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Optimal Tracking Over an Additive White Gaussian Noise Channel

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***Abstract*— This paper studies the optimal reference tracking problems of finite-dimensional, linear, time-invariant (LTI) systems with an additive white Gaussian noise (AWGN) channel between the controller and the plant. We consider two types of the reference signal: a random variable and a Brownian motion. The power of the tracking error is adopted as the measure of the performance and is to be minimized over all stabilizing two-parameter controllers. We assume the power of the channel input is limited and seek to solve the constrained optimization problem explicitly. It is shown that, besides the power constraint, the lowest power of the tracking error hinges closely on non-minimum phase zeros, the unstable poles and the plant gain.**

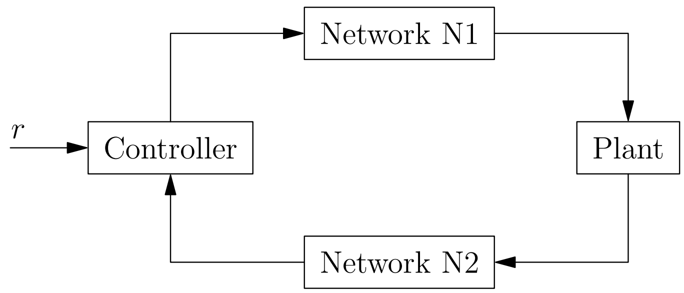

Fig. 1. General configuration of networked control systems

I. Introduction

A somewhat simplified, yet still rather typical configuration of networked control system is shown in Fig. 1. This scenario may occur when, for example, the controller and the plant are connected through data networks. It is clear that the limitations of the communication channel such as data-rate limit, quantization, time delays and data packet drop-out will affect the stability as well as performance of the system fundamentally. It then necessitates a tradeoff between the control performance and the figures of merit of the channel. This type of interaction between control theory and information theory gives rise to a new challenging topic that has attracted considerable attention.

Recently, numerous papers have been published dealing with the stabilization issues of the control system over communication channels [1]–[5]. For example, necessary and sufficient conditions on the smallest data rate for stabilization of discrete-time LTI systems have been derived for a noiseless digital channel model [1], [2] and further generalized to other noisy channels [5]. The same problem has been investigated with respect to stochastic linear systems [4] for noiseless digital channels. In addition, the authors of [6] have adopted an additive white Gaussian noise channel with power constraint, and obtained the minimum signal-to-noise ration (SNR) required to stabilize a given unstable plant. The AWGN channel in the feedback loop furnishes an appealing model that not only preserves the system's linearity, but also can readily translate the power constraint into one on the capacity of the channel.

In spite of the significant progress on stabilization issues, the more inspiring but difficult control performance questions remain open. And it is fairly essential to understand the relations between the control performance and communication channels before we are able to design a satisfactory networked control system. The problem poses a daunting challenge but is a crucial step toward exploring the connection between control and communication. The present paper thus focuses on the optimal tracking performance of single-input single-output (SISO), finite-dimensional, LTI systems with output feedback over an AWGN channel with power constraint.

For standard control systems, it is known that the minimal tracking error is determined by the non-minimum phase zeros of the plant [7]. For a control system with communication constraints, it is plausible that other factors such as the plant gain and the capacity of the communication channel [6] will play a role as well. In this paper, we derive the expression for the optimal unconstrained tracking performance in terms of the weighted sum between the power of the tracking error and the channel input. As a byproduct, we are able to obtain the stabilization requirement on the SNR of the channel which coincides with the earlier result found in [6]. More importantly, we solve the best achievable constrained tracking performance explicitly.

II. Notations and Problem Statement

We first introduce the notation used in this paper. $\bar{z}$ denotes the conjugate of a complex number z. The transpose and conjugate transpose of a vector $\boldsymbol{u}$ are denoted by $\boldsymbol{u}^T$ and $\boldsymbol{u}^H$ and the boldface type is used to denote vectors. The transpose and conjugate transpose of a matrix A are denoted by A^T and A^H. The open left, open right halves of the complex plane and the imaginary axis are denoted by $\mathbb{C}_-$, $\mathbb{C}_+$, and $\mathbb{C}_0$ respectively. The Hilbert Space we shall consider is

$$\mathcal{L}_2 \triangleq \left\{ f : f(s) \text{ measurable in } \mathbb{C}_0, \right.$$
$$\left. \|f\|_2^2 \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \|f(j\omega)\|^2 \, \mathrm{d}\omega < \infty \right\}$$

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in which the inner product is defined as

$$\langle f, g \rangle \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} f^H(j\omega) g(j\omega)\, \mathrm{d}\omega.$$

It is an important fact that $\mathcal{L}_2$ admits an orthogonal decomposition into the subspaces $\mathcal{H}_2$ and $\mathcal{H}_2^{\perp}$, where

$$\mathcal{H}_2 \triangleq \Big\{ f : f(s) \text{ analytic in } \mathbb{C}_+, \quad \|f\|_2^2 \triangleq \sup_{\epsilon>0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \|f(\epsilon + j\omega)\|^2 \, \mathrm{d}\omega < \infty \Big\}$$

and

$$\mathcal{H}_2^{\perp} \triangleq \Big\{ f : f(s) \text{ analytic in } \mathbb{C}_-, \quad \|f\|_2^2 \triangleq \sup_{\epsilon<0} \frac{1}{2\pi} \int_{-\infty}^{\infty} \|f(\epsilon + j\omega)\|^2 \, \mathrm{d}\omega < \infty \Big\}.$$

It follows that for any $f \in \mathcal{H}_2$ and $g \in \mathcal{H}_2^{\perp}$, we have $\langle f, g \rangle = 0$. It is worth pointing out that we shall use the same notation $\|\cdot\|_2$ to denote these norms, as the meaning of each of these norms will be clear from the context. Let $\mathbb{R}\mathcal{H}_{\infty}$ denote the set of all stable, proper, rational transfer function matrices. The expectation operator is denoted by $E[\cdot]$. Finally, the logarithm used throughout this paper has base e unless otherwise specified.

The SISO finite-dimensional, LTI unity feedback system under consideration is depicted in Fig. 2. Here, P is the plant model and $[K_1, K_2]$ is a general two-parameter controller. Their transfer functions are $P(s)$ and $K_1(s), K_2(s)$, respectively. We shall use the same symbol for the system and its transfer function and omit the frequency variable s whenever convenient. The signals r, u, n, y are the reference input, the channel input, the channel noise and the system output, respectively. The tracking error signal is defined as $e(t) \doteq r(t) - y(t)$ and its average power is $E[e^2(t)]$. In the configuration, the control input u accesses the reference input r and the output y via two independently designed controllers K_1 and K_2. Since the two-parameter controller represents the most general linear feedback structure available, the optimal tracking error obtained herein is the smallest achievable performance. It has two degrees of freedom and offers us advantage in dealing with tracking error as well as countering the channel noise. Lastly, the noise $n(t)$ is zero-mean white Gaussian noise with power spectral density Φ and the channel is assumed to be AWGN with infinite bandwidth. The power of the channel input signal is limited by

$$E[u^2(t)] \leq \Gamma \tag{1}$$

and the channel capacity of the AWGN channel is therefore given by [8] $\mathcal{C} = \frac{1}{2}(\log_2 e)\frac{\Gamma}{\Phi}$.

In the sequel, we shall focus on the case that all the signals are wide-sense stationary processes or that the system has reached its steady state. In other words, this formulation casts aside any transient behavior. As a result, from this point onward, we may drop the time variable t in the second moment expression whenever convenient.

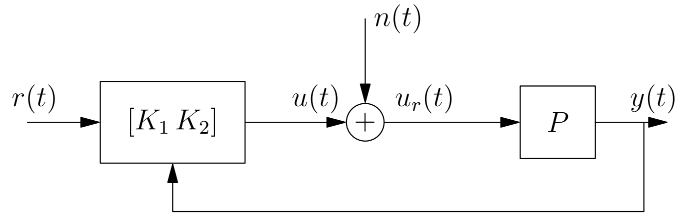

Fig. 2. Two-parameter tracker over an AWGN channel

Next we introduce some important factorizations that will be frequently used in the development of the result. First, let the coprime factorization of P be given by

$$P = NM^{-1} \tag{2}$$

where $N, M \in \mathbb{R}\mathcal{H}_{\infty}$ and satisfy the Bezout identity

$$MX - NY = 1 \tag{3}$$

for some $X, Y \in \mathbb{R}\mathcal{H}_{\infty}$. It is useful to factorize $N(s)$ as

$$N(s) = L(s) N_m(s) \tag{4}$$

where $N_m(s)$ represents the minimum phase part of $N(s)$, and $L(s)$ represents an all-pass factor which can be constructed as

$$L(s) = \prod_{i=1}^{N_z} \frac{\bar{z}_i}{z_i} \frac{z_i - s}{\bar{z}_i + s} \tag{5}$$

where $z_i, i = 1, \ldots, N_z$ are the non-minimum phase zeros of P. When so constructed, $L(0) = 1$. Similarly, a factorization of $M(s)$ yields

$$M(s) = B(s) M_m(s) \tag{6}$$

where $M_m(s)$ is minimum phase, and $B(s)$ is all-pass which can be constructed as

$$B(s) = \prod_{i=1}^{N_p} \frac{s - p_i}{s + \bar{p}_i} \tag{7}$$

where $p_i, i = 1, \ldots, N_p$ are the unstable poles of P. We also note that $M(\infty) = 1$. The set of all stabilizing two-parameter compensators can be characterized by the Youla parametrization

$$\mathcal{K} \doteq \{K : K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} = (X - RN)^{-1} \times \begin{bmatrix} Q & Y - RM \end{bmatrix}, Q, R \in \mathbb{R}\mathcal{H}_{\infty}\}. \tag{8}$$

The optimal tracking problem over an AWGN channel can be formulated as

$$\inf_{K \in \mathcal{K}} \mathrm{E}[e^2], \text{ subject to } \mathrm{E}[u^2] < \Gamma. \tag{9}$$

The two-parameter controller chosen from $\mathcal{K}$ is to be designed such that the power of the tracking error is minimized while the channel power constraint is satisfied.

III. Tracking a Random Variable

In this section, we shall derive an analytical expression for the best performance of tracking a random variable. The reference input r is a wide-sense stationary random process satisfying $r(t) = A$, $-\infty < t < \infty$ where A is a random variable with $E[A^2] = \sigma^2$. The power spectral density of $r(t)$ is therefore given by $S_r(\omega) = 2\pi\sigma^2\delta(\omega)$ where $\delta(\omega)$ denotes the Dirac delta function. With abuse of notation, in Fig. 2, the transfer functions from n and r to e and u are given by

$$\begin{aligned} e &= \left(1 - (1 - PK_2)^{-1} PK_1\right) r - (1 - PK_2)^{-1} Pn, \\ u &= (1 - PK_2)^{-1} K_1 r + (1 - PK_2)^{-1} PK_2 n. \end{aligned}$$

Since we are choosing the controllers from $\mathcal{K}$, we can further write the above transfer functions as

$$\begin{aligned} e &= (1 - NQ)\, r - N(X - RN)n, \\ u &= MQr + (-1 + M(X - RN))\, n. \end{aligned}$$

Because of r and n are uncorrelated, we could express the power of e and u as [9]

$$\begin{aligned} E[e^2] &= |1 - N(0)Q(0)|^2 \sigma^2 + \|N(X - RN)\|_2^2 \Phi, \\ E[u^2] &= |M(0)Q(0)|^2 \sigma^2 + \|1 - M(X - RN)\|_2^2 \Phi. \end{aligned}$$

A. Optimization of the Tracking Error and Channel Input Combined

We shall use convex optimization to address problem (9). To that end, we first adopt a performance measure that weighs the power of the tracking error and and the channel input jointly, that is,

$$H(\epsilon) \doteq (1 - \epsilon)E[e^2] + \epsilon E[u^2], \tag{10}$$

where $0 \le \epsilon \le 1$. $H(\epsilon)$ can also be considered as the Lagrangian.

It follows from (10) and a standard algebraic manipulation that

$$H = A + J$$

where

$$A \doteq (1 - \epsilon)\sigma^2 \left|1 - N(0)Q(0)\right|^2 + \epsilon\sigma^2 \left|M(0)Q(0)\right|^2, \tag{11}$$

$$J \doteq \left\| \begin{bmatrix} \sqrt{1-\epsilon}N(X - RN) \\ \sqrt{\epsilon}\,(1 - M(X - RN)) \end{bmatrix} \sqrt{\Phi} \right\|_2^2. \tag{12}$$

Let $H^*(\epsilon)$ denote the infimum of $H(\epsilon)$ over all stabilizing controllers. The optimization problem boils down to two independent cases

$$H^*(\epsilon) = \inf_{Q, R \in \mathbb{R}\mathcal{H}_\infty} H = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} A + \inf_{R \in \mathbb{R}\mathcal{H}_\infty} J. \tag{13}$$

Before stating the main theorem of this paper, we introduce an inner-outer factorization [10]

$$\begin{bmatrix} \sqrt{\frac{1-\epsilon}{\epsilon}} N_m \\ -M_m \end{bmatrix} = \Delta_i \Delta_o \tag{14}$$

where $\Delta_i \in \mathbb{R}\mathcal{H}_\infty$ is an inner matrix function, and $\Delta_o \in \mathbb{R}\mathcal{H}_\infty$ is an outer scalar function.

Theorem 1: Let r be a random variable with zero mean and variance σ^2. Suppose that $P(s)$ is a rational scalar transfer function which admits the factorization (2). Further assume that the non-minimum phase zeros of P are distinct. Let z_i, $i = 1, \ldots, N_z$ be the non-minimum phase zeros and p_i, $i = 1, \ldots, N_p$ be the unstable poles of P. Then,

$$H^*(\epsilon) = \sigma^2 \frac{\epsilon(1 - \epsilon)}{(1 - \epsilon)P(0)^2 + \epsilon} + \epsilon\Phi \Bigg\{ 2\sum_{i=1}^{N_p} p_i + \frac{1}{\pi}\int_0^\infty \log\left(1 + \frac{1-\epsilon}{\epsilon}\left|P(j\omega)\right|^2\right) \mathrm{d}\omega + \sum_{l=1}^{N_z}\sum_{i=1}^{N_z} \frac{\gamma_l \bar{\gamma}_i}{z_l + \bar{z}_i} \Bigg\}. \tag{15}$$

where

$$\gamma_i \doteq 2\mathrm{Re}(z_i)\left[1 - \Delta_o(z_i)M^{-1}(z_i)\right] \prod_{j=1, j\neq i}^{N_z} \frac{z_i + \bar{z}_j}{z_i - z_j} \tag{16}$$

Proof: We briefly state several key steps of this proof using the framework established in [11]. First, it is straightforward to calculate

$$A^* = \sigma^2 \frac{\epsilon(1 - \epsilon)}{(1 - \epsilon)P(0)^2 + \epsilon}.$$

Then we consider J. By virtue of the factorizations (4) and (6), we have

$$J = \left\| \begin{bmatrix} 0 \\ \sqrt{\epsilon}B^{-1} \end{bmatrix} + \sqrt{\epsilon} \begin{bmatrix} \sqrt{\frac{1-\epsilon}{\epsilon}} N_m(X - RN) \\ -M_m(X - RN) \end{bmatrix} \right\|_2^2 \Phi \tag{17}$$

With the aid of the inner-outer factorization (14) and by properly choosing R, we can further simplify J^* as

$$J^* = \left\|\sqrt{\epsilon}\left(B^{-1} - 1\right)\right\|_2^2 + \hat{J}^* \tag{18}$$

where $\sqrt{\epsilon}\left(B^{-1} - 1\right) \in \mathcal{H}_2^\perp$ and $\hat{J}^* \doteq \inf_{R \in \mathbb{R}\mathcal{H}_\infty} \hat{J}$ with

$$\hat{J} \doteq \left\| \begin{bmatrix} 0 \\ \sqrt{\epsilon} \end{bmatrix} + \sqrt{\epsilon}\Delta_i\Delta_o(X - RN) \right\|_2^2 \Phi \in \mathcal{H}_2.$$

To calculate $\hat{J}^*$, define

$$\Psi(s) \doteq \begin{bmatrix} \Delta_i^T(-s) \\ I - \Delta_i(s)\Delta_i^T(-s) \end{bmatrix},$$

which satisfies $\Psi^H(j\omega)\Psi(j\omega) = I$. Then we may pre-multiply $\hat{J}$ by Ψ, yielding

$$\hat{J} = \left\| \Psi \left\{ \begin{bmatrix} 0 \\ \sqrt{\epsilon} \end{bmatrix} + \sqrt{\epsilon}\Delta_i\Delta_o(X - RN) \right\} \right\|_2^2 \Phi$$

which, by further calculation, can be reduced to

$$\hat{J} = \left\|W_1 + \sqrt{\epsilon}\Delta_o(X - RN)\right\|_2^2 \Phi + \|W_2\|_2^2 \Phi.$$

where

$$\begin{aligned} W_1 &\doteq -\sqrt{\epsilon}\bar{\Delta}_o^{-1}\bar{M}_m, \\ W_2 &\doteq \begin{bmatrix} -\sqrt{1-\epsilon}N_m\Delta_o^{-1}\bar{\Delta}_o^{-1}\bar{M}_m \\ \sqrt{\epsilon}\left(1 - M_m\Delta_o^{-1}\bar{\Delta}_o^{-1}\bar{M}_m\right) \end{bmatrix}. \end{aligned}$$

The derivations of the first term makes use of the factorization (4) and a partial fraction procedure on

$\left(\bar{\Delta}_o^{-1}(\infty)\bar{M}_m(\infty) - \Delta_o X\right) L^{-1}$. After a proper choice of $R \in \mathbb{RH}_\infty$, we can express

$$\hat{J}^* = \Bigg(\|W_2\|_2^2 + \epsilon \left\|\bar{\Delta}_o^{-1}\bar{M}_m - \bar{\Delta}_o^{-1}(\infty)\bar{M}_m(\infty)\right\|_2^2 + \epsilon \left\| \sum_{i=1}^{N_z} \frac{\gamma_i}{z_i - s} \right\|_2^2 \Bigg)\Phi$$

The summation of the first two terms of the above equation is equal to [11]

$$\frac{\epsilon}{\pi}\int_0^\infty \log\left(1 + \frac{1-\epsilon}{\epsilon}|P(j\omega)|^2\right) \mathrm{d}\omega$$

and the last term leads to

$$\left\| \sum_{i=1}^{N_z} \frac{\gamma_i}{z_i - s} \right\|_2^2 = \sum_{l=1}^{N_z}\sum_{i=1}^{N_z} \frac{\gamma_l \bar{\gamma}_i}{z_l + \bar{z}_i}.$$

Observing that

$$\left\|\sqrt{\epsilon}\left(B^{-1} - 1\right)\right\|_2^2 = 2\epsilon \sum_{i=1}^{N_p} p_i$$

we proved the theorem. ■

We present here a brief analysis of Theorem 1. To begin with, notice that the noise power appears as a positive scaling factor and therefore amplifies the negative effect resulting from the unstable poles, the non-minimum phase zeros and the plant gain. This is unsurprising in that the greater the noise power, the harder it is to achieve a good performance. Second, the first term shows a large plant gain at DC is helpful in countering the power of the input reference signal. Thirdly, the unstable poles as well as the plant gain constrain the performance in a way represented by the second and third term of (15). It is clear that a small plant gain is desirable for lessening the effect of the channel noise. These expressions remain largely the same as the optimal regulation performance derived in [11], despite the fact that there the control energy is regulated while the performance index (15) considers the power of the channel input. Finally, the last term is positive and thus demonstrates the detrimental effect of the simultaneous presence of unstable poles and non-minimum phase zeros on the best achievable performance.

It is interesting to examine two extreme cases, i.e. $\epsilon = 0$ and $\epsilon = 1$. When $\epsilon = 0$, H^* defines the minimal tracking error without channel input power constraint and

$$H^*(0) = \Phi \sum_{l=1}^{N_z}\sum_{i=1}^{N_z} \frac{\beta_l \bar{\beta}_i}{z_l + \bar{z}_i} \tag{19}$$

where

$$\beta_i \doteq 2\mathrm{Re}(z_i) N_m(z_i) M^{-1}(z_i) \prod_{j=1, j\neq i}^{N_z} \frac{z_i + \bar{z}_j}{z_i - z_j}. \tag{20}$$

β_i is derived directly from (17) following the same procedure as in the proof. To achieve the best tracking performance bound, the channel input power is necessarily infinite, and thus requires infinite channel capacity.

On the other hand, $H^*(1)$ defines the minimal channel input power without taking tracking error into consideration and

$$H^*(1) = \Phi\left(2\sum_{i=1}^{N_p} p_i + \sum_{l=1}^{N_z}\sum_{i=1}^{N_z} \frac{\alpha_l \bar{\alpha}_i}{z_l + \bar{z}_i}\right) \tag{21}$$

where

$$\alpha_i \doteq 2\mathrm{Re}(z_i)\left[1 - B^{-1}(z_i)\right] \prod_{j=1, j\neq i}^{N_z} \frac{z_i + \bar{z}_j}{z_i - z_j}. \tag{22}$$

To obtain the above equation, notice that $\Delta_o = M_m$ when $\epsilon = 1$ and therefore in (16), $\Delta_o(z_i)M^{-1}(z_i) = B^{-1}(z_i)$. The equation (21) identifies the minimum SNR of the AWGN channel below which it is impossible to stabilize the plant [6]. Let $\Gamma_{th} \doteq H^*(1)$. Since the optimization is derived over all stabilizing two-parameter controller (8), the plant is stabilizable if and only if the channel input power constraint

$$\Gamma > \Gamma_{th} \tag{23}$$

or the capacity of the AWGN channel satisfies

$$\mathcal{C} > \frac{1}{2}(\log_2 e)\left(2\sum_{i=1}^{N_p} p_i + \sum_{l=1}^{N_z}\sum_{i=1}^{N_z} \frac{\alpha_l \bar{\alpha}_i}{z_l + \bar{z}_i}\right) \tag{24}$$

B. Tracking under Constraint on the Power of the Channel Input

Given the solution of the preceding problem, we can now treat the constrained optimization (9). Let $H_e^*(\Gamma)$ denote the optimal cost under the channel power constraint Γ. We seek to observe a relation between the power constraint of the channel and the minimal tracking error.

Define $\lambda \doteq \epsilon/(1-\epsilon)$. It follows from the inner-outer factorization (14) that Δ_o is a function of λ. Therefore γ_i is also a function of λ through the definition (16). Then we may further define

$$R(\lambda) \doteq \sum_{l=1}^{N_z}\sum_{i=1}^{N_z} \frac{\gamma_l(\lambda)\bar{\gamma}_i(\lambda)}{z_l + \bar{z}_i}. \tag{25}$$

The result is summarized in the following theorem.

Theorem 2: Suppose that $P(s)$ is a transfer function satisfying the assumption in Theorem 1. If the power of the channel input satisfies the constraint $\Gamma > H^*(1)$, then the smallest constrained tracking error is given as

$$H_e^*(\Gamma) = \sigma^2 \frac{\lambda^*}{P(0)^2 + \lambda^*} + \Phi\lambda^*\Bigg\{2\sum_{i=1}^{N_p} p_i + R(\lambda^*) + \frac{1}{\pi}\int_0^\infty \log\left(1 + \frac{|P(j\omega)|^2}{\lambda^*}\right)\mathrm{d}\omega\Bigg\} - \lambda^*\Gamma \tag{26}$$

where λ^* is the positive zero of

$$\frac{\sigma^2 P(0)^2}{(P(0)^2+\lambda)^2} + \Phi\left\{2\sum_{i=1}^{N_p} p_i + R(\lambda) + \lambda\frac{\mathrm{d}\,R(\lambda)}{\mathrm{d}\lambda}\right.$$
$$\left. + \frac{1}{\pi}\int_0^\infty \left[\log\left(1+\frac{|P(j\omega)|^2}{\lambda}\right) - \frac{|P(j\omega)|^2}{\lambda+|P(j\omega)|^2}\right]\mathrm{d}\omega\right\} - \Gamma. \tag{27}$$

Proof: As defined in (10), $\frac{1}{1-\epsilon}\left(H^*(\epsilon)-\epsilon\Gamma\right)$ gives a lower bound on $H_e^*(\Gamma)$ [12]. Since the feasibility region for the existence of a stabilizing controller in the $\mathrm{E}[e^2]$-$\mathrm{E}[u^2]$ space is convex [13], the bound is tight and the equality can be attained by some ϵ. More specifically,

$$H_e^*(\Gamma) = \sup_{0<\epsilon<1}\left\{\frac{1}{1-\epsilon}(H^*(\epsilon)-\epsilon\Gamma)\right\}.$$

It follows that

$$H_e^*(\Gamma) = \sup_{\lambda>0}\{\phi(\lambda)\} \tag{28}$$

where

$$\phi(\lambda) \doteq \frac{\sigma^2\lambda}{P(0)^2+\lambda} + \Phi\lambda\left\{2\sum_{i=1}^{N_p} p_i + R(\lambda) + \right.$$
$$\left. \frac{1}{\pi}\int_0^\infty \log\left(1+\frac{|P(j\omega)|^2}{\lambda}\right)\mathrm{d}\omega\right\} - \lambda\Gamma. \tag{29}$$

which is a convex function with respect to λ. The derivative of $\phi(\lambda)$ is then a monotonically decreasing function given by (27). It can be shown that $\lim_{\lambda\to\infty}\phi(\lambda) = H^*(1)-\Gamma$ and $\lim_{\lambda\to 0}\phi(\lambda)=\infty$. Thus the positive solution of

$$\frac{\mathrm{d}\,\phi(\lambda)}{\mathrm{d}\lambda} = 0$$

exists when $H^*(1)-\Gamma<0$ and it will be the maximizer λ^* of $\phi(\lambda)$. We thus arrive at (26) which is valid for $\Gamma>\Gamma_{th}$, i.e. when the system is stable. ■

Theorem 2 sheds light on the effect of the AWGN channel on the performance of the control system. The channel SNR is related to the tracking performance in a quite intriguing way. To better understand it, we may assume that $P(0)=0$ and P does not have non-minimum phase zeros. We thus obtain from (27) that λ^* is the positive solution of

$$\frac{1}{\pi}\int_0^\infty \left[\log\left(1+\frac{|P(j\omega)|^2}{\lambda}\right) - \frac{|P(j\omega)|^2}{\lambda+|P(j\omega)|^2}\right]\mathrm{d}\omega$$
$$= \frac{\Gamma}{\Phi} - 2\sum_{i=1}^{N_p} p_i. \tag{30}$$

And therefore λ^* depends on the SNR of the channel and the the plant gain. Then, by inserting λ^* to (26), we obtain the best constrained tracking performance.

Numerically, λ^* is easy to calculate because of the monotonicity of the function $(\mathrm{d}/(\mathrm{d}\lambda))\phi(\lambda)$. Once λ^* is found, $H_e^*(\Gamma)$ can be calculated directly using (26).

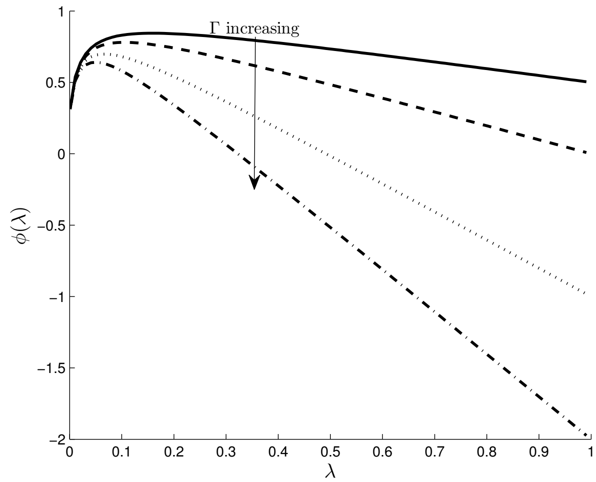

Fig. 3. Constrained tracking error when $\Gamma > \Gamma_{th}$

C. Numerical Example

We now use a simple example to illustrate the preceding result.

Example 1: Consider the plant

$$P(s) = \frac{s-1}{s^2-s-6}. \tag{31}$$

Assume that the reference input signal has unit variance and the power spectral density of the white noise $\Phi=1$. When the channel input power constraint satisfies $\Gamma>\Gamma_{th}$, by computing the expression (29) we obtain the curves in Fig. 3. According to (28), the maxima of the concave curves gives the constrained optimal tracking errors. The figure also shows that as the channel input power constraint increases, the tracking error decreases, which is expected.

IV. Tracking a Brownian Motion

In this section, we shall deal with the problem of tracking a slowly varying "constant". To be more specific, we assume that the reference input r is the integral of a standard white noise which can be non-rigorously considered as a Brownian motion [14]. Then, the problem resembles the tracking of a deterministic step signal.

Assuming that r and n are uncorrelated, we have

$$E[e^2] = \|(1-NQ)\,\hat{r}\|_2^2 + \|N(X-RN)\|_2^2\,\Phi,$$
$$E[u^2] = \|MQ\hat{r}\|_2^2 + \|1-M(X-RN)\|_2^2\,\Phi,$$

where $\hat{r}=1/s$ is the transfer function of the integrator. The performance index (10) is then given by

$$H = B + J$$

in which

$$B \doteq \left\|\begin{bmatrix}\sqrt{1-\epsilon}(1-NQ)\\ \sqrt{\epsilon}MQ\end{bmatrix}\hat{r}\right\|_2^2 \tag{32}$$

and J is the same as (12). It is then clear that in order for H to be finite, we need to have $M(0)=0$ and thus P needs to have a pole at $s = 0$ and no zero at the origin. The optimal performance becomes $H^* = \inf_{Q \in \mathbb{R}\mathcal{H}_\infty} B + \inf_{R \in \mathbb{R}\mathcal{H}_\infty} J$. We then state the following result without proof, as it is analogous to that for Theorem 1 in the previous section and Theorem 4 in [11].

Theorem 3: Let r be the random process specified above. Suppose that

$$P(s) = \frac{P_0(s)}{s^n}$$

for some integer $n \geq 1$, such that $P_0(s)$ is proper and has no zero at $s = 0$. Let z_i, $i = 1, \ldots, N_z$ be the non-minimum phase zeros and p_i, $i = 1, \ldots, N_p$ be the unstable poles of P. Assume further that the non-minimum phase zeros of P are distinct. Then,

$$H^* = (1-\epsilon)\left\{\frac{1}{\pi}\int_0^\infty \frac{1}{\omega^2}\log\left(1 + \frac{\epsilon}{(1-\epsilon)\left|P(j\omega)\right|^2}\right)\mathrm{d}\omega + 2\sum_{i=1}^{N_z}\frac{1}{z_i}\right\} + \epsilon\Phi\left\{\frac{1}{\pi}\int_0^\infty \log\left(1 + \frac{1-\epsilon}{\epsilon}\left|P(j\omega)\right|^2\right)\mathrm{d}\omega + 2\sum_{i=1}^{N_p} p_i + \sum_{l=1}^{N_z}\sum_{i=1}^{N_z}\frac{\gamma_l\bar{\gamma}_i}{z_l + \bar{z}_i}\right\}, \quad (33)$$

where γ_i is given by (16).

Theorem 3 shows that besides the effect of non-minimum phase zeros and unstable poles, the best achievable tracking performance exhibits a tradeoff between the high and low plant gain. Because of the factor $1/\omega^2$, we may expect that a plant with large gain at low frequency and small gain at high frequency is beneficial for our purposes. Tracking and constraining the channel input power are thus competing objectives that depend heavily on the plant gain.

In an analogous fashion to Theorem 2, we solve the corresponding constrained optimization problem (9) and summarize the result in the following theorem.

Theorem 4: Suppose that $P(s)$ satisfies the assumption in Theorem 3. Let $\Gamma > \Gamma_{th}$, then the optimal constrained tracking error is given by

$$H_e^*(\Gamma) = 2\sum_{i=1}^{N_z}\frac{1}{z_i} + \frac{1}{\pi}\int_0^\infty \frac{1}{\omega^2}\log\left(1 + \frac{\lambda^*}{\left|P(j\omega)\right|^2}\right)\mathrm{d}\omega + \lambda^*\Phi\left\{2\sum_{i=1}^{N_p} p_i + \frac{1}{\pi}\int_0^\infty \log\left(1 + \frac{\left|P(j\omega)\right|^2}{\lambda^*}\right)\mathrm{d}\omega + R(\lambda^*)\right\} - \lambda^*\Gamma \quad (34)$$

where λ^* is the positive solution of

$$\int_0^\infty \frac{1}{\omega^2(\left|P(j\omega)\right|^2 + \lambda)}\mathrm{d}\omega + \Phi\left\{2\sum_{i=1}^{N_p} p_i + \int_0^\infty \left[\log\left(1 + \frac{\left|P(j\omega)\right|^2}{\lambda}\right) - \frac{\left|P(j\omega)\right|^2}{\lambda + \left|P(j\omega)\right|^2}\right]\mathrm{d}\omega + R(\lambda) + \lambda\frac{\mathrm{d}\,R(\lambda)}{\mathrm{d}\lambda}\right\} = \pi\Gamma. \quad (35)$$

V. Conclusion

In this paper we have investigated the best tracking performance of a linear system over an AWGN channel with constraint on the power of the input. We have derived explicit expressions for both unconstrained and constrained optimal tracking performance using $\mathcal{H}_2$ optimization techniques. Not surprisingly, the constrained tracking performance depends on the unstable poles, non-minimum phase zeros, the plant frequency response and the SNR of the channel. The simultaneous presence of the unstable poles and non-minimum zeros plays an important role in not only the performance achievable but also the stabilization requirement on the SNR of the channel. Besides, in the case of tracking a Brownian motion, we have observed the tradeoff caused by the plant gain.

The current work can be extended to deal with the performance issues over more general additive noise channels such as bandlimited AWGN or additive channel with colored noise. Although much more complicated, it is interesting to derive the similar results for multivariable plants with parallel channels in the feedback loop where certain directional characteristics will come into play.

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