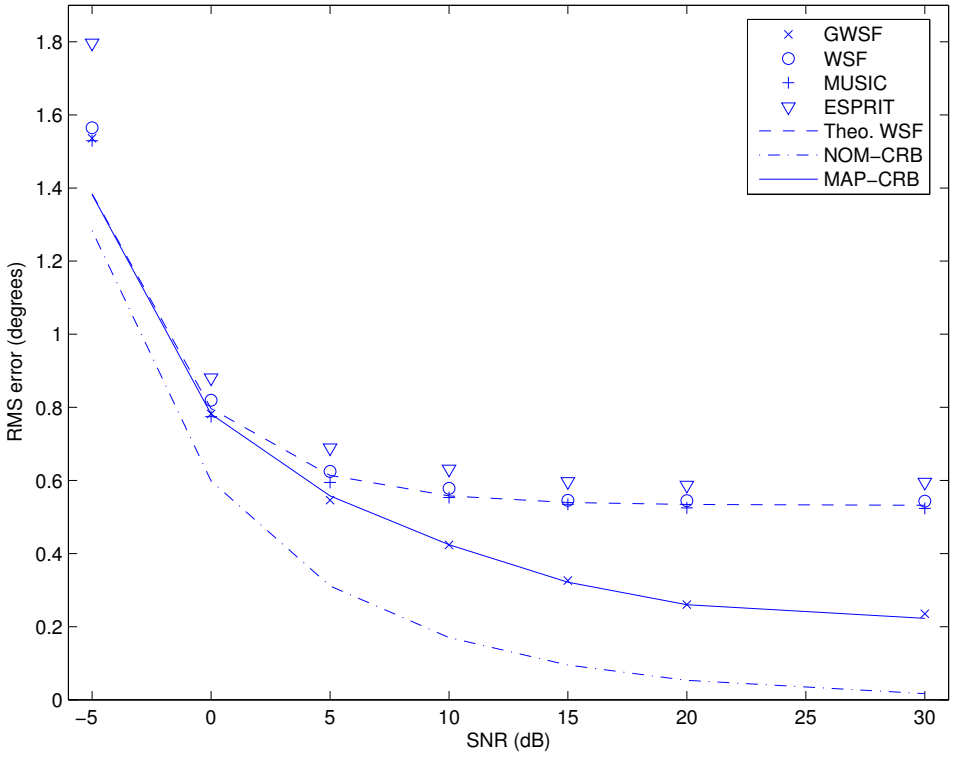

Fig. 1.1. The RMS errors for $\boldsymbol{\theta}_1$ versus the SNR.

performance analysis of WSF for the case under study (i.e., for small model errors and large N or high SNR) [Viberg and Swindlehurst, 1994b; Jansson et al., 1998].

In Fig. 1.1 it can be seen that for lower SNRs, when the measurement noise is the dominating error source, all methods have a very similar performance in this scenario since the emitter signals are uncorrelated. For higher SNRs, the gain and phase errors dominate and, as expected, the GWSF method outperforms the other methods that do not take the model errors into account. It can also be seen that the RMS error of GWSF attains the MAP-CRB as predicted by theory. Similarly, the theoretical RMS curve for WSF predicts the corresponding empirical RMS values very well.

1.7 Concluding remarks

The subspace approach to signal parameter estimation has been successful in providing high accuracy at a reasonable cost. This chapter has reviewed

a class of optimal subspace methods, which maintains the asymptotic performance of maximum likelihood and similar techniques. The price for the statistical efficiency is generally that the complexity approaches that of maximum likelihood. However, in special cases computationally more attractive implementations are available, see e.g. [Stoica and Sharman, 1990a; Jansson et al., 1998]. We have also shown how the original MUSIC algorithm can be obtained from the general GWSF formulation. The GWSF framework can therefore be said to provide a clear link between classical statistical estimation and a wide class of practically useful subspace-based methods. Similar ideas as those presented here for the direction estimation problem can also be used in other related applications, e.g., frequency estimation [Eriksson et al., 1994; Kristensson et al., 2001], subspace based system identification [Van Overschee and De Moor, 1996; Verhaegen, 1994; Viberg et al., 1997], and blind channel identification [Moulines et al., 1995; Kristensson and Ottersten, 1998].

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