Optimal Surrender Policy for Variable Annuity Guarantees

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Outline

Variable annuities and financial options

- Variable annuities: similar to mutual funds with additional guarantees
- VA guarantees paid for via a fixed fee rate throughout the life of the contract (not paid upfront)
  - Decreases return on fund
  - Impacts value of option
- VA contract can be surrendered (Knoller, Kraut, and Schoenmaekers (2011))
  - Financial needs
  - Higher costs of opportunity
  - Moneyness of the option

Continuous fee and the surrender option

- Simple payoff at maturity:

$$\max(G, F_T) = F_T + (G - F_T)^+$$

- Option paid by continuous fee set as percentage of fund:
  - Fee is low when option value is high
  - *Incentive to surrender* when fund value is high

- Surrender region: surrender benefit higher than payoff expected if contract is kept

Setting

Index value S_t:

$$\frac{dS_t}{S_t} = r\ dt + \sigma dW_t$$

Account value F_t based on index:

$$\frac{dF_t}{F_t} = (r-c)dt + \sigma dW_t$$

$$\Rightarrow F_t|F_s \sim \mathcal{LN}(\log(F_t) + (r-c-\frac{\sigma^2}{2})(t-s), \sigma^2(t-s))$$

Contract

- Accumulation benefit:

$$\max(F_T, G), \qquad G = F_0 e^{gT}, g < r$$

- Surrender benefit:

$$e^{-\kappa(T-t)} F_t$$

Integral representation for surrender option

- Similarities between surrender option and American option
- Can use techniques developed for American options (Kim (1990), Kim and Yu (1996), Carr, Jarrow, and Myneni (1992), Wu and Fu (2003))
- Integral can be obtained in different ways:
  - Finite number of surrender times (as in Kim (1990))
  - No-arbitrage arguments (as in Kim and Yu (1996))

Trading strategy for surrender option

- Confirm that optimal strategy is a **threshold** strategy
- **Hold the VA** below the surrender boundary B_t
- When F_t crosses B_t from below, **sell the VA** and invest the proceeds
- When F_t crosses B_t from above, use the investment to **buy the VA**
- Payoff of portfolio is payoff of VA

Gain from surrender

Proposition

The benefit associated with the surrender option between $[t, t + dt]$ *for an infinitesimal time step* dt *is given by*

$$e^{-\kappa(T-t)}(c-\kappa)F_t dt$$

Proof of Proposition

- Suppose surrender at t.
- Policyholder receives

$$e^{-\kappa(T-t)}F_t = e^{-\kappa(T-t)}e^{-ct}S_t = e^{-\kappa T}e^{-(c-\kappa)t}S_t$$

- To buy the VA at time $t + dt$, policyholder only needs

$$e^{-\kappa(T-(t+dt))}F_{t+dt} = e^{-\kappa T}e^{-(c-\kappa)(t+dt)}S_{t+dt}$$

- Investment made at t becomes

$$\begin{aligned} &e^{-\kappa T}e^{-(c-\kappa)(t+dt)}S_{t+dt} + e^{-\kappa T}e^{-(c-\kappa)t}S_t e^{rdt}(1 - e^{-(c-\kappa)dt}) \\ &= e^{-\kappa(T-(t+dt))}F_{t+dt} + e^{-\kappa(T-t)}(c-\kappa)F_t dt + o(dt) \end{aligned}$$

Price of VA with surrender

Price of VA contract

$$V(t,F_t) = \underbrace{E[e^{-rt}\max(G,F_T)|\mathcal{F}_t]}_{\text{Maturity benefit}} + \underbrace{e^{-\kappa T}(c-\kappa)F_t e^{ct}\int_t^T e^{-(c-\kappa)u}\Phi\big(d_1(F_t,B_u,u,t)\big)du}_{\text{Surrender option}}$$

- Maturity benefit: Similar to vanilla option under Black-Scholes
- Surrender option:

$$\int_t^T e^{-r(u-t)}\int_{B_u}^{\infty} e^{-\kappa(T-u)}(c-\kappa)x f_{F_u}(x|F_t)\,dx\,du.$$

Optimal exercise boundary condition

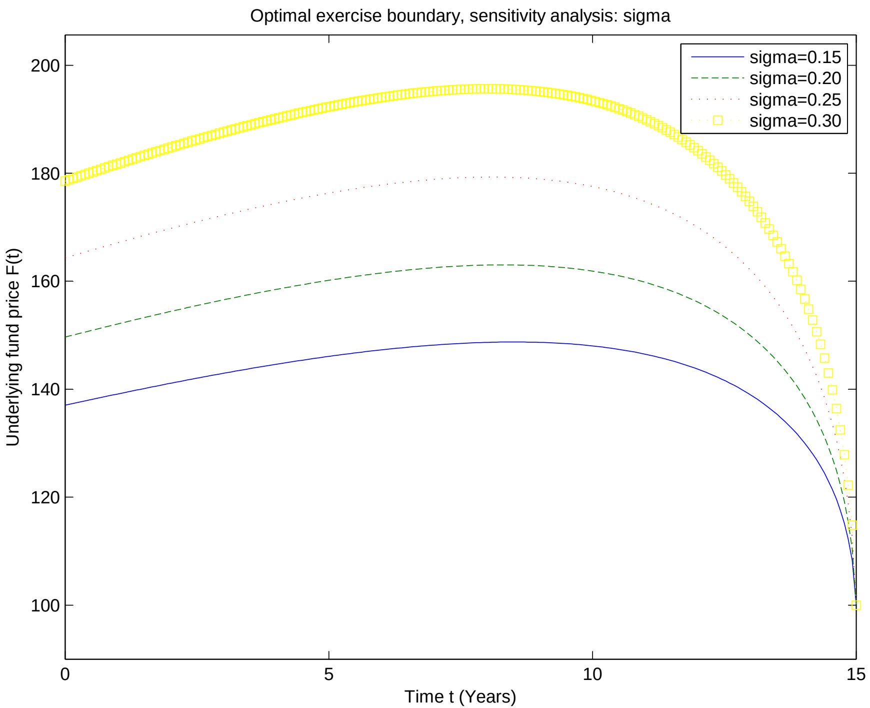
- At maturity $B_T = G_T$ and along the surrender boundary, $V(F_t, t) = e^{-\kappa(T-t)} F_t = B_t$.
- Work backwards to solve for B_t

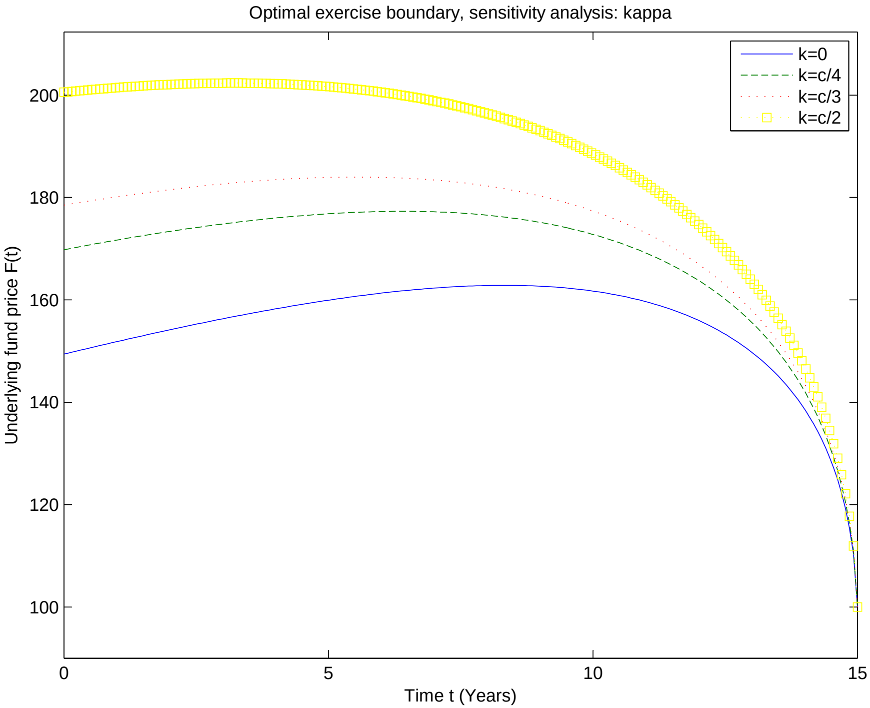
$$
\begin{aligned}
B_t =& v(F_t, t) + e(F_t, t) \\
=& e^{-c(T-t)} B_t e^{\kappa(T-t)} \Phi(d_1(B_t e^{\kappa(T-t)}, G_T, T, t)) \\
& + e^{-r(T-t)} G_T \Phi(d_2(B_t e^{\kappa(T-t)}, G_T, T, t)) \\
& + (c-\kappa) B_t e^{(c-\kappa)t} \int_t^T e^{-(c-\kappa)u} \Phi(d_1(B_t e^{\kappa(T-t)}, B_u, u, t)) du
\end{aligned}
$$

Numerical example

Assumptions:

- $T = 15$
- $G = F_0 = 100$
- $\kappa = 0$, unless otherwise indicated
- $c = 0.91\%$ (fair fee for maturity benefit)
- $r = 0.03$
- $\sigma = 0.2$, unless otherwise indicated


Optimal exercise boundary, sensitivity analysis: sigma
sigma=0.15
sigma=0.20
sigma=0.25
sigma=0.30
Underlying fund price F(t)
Time t (Years)
200
180
160
140
120
100
0
5
10
15



Optimal exercise boundary, sensitivity analysis: kappa
k=0
k=c/4
k=c/3
k=c/2
200
180
160
140
120
100
Underlying fund price F(t)
0
5
10
15
Time t (Years)

Geometric average

- Consider the payoff $\max(G_T, Y_T)$, where Y_T is the geometric average defined as

$$Y_t = \exp\left(\frac{1}{t}\int_0^t \ln F_s ds\right)$$

- The conditional distribution of $Y_u|(Y_t, F_t)$ for $u > t$ is again log-normal with mean and variance given by

$$M_t^g = \frac{t}{u}\ln Y_t + \frac{u-t}{u}\ln F_t + \frac{r - c - \frac{\sigma^2}{2}}{2u}(u-t)^2$$
$$V_t^g = \frac{\sigma^2}{3u^2}(u-t)^3$$

Pricing formula

Theorem

Let $V^g(Y_t, F_t, t)$ denote the price at time t of the VA with guarantee G_T and a surrender benefit equal to $e^{-\kappa(T-t)}Y_t$. Then $V^g(Y_t, F_t, t)$ can be decomposed into a European part $v^g(Y_t, F_t, t)$ and an early exercise premium $e^g(Y_t, F_t, t)$

$$V^g(Y_t, F_t, t) = v^g(Y_t, F_t, t) + e^g(Y_t, F_t, t),$$

where

$$v^g(Y_t, F_t, t) = e^{-r(T-t)} e^{M_t^g + \frac{V_t^g}{2}} \Phi\left(\frac{-\ln(G_T) + M_t^g + V_t^g}{\sqrt{V_t^g}}\right) + e^{-r(T-t)} G_T \Phi\left(\frac{\ln(G_T) - M_t^g}{\sqrt{V_t^g}}\right),$$

$$e^g(Y_t, F_t, t) = e^{-\kappa T} e^{rt} \int_t^T e^{u(\kappa - r)} e^{\frac{\widehat{V}_{u,t}}{2}} Y_t^{\frac{t}{u}} F_t^{\frac{u-t}{2u}} \mathbb{E}\left[k(u, F_u, t)\right] du$$

Particularities of the path-dependent case

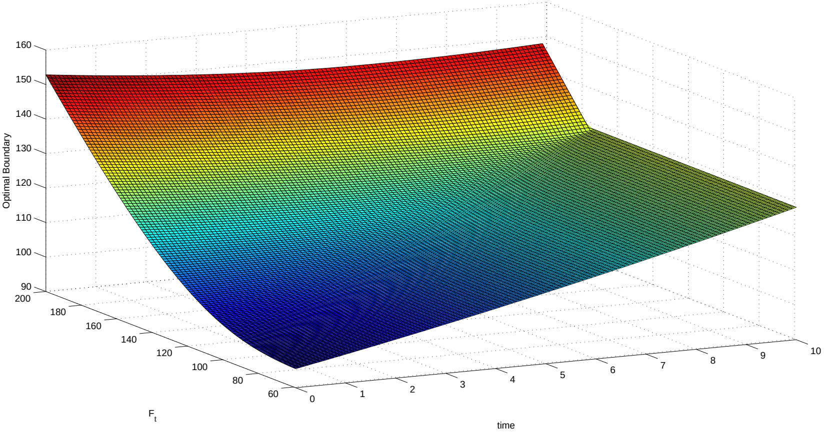
- Optimal surrender behaviour depends on account value F_t and geometric average Y_t.
  $\Rightarrow$ Optimal surrender surface
- To solve for optimal surrender surface, need to consider many values F_t at each time t.
- To simplify calculations, assume that $B_t(F_t)$ has the form

$$B_t(F_t) = \max(G_T e^{-r(T-t)}, a_t + b_t F_t)$$

Numerical example

Additional assumptions:

- $T = 10$
- Payoff: $\max(Y_T, G_T)$
- $G_T = F_0 e^{0.025T}$
- $c = 0.0197$



160
150
140
130
120
110
100
90
Optimal Boundary
200
180
160
140
120
100
80
60
F_t
0
1
2
3
4
5
6
7
8
9
10
time

Conclusion

- Integral representation for the surrender option
- Can retrieve optimal surrender boundary
- Can be used for path independent and path dependent payoffs

Future work:

- Use for other types of fee structures
- Consider flexible premium (as in Chi and Lin (2013))

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Thank you for your attention!