Optimal time travel in the Gödel universe

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Outline

- Newtonian cosmology
- Rotating Newtonian universe
- Newton-Cartan theory
- The Gödel universe
- Total integrated acceleration
- Rocket theory
- Optimal time travel

Newtonian cosmology

- Choose a galaxy as the center of the universe; all other galaxies are at $\mathbf{r} = a(t)\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^3$.

- Equation of motion:

$$\ddot{\mathbf{r}} = -\frac{4\pi}{3} r^3 \left(\rho(t) - \frac{\Lambda}{4\pi} \right) \frac{\mathbf{r}}{r^3} \qquad \Leftrightarrow \qquad \frac{\ddot{a}}{a} = -\frac{4\pi\rho}{3} + \frac{\Lambda}{3}$$

- Could have chosen any other galaxy to be the center of the universe: $\mathbf{r} = a(t)(\mathbf{x} - \mathbf{x_0})$ leads to the same equation.

Rotating Newtonian universe

- Choose an axis $\mathbb{R}\mathbf{e_z}$ to be the rotation axis of the universe; all galaxies move with velocity $\dot{\mathbf{r}} = \omega\,\mathbf{e_z} \times \mathbf{r}$.

- Equation of motion:

$$\omega^2\,\mathbf{e_z} \times (\mathbf{e_z} \times \mathbf{r}) = 2\pi\left(\rho - \frac{\Lambda}{4\pi}\right)\mathbf{e_z} \times (\mathbf{e_z} \times \mathbf{r}) \quad \Leftrightarrow \quad \boxed{2\omega^2 = 4\pi\rho - \Lambda}$$

- Could have chosen any other parallel axis to be the rotation axis of the universe: $\dot{\mathbf{r}} = \omega\,\mathbf{e_z} \times (\mathbf{r} - \mathbf{r_0})$ leads to the same equation.

Newton-Cartan theory

- What are inertial frames in Newtonian cosmology?

- Newtonian gravity does not respect the principle of equivalence – must introduce Coriolis field $\mathbf{H}$.

- Equation of motion for a free-falling particle:

$$\ddot{\mathbf{r}} = \mathbf{G} + \dot{\mathbf{r}} \times \mathbf{H}$$

- Field equations:

$$\begin{cases} \nabla \times \mathbf{G} = -\dfrac{\partial \mathbf{H}}{\partial t} \\ \nabla \cdot \mathbf{H} = 0 \\ \nabla \cdot \mathbf{G} = \dfrac{1}{2}\mathbf{H}^2 - 4\pi\rho + \Lambda \\ \nabla \times \mathbf{H} = \mathbf{0} \end{cases}$$

- Rotating universe in co-rotating frame: $\mathbf{G} = \mathbf{0}$, $\mathbf{H} = 2\omega\,\mathbf{e_z}$.

The Gödel universe

- $ds^2 = \frac{1}{2\omega^2} \{-[dt - \sqrt{2}(\cosh(r) - 1)d\varphi]^2 + dr^2 + \sinh^2(r)d\varphi^2 + dz^2\}$

- Solution of the Einstein equations

$$Ric - \frac{1}{2}Rg + \Lambda g = 8\pi T$$

with

$$\Lambda = -\omega^2, \qquad T = \frac{\omega^2}{4\pi} U \otimes U$$

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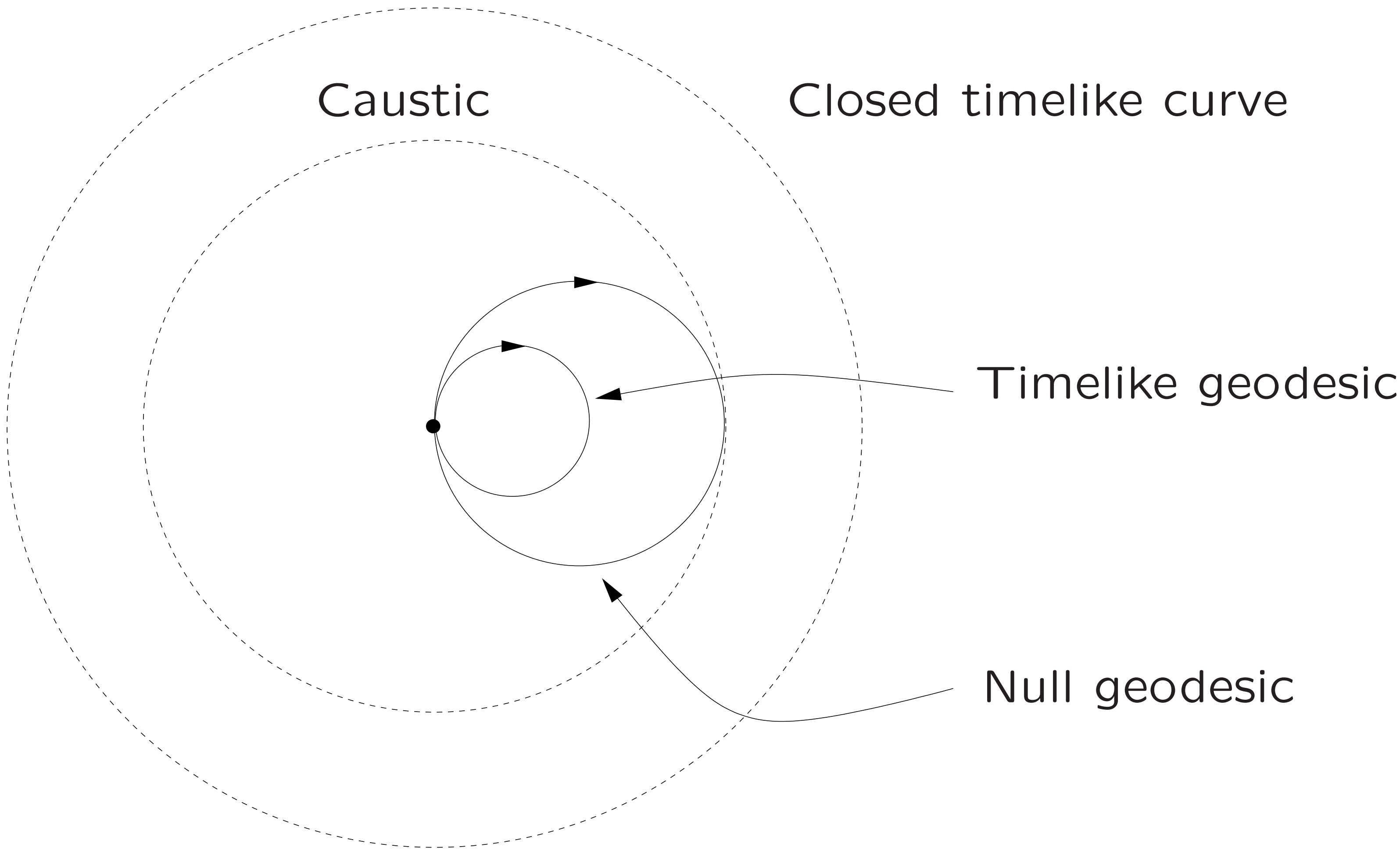
- $ds^2 = \qquad -[dt - \sqrt{2}\underbrace{(\cosh(r) - 1)d\varphi}_{\theta}]^2 + \underbrace{dr^2 + \sinh^2(r)d\varphi^2}_{\text{hyperbolic plane}}$

- $d\theta = \sinh(r)\, dr \wedge d\varphi = dA$


Caustic
Closed timelike curve
Timelike geodesic
Null geodesic

- $\Delta t = \oint_\gamma dt = \oint_\gamma \sqrt{2}\,\theta + \sqrt{d\tau^2 + dl^2} = \sqrt{2}\,A + \oint_\gamma \frac{dl}{v}$

- So the projection γ of a closed timelike curve must be traversed clockwise and satisfy

$$l < \sqrt{2}|A|$$

and so (using the isoperimetric inequality)

$$|A| > 4\pi, \qquad l > 4\pi\sqrt{2}$$

and

$$v_{max} > \frac{\sqrt{2}}{2}$$

Total integrated acceleration

- Rocket equation: $m(\tau_1) = m(\tau_0)\,\mathsf{exp}\left(-\frac{1}{v_e}\int_{\tau_0}^{\tau_1} a(\tau)d\tau\right)$

- $\frac{m(\tau_0)}{m(\tau_1)} \geq \mathsf{exp}(TA), \qquad TA = \int_{\tau_0}^{\tau_1} a(\tau)d\tau$

- Malament (1985) proved that

$$TA \geq \ln(2+\sqrt{5}) \simeq 1.4436$$

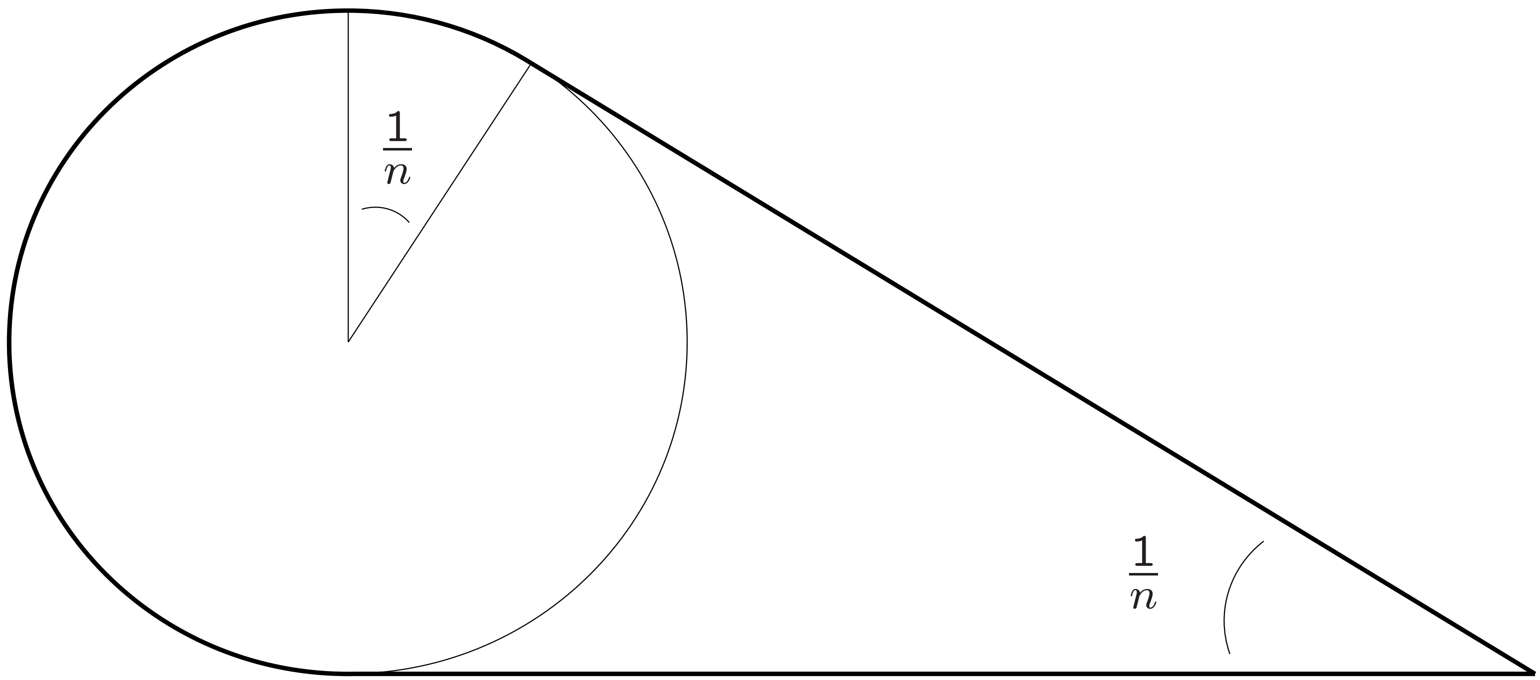
for any CTC, and conjectured that in fact

$$TA \geq 2\pi\sqrt{9+6\sqrt{3}} \simeq 27.6691$$

This would mean (Gödel, 1949)

$$\frac{m(\tau_0)}{m(\tau_1)} \gtrsim 10^{12}$$

- Manchak (2011) gave a non-periodic counter-example.



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$\frac{1}{n}$

Rocket theory

- Problem: minimize $TA = \int_{\tau_0}^{\tau_1} a(\tau) d\tau$

- Issue: $L = a = |\nabla_U U|$ is not differentiable along geodesics

- Issue: one has to deal with instantaneous accelerations, unlike for, say, $L = a^2$ (elastic curves)

- $$\begin{cases} \nabla_U U^\mu = aP^\mu \\ \nabla_U P^\mu = -q^\mu + aU^\mu \\ \nabla_U q_\mu = R_{\mu\alpha\beta\gamma} U^\alpha P^\beta U^\gamma \end{cases}$$ (Henriques and Natário, 2012)

- $P^\mu P_\mu \leq 1$ with $P^\mu P_\mu = 1$ if $a \neq 0$ and at instantaneous accelerations

- If the initial/final U^μ is not specified then $P^\mu = 0$ at the beginning/end

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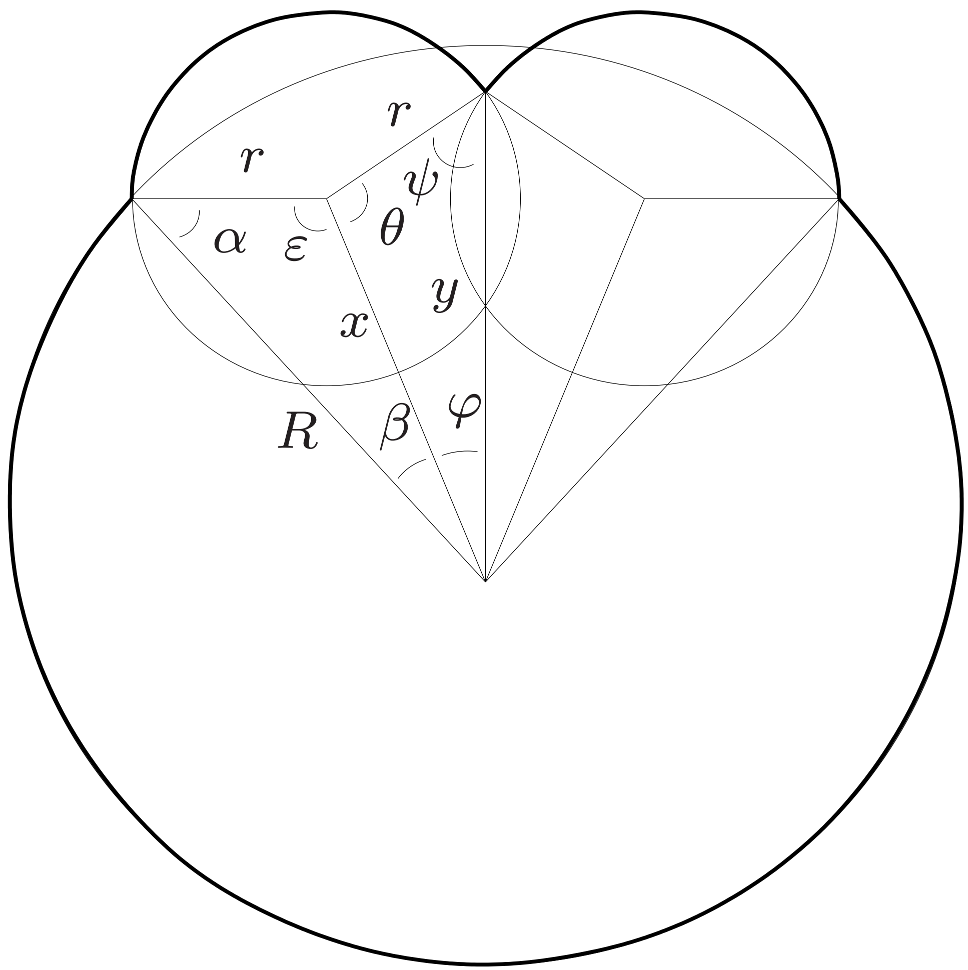
Optimal time travel

- $P^\mu = 0$ at the beginning/end means that we must begin/end with a geodesic arc

- Start with an ansatz depending on finitely many parameters, (numerically) minimize TA with respect to those parameters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- Satisfies the minimum conditions
- $TA \simeq 24.9927$
- $TA \simeq 28.6085$ if we make it periodic
- $TA \simeq 27.6691$ for Malament's conjecture

- Is $TA \simeq 24.9927$ the minimum?

- Does Malament's conjecture still hold for periodic CTCs?

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