J. DIFFERENTIAL GEOMETRY
80 (2008) 501-523

OPTIMAL TEST-CONFIGURATIONS FOR TORIC VARIETIES

Gábor Székelyhidi

Abstract

On a K-unstable toric variety we show the existence of an optimal destabilising convex function. We show that if this is piecewise linear then it gives rise to a decomposition into semistable pieces analogous to the Harder-Narasimhan filtration of an unstable vector bundle. We also show that if the Calabi flow exists for all time on a toric variety then it minimizes the Calabi functional. In this case the infimum of the Calabi functional is given by the supremum of the normalized Futaki invariants over all destabilising test-configurations, as predicted by a conjecture of Donaldson.

1. Introduction

The Harder-Narasimhan filtration of an unstable vector bundle is a canonical filtration with semistable quotient sheaves. It arises for example when computing the infimum of the Yang-Mills functional (see Atiyah-Bott [**2**]), which is analogous to the Calabi functional on a Kähler manifold. Bruasse and Teleman [**3**] have shown that the Harder-Narasimhan filtration arises in other moduli problems as well, when one looks at the optimal destabilising one-parameter subgroup for a non-semistable point. The notion of optimal one-parameter subgroups is well known in geometric invariant theory, see for example Kirwan [**16**].

In the meantime, much progress has been made in studying the stabiliy of manifolds in relation to the existence of canonical metrics. Such a relationship was originally conjectured by Yau [**21**] in the case of Kähler-Einstein metrics. Tian [**20**] and Donaldson [**10**], [**11**] made great progress on this problem, and by now there is a large relevant literature. For us the important work is [**11**], through which we have a good understanding of stability for toric varieties (for further work on toric varieties see also [**12**], [**9**]). In particular, we can construct a large family of test-configurations, which are analogous to one-parameter subgroups, in terms of data on the moment polytope. In this paper we use this to study the optimal destabilising test-configuration on an unstable toric

Received 06/14/2007.

variety and the Harder-Narasimhan type decomposition that it gives rise to.

Recall that a compact polarised toric variety (X, L) corresponds to a polytope $P \subset \mathbf{R}^n$, which is equipped with a canonical measure $d\sigma$ on the boundary ∂P (for details see Section 2). We also let $d\mu$ denote the Lebesgue measure on the interior of P, and write $\hat{S}$ for the quotient $\mathrm{Vol}\,(\partial P, d\sigma)/\mathrm{Vol}\,(P, d\mu)$. This is essentially the average scalar curvature of metrics on the toric variety. Let us define the functional

$$\mathcal{L}(f) = \int_{\partial P} f\, d\sigma - \hat{S} \int_P f\, d\mu,$$

which by the choice of $\hat{S}$ vanishes on constant functions. Donaldson shows that given a rational piecewise linear convex function f on P, one can define a test-configuration for (X, L) with generalised Futaki invariant $\mathcal{L}(f)$ (if we scale the Futaki invariant in the right way). We will say that the toric variety is unstable if for some convex function f we have $\mathcal{L}(f) < 0$. The natural norm for the test-configuration is given by the L^2-norm of f at least if we consider f with zero mean. This means that the optimal destabilizing test-configuration we are looking for in the unstable case should minimize the functional

$$W(f) = \frac{\mathcal{L}(f)}{\|f\|_{L^2}},$$

defined for non-zero convex functions in the space $\mathcal{C}_1 \cap L^2(P)$. Here $\mathcal{C}_1$ is the set of continuous convex functions on P^*, integrable on ∂P, where P^* is the union of P and the interiors of its codimension one faces. This ensures that $\mathcal{L}(f)$ is finite. Note that the minimum will be negative and the minimiser automatically has zero mean. Our first result in Section 3 is

Theorem 5. *Let the toric variety with moment polytope P be unstable. Then there exists a convex minimiser $\Phi \in \mathcal{C}_1 \cap L^2(P)$ for W which is unique up to scaling. Let us fix the scaling by requiring that*

$$\mathcal{L}(\Phi) = -\|\Phi\|_{L^2}^2.$$

Letting $B = \hat{S} - \Phi$, we then have $\mathcal{L}_B(f) \geqslant 0$ for all convex functions f, and $\mathcal{L}_B(\Phi) = 0$. Conversely these two conditions characterise Φ.

Here we define

$$\mathcal{L}_B(f) = \int_{\partial P} f\, d\sigma - \int_P Bf\, d\mu.$$

Note that Φ would only define a test-configuration if it were piecewise linear. This is not known and perhaps not true in general so instead we may think of Φ as a limit of test-configurations. The proof is based on a compactness theorem for convex functions in $\mathcal{C}_1$ due to Donaldson.

We also give an alternative description of the optimal destabiliser:

Theorem 9. *Consider the set $E \subset L^2(P)$ defined by*

$$E = \{h \in L^2 \mid \mathcal{L}_h(f) \geqslant 0 \text{ for all convex } f\}.$$

If Φ is the optimal destabilising convex function we found above, then $B = \hat{S} - \Phi$ is the unique minimiser of the L^2 norm for functions in E.

The above two results show that

$$\inf_{h\in E} \|h - \hat{S}\|_{L^2} = \sup_{f \text{ convex}} \frac{-\mathcal{L}(f)}{\|f\|_{L^2}}. \tag{1}$$

In view of a conjecture of Donaldson's in [**11**] (see Conjecture 4 in the next section), one can think of E as the closure in L^2 of the possible scalar curvature functions of torus invariant metrics on the toric variety. Thus Equation (1) should be compared to another conjecture of Donaldson's (see [**13**]) saying that the infimum of the Calabi functional is given by the supremum of the normalized Futaki invariants over all test-configurations. Recall that the Calabi functional is defined to be the L^2-norm of $S(\omega) - \hat{S}$ where $S(\omega)$ is the scalar curvature of a Kähler metric ω and $\hat{S}$ is its average. In our toric setting this conjecture is

Conjecture 1. For a polarised toric variety (X, L) we have

$$\inf_{\omega\in c_1(L)} \|S(\omega) - \hat{S}\|_{L^2} = \sup_{f \text{ convex}} \frac{-\mathcal{L}(f)}{\|f\|_{L^2}},$$

where f runs over convex functions in $\mathcal{C}_1 \cap L^2(P)$ on the moment polytope P.

Instead of trying to show that Conjecture 4 implies this conjecture, we will show in Section 5 that it holds if the Calabi flow exists for all time.

In Section 4 we show that if the optimal convex function Φ that we found above is piecewise linear, then we obtain a canonical decomposition of the polytope into semistable pieces, i.e., an analogue of the Harder-Narasimhan filtration. The pieces are given by the maximal subpolytopes on which Φ is linear. For the precise statement see Theorem 14. When Φ is not piecewise linear then in the same way it defines a decomposition into infinitely many pieces. We discuss the conjectured relationship between these decompositions and the Calabi flow.

In the final Section 5 we study the Calabi flow on a toric variety. This is a fourth order parabolic flow in a fixed Kähler class defined by

$$\frac{\partial \phi_t}{\partial t} = S(\omega_t),$$

where $\omega_t = \omega + i\partial\overline{\partial}\phi_t$ is a path of Kähler metrics and $S(\omega_t)$ is the scalar curvature. It was introduced by Calabi in [**4**] in order to find extremal Kähler metrics. It is known that the flow exists for a short time (see Chen-He [**7**]), but the long time existence has only been shown in special cases. For the case of Riemann surfaces see Chruściel [**8**] (and also [**6**] and [**17**]). For ruled manifolds, restricting to metrics of cohomogeneity one see [**14**]. For general Kähler manifolds long time existence has been shown in [**7**], assuming that the Ricci curvature remains bounded.

Under the assumption that it exists for all time, we show that the Calabi flow minimizes the Calabi functional. More precisely, we show

Theorem 17. *Suppose that u_t is a solution of the Calabi flow for all $t \in [0, \infty)$. Then*

$$\lim_{t\to\infty} \|S(u_t) - \hat{S} + \Phi\|_{L^2} = 0,$$

where Φ is the optimal destabilising convex function from Theorem 5. *Moreover,*

$$\|\Phi\|_{L^2} = \inf_{u\in\mathcal{S}} \|S(u) - \hat{S}\|_{L^2}.$$

Here the u_t are symplectic potentials on the polytope defining torus invariant metrics on the toric variety. It follows from this result that existence of the Calabi flow for all time implies Conjecture 1. The proof of the result relies on studying the behavior of some functionals introduced in [**11**] generalizing the well known Mabuchi functional, and is similar to a previous result by the author on ruled surfaces (see [**18**]).

Acknowledgements. I would like to thank Simon Donaldson, Dimitri Panov, Jacopo Stoppa and Valentino Tosatti for helpful conversations. I am also grateful for the referee's helpful suggestions. This work was supported by an EPSRC Postdoctoral Fellowship.

2. Preliminaries

In this section we present some of the definitions and results following Donaldson [**11**] that we will need in the paper. We first describe how to write metrics on a toric variety in terms of symplectic potentials (see Guillemin [**15**]). Let (X, L) be a polarized toric variety of dimension n. There is a dense free open orbit of $(\mathbf{C}^*)^n$ inside X which we denote by X_0. Let us choose complex coordinates $w_1, \ldots, w_n \in \mathbf{C}^*$. On the covering space $\mathbf{C}^n$ we have coordinates $z_i = \log w_i = \xi_i + \sqrt{-1}\eta_i$. A $T^n = (S^1)^n$-invariant metric on $\mathbf{C}^n$ can be written as $\omega = 2i\overline{\partial}\partial\phi$ where ϕ is a function of $\xi_1, \ldots, \xi_n$. This means that

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j} \frac{\partial^2 \phi}{\partial\xi_i \partial\xi_j} dz_i \wedge d\overline{z}_j,$$

so we need ϕ to be strictly convex.

The T^n action on $\mathbf{C}^n$ is Hamiltonian with respect to ω and has moment map

$$m(z_1, \ldots, z_n) = \left(\frac{\partial\phi}{\partial\xi_i}\right).$$

If ω compactifies to give a metric representing the first Chern class $c_1(L)$ then the image of m is an integral polytope $P \subset \mathbf{R}^n$. The *symplectic potential* of the metric is defined to be the Legendre transform of ϕ: for $\underline{x} \in P$ there is a unique point $\underline{\xi} = \underline{\xi}(\underline{x}) \in \mathbf{R}^n$ where $\frac{\partial \phi}{\partial \xi_i} = x_i$, and the Legendre transform u of ϕ is

$$u(\underline{x}) = \sum_i x_i \xi_i - \phi(\underline{\xi}). \tag{2}$$

This is a strictly convex function and the metric in the coordinates x_i, η_i is given by

$$u_{ij} dx^i dx^j + u^{ij} d\eta^i d\eta^j, \tag{3}$$

where u^{ij} is the inverse of the Hessian matrix u_{ij}.

It is important to study the behavior of u near the boundary of P. Suppose that P is defined by linear inequalities $h_k(x) > c_k$, where each h_k induces a primitive integral function $\mathbf{Z}^n \to \mathbf{Z}$. Write $\delta_k(x) = h_k(x) - c_k$ and define the function

$$u_0(x) = \sum_k \delta_k(x) \log \delta_k(x),$$

which is a continuous function on $\overline{P}$, smooth in the interior. It turns out that the boundary behavior of u_0 models the required boundary behavior for a symplectic potential u to give a metric on X in the class $c_1(L)$. More precisely, let $\mathcal{S}$ be the set of continuous, convex functions u on $\overline{P}$ such that $u - u_0$ is smooth on $\overline{P}$. Then (see Guillemin [**15**]) there is a one-to-one correspondence between T-invariant Kähler potentials ψ on X, and symplectic potentials u in $\mathcal{S}$.

The scalar curvature of the metric defined by $u \in \mathcal{S}$ was computed by Abreu [**1**], and up to a factor of two is given by

$$S(u) = -\frac{\partial^2 u^{ij}}{\partial x^i \partial x^j},$$

where u^{ij} is the inverse of the Hessian of u, and we sum over the indices i, j.

Define the measure $d\mu$ on P to be the n-dimensional Lebesgue measure. Let us also define a measure $d\sigma$ on the boundary ∂P as follows. On the face of P defined by $h_k(x) = c_k$, we choose $d\sigma$ so that $d\sigma \wedge dh_k = \pm d\mu$. For example, if the face is parallel to a coordinate hyperplane, then the measure $d\sigma$ on it is the standard $n-1$-dimensional Lebesgue measure. Let us write P^* for the union of P and its codimension one faces and write $\mathcal{C}_1$ for the set of continuous convex functions on P^* which are integrable on ∂P. For a function $A \in L^2(P)$ let us define the functional

$$\mathcal{L}_A(f) = \int_{\partial P} f \, d\sigma - \int_P A f \, d\mu,$$

defined for convex functions $f \in \mathcal{C}_1 \cap L^2$. Let us recall the following integration by parts result from [**12**].

Lemma 2. *Let $u \in \mathcal{S}$ and let f be a continuous function on $\overline{P}$ smooth in the interior, such that $\nabla f = o(d^{-1})$, where d is the distance to the boundary of P. Then $u^{ij} f_{ij}$ is integrable on P and*

$$\int_P u^{ij} f_{ij}\, d\mu = \int_P (u^{ij})_{ij} f\, d\mu + \int_{\partial P} f\, d\sigma.$$

In [**12**] this result was stated for the case when f is convex, but the same proof works for general f as well. In Section 5 we will want to use this lemma with $f = S(u)$ for some symplectic potential u, so we need to check the following.

Lemma 3. *For $u \in \mathcal{S}$ we have that $\nabla S(u) = o(d^{-1})$, where d is the distance to the boundary of P.*

Proof. We want to use the fact that u defines a smooth metric on the compact toric variety X corresponding to P, so the derivatives of $S(u)$ on X are bounded. For this we need to relate the length of vectors on P with respect to the Euclidean metric to their length with respect to the metric g on X induced by u.

Consider the vector field $V = \partial/\partial x^i$ on P where the x^i are the Euclidean coordinates on P. By the expression (3) for the metric on X induced by u we see that the norm squared of V is $\|V\|_g^2 = u_{ii}$. Choose a point $p \in P$, and write q for the closest point to p on the boundary of P. Then q lies on a codimension one face, and we recall from [**12**] that we can choose adapted coordinates $y_1, \ldots, y_n$ around q. This means that a neighborhood of q in P is defined by the inequality $y_1 > 0$, and in addition u in this neighborhood has the form

$$u = y_1 \log y_1 + f,$$

where f is smooth up to the boundary. Therefore $u_{ii} \leqslant C y_1^{-1}$. The distance of p to the boundary is given by a constant times $y_1(p)$, where the constant depends only on which codimension one face q lies on. In sum we obtain that

$$\|V\|_g^2 \leqslant C' d^{-1},$$

where the constant C' does not depend on the point p. Therefore, $\sqrt{d}\cdot V$ gives a bounded vector field on X, so $\sqrt{d}\cdot \nabla_V S(u)$ is a bounded function. This implies that $\nabla S(u) = O(d^{-1/2}) = o(d^{-1})$, which is what we wanted to prove. q.e.d.

Also note that if f is convex, smooth on the interior of P and continuous on $\overline{P}$, then $\nabla f = o(d^{-1})$ holds automatically. It follows that if we let $A = S(u)$ for some $u \in \mathcal{S}$ then

$$\mathcal{L}_A(f) = \int_P u^{ij} f_{ij}\, d\mu.$$

In particular $\mathcal{L}_A(f) \geqslant 0$ for all convex f with equality only if f is affine linear. The converse is conjectured by Donaldson.

Conjecture 4 (see [**11**])**.** Let A be a smooth bounded function on P. If $\mathcal{L}_A(f) > 0$ for all non affine linear convex functions $f \in \mathcal{C}_1$ then there exists a symplectic potential $u \in \mathcal{S}$ with $S(u) = A$.

In the special case when $A = \hat{S}$ we simply write $\mathcal{L}$ instead of $\mathcal{L}_A$. The condition $\mathcal{L}(f) \geqslant 0$ for all convex f is called K-semistability. If in addition we require that equality only holds for affine linear f then it is called K-polystability. Technically we should say "with respect to toric test-configurations," but since we only deal with toric varieties we will neglect this. For more details on stability, in particular on how to construct a test-configuration given a rational piecewise-linear convex function and how to compute the Futaki invariant, see [**11**].

3. Optimal destabilising convex functions

The aim of this section is to show that for an unstable toric variety there exists a "worst destabilising test-configuration". We introduce the normalised Futaki invariant

$$W(f) = \frac{\mathcal{L}(f)}{\|f\|_{L^2}}$$

for non-zero convex functions $f \in \mathcal{C}_1 \cap L^2(P)$ and let $W(0) = 0$. The worst destabilizing test-configuration is a convex function minimizing W. It will only define a genuine test-configuration if it is rational and piecewise linear, so in general we should think of it as a limit of test-configurations.

Theorem 5. *Let the toric variety with moment polytope P be unstable. Then there exists a convex minimizer $\Phi \in \mathcal{C}_1 \cap L^2(P)$ for W which is unique up to scaling. Let us fix the scaling by requiring that*

$$\mathcal{L}(\Phi) = -\|\Phi\|^2_{L^2}.$$

Letting $B = \hat{S} - \Phi$, we then have $\mathcal{L}_B(f) \geqslant 0$ for all convex functions f and $\mathcal{L}_B(\Phi) = 0$. Conversely these two conditions characterize Φ.

Proof. Let A be the unique affine linear function so that $\mathcal{L}_A(f) = 0$ for all affine linear f. We will show in Proposition 6 the existence of a convex $\phi \in \mathcal{C}_1 \cap L^2$ such that letting $B = A - \phi$ we have

$$\begin{aligned} &\mathcal{L}_B(f) \geqslant 0 \quad \text{for all convex } f \\ &\mathcal{L}_B(\phi) = 0. \end{aligned}$$

In addition, ϕ is L^2-orthogonal to the affine linear functions. Let $\Phi = \phi + \hat{S} - A$. We show that this Φ satisfies the requirements of the theorem.

Note that $B = \hat{S} - \Phi$ with the same B as above, and we also have $\mathcal{L}_B(\Phi) = 0$. By definition we have

$$\mathcal{L}(f) = \mathcal{L}_B(f) + \langle B - \hat{S}, f\rangle.$$

In particular, for all convex f

$$\mathcal{L}(f) \geqslant \langle B - \hat{S}, f\rangle \geqslant -\|B - \hat{S}\|_{L^2}\|f\|_{L^2},$$

i.e., $W(f) \geqslant -\|\Phi\|_{L^2}$. On the other hand $W(\Phi) = -\|\Phi\|_{L^2}$, so that Φ is indeed a minimizer for W.

To show uniqueness, suppose that there are two minimizers Φ_1 and Φ_2, and normalize them so that $\|\Phi_1\|_{L^2} = \|\Phi_2\|_{L^2}$, which in turn implies $\mathcal{L}(\Phi_1) = \mathcal{L}(\Phi_2)$. If Φ_1 is not a scalar multiple of Φ_2, then we have

$$\|\Phi_1 + \Phi_2\|_{L^2} < 2\|\Phi_1\|_{L^2},$$

so that

$$W(\Phi_1 + \Phi_2) = \frac{2\mathcal{L}(\Phi_1)}{\|\Phi_1 + \Phi_2\|_{L^2}} < \frac{\mathcal{L}(\Phi_1)}{\|\Phi_1\|_{L^2}},$$

contradicting that Φ_1 was a minimizer (note that $\mathcal{L}(\Phi_1) < 0$). q.e.d.

Proposition 6. *There exists a convex function ϕ such that $B = A - \phi$ (where A is as in the previous proof) satisfies*

$$\mathcal{L}_B(f) \geqslant 0 \textit{ for all convex } f \textit{ and } \mathcal{L}_B(\phi) = 0.$$

In addition ϕ is L^2-orthogonal to the affine linear functions.

The proof of this will take up most of this section. Suppose the origin is contained in the interior of P. We call a convex function normalized if it is non-negative and vanishes at the origin. The key to our proof is a compactness result for normalized convex functions given by Donaldson in [**11**]. In order to apply it we need to reduce our minimization problem to one where we can work with normalized convex functions. Let A be the unique affine linear function so that $\mathcal{L}_A(f) = 0$ for all affine linear f as before, and let us introduce the functional

$$W_A(f) = \frac{\mathcal{L}_A(f)}{\|f\|_{L^2}}.$$

Proposition 7. *Suppose that $\mathcal{L}_A(f) < 0$ for some convex f. Then there exists a convex minimizer $\phi \in \mathcal{C}_1 \cap L^2$ for W_A.*

Proof. We introduce one more functional

$$\tilde{W}_A(f) = \frac{\mathcal{L}_A(f)}{\|f - \pi(f)\|_{L^2}},$$

where π is the L^2-orthogonal projection onto affine linear functions. We define $\tilde{W}_A(f) = 0$ for affine linear f. The advantage of $\tilde{W}_A$ is that it is invariant under adding affine linear functions to f, so we can restrict to looking at normalized convex functions. In addition if we find a minimizer g for $\tilde{W}_A$, then clearly $g - \pi(g)$ is a minimizer for W_A.

The first task is to show that $\tilde{W}_A$ is bounded from below. For this note that for a normalized convex function f we have

$$\mathcal{L}_A(f) \geqslant -\int_P Af\,d\mu \geqslant -\|A\|_{L^2}\|f\|_{L^2}.$$

By Lemma 8 below this implies

$$\mathcal{L}_A(f) \geqslant -C\|A\|_{L^2}\|f-\pi(f)\|_{L^2},$$

so that $\tilde{W}_A(f) \geqslant -C\|A\|_{L^2}$.

Now we can choose a minimizing sequence f_k for $\tilde{W}_A$, where each f_k is a normalized convex function. In addition, we can scale each f_k so that

$$\int_{\partial P} f_k\,d\sigma = 1. \tag{4}$$

According to Proposition 5.2.6. in [**11**] we can choose a subsequence which converges uniformly over compact subsets of P to a convex function which has a continuous extension to a function ϕ on P^* with

$$\int_{\partial P}\phi\,d\sigma \leqslant \liminf \int_{\partial P} f_k\,d\sigma.$$

As in [**11**], we find that this implies

$$\mathcal{L}_A(\phi) \leqslant \liminf \mathcal{L}_A(f_k). \tag{5}$$

If we can show that at the same time

$$\|\phi-\pi(\phi)\|_{L^2} \leqslant \liminf \|f_k-\pi(f_k)\|_{L^2}, \tag{6}$$

then together with the previous inequality this will imply that ϕ is a minimizer of $\tilde{W}_A$ and also $\phi \in L^2$.

In order to show Inequality 6 we first show that the $f_k - \pi(f_k)$ are uniformly bounded in L^2. To see this, note that

$$|\mathcal{L}_A(f_k)| \leqslant \int_{\partial P} f_k\,d\sigma + \|A\|_{L^\infty}\int_P f_k\,d\mu \leqslant C\int_{\partial P} f_k\,d\sigma = C$$

for some $C>0$ depending on A, since the boundary integral of a normalized convex function controls the integral on P as can be seen by using polar coordinates centered at the origin. Since f_k is a minimizing sequence for $\tilde{W}_A$, this implies that for some constant C_1 we have

$$\|f_k-\pi(f_k)\|_{L^2} \leqslant C_1.$$

Now from the fact that $f_k \to \phi$ uniformly on compact sets $K \subset\subset P$ we have

$$\|\phi-\pi(\phi)\|_{L^2(K)} = \lim_k \|f_k-\pi(f_k)\|_{L^2(K)} \leqslant \liminf_k \|f_k-\pi(f_k)\|_{L^2(P)},$$

and taking the limit over compact subsets K, we get the Inequality 6.

q.e.d.

We now prove a lemma that we have used in this proof.

Lemma 8. *There is a constant $C > 0$ such that for all normalized convex functions f we have*

$$\|f\|_{L^2} \leqslant C\|f - \pi(f)\|_{L^2}.$$

Proof. We will prove that for some $\epsilon > 0$ we have

$$\|\pi(f)\|_{L^2} \leqslant (1-\epsilon)\|f\|_{L^2}. \tag{7}$$

The result follows from this, with $C = \epsilon^{-1}$.

Suppose Inequality 7 does not hold so that there is a sequence of normalised convex functions f_k such that $\|f_k\|_{L^2} = 1$ and $\|\pi(f_k)\|_{L^2} \to 1$. By possibly taking a subsequence we can assume that f_k converges weakly to f. The projection π onto a finite dimensional space is compact, so $\pi(f_k) \to \pi(f)$ in norm. In particular, $\|\pi(f)\|_{L^2} = 1$. It follows that $\|f\|_{L^2} = 1$ since the norm is lower semicontinuous. Hence $f = \pi(f)$, i.e., f is affine linear and also the convergence $f_k \to f$ is strong. Then there is a subsequence which we also denote by f_k which converges pointwise almost everywhere to f. Since the f_k are normalized convex functions it is easy to see that f must be zero, which is a contradiction, so Inequality 7 holds. q.e.d.

Finally we can prove Proposition 6, which then completes the proof of Theorem 5.

Proof of Proposition 6. If $\mathcal{L}_A(f) \geqslant 0$ for all convex f, then we take $\phi = 0$. Otherwise Proposition 7 implies that there is a minimizer ϕ for W_A, and by rescaling ϕ we can ensure that

$$\mathcal{L}_A(\phi) = -\|\phi\|^2_{L^2}.$$

Note that ϕ is L^2-orthogonal to the affine linear functions because it minimizes W_A. By definition we have that for all f

$$\mathcal{L}_B(f) = \mathcal{L}_A(f) + \langle A - B, f\rangle_{L^2} = \mathcal{L}_A(f) + \langle \phi, f\rangle_{L^2}.$$

It follows that

$$\mathcal{L}_B(\phi) = \mathcal{L}_A(\phi) + \|\phi\|^2_{L^2} = 0.$$

Now consider perturbations of the form $\phi_t = \phi + t\psi$ which are convex for sufficiently small t, $\langle \phi, \psi\rangle_{L^2} = 0$, but ψ is not necessarily convex. Since ϕ minimizes W_A, we must have

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{L}_A(\phi_t) \geqslant 0,$$

i.e., $\mathcal{L}_A(\psi) \geqslant 0$.

We can write any convex function f as $f = c\cdot\phi + \psi$, where $c \in \mathbf{R}$ and $\langle \phi, \psi\rangle_{L^2} = 0$. Since ϕ is convex, we have that for all $K > \max\{-c, 0\}$ the function

$$\frac{f + K\phi}{c+K} = \phi + \frac{1}{c+K}\psi$$

is convex, so by the previous argument we must have $\mathcal{L}_A(\psi) \geqslant 0$. This means that

$$\mathcal{L}_B(f) = c \cdot \mathcal{L}_B(\phi) + \mathcal{L}_A(\psi) + \langle \phi, \psi \rangle = \mathcal{L}_A(\psi) \geqslant 0.$$

This is what we wanted to show. q.e.d.

We finally give a slightly different variational characterization of Φ.

Proposition 9. *Consider the set $E \subset L^2(P)$ defined by*

$$E = \{h \in L^2 \,|\, \mathcal{L}_h(f) \geqslant 0 \text{ for all convex } f\}.$$

If Φ is the optimal destabilizing convex function we found above, then $B = \hat{S} - \Phi$ is the unique minimizer of the L^2 norm for functions in E.

Proof. Suppose that $h \in E$. Since Φ is convex we have

$$0 \leqslant \mathcal{L}_h(\Phi) = \mathcal{L}(\Phi) + \langle \hat{S} - h, \Phi \rangle. \tag{8}$$

Since we have $\mathcal{L}(\Phi) = -\|\Phi\|^2_{L^2}$, we get

$$\|\Phi\|^2_{L^2} \leqslant \langle \hat{S} - h, \Phi \rangle \leqslant \|\hat{S} - h\|_{L^2} \|\Phi\|_{L^2}, \tag{9}$$

i.e.,

$$\|\Phi\|_{L^2} \leqslant \|\hat{S} - h\|_{L^2}.$$

Since $\mathcal{L}_h(1) = 0$, it follows from (8) that $\hat{S} - h$ is orthogonal to constants. So is Φ; therefore the previous inequality implies

$$\|B\|_{L^2} = \|\hat{S} - \Phi\|_{L^2} \leqslant \|h\|_{L^2}.$$

Equality in (9) can only occur if $\hat{S} - h$ is a positive scalar multiple of Φ, but then it must be equal to Φ by (8). q.e.d.

Note that we can rewrite the definition of the set E as saying that $h \in E$ if and only if for all convex $f \in \mathcal{C}_1 \cap L^2$ we have

$$\langle h, f \rangle \leqslant \int_{\partial P} f \, d\sigma.$$

Thus E is the intersection of a collection of closed affine half spaces, and is therefore a closed convex set in L^2. It follows that there exists a unique minimizer for the L^2-norm in E. From this point of view the content of Theorem 5 is that this minimizer is concave.

Also note that Theorem 5 still holds when we use a different boundary measure $d\sigma$ in defining the functional $\mathcal{L}$. In particular, when $d\sigma$ is zero on some faces, which is a situation we encounter in the next section. The proof is identical, except in the normalization (4) we still use the old $d\sigma$.

4. Harder-Narasimhan filtration

In this section, we would like to study the problem of decomposing an unstable toric variety into semistable pieces. This is analogous to the Harder-Narasimhan filtration of an unstable vector bundle. After making the problem more precise, we will show that we obtain such a decomposition when the optimal destabilizing convex function found in Section 3 is piecewise linear. After that we discuss the implications of such a decomposition and we also look at the case when the optimal destabilizer is not piecewise linear. For convenience, we introduce the following terminology.

Definition 10. Let $Q \subset \mathbf{R}^n$ be a polytope, and let $d\sigma$ be a measure on the boundary ∂Q. It may well be zero on some edges. Let A be the unique affine linear function on Q such that $\mathcal{L}_A(f) = 0$ for all affine linear functions f, where

$$\mathcal{L}_A(f) = \int_{\partial Q} f \, d\sigma - \int_Q Af \, d\mu$$

as before with $d\mu$ being the standard Lebesgue measure (but $d\sigma$ can be different from the one we used before).

We say that $(Q, d\sigma)$ is *semistable*, if $\mathcal{L}_A(f) \geqslant 0$ for all convex functions. It is *stable* if in addition $\mathcal{L}_A(f) = 0$ only for affine linear f.

Let us say that a concave $B \in L^2$ is the *optimal density function* for $(Q, d\sigma)$ if $\mathcal{L}_B(f) \geqslant 0$ for all convex f, and $\mathcal{L}_B(B) = 0$. Note that such a B exists and is unique by the results in Section 3.

Remark.

1) If in the above definition Q is the moment polytope of a toric variety and $d\sigma$ is the canonical boundary measure we have defined before, then $(Q, d\sigma)$ is stable if and only if the toric variety is relatively K-stable (see [**19**]). It is conjectured that in this case the toric variety admits an extremal metric (see [**11**]).
2) If the measure $d\sigma$ is the canonical measure on *some* edges but zero on some others corresponding to a divisor D, then it is conjectured (see [**11**]) that stability of $(Q, d\sigma)$ implies that the toric variety admits a complete extremal metric on the complement of D.
3) Also note that $(Q, d\sigma)$ is semistable precisely when its optimal density function is affine linear.

With this terminology we can state precisely what we would like to show (see also Donaldson [**11**]).

Conjecture 11. Let $(P, d\sigma)$ be the moment polytope of a polarized toric variety with the canonical boundary measure $d\sigma$. If $(P, d\sigma)$ is not semistable, then it has a subdivision into finitely many polytopes Q_i, such that if $d\sigma_i$ is the restriction of $d\sigma$ to the faces of Q_i, then each $(Q_i, d\sigma_i)$ is semistable.

Our main tool is the theorem of Cartier-Fell-Meyer [**5**] about measure majorization. We state it in a slightly different form from the original one.

Theorem 12 (Cartier-Fell-Meyer)**.** *Suppose $d\lambda$ is a signed measure supported on the closed convex set P. Then*

$$\int_P f\, d\lambda \geqslant 0 \tag{10}$$

for all convex functions f if and only if $d\lambda$ can be decomposed as

$$d\lambda = \int_P (T_x - \delta_x)\, d\nu(x),$$

where each T_x is a probability measure with barycentre x, the measure δ_x is the point mass at x and $d\nu(x)$ is a non-negative measure on P.

Note that the converse of the theorem follows easily from Jensen's inequality:

Lemma 13 (Jensen's inequality)**.** *Let T_x be a probability measure with barycentre x. Then for all convex functions f we have*

$$f(x) \leqslant \int f(y)\, dT_x(y).$$

Equality holds if and only if f is affine linear on the convex hull of the support of T_x.

Our result is the following:

Theorem 14. *Suppose $(P, d\sigma)$ is not semistable, and let Φ be the optimal destabilizing convex function found in Section* 3*. If Φ is piecewise linear, then the maximal subpolytopes of P on which Φ is linear give the decomposition of P into semistable pieces required by Conjecture* 11*.*

Proof. Let Φ be the optimal destabilizing convex function, and assume that it is piecewise linear. Let us write $(Q_i, d\sigma_i)$ for the maximal subpolytopes of P on which Φ is linear, with $d\sigma_i$ being the restriction of $d\sigma$ to the boundary of Q_i. According to Theorem 5 we have

$$\mathcal{L}_B(f) \geqslant 0$$

for all convex f, where $B = \hat{S} - \Phi$. This means that the signed measure $d\sigma - B\, d\mu$ satisfies (10). It follows that there is a decomposition

$$d\sigma - B\, d\mu = \int_P (T_x - \delta_x)\, d\nu(x).$$

Since in addition $\mathcal{L}_B(\Phi) = 0$, we have that for almost every x with respect to $d\nu$, the restriction of Φ to the convex hull of the support of

T_x is linear. This means that for almost every x (w.r.t. $d\nu$) the support of T_x is contained in some Q_i, so that for each i we have

$$d\sigma_i - B\, d\mu|_{Q_i} = \int_{Q_i} (T_x - \delta_x)\, d\nu(x).$$

The Jensen inequality implies that for every convex function f on Q_i we have

$$\int_{\partial Q_i} f\, d\sigma - \int_{Q_i} Bf\, d\mu \geqslant 0.$$

Since B is linear when restricted to Q_i this means that $(Q_i, d\sigma_i)$ is semistable. q.e.d.

Remark. Note that by the uniqueness of the optimal density function we get a canonical decomposition into semistable pieces Q_i if we require that the affine linear densities corresponding to the Q_i fit together to form a concave function on P. This corresponds to the condition that in the Harder-Narasimhan filtration of an unstable vector bundle the slope of the successive quotients is decreasing.

Suppose as in the theorem that Φ is piecewise linear and that in addition all the pieces Q_i that we obtain are in fact stable (not just semistable). Then conjecturally they admit complete extremal metrics. We think of this purely in terms of symplectic potentials on polytopes, and not in terms of the complex geometry because when the pieces are not *rational* polytopes then they do not correspond to complex varieties. So an extremal metric on a piece Q is a strictly smooth convex function u on Q which has the same asymptotics as a symplectic potential near faces of Q that lie on ∂P, but which has the asymptotics $-a \log d$ near interior faces. Here $a > 0$ is a function on the face and d is the distance to the face. Piecing together these functions, we obtain a "symplectic potential" u on P, which is singular along the interior boundaries of the pieces Q_i, i.e., along the codimension one locus where Φ is not smooth. Conjecturally the Calabi flow should converge to this singular symplectic potential. More precisely, if u_t is a solution to the Calabi flow, then the sequence of functions $u_t - tB$ should converge to u up to addition of an affine linear function, where $B = \hat{S} - \Phi$ as usual. A decisive step in this direction would be to show that along the flow the scalar curvature converges uniformly to B. In the next section we show the much weaker result that this is true in L^2, assuming that the flow exists for all time.

Suppose now that some of the pieces we obtain are semistable. In some cases it may be possible to decompose these into a finite number of stable pieces, to which the previous discussion applies. There may be some semistable pieces, though, which do not have a decomposition into finitely many stable pieces. For example, suppose that Q is a trapezium, and that the measure $d\sigma$ is only non-zero on the two parallel edges. Let us suppose for simplicity that Q is the trapezium in $\mathbf{R}^2$ with vertices $(0,0),(1,0),(1,l),(0,1)$ for some $l>0$ and that $d\sigma$ is the Lebesgue measure on the vertical edges.

Proposition 15. *The trapezium* $(Q,d\sigma)$ *is semistable in the sense of Definition* 10. *Moreover,* $\mathcal{L}_A(f)=0$ *for all simple piecewise linear* f *with crease joining the points* $(0,u),(1,ul)$ *for* $0<u<1$.

Recall that a simple piecewise linear function is $\max\{h,0\}$ where h is affine linear. The line $h=0$ is called the crease.

Proof. The first task is to compute the linear function A. This can be done easily by writing $A(x,y)=ax+by+c$ and solving the linear system of equations $\mathcal{L}_A(1),\mathcal{L}_A(x),\mathcal{L}_A(y)=0$ for a,b,c. As a result we obtain

$$A(x,y)=\frac{1}{l^2+4l+1}\Big[12(l^2-1)x-6(l^2-2l-1)\Big].$$

It follows that

$$\begin{aligned}\int_Q Af\,d\mu &= \int_0^1\int_0^{1+(l-1)x} Af\,dy\,dx\\ &= \int_0^1\int_0^1 [1+(l-1)x]\,A(x)\,f\big(x,(1+(l-1)x)y'\big)\,dx\,dy',\end{aligned}$$

where we have made the substitution $y'=y/(1+(l-1)x)$. Since for a fixed y' the function $f\big(x,(1+(l-1)x)y'\big)$ is convex in x, the following lemma tells us that

$$\int_0^1 [1+(l-1)x]\,A(x)\,f\big(x,(1+(l-1)x)y'\big)\,dx \leqslant f(0,y')+l\cdot f(1,ly').$$

Integrating over y' as well, we get

$$\mathcal{L}_A(f)=\int_{\partial Q} f\,d\sigma-\int_Q Af\,d\mu\geqslant 0,$$

which shows that $(Q,d\sigma)$ is semistable. It is clear from the proof that if f is linear when restricted to the line segments $y=u+u(l-1)x$ for $0<u<1$, then $\mathcal{L}_A(f)=0$, which gives the second statement in the proposition. q.e.d.

Lemma 16. *Let* $g:[0,1]\to\mathbf{R}$ *be convex. Then we have*

$$\int_0^1 [1+(l-1)x]\,A(x)\,g(x)\,dx\leqslant g(0)+l\cdot g(1), \tag{11}$$

where $A(x)$ *is as in the previous proposition. Moreover, equality holds only if* g *is affine linear.*

Proof. By an approximation argument we can assume that g is smooth. It can be checked directly that when g is affine linear, we have equality in (11), so we can also assume that $g(0) = 0$ and $g'(0) = 0$. We can then write

$$g(x) = \int_0^x g''(t) \cdot (x-t)\, dt = \int_0^1 g''(t) \cdot \max\{0, x-t\}\, dt.$$

It follows that it is enough to check (11) for the functions $g(x) = \max\{0, x-t\}$ for $0 \leqslant t \leqslant 1$. In other words, we need to show that

$$\int_t^1 [1 + (l-1)x]\, A(x)\, (x-t)\, dx - l(1-t) \leqslant 0,$$

for $0 \leqslant t \leqslant 1$. This expression is a quartic in t, whose roots include $t = 0$ and $t = 1$. It is then easy to see by explicit computation that the inequality holds, and equality only holds for $t = 0, 1$. This means that in (11) equality can only hold if $g''(t) = 0$ for almost every $t \in (0, 1)$, i.e., if g is affine linear. q.e.d.

As a consequence of the proposition we see that if we decompose the measure $d\sigma - A d\mu$ according to Theorem 12, then for almost every x the T_x that we obtain has support contained in one of the line segments joining $(0, u), (1, ul)$ for some $0 < u < 1$. It is then clear that $(Q, d\sigma)$ does not have a decomposition into finitely many stable pieces. On such semistable pieces the Calabi flow is expected to collapse an S^1 fibration. This was predicted in [**11**] for the case when Q is a parallelogram. Note that parallelograms correspond to product fibrations, whereas other rational trapeziums correspond to non-trivial S^1 fibrations.

Finally, let us see what we can say when Φ is not piecewise linear. We can still decompose P into the maximal subsets Q_i on which Φ is linear, but now we get infinitely many such pieces and many will have dimension lower than that of P. We still have a decomposition

$$d\sigma - B\, d\mu = \int_P (T_x - \delta_x)\, d\nu,$$

as in the proof of the theorem, but if Q is a lower dimensional piece, then we cannot simply restrict the measures $d\sigma$ and $B\, d\mu$ to ∂Q and Q respectively. This is similar to the case of trapeziums above where the Q_i are the line segments joining the points $(0, u), (1, ul)$. The correct measure on the line segment is given by $[1 + (l-1)x] A(x)\, d\mu$ and on the boundary it's a weighted sum of the values at the endpoints. The lemma shows that with respect to these measures the line segments are stable. This is what we try to imitate in the general case.

Suppose then that Q is such a lower dimensional piece and that we can find a closed convex neighbourhood K of Q with non-empty interior such that $K \cap \partial P$ also has nonempty interior, and for almost every $x \in K$ the support of T_x is contained in K. For each such K we have

$$\int_{\partial K} f\, d\sigma - \int_K Bf\, d\mu \geqslant 0,$$

for all convex f. Suppose we have a sequence of such neighborhoods K_i such that $\bigcap_i K_i = Q$. Then, after perhaps choosing a subsequence of the K_i, we can define a measure $d\tilde{\sigma}$ on ∂Q by

$$\int_{\partial Q} f\, d\tilde{\sigma} = \lim_i \frac{1}{\mathrm{Vol}\,(K_i, d\mu)} \int_{\partial K_i} \tilde{f}\, d\sigma,$$

where $\tilde{f}$ is a continuous extension of a continuous function f on Q. By choosing a further subsequence we can similarly define $\tilde{B}\, d\mu$ and we have that for every convex function f on Q,

$$\int_{\partial Q} f\, d\tilde{\sigma} - \int_Q f\tilde{B}\, d\mu \geqslant 0,$$

since the corresponding inequality holds for each K_i. Note, however, that $\tilde{B}$ is not necessarily linear on Q, and also $d\tilde{\sigma}$ is not necessarily a constant multiple of the Lebesgue measure on the faces of Q. We thus obtain a decomposition of P into infinitely many pieces which are semistable in a suitable sense. As in the case of semistable trapeziums we discussed above, one expects collapsing to occur along the Calabi flow. See the end of the next section for an indication of why such collapsing must occur.

We have not said how to construct a suitable sequence of closed neighborhoods K_i. One way is to look at the subdifferential of Φ. At a point x we write $D\Phi(x) \subset (\mathbf{R}^n)^*$ for the closed set of supporting hyperplanes to Φ at x. Choose x_0 in the interior of Q, i.e., in $Q \setminus \partial Q$. Note that for all interior points $D\Phi(x_0)$ is the same set, and for points on the boundary of Q it is strictly larger since Q is a maximal subset on which Φ is linear. Now we can simply define

$$K_i = \{x \in P \mid D\Phi(x) \cap \overline{B}_{1/i}(D\Phi(x_0)) \neq 0\},$$

where $\overline{B}_{1/i}(D\Phi(x_0))$ denotes the points of distance at most $1/i$ from $D\Phi(x_0)$. So K_i is the set of points with supporting hyperplanes sufficiently close to those at x_0. These are necessarily closed sets with nonempty interior (here we use that Q is of strictly lower dimension than P, so we can choose a sequence of points not in Q approaching an interior point of Q) and the intersection of all of them is Q. Also note that for almost every x, any y in the support of T_x satisfies $D\Phi(x) \subset D\Phi(y)$ since Φ is linear on the convex hull of $\mathrm{supp}(T_x)$. This means that if $x \in K_i$ then also $y \in K_i$.

5. The Calabi flow

In this section we study the Calabi flow on toric varieties, assuming that it exists for all time. In terms of symplectic potentials the Calabi flow is given by the equation

$$\frac{\partial}{\partial t}u_t = -S(u_t) = (u_t^{ij})_{ij},$$

where $u_t \in \mathcal{S}$ for $t \in [0, \infty)$. This can be seen by differentiating the expression (2) defining the symplectic potential and using the definition of the Calabi flow.

The aim of this section is to prove the following.

Theorem 17. *Suppose that u_t is a solution of the Calabi flow for all $t \in [0, \infty)$. Then*

$$\lim_{t\to\infty} \|S(u_t) - \hat{S} + \Phi\|_{L^2} = 0,$$

where Φ is the optimal destabilizing convex function from Theorem 5. Moreover,

$$\|\Phi\|_{L^2} = \inf_{u\in\mathcal{S}} \|S(u) - \hat{S}\|_{L^2}.$$

The first thing to note is that the Calabi functional is decreased under the flow, i.e., $\|S(u_t)\|_{L^2}$ is monotonically decreasing. This is well-known and can be seen easily by computing the derivative.

Recall that for $A \in L^\infty(P)$ we have defined the functional

$$\mathcal{L}_A(u) = \int_{\partial P} u\, d\sigma - \int_P Au\, d\mu.$$

Following [**11**] let us also define

$$\mathcal{F}_A(u) = -\int_P \log\det(u_{ij}) + \mathcal{L}_A(u),$$

for $u \in \mathcal{S}$. That this is well defined for all $u \in \mathcal{S}$ is shown in [**11**]. In the special case when $A = \hat{S}$, the functional $\mathcal{F}_{\hat{S}}$ is the same as the well known Mabuchi functional and is also monotonically decreasing under the flow. For general A it is not monotonic, but will nevertheless be useful.

Finally recall that by Lemma 2, for $u, v \in \mathcal{S}$ we have

$$\mathcal{L}_{S(v)}(u) = \int_P v^{ij}u_{ij}\, d\mu. \tag{12}$$

The proof of Theorem 17 relies on the following two lemmas.

Lemma 18. *Choose some $v \in \mathcal{S}$. If u_t is a solution of the Calabi flow, we have*

$$\mathcal{L}_{S(v)}(u_t) \leqslant C(1+t),$$

for some constant $C > 0$.

Proof. Write $A = S(v)$. Along the flow we have

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_A(u_t) &= \int_P u_t^{ij} S(u_t)_{ij}\, d\mu - \mathcal{L}_A(S(u_t)) \\ &= \int_P (u_t^{ij})_{ij} S(u_t)\, d\mu + \int_P AS(u_t)\, d\mu \\ &= \int_P (A - S(u_t))S(u_t)\, d\mu \leqslant C,\end{aligned}$$

because the Calabi flow decreases the L^2-norm of $S(u_t)$. We have also used Lemmas 2 and 3 in the second line. This implies that

$$\mathcal{F}_A(u_t) \leqslant C(1+t) \tag{13}$$

for some constant C.

Now we use that $A = -(v^{ij})_{ij}$. We can write

$$\begin{aligned}\mathcal{F}_A(u) &= -\int_P \log\det(v^{ik}u_{kj})\, d\mu + \mathcal{L}_A(u) + C_1 \\ &= -\int_P \log\det(v^{ik}u_{kj})\, d\mu + \int_P v^{ij}u_{ij}\, d\mu + C_1,\end{aligned}$$

for some constant C_1. For a positive definite symmetric matrix M we have $\log\det(M) \leqslant \frac{1}{2}\mathrm{Tr}(M)$, applying the inequality $\log x < x/2$ to each eigenvalue. This implies that

$$\mathcal{F}_A(u) \geqslant \frac{1}{2}\mathcal{L}_A(u) + C_1.$$

Together with (13) this implies the result. q.e.d.

Lemma 19. *Fix some $v \in \mathcal{S}$, and write $A = S(v)$. For any $u \in \mathcal{S}$ we have*

$$-\int_P \log\det(u_{ij})\, d\mu \geqslant -C_1 \log \mathcal{L}_A(u) - C_2,$$

for some constants $C_1, C_2 > 0$.

Proof. Observe that

$$-\int_P \log\det(u_{ij}) = -\int_P \log\det(v^{ik}u_{kj})\, d\mu + C.$$

The convexity of $-\log$ implies

$$-\log\det(v^{ik}u_{kj}) \geqslant -C_1 \log \mathrm{Tr}(v^{ik}u_{kj}) - C_2 = -C_1 \log v^{ij}u_{ij} - C_2.$$

Therefore, using the convexity of $-\log$ again,

$$\begin{aligned}-\int_P \log\det(u_{ij})\, d\mu &\geqslant -C_1 \int_P \log v^{ij}u_{ij}\, d\mu - C_2 \\ &\geqslant -C_1' \log \int_P v^{ij}u_{ij}\, d\mu - C_2' \\ &= -C_1' \log \mathcal{L}_A(u) - C_2'.\end{aligned}$$

q.e.d.

We are now ready to prove our theorem.

Proof of Theorem 17. Let us write $B = \hat{S} - \Phi$ as usual. Recall that B satisfies $\mathcal{L}_B(f) \geqslant 0$ for all convex functions f, so that

$$\mathcal{F}_B(u_t) \geqslant -\int_P \log\det(u_{t,ij})\, d\mu.$$

The previous two lemmas combined imply that

$$\mathcal{F}_B(u_t) \geqslant -C_1 \log(1+t) - C_2.$$

At the same time we have

$$\begin{aligned}(14) \qquad \frac{d}{dt}\mathcal{F}_B(u_t) &= -\int_P (B - S(u_t))^2\, d\mu + \int_P B^2\, d\mu - \int_P BS(u_t)\, d\mu \\ &= -\int_P (B - S(u_t))^2\, d\mu + \int_P u_t^{ij} B_{ij}\, d\mu - \mathcal{L}_B(B) \\ &\leqslant -\int_P (B - S(u_t))^2\, d\mu\end{aligned}$$

since B is concave and $\mathcal{L}_B(B) = 0$. Note that here we interpret $\int u_t^{ij} B_{ij}\, d\mu$ in the distributional sense, using the integration by parts formula of Lemma 2. The fact that it is non-positive follows from approximating B by smooth concave functions. Together these inequalities imply that along some subsequence u_k we have

$$\|S(u_k) - B\|_{L^2} \to 0.$$

Since $\|S(u_t)\|_{L^2}$ is monotonically decreasing under the flow, this implies that

$$\|S(u_t)\|_{L^2} \to \|B\|_{L^2}.$$

In order to show that $S(u_t) \to B$ in L^2 not just along a subsequence, note that for $u \in \mathcal{S}$ we have

$$\mathcal{L}_{S(u)}(f) = \int_P u^{ij} f_{ij}\, d\mu \geqslant 0$$

for all continuous convex f, so that $S(u)$ is in the set E defined in Proposition 9. Since E is convex, we have that

$$\frac{1}{2}(S(u_t) + B) \in E,$$

so since B minimizes the L^2-norm in E, we have (suppressing the L^2 from the notation)

$$\|S(u_t) + B\| \geqslant 2\|B\|.$$

It follows that

$$\begin{aligned}\|S(u_t) - B\|^2 &= 2(\|S(u_t)\|^2 + \|B\|^2) - \|S(u_t) + B\|^2 \\ &\leqslant 2(\|S(u_t)\|^2 + \|B\|^2) - 4\|B\|^2 \\ &= 2(\|S(u_t)\|^2 - \|B\|^2) \to 0.\end{aligned}$$

This proves the first part of the theorem.

For the second part simply note that for $u \in \mathcal{S}$ we have $S(u) \in E$ as above, so that Proposition 9 implies that

$$\|S(u)\|_{L^2} \geqslant \|\hat{S} - \Phi\|_{L^2}.$$

Hence, by the previous argument $\|\hat{S} - \Phi\|$ is in fact the infimum of $\|S(u)\|$ over $u \in \mathcal{S}$. q.e.d.

We remark that Donaldson's theorem in [**13**] implies that we can take the infimum over all metrics in the Kähler class, not just the torus invariant ones. In other words, we obtain

$$\inf_{\omega \in c_1(L)} \|S(\omega) - \hat{S}\|_{L^2} = \|\Phi\|_{L^2},$$

where L is the polarization that we chose. This shows that existence of the Calabi flow for all time implies Conjecture 1 for toric varieties.

Let us also observe that from Equation (14) it follows that if the flow exists for all time, then along a subsequence u_k we have

$$\int_P u_k^{ij} B_{ij}\, d\mu \to 0.$$

In particular, at almost every point where B is strictly concave, we must have $u_k^{ij} \to 0$. On the other hand, suppose that B is piecewise linear and one of its creases is parallel to the plane $x_1 = 0$. This means that B_{11} is a delta function along that crease, and B_{ij} vanishes for other i, j. It follows that along the subsequence u_k we have $u_k^{11} \to 0$ on this crease. In view of the formula (3) for the metric given by u, this means that along the creases of B an S^1 fibration collapses. This suggests that the Calabi flow breaks up the toric variety into the pieces given by the Harder-Narasimhan filtration.

We hope that the calculations here will be useful for showing that the Calabi flow exists for all time. In particular, note that it follows from Proposition 5.2.2. in [**11**] that for $v \in \mathcal{S}$ there is a constant $\lambda > 0$ such that for all normalized convex functions $f \in \mathcal{C}_1$ on the polytope we have

$$\mathcal{L}_{S(v)}(f) \geqslant \lambda \int_{\partial P} f\, d\sigma.$$

Together with Lemma 18, this implies that for a solution u_t of the Calabi flow we have a bound of the form

$$\int_{\partial P} \tilde{u}_t\, d\sigma \leqslant C(1+t), \tag{15}$$

where $\tilde{u}_t$ is the normalization of u_t. In addition, one would need much better control of the scalar curvature along the flow in order to use Donaldson's results ([**9**] and unpublished work in progress) to control the metrics under the flow at least in the two dimensional case.

References

[1] M. Abreu, *Kähler geometry of toric varieties and extremal metrics*, Internat. J. Math. **9** (1998) 641–651, MR 1644291, Zbl 0932.53043.

[2] M.F. Atiyah & R. Bott, *The Yang-Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1983) 523–615, MR 0286136, Zbl 0509.14014.

[3] L. Bruasse & A. Teleman, *Harder-Narasimhan filtrations and optimal destabilizing vectors in complex geometry*, Ann. Inst. Fourier (Grenoble) **55**(**3**) (2005) 1017–1053, MR 2149409, Zbl 1093.32009.

[4] E. Calabi, *Extremal Kähler metrics*, in 'Seminar on Differential Geometry' (S.-T. Yau, editor), Princeton, 1982, MR 0645743, Zbl 0487.53057.

[5] P. Cartier, J. Fell, & P. Meyer, *Comparaison des mesures portées par un ensemble convexe compact*, Bull. Soc. Math. France **92** (1964) 435–445, MR 0206193, Zbl 0154.39402.

[6] X.X. Chen, *Calabi flow in Riemann surfaces revisited: a new point of view*, Internat. Math. Res. Notices **6** (2001) 275–297, MR 1820328, Zbl 1078.53065.

[7] X.X. Chen & W. He, *On the Calabi flow*, math.DG/0603523.

[8] P.T. Chruściel, *Semi-global existence and convergence of solutions of the Robinson-Trautman* (*2-dimensional Calabi*) *equation*, Comm. Math. Phys. **137**(**2**) (1991) 289–313, MR 1101689, Zbl 0729.53071.

[9] S.K. Donaldson, *Extremal metrics on toric surfaces*, I, math.DG/0612120.

[10] ———, *Scalar curvature and projective embeddings*, I, J. Differential Geom. **59** (2001) 479–522, MR 1916953, Zbl 1052.32017.

[11] ———, *Scalar curvature and stability of toric varieties*, J. Differential Geom. **62** (2002) 289–349, MR 1988506, Zbl 1074.53059.

[12] ———, *Interior estimates for solutions of Abreu's equation*, Collect. Math. **56**(**2**) (2005) 103–142, MR 2154300, Zbl 1085.53063.

[13] ———, *Lower bounds on the Calabi functional*, J. Differential Geom. **70**(**3**) (2005) 453–472, MR 2192937.

[14] D. Guan, *Extremal-solitons and exponential C^∞ convergence of the modified Calabi flow on certain $\mathbf{CP}^1$ bundles*, preprint, 2005.

[15] V. Guillemin, *Kaehler structures on toric varieties*, J. Differential Geom. **40** (1994) 285–309, MR 1293656, Zbl 0813.53042.

[16] F.C. Kirwan, *Cohomology of Quotients in Symplectic and Algebraic Geometry*, Princeton University Press, 1984, MR 0766741, Zbl 0553.14020.

[17] M. Struwe, *Curvature flows on surfaces*, Ann. Sc. Norm. Super. Pisa. Cl. Sci. (5) **1**(**2**) (2002) 247–274, MR 1991140.

[18] G. Székelyhidi, *The Calabi functional on a ruled surface*, math.DG/0703562.

[19] ———, *Extremal metrics and K-stability*, Ph.D. thesis, Imperial College, London, 2006, math.DG/0611002.

[20] G. Tian, *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math. **137** (1997) 1–37, MR 1471884, Zbl 0892.53027.

[21] S.-T. Yau, *Open problems in geometry*, Proc. Symposia Pure Math. **54** (1993) 1–28, MR 1216573, Zbl 0801.53001.

Mathematics Department
Imperial College London
SW7 2AZ, United Kingdom
E-mail address: gaborsz@gmail.com