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Calculus of Variations

Optimal transport and Perelman's reduced volume

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Abstract We show that a certain entropy-like function is convex, under an optimal transport problem that is adapted to Ricci flow. We use this to reprove the monotonicity of Perelman's reduced volume.

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1 Introduction

One of the major tools introduced by Perelman is his reduced volume $\widetilde{V}$ [21, Sect. 7]. This is a certain geometric quantity which is monotonically nondecreasing in time when one has a Ricci flow solution. Perelman's main use of the reduced volume was to rule out local collapsing in a Ricci flow.

Before giving his rigorous proof that $\widetilde{V}$ is monotonic, Perelman gave a heuristic argument [21, Sect. 6]. Given a Ricci flow solution $(M, g(\tau))$ on a compact manifold M, where τ is backward time, Perelman considered the manifold $\widetilde{M} = M \times S^N \times \mathbb{R}^+$ with the Riemannian metric

$$\widetilde{g} = g(\tau) + 2N\tau g_{S^N} + \left(\frac{N}{2\tau} + R\right) d\tau^2. \tag{1.1}$$

Here R denotes the scalar curvature and g_{S^N} is the metric on S^N with constant sectional curvature 1. Perelman showed that the Ricci curvatures of $\widetilde{M}$ vanish to leading order in N. Now the Bishop–Gromov inequality says that if a complete Riemannian manifold Z has nonnegative Ricci curvature then $r^{-\dim(Z)}\ \mathrm{vol}(B_r(z))$ is nonincreasing in r. Perelman formally

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applied the Bishop-Gromov inequality to $\widetilde{M}$, translated the result back down to M and took the limit when $N \to \infty$, to get the monotonicity of $\widetilde{V}$.

In another direction, there has been recent work showing the equivalence between the nonnegative Ricci curvature of a Riemannian manifold M, and the convexity (in time) of certain entropy functions in an optimal transport problem on M [4,15,19,23–25]. A survey is in [13] and a detailed exposition is in Villani's book [28] (background information on optimal transport is in Villani's books [27,28]). In view of Perelman's heuristic argument, it is natural to wonder whether having a Ricci flow solution $(M, g(t))$ implies the convexity of an entropy in some optimal transport problem on M. The idea is that the asymptotic nonnegative Ricci curvature on $\widetilde{M}$ should imply the asymptotic convexity of the entropy in an optimal transport problem on $\widetilde{M}$, which should then translate to a statement about optimal transport on M.

It turns out that this can be done. The optimal transport problem on M has a cost function coming from Perelman's $\mathcal{L}$-functional. This sort of transport problem was introduced by Topping [26], as described below, with the purpose of constructing certain monotonic quantities for a Ricci flow. Bernard–Buffoni [2] and Villani [28, Chaps. 7,10,13] gave analytic results for general time-dependent cost functions.

In fact, there are three relevant costs for Ricci flow: one corresponding to Perelman's $\mathcal{L}$-functional (which we will call $\mathcal{L}_-$), one corresponding to the Feldman–Ilmanen–Ni $\mathcal{L}_+$-functional [6] and a third one which we call $\mathcal{L}_0$. In the case of the $\mathcal{L}_-$-cost, the main result of the paper is the following.

Theorem 1 *Suppose that $(M, g(\tau))$ is a Ricci flow solution on a connected closed n-dimensional manifold M, where τ denotes backward time. Let $c(\tau)$ be the displacement interpolation in an optimal transport problem between absolutely continuous probability measures $c(\tau_0)$ and $c(\tau_1)$, with $\mathcal{L}_-$-cost. Then $\mathcal{E}(c(\tau)) + \int_M \phi(\tau) dc(\tau) + \frac{n}{2}\log(\tau)$ is convex in the variable $s = \tau^{-\frac{1}{2}}$.*

Here $\mathcal{E}(c(\tau))$ is the (negative) relative entropy of $c(\tau)$ with respect to the time-τ Riemannian volume density. The function $\phi(\tau)$ is the potential for the velocity field in the displacement interpolation.

We show that the monotonicity of Perelman's reduced volume $\widetilde{V}$ is a consequence of Theorem 1; see Corollary 8.

There are two main approaches to optimal transport problems: the Eulerian approach and the Lagrangian approach. Let $P(M)$ denote the Borel probability measures on a static Riemannian manifold M and let $P^\infty(M)$ denote those with a smooth positive density. The Eulerian approach of Benamou–Brenier considers smooth maps $c : [t_0, t_1] \to P^\infty(M)$ that minimize an action $E(c)$, among all such curves with the same endpoints [3]. In the associated Otto calculus, one considers $P^\infty(M)$ to be an infinite-dimensional Riemannian manifold and $E(c)$ to be the corresponding energy of the curve c, so the Euler–Lagrange equation for E becomes the geodesic equation on $P^\infty(M)$ [18]. Otto and Villani used this approach to compute the time-derivatives of the entropy function $\mathcal{E}$ along the curve c [19].

The Lagrangian approach to optimal transport considers a displacement interpolation c, i.e. a geodesic in the Eulerian approach, to be specified by the family of geodesics in M that describe the trajectories taken by particles in the original mass distribution $c(t_0)$, when transporting it to the final mass distribution $c(t_1)$. In the case of optimal transport on Riemannian manifolds, the Lagrangian approach was developed by McCann [16] and Cordero-Erausquin et al. [4].

Comparing the two approaches, the Eulerian approach is perhaps more insightful whereas the Lagrangian approach is better suited to deal with the regularity issues that arise in optimal transport (see, however, the papers of Daneri–Savaré [5] and Otto–Westdickenberg [20], which prove results about optimal transport in $P(M)$ using the Eulerian approach along with density arguments). Much of the present paper consists of describing an Otto calculus which is adapted for the optimal transport of measures under a Ricci flow background.

There has been earlier work relating optimal transport to Ricci flow. The author [10] and McCann–Topping [17] observed that under a Ricci flow background, if $c_1(t)$ and $c_2(t)$ are solutions of the backward heat equation on $P(M)$ then the Wasserstein distance $W_2(c_1(t), c_2(t))$ is monotonically nondecreasing in t. A detailed proof using the Lagrangian approach appears in [17]. McCann–Topping noted that this monotonicity property characterizes supersolutions to the Ricci flow equation. In follow-up work, Topping considered optimal transport with the $\mathcal{L}_-$-cost function and showed the monotonicity of a certain distance function between the measures c_1 and c_2, when taken at different but related times. We refer to [26] for the precise statement. He then used this to rederive the monotonicity of Perelman's $\mathcal{W}$-functional. In the Lagrangian proof of Theorem 1 we use Topping's calculations for the τ-derivatives of $\mathcal{E}(c(\tau))$; see Remark 7.

The outline of this paper is as follows. In Sect. 2, we review the Otto calculus for optimal transport on a manifold with a time-independent Riemannian metric. In Sect. 3, we use the Otto calculus to prove that if $(M, g(t))$ is a Ricci flow solution and $c_1(t)$, $c_2(t)$ are solutions of the backward heat equation in $P^\infty(M)$ then the Wasserstein distance $W_2(c_1(t), c_2(t))$ is monotonically nondecreasing in t. In Sect. 4, we introduce the $\mathcal{L}_0$-cost. We give an Otto calculus for optimal transport with $\mathcal{L}_0$-cost, under a background Ricci flow solution. We then show the $\mathcal{L}_0$-analog of Theorem 1 above. In Sect. 5, we give the $\mathcal{L}_0$-analog of Topping's monotonicity statement regarding the distance between two solutions of the backward heat equation on measures. We use this to reprove the monotonicity of Perelman's $\mathcal{F}$-functional. In Sect. 6, we give an Otto calculus for optimal transport with $\mathcal{L}_-$-cost, under a background Ricci flow solution. In Sect. 7, we prove Theorem 1 and we use it to reprove the monotonicity of Perelman's reduced volume. In Sect. 8, we discuss what Ricci flow should mean on a smooth metric-measure space. In Appendix 8, we indicate how the results of Sects. 6 and 7 extend to the $\mathcal{L}_+$-cost.

Regarding the overall method of proof in this paper, calculations in the Eulerian formalism can be considered to be either rigorous statements on $P^\infty(M)$ or formal statements on $P(M)$. When a suitable density result is available, one can use the Eulerian methods to give rigorous proofs on $P(M)$. In this way, we give rigorous Eulerian proofs on $P(M)$ of the statements in Sects. 2 and 3, making use of the nontrivial Otto–Westdickenberg density result [20]. Sections 4–7 contain calculations in the Eulerian framework under a Ricci flow background. We expect that one can extend these calculations to rigorous proofs on $P(M)$, by adapting the density methods of [5] or [20] to the setting of time-dependent cost functions. We do not address this issue here. Consequently, we revert to Lagrangian methods when we want to give rigorous proofs in $P(M)$ of the statements in Sects. 4–7.

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2 Otto calculus

This section is mostly concerned with known results about optimal transport on a fixed Riemannian manifold M. It is a warmup for the later sections, which extend the results to the case when the Riemannian metric evolves under the Ricci flow.

We use the Otto calculus to give rigorous proofs of certain statements about the space of smooth probability measures $P^\infty(M)$. These proofs can then be considered as formal proofs of the analogous statements on the space of all probability measures $P(M)$. The rigorous proofs of the statements on $P(M)$ are usually done by the Lagrangian approach, but one can also use the density of $P^\infty(M)$ in $P(M)$ [5,20]. Most of the calculations in this section can be extracted from [19] and [20].

In what follows, we use the Einstein summation convention freely.

Let (M, g) be a smooth connected closed (= compact boundaryless) Riemannian manifold of dimension $n > 0$. We denote the Riemannian density by dvol_M. Let $P(M)$ denote the space of Borel probability measures on M, equipped with the Wasserstein metric W_2. For relevant results about optimal transport and the Wasserstein metric, we refer to [15, Sects. 1 and 2] and references therein. A fuller exposition is in the books [27] and [28]. As $P(M)$ is a length space, it makes sense to talk about geodesics in $P(M)$, which we will always take to be minimizing and parametrized proportionately to arc-length.

Put

$$P^\infty(M) = \left\{ \rho \, \mathrm{dvol}_M : \rho \in C^\infty(M), \rho > 0, \int_M \rho \, \mathrm{dvol}_M = 1 \right\}. \tag{2.1}$$

Then $P^\infty(M)$ is a dense subset of $P(M)$, as is the complement of $P^\infty(M)$ in $P(M)$. For the purposes of this paper, we give $P^\infty(M)$ the smooth topology (this differs from the subspace topology on $P^\infty(M)$ coming from its inclusion in $P(M)$). Then $P^\infty(M)$ has the structure of an infinite-dimensional smooth manifold in the sense of [8]. The formal calculations in this section are rigorous calculations on the smooth manifold $P^\infty(M)$.

Given $\phi \in C^\infty(M)$, define a vector field V_ϕ on $P^\infty(M)$ by saying that for $F \in C^\infty(P^\infty(M))$,

$$\begin{aligned} (V_\phi F)(\rho \, \mathrm{dvol}_M) &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} F\left(\rho \, \mathrm{dvol}_M - \epsilon \, \nabla^i(\rho \nabla_i \phi) \, \mathrm{dvol}_M\right) \\ &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} F\left(\Phi^\epsilon_*(\rho \, \mathrm{dvol}_M)\right), \end{aligned} \tag{2.2}$$

where $\Phi^\epsilon(m) = \exp_m(\epsilon \nabla_m \phi)$. The map $\phi \to V_\phi$ passes to an isomorphism $C^\infty(M)/\mathbb{R} \to T_{\rho \, \mathrm{dvol}_M} P^\infty(M)$. This parametrization of $T_{\rho \, \mathrm{dvol}_M} P^\infty(M)$ goes back to Otto's paper [18]; see [1] for further discussion. Otto's Riemannian metric G on $P^\infty(M)$ is given [18] by

$$\begin{aligned} G(V_{\phi_1}, V_{\phi_2})(\rho \, \mathrm{dvol}_M) &= \int_M \langle \nabla \phi_1, \nabla \phi_2 \rangle \rho \, \mathrm{dvol}_M \\ &= - \int_M \phi_1 \nabla^i(\rho \nabla_i \phi_2) \, \mathrm{dvol}_M \, . \end{aligned} \tag{2.3}$$

In view of (2.2), we write $\delta_{V_\phi} \rho = -\nabla^i(\rho \nabla_i \phi)$. Then

$$G(V_{\phi_1}, V_{\phi_2})(\rho \, \mathrm{dvol}_M) = \int_M \phi_1 \delta_{V_{\phi_2}} \rho \, \mathrm{dvol}_M = \int_M \phi_2 \delta_{V_{\phi_1}} \rho \, \mathrm{dvol}_M \, . \tag{2.4}$$

We now relate the Riemannian metric G to the Wasserstein metric W_2. In [19] it was heuristically shown that the geodesic distance coming from (2.4) equals the Wasserstein metric.

To give a rigorous relation, we recall that a curve $c : [0, 1] \to P(M)$ has a length given by

$$L(c) = \sup_{J \in \mathbb{N}} \sup_{0=s_0 \le s_1 \le \cdots \le s_J = 1} \sum_{j=1}^{J} W_2\left(c(s_{j-1}), c(s_j)\right). \tag{2.5}$$

From the triangle inequality, the expression $\sum_{j=1}^{J} W_2\left(c(s_{j-1}), c(s_j)\right)$ is nondecreasing under a refinement of the partition $0 = s_0 \le s_1 \le \cdots \le s_J = 1$.

If $c : [0, 1] \to P^\infty(M)$ is a smooth curve in $P^\infty(M)$ then we write $c(s) = \rho(s)\ \mathrm{dvol}_M$ and let $\phi(s) \in C^\infty(M)$ satisfy

$$\frac{\partial \rho}{\partial s} = -\nabla^i \left(\rho \nabla_i \phi\right). \tag{2.6}$$

It is easy to see, using the spectral theory of the weighted Laplacian on $L^2(M, \rho(s)\ \mathrm{dvol}_M)$, that $\phi(s)$ exists. Note that $\phi(s)$ is uniquely defined up to an additive constant. The Riemannian length of c, as computed using (2.3), is

$$\int_0^1 \sqrt{G(c'(s), c'(s))} ds = \int_0^1 \left(\int_M |\nabla \phi(s)|^2 \rho(s)\, \mathrm{dvol}_M \right)^{\frac{1}{2}} ds. \tag{2.7}$$

Theorem 2 [12, Proposition 1] *If $c : [0, 1] \to P^\infty(M)$ is a smooth immersed curve then the two notions of length agree, in the sense that*

$$L(c) = \int_0^1 \sqrt{G(c'(s), c'(s))} ds. \tag{2.8}$$

Next, consider the Lagrangian

$$E(c) = \frac{1}{2} \int_0^1 G\left(c'(s), c'(s)\right) ds = \frac{1}{2} \int_0^1 \int_M |\nabla \phi(s)|^2 \rho(s)\, \mathrm{dvol}_M\, ds. \tag{2.9}$$

Theorem 3 [20, Proposition 4.3] *Fix measures $\rho_0\, \mathrm{dvol}_M, \rho_1\, \mathrm{dvol}_M \in P^\infty(M)$. Then the infimum of E, over smooth paths in $P^\infty(M)$ with those endpoints, is $\frac{1}{2} W_2(\rho_0\, \mathrm{dvol}_M, \rho_1\, \mathrm{dvol}_M)^2$.*

In general we cannot replace the "inf" in the statement of Theorem 3 by "min", since the Wasserstein geodesic connecting $\rho_0\, \mathrm{dvol}_M$ and $\rho_1\, \mathrm{dvol}_M$ may not lie entirely in $P^\infty(M)$.

We now compute the first variation of E.

Proposition 1 *Let*

$$\rho\, \mathrm{dvol}_M : [0, 1] \times [t_0 - \epsilon, t_0 + \epsilon] \to P^\infty(M) \tag{2.10}$$

be a smooth map, with $\rho \equiv \rho(s, t)$. Let

$$\phi : [0, 1] \times [t_0 - \epsilon, t_0 + \epsilon] \to C^\infty(M) \tag{2.11}$$

be a smooth map that satisfies (2.6), with $\phi \equiv \phi(s, t)$. Then

$$\left.\frac{dE}{dt}\right|_{t=t_0} = \left. \int_M \phi \frac{\partial \rho}{\partial t}\, \mathrm{dvol}_M \right|_{s=0}^{1} - \int_0^1 \int_M \left(\frac{\partial \phi}{\partial s} + \frac{1}{2} |\nabla \phi|^2 \right) \frac{\partial \rho}{\partial t}\, \mathrm{dvol}_M\, ds, \tag{2.12}$$

where the right-hand side is evaluated at time $t = t_0$.

Proof We have

$$\frac{dE}{dt} = \int_0^1 \int_M \left(\left\langle \nabla\phi, \nabla \frac{\partial \phi}{\partial t} \right\rangle \rho + \frac{1}{2} |\nabla \phi|^2 \frac{\partial \rho}{\partial t} \right) \mathrm{dvol}_M \, ds. \tag{2.13}$$

For a fixed $f \in C^\infty(M)$, from (2.6),

$$\int_M f \frac{\partial \rho}{\partial s} \mathrm{dvol}_M = \int_M \langle \nabla f, \nabla \phi \rangle \rho \, \mathrm{dvol}_M \, . \tag{2.14}$$

Hence

$$\int_M f \frac{\partial^2 \rho}{\partial s \partial t} \mathrm{dvol}_M = \int_M \left(\left\langle \nabla f, \nabla \frac{\partial \phi}{\partial t} \right\rangle \rho + \langle \nabla f, \nabla \phi \rangle \frac{\partial \rho}{\partial t} \right) \mathrm{dvol}_M \, . \tag{2.15}$$

Taking $f = \phi$ gives

$$\int_M \phi \frac{\partial^2 \rho}{\partial s \partial t} \mathrm{dvol}_M = \int_M \left(\left\langle \nabla \phi, \nabla \frac{\partial \phi}{\partial t} \right\rangle \rho + |\nabla \phi|^2 \frac{\partial \rho}{\partial t} \right) \mathrm{dvol}_M \, . \tag{2.16}$$

Equations (2.13) and (2.16) give

$$\begin{aligned} \frac{dE}{dt} &= \int_0^1 \int_M \left(\phi \frac{\partial^2 \rho}{\partial s \partial t} - \frac{1}{2} |\nabla \phi|^2 \frac{\partial \rho}{\partial t} \right) \mathrm{dvol}_M \, ds \\ &= \int_0^1 \int_M \left(\frac{\partial}{\partial s} \left(\phi \frac{\partial \rho}{\partial t} \right) - \left(\frac{\partial \phi}{\partial s} + \frac{1}{2} |\nabla \phi|^2 \right) \frac{\partial \rho}{\partial t} \right) \mathrm{dvol}_M \, ds, \end{aligned} \tag{2.17}$$

from which the proposition follows. □

From (2.12), the Euler–Lagrange equation for E is

$$\frac{\partial \phi}{\partial s} = - \frac{1}{2} |\nabla \phi|^2 + \alpha(s), \tag{2.18}$$

where $\alpha \in C^\infty([0, 1])$. Changing ϕ by a spatially constant function, we can assume that $\alpha = 0$, so the Euler–Lagrange equation for E becomes the Hamilton–Jacobi equation

$$\frac{\partial \phi}{\partial s} = - \frac{1}{2} |\nabla \phi|^2. \tag{2.19}$$

If a geodesic in $P(M)$ happens to be a smooth curve in $P^\infty(M)$ then it will satisfy (2.19).

For any $0 \le s' < s'' \le 1$, the viscosity solution of (2.19) satisfies

$$\phi(s'')(m'') = \inf_{m' \in M} \left(\phi(s')(m') + \frac{d_M(m', m'')^2}{s'' - s'} \right). \tag{2.20}$$

Then the solution of (2.6) satisfies

$$\rho(s'') \, \mathrm{dvol}_M = (F_{s', s''})_* (\rho(s') \, \mathrm{dvol}_M), \tag{2.21}$$

where the transport map $F_{s', s''} : M \to M$ is given by

$$F_{s', s''}(m') = \exp_{m'} \left((s'' - s') \nabla_{m'} \phi(s') \right). \tag{2.22}$$

We now give some simple results in the Otto calculus.

Proposition 2 *Assuming* (2.6) *and* (2.19), *we have*

$$\frac{d}{ds}\int_M \phi\rho \,\mathrm{dvol}_M = \frac{1}{2}\int_M |\nabla\phi|^2\rho \,\mathrm{dvol}_M \tag{2.23}$$

and

$$\frac{1}{2}\frac{d}{ds}\int_M |\nabla\phi|^2 \rho \,\mathrm{dvol}_M = 0. \tag{2.24}$$

Proof First,

$$\begin{aligned}\frac{d}{ds}\int_M \phi\,\rho\,\mathrm{dvol}_M &= -\frac{1}{2}\int_M |\nabla\phi|^2\rho\,\mathrm{dvol}_M - \int_M \phi\,\nabla^i(\rho\nabla_i\phi)\,\mathrm{dvol}_M \\ &= \frac{1}{2}\int_M |\nabla\phi|^2\rho\,\mathrm{dvol}_M\,. \end{aligned}\tag{2.25}$$

Next, using (2.18),

$$\begin{aligned}&\frac{1}{2}\frac{d}{ds}\int_M |\nabla\phi|^2\rho\,\mathrm{dvol}_M \\ &= -\frac{1}{2}\int_M \langle\nabla\phi,\ \nabla(|\nabla\phi|^2)\rangle\,\rho\,\mathrm{dvol}_M - \frac{1}{2}\int_M |\nabla\phi|^2\,\nabla^i(\rho\nabla_i\phi)\,\mathrm{dvol}_M = 0.\end{aligned}\tag{2.26}$$

This proves the proposition. □

Equation (2.24) is just the statement that a geodesic in $P^\infty(M)$ has constant speed. Equation (2.23) says that $\int_M \phi\,\rho\,\mathrm{dvol}_M$ is proportionate to the arc length along the geodesic.

The (negative) entropy $\mathcal{E} : P^\infty(M) \to \mathbb{R}$ is given by

$$\mathcal{E}(\rho\,\mathrm{dvol}_M) = \int_M \rho\,\log(\rho)\,\mathrm{dvol}_M\,. \tag{2.27}$$

We now compute its first two derivatives along a curve in $P^\infty(M)$.

Proposition 3 *Assuming* (2.6), *we have*

$$\frac{d\mathcal{E}}{ds} = \int_M \langle\nabla\phi,\nabla\rho\rangle\,\mathrm{dvol}_M = -\int_M \nabla^2\phi\,\rho\,\mathrm{dvol}_M \tag{2.28}$$

and

$$\begin{aligned}\frac{d^2\mathcal{E}}{ds^2} &= -\int_M \left(\frac{\partial\phi}{\partial s} + \frac{1}{2}|\nabla\phi|^2\right)\nabla^2\rho\,\mathrm{dvol}_M \\ &\quad + \int_M \left(|\,\mathrm{Hess}\,\phi|^2 + \mathrm{Ric}(\nabla\phi,\nabla\phi)\right)\rho\,\mathrm{dvol}_M\,. \end{aligned}\tag{2.29}$$

Proof First,

$$\frac{d\mathcal{E}}{ds} = -\int_M (\log(\rho)+1)\,\nabla^i(\rho\,\nabla_i\phi)\,\mathrm{dvol}_M = \int_M \langle\nabla\phi,\nabla\rho\rangle\,\mathrm{dvol}_M\,. \tag{2.30}$$

Then

$$\frac{d^2\mathcal{E}}{ds^2} = \int_M \left\langle \nabla\left(\frac{\partial\phi}{\partial s} + \frac{1}{2}|\nabla\phi|^2\right), \nabla\rho\right\rangle \mathrm{dvol}_M$$
$$-\frac{1}{2}\int_M \langle\nabla\left(|\nabla\phi|^2\right), \nabla\rho\rangle\ \mathrm{dvol}_M + \int_M \left\langle\nabla\phi, \nabla\left(-\nabla^i(\rho\nabla_i\phi)\right)\right\rangle \mathrm{dvol}_M$$
$$= -\int_M \left(\frac{\partial\phi}{\partial s} + \frac{1}{2}|\nabla\phi|^2\right)\nabla^2\rho\ \mathrm{dvol}_M$$
$$+\frac{1}{2}\int_M \nabla^2\left(|\nabla\phi|^2\right)\rho\,\mathrm{dvol}_M - \int_M \langle\nabla\nabla^2\phi, \nabla\phi\rangle\,\rho\ \mathrm{dvol}_M$$
$$= -\int_M \left(\frac{\partial\phi}{\partial s} + \frac{1}{2}|\nabla\phi|^2\right)\nabla^2\rho\ \mathrm{dvol}_M + \int_M \left(|\operatorname{Hess}\phi|^2 + \operatorname{Ric}(\nabla\phi, \nabla\phi)\right)\rho\,\mathrm{dvol}_M, \tag{2.31}$$

where we used the Bochner identity in the last line. This proves the proposition. □

Corollary 1 *Assuming* (2.6) *and* (2.19), *if* $\operatorname{Ric}(M, g) \geq 0$ *then* $\frac{d^2\mathcal{E}}{ds^2} \geq 0$. *That is,* $\mathcal{E}$ *is convex along geodesics in* $P^\infty(M)$.

Remark 1 In view of Proposition 2, Corollary 1 would still hold if we replaced $\mathcal{E}(\rho(s)\ \mathrm{dvol}_M)$ by $\mathcal{E}(\rho(s)\ \mathrm{dvol}_M) \pm \int_M \phi(s)\,\rho(s)\ \mathrm{dvol}_M$. This modification will be crucial in later sections.

Corollary 1 was proven in [19]. The extension of Corollary 1 to $P(M)$ was proven in [4]. We now give a slight refinement of the first variation result.

Proposition 4 *Under the assumptions of Proposition* 1,

$$\left.\frac{dE}{dt}\right|_{t=t_0} = \left.\int_M \phi\left(\frac{\partial\rho}{\partial t} - \nabla^2\rho\right)\mathrm{dvol}_M\right|_{s=0}^{1}$$
$$-\int_0^1\int_M \left(\frac{\partial\phi}{\partial s} + \frac{1}{2}|\nabla\phi|^2\right)\left(\frac{\partial\rho}{\partial t} - \nabla^2\rho\right)\mathrm{dvol}_M\,ds$$
$$-\int_0^1\int_M \left(|\operatorname{Hess}\phi|^2 + \operatorname{Ric}(\nabla\phi, \nabla\phi)\right)\rho\,\mathrm{dvol}_M\,ds, \tag{2.32}$$

where the right-hand side is evaluated at time $t = t_0$.

Proof Integrating (2.29) with respect to s gives

$$-\left.\int_M \phi\nabla^2\rho\,\mathrm{dvol}_M\right|_{s=0}^{1} = -\int_0^1\int_M \left(\frac{\partial\phi}{\partial s} + \frac{1}{2}|\nabla\phi|^2\right)\nabla^2\rho\,\mathrm{dvol}_M\,ds$$
$$+\int_0^1\int_M \left(|\operatorname{Hess}\phi|^2 + \operatorname{Ric}(\nabla\phi, \nabla\phi)\right)\rho\,\mathrm{dvol}_M\,ds. \tag{2.33}$$

The proposition follows from combining (2.12) and (2.33). □

Corollary 2 [18,20,25] *Suppose that* $\mathrm{Ric}(M, g) \geq 0$. *Let* $e^{t\nabla^2}$ *be the heat flow on* $P^\infty(M)$. *Then for* $\mu_0, \mu_1 \in P^\infty(M)$ *and* $t \geq 0$,

$$W_2\left(e^{t\nabla^2}\mu_0, e^{t\nabla^2}\mu_1\right) \leq W_2(\mu_0, \mu_1). \tag{2.34}$$

Proof Using Theorem 3, given $\epsilon > 0$, choose a smooth curve $c : [0, 1] \to P^\infty(M)$ with $c(0) = \mu_0$ and $c(1) = \mu_1$ so that $E(c) \leq \frac{1}{2} W_2(\mu_0, \mu_1)^2 + \epsilon$. Define $c_t : [0, 1] \to P^\infty(M)$ by $c_t(s) = e^{t\nabla^2} c(s)$. By Proposition 4, $E(c_t)$ is nonincreasing in t. Hence $\frac{1}{2} W_2(c_t(0), c_t(1))^2 \leq E(c_t) \leq E(c_0) \leq \frac{1}{2} W_2(\mu_0, \mu_1)^2 + \epsilon$. As ϵ was arbitrary, the corollary follows. □

We recall that $n = \dim(M)$. We now give a new convexity result concerning Wasserstein geodesics.

Proposition 5 *If* $\mathrm{Ric}(M, g) \geq 0$ *then* $s\mathcal{E} + ns \log(s)$ *is convex along a Wasserstein geodesic in* $P^\infty(M)$, *defined for* $s \in [0, 1]$.

Proof From (2.29), $\frac{d^2\mathcal{E}}{ds^2} \geq 0$. As

$$\frac{d^2}{ds^2}(s\mathcal{E} + ns \log(s)) = s\frac{d^2\mathcal{E}}{ds^2} + 2\frac{d\mathcal{E}}{ds} + \frac{n}{s}, \tag{2.35}$$

it suffices to show that

$$\left(\frac{d\mathcal{E}}{ds}\right)^2 \leq n\frac{d^2\mathcal{E}}{ds^2}. \tag{2.36}$$

Now

$$\begin{aligned}\left(\frac{d\mathcal{E}}{ds}\right)^2 &= \left(\int_M \nabla^2\phi\rho \,\mathrm{dvol}_M\right)^2 \leq \int_M (\nabla^2\phi)^2 \rho \,\mathrm{dvol}_M \\ &\leq n \int_M |\,\mathrm{Hess}\,\phi|^2 \rho \,\mathrm{dvol}_M \leq n\frac{d^2\mathcal{E}}{ds^2},\end{aligned} \tag{2.37}$$

which proves the proposition. □

Remark 2 More generally, suppose that a background measure $\nu = e^{-\Psi}\,\mathrm{dvol}_M \in P^\infty(M)$ is such that (M, ν) has $\mathrm{Ric}_N \geq 0$ in the sense of [15, Definition 0.10]. Recall the class of functions DC_∞ in [15, Equation (0.5)]. Given $U \in DC_\infty$, define $U_\nu : P^\infty(M) \to \mathbb{R}$ as in [15, Equation (0.1)]. Then using the calculations of [15, Appendix D], one can show that $sU_\nu + Ns \log(s)$ is convex along a Wasserstein geodesic in $P^\infty(M)$.

Now define $\mathcal{E} : P(M) \to \mathbb{R} \cup \{\infty\}$ by

$$\mathcal{E}(\mu) = \begin{cases} \int_M \rho \log(\rho)\,\mathrm{dvol}_M & \text{if } \mu = \rho\,\mathrm{dvol}_M, \\ \infty & \text{if } \mu \text{ is not absolutely continuous with respect to } \mathrm{dvol}_M. \end{cases} \tag{2.38}$$

Proposition 6 *If* $\mathrm{Ric}(M, g) \geq 0$ *then* $s\mathcal{E} + ns \log(s)$ *is convex along a Wasserstein geodesic in* $P(M)$.

Proof The proof uses the Lagrangian formulation of optimal transport [15, Pf. of Theorem 7.3]. We omit the details. □

Remark 3 Similarly, in the setup of Remark 2, one has that $sU_\nu + Ns\log(s)$ is convex along a Wasserstein geodesic in $P(M)$. It appears that most of the results of [15] could be derived using the class of functions DC_∞ and the functional $sU_\nu + Ns\log(s)$. The paper [15] used instead the class of functions DC_N and the function U_ν.

3 Wasserstein distance and Ricci flow

In this section we discuss a first monotonicity relation between Ricci flow and optimal transport. Namely, suppose that the Ricci flow equation is satisfied and we have two solutions $c_0(t), c_1(t)$ of the backward heat flow, acting on probability measures on M. Then the Wasserstein distance $W_2(c_0(t), c_1(t))$ is nondecreasing in t. We first give a quick formal proof. We then write out a rigorous proof using the Otto calculus. A proof using the Lagrangian approach appears in [17].

Let $(M, g(\cdot))$ be a solution to the Ricci flow equation

$$\frac{dg}{dt} = -2\,\mathrm{Ric}\,. \tag{3.1}$$

Then

$$\frac{d(\mathrm{dvol}_M)}{dt} = -R\,\mathrm{dvol}_M\,. \tag{3.2}$$

The metric G on $P^\infty(M)$, from (2.3), is also t-dependent. Fix $\mu \in P^\infty(M)$ and $\delta\mu \in T_\mu P^\infty(M)$. At time t, we can write $\mu = \rho\,\mathrm{dvol}_M$ and $\delta\mu = V_\phi$ where ρ and ϕ are t-dependent.

We now compute the first derivative of G with respect to t.

Proposition 7

$$\frac{dG}{dt}(\delta\mu, \delta\mu) = -2\int_M \mathrm{Ric}(\nabla\phi, \nabla\phi)d\mu. \tag{3.3}$$

Proof Letting g^* denote the dual inner product on T^*M, we can write

$$G(\delta\mu, \delta\mu) = \int_M g^*(d\phi, d\phi)d\mu. \tag{3.4}$$

Since the differential d is invariantly defined, we have $\frac{d}{dt}d\phi = d\frac{d\phi}{dt}$. Then

$$\frac{dG}{dt}(\delta\mu, \delta\mu) = 2\int_M \mathrm{Ric}(\nabla\phi, \nabla\phi)d\mu + 2\int_M g^*\left(d\phi, d\frac{d\phi}{dt}\right)d\mu. \tag{3.5}$$

For any fixed $f \in C^\infty(M)$, we have

$$\int_M f\,d(\delta\mu) = \int_M g^*(df, d\phi)d\mu. \tag{3.6}$$

Differentiating with respect to t gives

$$0 = 2\int_M \mathrm{Ric}(\nabla f, \nabla\phi)d\mu + \int_M g^*\left(df, d\frac{d\phi}{dt}\right)d\mu. \tag{3.7}$$

Putting $f = \phi$ gives

$$0 = 2 \int_M \mathrm{Ric}(\nabla\phi, \nabla\phi) d\mu + \int_M g^* \left(d\phi, d\frac{d\phi}{dt} \right) d\mu. \tag{3.8}$$

Equation (3.3) follows from combining (3.5) and (3.8). □

Let $\mathrm{grad}\, \mathcal{E}$ denote the formal gradient of $\mathcal{E}$ on $P^\infty(M)$ and let Hess $\mathcal{E}$ denote its Hessian. Now the Lie derivative of the metric G with respect to the vector field $\mathrm{grad}\, \mathcal{E}$ is $\mathcal{L}_{\mathrm{grad}\, \mathcal{E}} G = 2\, \mathrm{Hess}\ \mathcal{E}$. From Proposition 3,

$$(\mathrm{Hess}\ \mathcal{E})(V_\phi, V_\phi) = \int_M \left(|\,\mathrm{Hess}\ \phi|^2 + \mathrm{Ric}(\nabla\phi, \nabla\phi) \right) \rho\, \mathrm{dvol}_M\,. \tag{3.9}$$

Then from (3.3) and (3.9),

$$\frac{dG}{dt} + \mathcal{L}_{\mathrm{grad}\, \mathcal{E}} G \geq 0. \tag{3.10}$$

Let $\{\phi_t\}$ be the 1-parameter group generated by $\mathrm{grad}\, \mathcal{E}$. Equation (3.10) implies that $\phi_t^* G(t)$ is nondecreasing in t. In particular, for any $\mu_0, \mu_1 \in P^\infty(M)$ the Wasserstein distance $d_W(\phi_t(\mu_0), \phi_t(\mu_1))$ is nondecreasing in t.

It remains to compute the flow $\{\phi_t\}$. This is a well-known calculation.

Lemma 1 *In $T_{\rho\ \mathrm{dvol}_M} P^\infty(M)$,*

$$\mathrm{grad}\, \mathcal{E} = V_{\log \rho}. \tag{3.11}$$

Proof From (2.28), for all $V_\phi \in T_{\rho\ \mathrm{dvol}_M} P^\infty(M)$, we have

$$\begin{aligned} G(V_\phi, \mathrm{grad}\, \mathcal{E})(\rho\, \mathrm{dvol}_M) &= (V_\phi \mathcal{E})(\rho\, \mathrm{dvol}_M) = \int_M \langle \nabla\phi, \nabla\rho \rangle\, \mathrm{dvol}_M \\ &= \int_M \langle \nabla\phi, \nabla \log \rho \rangle \rho\, \mathrm{dvol}_M = G(V_\phi, V_{\log \rho})(\rho\, \mathrm{dvol}_M), \end{aligned} \tag{3.12}$$

from which the lemma follows. □

Lemma 2 *For $\mu \in P^\infty(M)$, if $\mu_t = \phi_t(\mu)$ then*

$$\frac{d\mu_t}{dt} = -\nabla^2 \mu_t. \tag{3.13}$$

Equivalently, writing $\mu_t = \rho_t\, \mathrm{dvol}_M$, we have

$$\frac{d\rho_t}{dt} = -\nabla^2 \rho_t + R\rho_t. \tag{3.14}$$

Proof Given $\mu = \rho\, \mathrm{dvol}_M$, we can write

$$-\nabla^i (\rho \nabla_i \log \rho)\, \mathrm{dvol}_M = -(\nabla^2 \rho)\, \mathrm{dvol}_M = -\nabla^2 \mu. \tag{3.15}$$

Then (3.13) follows from (3.11) and (3.15). Equation (3.14) follows from (3.2). □

Thus we have formally shown that if $g(t)$ satisfies the Ricci flow equation (3.1) and $\rho_{i,t}$ satisfies the backward heat equation

$$\frac{d\rho_{i,t}}{dt} = -\nabla^2 \rho_{i,t} + R\rho_{i,t} \tag{3.16}$$

for $i \in \{0, 1\}$ then the time-dependent Wasserstein distance $d_W(\rho_{0,t}\, \mathrm{dvol}_M, \rho_{1,t}\, \mathrm{dvol}_M)$ is nondecreasing in t.

We now translate this into a rigorous proof using the Otto calculus. We first derive a general formula for the derivative of the energy functional E along a 1-parameter family of smooth curves in $P^\infty(M)$.

Proposition 8 *Let $g(\cdot)$ solve the Ricci flow equation (3.1) for $t \in [t_0 - \epsilon, t_0 + \epsilon]$. Let*

$$\rho\, \mathrm{dvol}_M : [0, 1] \times [t_0 - \epsilon, t_0 + \epsilon] \to P^\infty(M) \tag{3.17}$$

be a smooth map, with $\rho \equiv \rho(s, t)$. Let

$$\phi : [0, 1] \times [t_0 - \epsilon, t_0 + \epsilon] \to C^\infty(M) \tag{3.18}$$

be a smooth map that satisfies (2.6), with $\phi \equiv \phi(s, t)$. Put

$$E(t) = \frac{1}{2} \int_0^1 \int_M |\nabla \phi|^2 \rho\, \mathrm{dvol}_M\, ds. \tag{3.19}$$

Then

$$\begin{aligned} \left.\frac{dE}{dt}\right|_{t=t_0} &= \left.\int_M \phi \left(\frac{\partial \rho}{\partial t} + \nabla^2 \rho - R\rho \right) \mathrm{dvol}_M \right|_{s=0}^{1} \\ &\quad - \int_0^1 \int_M \left(\frac{\partial \phi}{\partial s} + \frac{1}{2} |\nabla \phi|^2 \right) \left(\frac{\partial \rho}{\partial t} + \nabla^2 \rho - R\rho \right) \mathrm{dvol}_M \\ &\quad + \int_0^1 \int_M |\operatorname{Hess} \phi|^2 \rho\, \mathrm{dvol}_M\ ds, \end{aligned} \tag{3.20}$$

where the right-hand side is evaluated at time $t = t_0$.

Proof We have

$$\frac{dE}{dt} = \int_0^1 \int_M \left(\mathrm{Ric}(\nabla\phi, \nabla\phi)\rho + \left\langle \nabla\phi, \nabla \frac{\partial \phi}{\partial t} \right\rangle \rho + \frac{1}{2} |\nabla\phi|^2 \frac{\partial \rho}{\partial t} - \frac{1}{2} R |\nabla \phi|^2 \rho \right) \mathrm{dvol}_M\, ds. \tag{3.21}$$

For a fixed $f \in C^\infty(M)$,

$$\int_M f \frac{\partial \rho}{\partial s}\, \mathrm{dvol}_M = \int_M \langle \nabla f, \nabla \phi \rangle \rho\, \mathrm{dvol}_M\, . \tag{3.22}$$

Hence

$$\begin{aligned}\int_M f \left(\frac{\partial^2 \rho}{\partial s \partial t} - R \frac{\partial \rho}{\partial s} \right) \mathrm{dvol}_M \\ = \int_M \left(2 \operatorname{Ric}(\nabla f, \nabla \phi) \rho + \left\langle \nabla f, \nabla \frac{\partial \phi}{\partial t} \right\rangle \rho + \langle \nabla f, \nabla \phi \rangle \frac{\partial \rho}{\partial t} - R \langle \nabla f, \nabla \phi \rangle \rho \right) \mathrm{dvol}_M .\end{aligned} \tag{3.23}$$

Taking $f = \phi$ gives

$$\begin{aligned}\int_M \phi \left(\frac{\partial^2 \rho}{\partial s \partial t} - R \frac{\partial \rho}{\partial s} \right) \mathrm{dvol}_M \\ = \int_M \left(2 \operatorname{Ric}(\nabla \phi, \nabla \phi) \rho + \left\langle \nabla \phi, \nabla \frac{\partial \phi}{\partial t} \right\rangle \rho + |\nabla \phi|^2 \frac{\partial \rho}{\partial t} - R |\nabla \phi|^2 \rho \right) \mathrm{dvol}_M .\end{aligned} \tag{3.24}$$

Equations (3.21) and (3.24) give

$$\begin{aligned}\frac{dE}{dt} &= \int_0^1 \int_M \left(\phi \frac{\partial^2 \rho}{\partial s \partial t} - R \phi \frac{\partial \rho}{\partial s} - \frac{1}{2} |\nabla \phi|^2 \frac{\partial \rho}{\partial t} - \operatorname{Ric}(\nabla \phi, \nabla \phi) \rho + \frac{1}{2} R |\nabla \phi|^2 \rho \right) \mathrm{dvol}_M \, ds \\ &= \int_0^1 \int_M \left(\frac{\partial}{\partial s} \left(\phi \frac{\partial \rho}{\partial t} \right) - R \phi \frac{\partial \rho}{\partial s} - \left(\frac{\partial \phi}{\partial s} + \frac{1}{2} |\nabla \phi|^2 \right) \frac{\partial \rho}{\partial t} - \operatorname{Ric}(\nabla \phi, \nabla \phi) \rho \right. \\ &\quad \left. + \frac{1}{2} R |\nabla \phi|^2 \rho \right) \mathrm{dvol}_M \, ds \\ &= \int_M \phi \frac{\partial \rho}{\partial t} \mathrm{dvol}_M \Big|_{s=0}^{1} + \int_0^1 \int_M \left(- R \phi \frac{\partial \rho}{\partial s} - \left(\frac{\partial \phi}{\partial s} + \frac{1}{2} |\nabla \phi|^2 \right) \frac{\partial \rho}{\partial t} - \operatorname{Ric}(\nabla \phi, \nabla \phi) \rho \right. \\ &\quad \left. + \frac{1}{2} R |\nabla \phi|^2 \rho \right) \mathrm{dvol}_M \, ds.\end{aligned} \tag{3.25}$$

From (2.33),

$$\begin{aligned}0 = \int_M \phi \nabla^2 \rho \, \mathrm{dvol}_M \Big|_{s=0}^{1} + \int_0^1 \int_M \left[|\operatorname{Hess} \phi|^2 + \operatorname{Ric}(\nabla \phi, \nabla \phi) \right] \rho \, \mathrm{dvol}_M \, ds \\ - \int_0^1 \int_M \nabla^2 \rho \left(\frac{\partial \phi}{\partial s} + \frac{1}{2} |\nabla \phi|^2 \right) \mathrm{dvol}_M \, ds.\end{aligned} \tag{3.26}$$

Finally,

$$\frac{\partial}{\partial s} \int_M R \phi \rho \, \mathrm{dvol}_M = \int_M \left(R \frac{\partial \phi}{\partial s} \rho + R \phi \frac{\partial \rho}{\partial s} \right) \mathrm{dvol}_M, \tag{3.27}$$

so

$$0 = -\int_M R\phi\rho \,\mathrm{dvol}_M \Big|_{s=0}^{1} + \int_0^1 \int_M \left(R\frac{\partial \phi}{\partial s}\rho + R\phi\frac{\partial \rho}{\partial s} \right) \mathrm{dvol}_M \, ds. \tag{3.28}$$

Adding (3.25), (3.26) and (3.28) gives the proposition. □

Corollary 3 *For $i \in \{0, 1\}$, let $c_i(t)$ be a solution of the backward heat equation (3.13) in $P^\infty(M)$. Then $W_2(c_0(t), c_1(t))$ is nondecreasing in t.*

Proof Fix t_0. Using Theorem 3, given $\epsilon > 0$, choose a smooth curve $c : [0, 1] \to P^\infty(M)$ so that $c(0) = c_0(t_0)$, $c(1) = c_1(t_0)$ and $E(c) \le \frac{1}{2} W_2(c_0(t_0), c_1(t_0))^2 + \epsilon$. For $t \le t_0$, define $c_t : [0, 1] \to P^\infty(M)$ by saying that $c_{t_0}(s) = c(s)$ and $c_t(s)$ satisfies Eq. (3.13) in t. By Proposition 8, $E(c_t)$ is nondecreasing in t. Hence $\frac{1}{2} W_2(c_0(t), c_1(t))^2 \le E(c_t) \le E(c_0) \le \frac{1}{2} W_2(c_0(t_0), c_1(t_0))^2 + \epsilon$. Since ϵ was arbitrary, the corollary follows. □

Remark 4 To see the relation between Corollaries 2 and 3, suppose that M is Ricci flat, in which case the Ricci flow on M is constant. Put $\tau = t_0 - t$. Then the backward heat equation (3.13) in t becomes a forward heat equation in τ. Corollary 2 says that the Wasserstein distance between the heat flows is nonincreasing in τ, i.e. nondecreasing in t.

Corollary 3 was proven using Lagrangian methods in [17].

4 Convexity of the $\mathcal{L}_0$-entropy

In this section we consider an analog $\mathcal{L}_0$ of Perelman's $\mathcal{L}$-functional, which has the same relationship to steady solitons as Perelman's $\mathcal{L}$-functional has to shrinking solitons. Under a Ricci flow, we consider the transport equation associated to the problem of minimizing the $\mathcal{L}_0$-cost. We show the convexity of a modified entropy functional.

Let M be a connected closed manifold and let $g(\cdot)$ be a Ricci flow solution on M.

Definition 1 If $\gamma : [t', t''] \to M$ is a smooth curve then its $\mathcal{L}_0$-length is

$$\mathcal{L}_0(\gamma) = \frac{1}{2} \int_{t'}^{t''} \left(g\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right) + R(\gamma(t), t) \right) dt, \tag{4.1}$$

where the time-t metric $g(t)$ is used to define the integrand.

Let $L_0^{t',t''}(m', m'')$ be the infimum of $\mathcal{L}_0$ over curves γ with $\gamma(t') = m'$ and $\gamma(t'') = m''$.

The Euler–Lagrange equation for the $\mathcal{L}_0$-functional is easily derived to be

$$\nabla_{\frac{d\gamma}{dt}} \left(\frac{d\gamma}{dt} \right) - \frac{1}{2} \nabla R - 2\,\mathrm{Ric}\left(\frac{d\gamma}{dt}, \cdot \right) = 0. \tag{4.2}$$

The $\mathcal{L}_0$-exponential map $\mathcal{L}_0 \exp_{m'}^{t',t''} : T_{m'}M \to M$ is defined by saying that for $V \in T_{m'}M$, one has

$$\mathcal{L}_0 \exp_{m'}^{t',t''}(V) = \gamma(t'') \tag{4.3}$$

where γ is the solution to (4.2) with $\gamma(t') = m'$ and $\frac{d\gamma}{dt}\Big|_{t=t'} = V$.

Definition 2 Given $\mu', \mu'' \in P(M)$, put

$$C_0^{t',t''}\left(\mu', \mu''\right) = \inf_{\Pi} \int\limits_{M \times M} L_0^{t',t''}(m', m'') d\Pi(m', m''), \tag{4.4}$$

where Π ranges over the elements of $P(M \times M)$ whose pushforward to M under projection onto the first (resp. second) factor is μ' (resp. μ''). Given a continuous curve $c : [t', t''] \to P(M)$, put

$$\mathcal{A}_0(c) = \sup_{J \in \mathbb{Z}^+} \sup_{t'=t_0 \le t_1 \le \cdots \le t_J = t''} \sum_{j=1}^{J} C_0^{t_{j-1}, t_j}\left(c(t_{j-1}), c(t_j)\right). \tag{4.5}$$

We can think of $\mathcal{A}_0$ as a generalized energy functional associated to the generalized metric C_0. By [28, Theorem 7.21], $\mathcal{A}_0$ is a coercive action on $P(M)$ in the sense of [28, Definition 7.13]. In particular,

$$C_0^{t',t''}\left(\mu', \mu''\right) = \inf_{c} \mathcal{A}_0(c), \tag{4.6}$$

where c ranges over continuous curves $c : [t', t''] \to P(M)$ with $c(t') = \mu'$ and $c(t'') = \mu''$.

We now consider the equations that come from minimizing the generalized energy functional $\mathcal{A}_0$, when restricted to smooth curves in $P^\infty(M)$. If $c : [t_0, t_1] \to P^\infty(M)$ is a smooth curve in $P^\infty(M)$ then we write $c(t) = \rho(t) \operatorname{dvol}_M$ and let $\phi(t) \in C^\infty(M)$ satisfy

$$\frac{\partial \rho}{dt} = -\nabla^i \left(\rho \nabla_i \phi\right) + R\rho. \tag{4.7}$$

Note that $\phi(t)$ is uniquely defined up to an additive constant. Using (3.2), the scalar curvature term in (4.7) ensures that

$$\frac{d}{dt} \int\limits_M \rho \operatorname{dvol}_M = 0. \tag{4.8}$$

Consider the Lagrangian

$$E_0(c) = \frac{1}{2} \int\limits_{t_0}^{t_1} \int\limits_M \left(|\nabla \phi|^2 + R\right) \rho \operatorname{dvol}_M dt, \tag{4.9}$$

where the integrand at time t is computed using $g(t)$.

Proposition 9 *Let*

$$\rho \operatorname{dvol}_M : [t_0, t_1] \times [-\epsilon, \epsilon] \to P^\infty(M) \tag{4.10}$$

be a smooth map, with $\rho \equiv \rho(t, u)$. Let

$$\phi : [t_0, t_1] \times [-\epsilon, \epsilon] \to C^\infty(M) \tag{4.11}$$

be a smooth map that satisfies (4.7), with $\phi \equiv \phi(t, u)$. Then

$$\left.\frac{dE_0}{du}\right|_{u=0} = \left.\int\limits_M \phi \frac{\partial \rho}{\partial u} \operatorname{dvol}_M \right|_{t=t_0}^{t_1} - \int\limits_{t_0}^{t_1} \int\limits_M \left(\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla \phi|^2 - \frac{1}{2} R\right) \frac{\partial \rho}{\partial u} \operatorname{dvol}_M dt, \tag{4.12}$$

where the right-hand side is evaluated at $u = 0$.

Proof The proof is similar to that of Proposition 1. We omit the details. □

From (4.12), the Euler–Lagrange equation for E_0 is

$$\frac{\partial \phi}{\partial t} = -\frac{1}{2}|\nabla \phi|^2 + \frac{1}{2} R + \alpha(t), \tag{4.13}$$

where $\alpha \in C^\infty([t_0, t_1])$. Changing ϕ by a spatially constant function, we can assume that $\alpha = 0$, so

$$\frac{\partial \phi}{\partial t} = -\frac{1}{2}|\nabla \phi|^2 + \frac{1}{2} R. \tag{4.14}$$

If a smooth curve in $P^\infty(M)$ minimizes E_0, relative to its endpoints, then it will satisfy (4.14). For each $t_0 \le t' < t'' \le t_1$, the viscosity solution of (4.14) satisfies

$$\phi(t'')(m'') = \inf_{m' \in M} \left(\phi(t')(m') + L_0^{t',t''}(m', m'') \right). \tag{4.15}$$

Then the solution of (4.7) satisfies

$$\rho(t'') \, \mathrm{dvol}_M = (F_{t',t''})_* \left(\rho(t') \, \mathrm{dvol}_M \right), \tag{4.16}$$

where the transport map $F_{t',t''} : M \to M$ is given by

$$F_{t',t''}(m') = \mathcal{L}_0 \exp_{m'}^{t',t''} \left(\nabla_{m'} \phi(t') \right). \tag{4.17}$$

We now do certain calculations in an Otto calculus that is adapted to the Ricci flow background.

Proposition 10 *Suppose that* (4.7) *and* (4.14) *are satisfied. Then*

$$\frac{d}{dt} \int_M \phi \rho \, \mathrm{dvol}_M = \frac{1}{2} \int_M \left(|\nabla \phi|^2 + R \right) \rho \, \mathrm{dvol}_M, \tag{4.18}$$

$$\frac{1}{2} \frac{d}{dt} \int_M |\nabla \phi|^2 \rho \, \mathrm{dvol}_M = \int_M \left(\mathrm{Ric}(\nabla \phi, \nabla \phi) + \frac{1}{2} \langle \nabla R, \nabla \phi \rangle \right) \rho \, \mathrm{dvol}_M, \tag{4.19}$$

$$\frac{d}{dt} \int_M \rho \log(\rho) \, \mathrm{dvol}_M = \int_M \left(\langle \nabla \rho, \nabla \phi \rangle + R\rho \right) \mathrm{dvol}_M, \tag{4.20}$$

$$\frac{d}{dt} \int_M R\rho \, \mathrm{dvol}_M = \int_M \left(R_t + \langle \nabla R, \nabla \phi \rangle \right) \rho \, \mathrm{dvol}_M \tag{4.21}$$

and

$$\frac{d}{dt} \int_M \langle \nabla \rho, \nabla \phi \rangle \, \mathrm{dvol}_M$$
$$= \int_M \left(|\,\mathrm{Hess}\ \phi|^2 + \mathrm{Ric}(\nabla \phi, \nabla \phi) - 2 \langle \mathrm{Ric}, \mathrm{Hess}\ \phi \rangle - \frac{1}{2} \nabla^2 R \right) \rho \, \mathrm{dvol}_M. \tag{4.22}$$

Proof For (4.18),

$$
\begin{aligned}
&\frac{d}{dt}\int_M \phi\rho \,\mathrm{dvol}_M \\
&= \int_M \left(\left(-\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}R \right)\rho + \phi\left(-\nabla^i\left(\rho\nabla_i\phi\right) + R\rho \right) - R\phi\rho \right) \mathrm{dvol}_M \\
&= \frac{1}{2}\int_M \left(|\nabla\phi|^2 + R \right)\rho \,\mathrm{dvol}_M \,. \qquad (4.23)
\end{aligned}
$$

For (4.19),

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_M |\nabla\phi|^2\rho \,\mathrm{dvol}_M \\
&= \int_M \left(\mathrm{Ric}(\nabla\phi,\nabla\phi)\rho + \left\langle \nabla\phi, \nabla\left(-\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}R \right) \right\rangle \rho \right. \\
&\qquad \left. + \frac{1}{2}|\nabla\phi|^2\left(-\nabla^i\left(\rho\nabla_i\phi\right) + R\rho \right) - \frac{1}{2}R|\nabla\phi|^2\rho \right) \mathrm{dvol}_M \\
&= \int_M \left(\mathrm{Ric}(\nabla\phi,\nabla\phi) + \frac{1}{2}\langle \nabla R, \nabla\phi\rangle \right)\rho \,\mathrm{dvol}_M \,. \qquad (4.24)
\end{aligned}
$$

For (4.20),

$$
\begin{aligned}
&\frac{d}{dt}\int_M \rho\log(\rho) \,\mathrm{dvol}_M \\
&= \int_M \left((\log(\rho)+1)\left(-\nabla^i\left(\rho\nabla_i\phi\right) + R\rho \right) - \rho\log(\rho)R \right) \mathrm{dvol}_M \\
&= \int_M \left(\langle\nabla\rho,\nabla\phi\rangle + R\rho \right) \mathrm{dvol}_M \,. \qquad (4.25)
\end{aligned}
$$

For (4.21),

$$
\begin{aligned}
\frac{d}{dt}\int_M R\rho \,\mathrm{dvol}_M &= \int_M \left(R_t\rho + R\left(-\nabla^i\left(\rho\nabla_i\phi\right) + R\rho \right) - R^2\rho \right) \mathrm{dvol}_M \\
&= \int_M \left(R_t + \langle \nabla R, \nabla\phi\rangle \right)\rho \,\mathrm{dvol}_M \,. \qquad (4.26)
\end{aligned}
$$

For (4.22),

$$
\begin{aligned}
&\frac{d}{dt}\int_M \langle\nabla\rho,\nabla\phi\rangle \,\mathrm{dvol}_M \\
&= \int_M \left(2\,\mathrm{Ric}(\nabla\rho,\nabla\phi) + \left\langle \nabla\left(-\nabla^i\left(\rho\nabla_i\phi\right) + R\rho \right), \nabla\phi \right\rangle \right. \\
&\qquad \left. + \left\langle \nabla\rho, \nabla\left(-\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}R \right) \right\rangle - R\langle\nabla\rho,\nabla\phi\rangle \right) \mathrm{dvol}_M \,. \qquad (4.27)
\end{aligned}
$$

Now

$$2\int_M \mathrm{Ric}(\nabla\rho, \nabla\phi)\,\mathrm{dvol}_M = -\int_M \left(\langle\nabla R, \nabla\phi\rangle + 2\langle\mathrm{Ric}, \mathrm{Hess}\ \phi\rangle\right)\rho\,\mathrm{dvol}_M \tag{4.28}$$

and

$$\begin{aligned}&\int_M \left(\left\langle\nabla\left(-\nabla^i\left(\rho\nabla_i\phi\right)\right), \nabla\phi\right\rangle + \left\langle\nabla\rho, \nabla\left(-\frac{1}{2}|\nabla\phi|^2\right)\right\rangle\right)\mathrm{dvol}_M \\ &\quad= \int_M \left(-\langle\nabla\phi, \nabla(\nabla^2\phi)\rangle + \frac{1}{2}\nabla^2|\nabla\phi|^2\right)\rho\,\mathrm{dvol}_M \\ &\quad= \int_M \left(|\,\mathrm{Hess}\ \phi|^2 + \mathrm{Ric}(\nabla\phi, \nabla\phi)\right)\rho\,\mathrm{dvol}_M\,. \end{aligned}\tag{4.29}$$

Thus

$$\begin{aligned}&\frac{d}{dt}\int_M \langle\nabla\rho, \nabla\phi\rangle\,\mathrm{dvol}_M \\ &= \int_M \left(|\,\mathrm{Hess}\ \phi|^2 + \mathrm{Ric}(\nabla\phi, \nabla\phi) - 2\langle\mathrm{Ric}, \mathrm{Hess}\ \phi\rangle\right)\rho\,\mathrm{dvol}_M \\ &\quad + \int_M \left(-\langle\nabla R, \nabla\phi\rangle\rho + \langle\nabla(R\rho), \nabla\phi\rangle + \frac{1}{2}\langle\nabla\rho, \nabla R\rangle - R\langle\nabla\rho, \nabla\phi\rangle\right)\mathrm{dvol}_M \\ &= \int_M \left(|\,\mathrm{Hess}\ \phi|^2 + \mathrm{Ric}(\nabla\phi, \nabla\phi) - 2\langle\mathrm{Ric}, \mathrm{Hess}\ \phi\rangle - \frac{1}{2}\nabla^2 R\right)\rho\,\mathrm{dvol}_M\,. \end{aligned}\tag{4.30}$$

This proves the proposition. □

Corollary 4 *Under the hypotheses of Proposition 10,*

$$\frac{d^2}{dt^2}\int_M \rho\log(\rho)\,\mathrm{dvol}_M = \int_M \left(|\,\mathrm{Ric} - \mathrm{Hess}\ \phi|^2 + \frac{1}{2}H(\nabla\phi)\right)\rho\,\mathrm{dvol}_M, \tag{4.31}$$

where

$$H(X) = R_t + 2\langle\nabla R, X\rangle + 2\,\mathrm{Ric}(X, X) \tag{4.32}$$

is Hamilton's trace Harnack expression. Also,

$$\frac{d^2}{dt^2}\int_M (\rho\log(\rho) - \phi\rho)\,\mathrm{dvol}_M = \int_M |\,\mathrm{Ric} - \mathrm{Hess}\ \phi|^2\rho\,\mathrm{dvol}_M\,. \tag{4.33}$$

In particular, $\int_M (\rho\log(\rho) - \phi\rho)\,\mathrm{dvol}_M$ *is convex in* t.

Proof This follows from Proposition 10, along with the equation

$$R_t = \nabla^2 R + 2|\,\mathrm{Ric}\,|^2. \tag{4.34}$$

□

We now give the analog of Corollary 4 for $P(M)$, using results from [2] and [28, Chaps. 7, 10, 13]. Let $c : [t_0, t_1] \to P(M)$ be a minimizing curve for $\mathcal{A}_0$ relative to its endpoints, which we assume to be absolutely continuous probability measures. Then $c(t) = (F_{t_0,t})_* c(t_0)$, where there is a semiconvex function $\phi_0 \in C(M)$ so that $F_{t_0,t}(m_0) = \mathcal{L}_0 \exp_{m_0}^{t_0,t}(\nabla_{m_0}\phi_0)$. Define $\phi(t) \in C(M)$ by

$$\phi(t)(m) = \inf_{m_0 \in M} \left(\phi_0(m_0) + L_0^{t_0,t}(m_0, m)\right). \tag{4.35}$$

Define $\mathcal{E} : P(M) \to \mathbb{R} \cup \{\infty\}$ as in (2.38).

Proposition 11 *$\mathcal{E}(c(t)) - \int_M \phi(t)\, dc(t)$ is convex in t.*

Proof The proof is along the lines of the proof of Proposition 16 ahead. □

Remark 5 The function ϕ also enters as a solution of the dual Kantorovitch problem. See [28, Theorem 7.36] (where what we call ϕ is called ψ).

Remark 6 Suppose that the Ricci flow solution $(M, g(\cdot))$ is a gradient steady soliton, meaning that it is a Ricci flow solution with $\mathrm{Ric} + \mathrm{Hess}\,(f) = 0$, where f satisfies $\frac{\partial f}{\partial t} = |\nabla f|^2$. Differentiating spatially and temporally, one shows that $|\nabla f|^2 + R = C$ for some constant C. Then there is a solution of (4.13) with $\phi = -f$ and $\alpha = -\frac{1}{2}C$. If ρ is transported along the static vector field $-\nabla f$, i.e. satisfies (4.7), then (4.33) says that $\int_M (\rho \log(\rho) + f\rho)\, \mathrm{dvol}_M$ is linear in t.

5 Monotonicity of the $\mathcal{L}_0$-cost under a backward heat flow

In this section we discuss the $\mathcal{L}_0$-cost between two measures that each evolve under the backward heat flow. The results are analogs of results of Topping for the $\mathcal{L}$-cost [26]. We first compute the variation of E_0 with respect to a one-parameter family of curves that begin and end at shifted times. We use this to show, within the Otto calculus, that if measures $c'(\cdot)$ and $c''(\cdot)$ evolve under the backward heat flow then the $\mathcal{L}_0$-cost between $c'(t'+u)$ and $c''(t''+u)$ is nondecreasing in u. We then show that this implies the monotonicity of Perelman's $\mathcal{F}$-functional, in analogy to what Topping did for Perelman's $\mathcal{W}$-functional.

Proposition 12 *Take $t_0 < t' < t'' < t_1$. For small ϵ, suppose that $c : [t', t''] \times (-\epsilon, \epsilon) \to P^\infty(M)$ is a smooth map, where $c \equiv c(t, u)$. Define $c_u : [t'+u, t''+u] \to P^\infty(M)$ by $c_u(t) = c(t-u, u)$. Put $\mu' = c_0(t')$ and $\mu'' = c_0(t'')$. Suppose that c_0 is a minimizer for E_0 among curves from $[t', t'']$ to $P^\infty(M)$ whose endpoints are μ' and μ''. Put $V(t) = \left.\frac{\partial c}{\partial u}\right|_{u=0}$. Then*

$$\left.\frac{dE_0(c_u)}{du}\right|_{u=0} = \int_{t'}^{t''} \int_M |\mathrm{Ric} - \mathrm{Hess}\,\phi|^2 \rho\, \mathrm{dvol}_M\, dt + \left.\int_M \phi(t) \left(V(t) + \nabla^2 \rho\, \mathrm{dvol}_M\right)\right|_{t=t'}^{t''}. \tag{5.1}$$

Proof For any $u \in (-\epsilon, \epsilon)$, we can write

$$E_0(c_u) = \frac{1}{2} \int_{t'}^{t''} \left(G\left(\frac{\partial c}{\partial t}, \frac{\partial c}{\partial t}\right) + \int_M Rc(t, u) \right) dt, \tag{5.2}$$

where the integrand is evaluated using the metric at time $t+u$, and the $c(t,u)$ in the term $R\,c(t,u)$ is taken to be a measure on M. There is a well-defined notion of covariant derivative on $P^\infty(M)$ [12, Proposition 2]. Letting D denote directional covariant differentiation on $P^\infty(M)$,

$$\begin{aligned}
&\left.\frac{dE_0(c_u)}{du}\right|_{u=0} \\
&= \int_{t'}^{t''} \left(\frac{1}{2} G_t\left(\frac{dc_0}{dt}, \frac{dc_0}{dt}\right) + G\left(\frac{dc_0}{dt}, D_{\frac{dc_0}{dt}} V(t)\right) + \frac{1}{2}\int_M R_t c_0(t) + \frac{1}{2}\int_M R V(t) \right) dt \\
&= \int_{t'}^{t''} \left(\frac{1}{2} G_t\left(\frac{dc_0}{dt}, \frac{dc_0}{dt}\right) + G\left(D_{\frac{dc_0}{dt}}\left(\frac{dc_0}{dt}\right), V(t)\right) + \frac{1}{2}\int_M R_t c_0(t) \right. \\
&\quad \left. + \frac{1}{2}\int_M R V(t) \right) dt + \left. G\left(\frac{dc_0}{dt}, V(t)\right)\right|_{t=t'}^{t''}. \qquad (5.3)
\end{aligned}$$

As c_0 is a minimizer,

$$\left.\frac{dE_0(c_u)}{du}\right|_{u=0} = \int_{t'}^{t''} \left(\frac{1}{2} G_t\left(\frac{dc_0}{dt}, \frac{dc_0}{dt}\right) + \frac{1}{2}\int_M R_t c_0(t) \right) dt + \left. G\left(\frac{dc_0}{dt}, V(t)\right)\right|_{t=t'}^{t''}. \qquad (5.4)$$

For any $f \in C^\infty(M)$,

$$\frac{d}{dt}\int_M f\rho \,\mathrm{dvol}_M = \int_M \langle \nabla f, \nabla \phi \rangle \rho \,\mathrm{dvol}_M . \qquad (5.5)$$

This gives $\frac{dc_0}{dt}$ in terms of ϕ. Then from (2.4) and Proposition 7,

$$\left.\frac{dE_0(c_u)}{du}\right|_{u=0} = \int_{t'}^{t''}\int_M \left(-\operatorname{Ric}(\nabla\phi, \nabla\phi)\rho \,\mathrm{dvol}_M + \frac{1}{2}\int_M R_t c_0(t) \right) dt + \left. \int_M \phi(t) V(t) \right|_{t=t'}^{t''}. \qquad (5.6)$$

From (4.30),

$$\begin{aligned}
&\left.\int_M \langle \nabla\rho, \nabla\phi\rangle \,\mathrm{dvol}_M \right|_{t'}^{t''} \\
&= \int_{t'}^{t''}\int_M \left(|\operatorname{Hess}\phi|^2 + \operatorname{Ric}(\nabla\phi, \nabla\phi) - 2\langle \operatorname{Ric}, \operatorname{Hess}\phi\rangle - \frac{1}{2}\nabla^2 R \right) \rho \,\mathrm{dvol}_M \, dt. \qquad (5.7)
\end{aligned}$$

The proposition follows from the curvature evolution equation (4.34), (5.6) and (5.7). $\square$

Corollary 5 *Under the hypotheses of Proposition 12, suppose that each c_u is a minimizer for E_0 relative to its endpoints. Suppose that the endpoint measures $c_u(t'+u) = c(t', u)$ and $c_u(t''+u) = c(t'', u)$ each satisfy the backward heat equation, in the variable u:*

$$\frac{dc}{du} = -\nabla^2 c. \tag{5.8}$$

Then $C_0^{t'+u, t''+u}(c_u(t'+u), c_u(t''+u))$ is nondecreasing in u.

We now give the general statement about the monotonicity of the $\mathcal{L}_0$-cost for two measures that evolve under the backward heat flow, without the extra assumption in Corollary 5 that minimizers c_u stay in $P^\infty(M)$. Its proof is an analog of Topping's proof of the corresponding statement for the $\mathcal{L}$-cost [26].

Proposition 13 *Suppose that $c' : [t_0, t_1] \to P^\infty(M)$ and $c'' : [t_0, t_1] \to P^\infty(M)$ satisfy (3.13). Then $C_0^{t'+u, t''+u}(c'(t'+u), c''(t''+u))$ is nondecreasing in u.*

Using Proposition 13, we now reprove the fact that Perelman's $\mathcal{F}$-functional is monotonic [21]. The proof is along the lines of Topping's proof [26] of the corresponding result for Perelman's $\mathcal{W}$-functional.

Corollary 6 *Suppose that $\alpha : [t_0, t_1] \to P^\infty(M)$ is a solution of (3.13). Write $\alpha(t) = \rho(t)\,\mathrm{dvol}_M$. Then*

$$\mathcal{F} = \int_M \left(|\nabla \log(\rho)|^2 + R\right) \rho \,\mathrm{dvol}_M \tag{5.9}$$

is nondecreasing in t.

Proof Put $c' = c'' = \alpha$. Take $t'' > t'$. By Corollary 5, if $u > 0$ then $C_0^{t'+u, t''+u}(\alpha(t'+u), \alpha(t''+u)) \geq C_0^{t', t''}(\alpha(t'), \alpha(t''))$, so

$$\frac{C_0^{t'+u, t''+u}(\alpha(t'+u), \alpha(t''+u))}{t''-t'} \geq \frac{C_0^{t', t''}(\alpha(t'), \alpha(t''))}{t''-t'}. \tag{5.10}$$

From (4.4),

$$\lim_{t'' \to t'} \frac{1}{t''-t'} C_0^{t', t''}(\alpha(t'), \alpha(t'')) = \frac{1}{2} \int_M \left(|\nabla \phi|^2 + R\right) \rho \,\mathrm{dvol}_M, \tag{5.11}$$

where ϕ satisfies (4.7) and the right-hand side is evaluated at time t'. As ρ satisfies (3.14), we can take $\phi = \log(\rho)$. The corollary follows. □

6 Convexity of the $\mathcal{L}_-$-entropy

In this section we extend the results of Sect. 4 from the $\mathcal{L}_0$-functional to the $\mathcal{L}_-$-functional. Optimal transport with an $\mathcal{L}_-$-cost was considered in [26]. As the results of this section are analogs of those in Sect. 4, we only indicate the needed changes.

Let M be a connected closed manifold and let $g(\cdot)$ be a Ricci flow solution on M. We put $\tau = t_0 - t$ and write the Ricci flow equation in terms of τ, i.e.

$$\frac{dg}{d\tau} = 2\,\mathrm{Ric}(g(\tau)). \tag{6.1}$$

Definition 3 If $\gamma : [\tau', \tau''] \to M$ is a smooth curve with $\tau' > 0$ then its $\mathcal{L}_-$-length is

$$\mathcal{L}_-(\gamma) = \frac{1}{2} \int_{\tau'}^{\tau''} \sqrt{\tau} \left(g\left(\frac{d\gamma}{d\tau}, \frac{d\gamma}{d\tau} \right) + R(\gamma(\tau), \tau) \right) d\tau, \tag{6.2}$$

where the time-τ metric $g(\tau)$ is used to define the integrand.

Let $L_-^{\tau',\tau''}(m', m'')$ be the infimum of $\mathcal{L}_-$ over curves γ with $\gamma(\tau') = m'$ and $\gamma(\tau'') = m''$.

The Euler–Lagrange equation for the $\mathcal{L}_-$-functional is easily derived [21, (7.2)] to be

$$\nabla_{\frac{d\gamma}{d\tau}} \left(\frac{d\gamma}{d\tau} \right) - \frac{1}{2} \nabla R + \frac{1}{2\tau} \frac{d\gamma}{d\tau} + 2 \operatorname{Ric} \left(\frac{d\gamma}{d\tau}, \cdot \right) = 0. \tag{6.3}$$

The $\mathcal{L}_-$-exponential map $\mathcal{L}_- \exp_{m'}^{\tau',\tau''} : T_{m'}M \to M$ is defined by saying that for $V \in T_{m'}M$, one has

$$\mathcal{L}_- \exp_{m'}^{\tau',\tau''}(V) = \gamma(\tau'') \tag{6.4}$$

where γ is the solution to (6.3) with $\gamma(\tau') = m'$ and $\left.\frac{d\gamma}{d\tau}\right|_{\tau=\tau'} = V$. Note that our $\mathcal{L}_-$-exponential map differs slightly from Perelman's $\mathcal{L}$-exponential map.

Definition 4 Given $\mu', \mu'' \in P(M)$, put

$$C_-^{\tau',\tau''}(\mu', \mu'') = \inf_{\Pi} \int_{M \times M} L_-^{\tau',\tau''}(m', m'') d\Pi(m', m''), \tag{6.5}$$

where Π ranges over the elements of $P(M \times M)$ whose pushforward to M under projection onto the first (resp. second) factor is μ' (resp. μ''). Given a continuous curve $c : [\tau', \tau''] \to P(M)$, put

$$\mathcal{A}_-(c) = \sup_{J \in \mathbb{Z}^+} \sup_{\tau' = \tau_0 \le \tau_1 \le \cdots \le \tau_J = \tau''} \sum_{j=1}^{J} C_-^{\tau_{j-1},\tau_j}(c(\tau_{j-1}), c(\tau_j)). \tag{6.6}$$

We can think of $\mathcal{A}_-$ as a generalized length functional associated to the generalized metric C_-. By [28, Theorem 7.21], $\mathcal{A}_-$ is a coercive action on $P(M)$ in the sense of [28, Definition 7.13]. In particular,

$$C_-^{\tau',\tau''}(\mu', \mu'') = \inf_c \mathcal{A}_-(c), \tag{6.7}$$

where c ranges over continuous curves $c : [\tau', \tau''] \to P(M)$ with $c(\tau') = \mu'$ and $c(\tau'') = \mu''$.

If $c : [\tau_0, \tau_1] \to P^\infty(M)$ is a smooth curve in $P^\infty(M)$, with $\tau_0 > 0$, then we write $c(\tau) = \rho(\tau) \operatorname{dvol}_M$ and let $\phi(\tau)$ satisfy

$$\frac{\partial \rho}{d\tau} = -\nabla^i (\rho \nabla_i \phi) - R\rho. \tag{6.8}$$

Note that $\phi(\tau)$ is uniquely defined up to an additive constant. The scalar curvature term in (6.8) ensures that

$$\frac{d}{d\tau} \int_M \rho \operatorname{dvol}_M = 0. \tag{6.9}$$

Consider the Lagrangian

$$E_-(c) = \int_{\tau_0}^{\tau_1} \int_M \sqrt{\tau} \left(|\nabla \phi|^2 + R \right) \rho \, \mathrm{dvol}_M \, d\tau, \tag{6.10}$$

where the integrand at time τ is computed using $g(\tau)$.

Proposition 14 *Let*

$$\rho \, \mathrm{dvol}_M : [\tau_0, \tau_1] \times [-\epsilon, \epsilon] \to P^\infty(M) \tag{6.11}$$

be a smooth map, with $\rho \equiv \rho(\tau, u)$. *Let*

$$\phi : [\tau_0, \tau_1] \times [-\epsilon, \epsilon] \to C^\infty(M) \tag{6.12}$$

be a smooth map that satisfies (6.8), *with* $\phi \equiv \phi(\tau, u)$. *Then*

$$\begin{aligned} \left. \frac{dE_-}{du} \right|_{u=0} &= 2\sqrt{\tau} \int_M \phi \frac{\partial \rho}{\partial u} \, \mathrm{dvol}_M \Big|_{\tau=\tau_0}^{\tau_1} \\ &\quad -2 \int_{\tau_0}^{\tau_1} \int_M \sqrt{\tau} \left(\frac{\partial \phi}{\partial \tau} + \frac{1}{2} |\nabla \phi|^2 - \frac{1}{2} R + \frac{1}{2\tau} \phi \right) \frac{\partial \rho}{\partial u} \, \mathrm{dvol}_M \, d\tau, \end{aligned} \tag{6.13}$$

where the right-hand side is evaluated at $u = 0$.

Proof The proof is similar to that of Proposition 9. We omit the details. □

From (6.13), the Euler–Lagrange equation for E_- is

$$\frac{\partial \phi}{\partial \tau} = -\frac{1}{2} |\nabla \phi|^2 + \frac{1}{2} R - \frac{1}{2\tau} \phi + \alpha(\tau), \tag{6.14}$$

where $\alpha \in C^\infty([\tau_0, \tau_1])$. Changing ϕ by a spatially constant function, we can assume that $\alpha = 0$, so

$$\frac{\partial \phi}{\partial \tau} = -\frac{1}{2} |\nabla \phi|^2 + \frac{1}{2} R - \frac{1}{2\tau} \phi. \tag{6.15}$$

If a smooth curve in $P^\infty(M)$ minimizes E_-, relative to its endpoints, then it will satisfy (6.15). For each $\tau_0 \le \tau' < \tau'' \le \tau_1$, the viscosity solution of (6.15) satisfies

$$2\sqrt{\tau''} \phi(\tau'')(m'') = \inf_{m' \in M} \left(2\sqrt{\tau'} \phi(\tau')(m') + L_-^{\tau', \tau''}(m', m'') \right). \tag{6.16}$$

Then the solution of (6.8) satisfies

$$\rho(\tau'') \, \mathrm{dvol}_M = (F_{\tau', \tau''})_* \left(\rho(\tau') \, \mathrm{dvol}_M \right), \tag{6.17}$$

where the transport map $F_{\tau', \tau''} : M \to M$ is given by

$$F_{\tau', \tau''}(m') = \mathcal{L}_- \exp_{m'}^{\tau', \tau''} \left(\nabla_{m'} \phi(\tau') \right). \tag{6.18}$$

Our function ϕ is related to the function φ of [26] by $\phi = -\frac{\varphi}{2\sqrt{\tau}}$.

Proposition 15 *Suppose that* (6.8) *and* (6.15) *are satisfied. Then*

$$\frac{d}{d\tau}\int_M \phi\rho\, \mathrm{dvol}_M = \frac{1}{2}\int_M \left(|\nabla\phi|^2 + R\right)\rho\, \mathrm{dvol}_M - \frac{1}{2\tau}\int_M \phi\rho\, \mathrm{dvol}_M, \tag{6.19}$$

$$\frac{1}{2}\frac{d}{d\tau}\int_M |\nabla\phi|^2\rho\, \mathrm{dvol}_M = \int_M \left(-\mathrm{Ric}(\nabla\phi, \nabla\phi) + \frac{1}{2}\langle\nabla R, \nabla\phi\rangle\right)\rho\, \mathrm{dvol}_M - \frac{1}{2\tau}\int_M |\nabla\phi|^2\rho\, \mathrm{dvol}_M, \tag{6.20}$$

$$\frac{d}{d\tau}\int_M \rho\log(\rho)\, \mathrm{dvol}_M = \int_M (\langle\nabla\rho, \nabla\phi\rangle - R\rho)\, \mathrm{dvol}_M, \tag{6.21}$$

$$\frac{d}{d\tau}\int_M R\rho\, \mathrm{dvol}_M = \int_M (R_\tau + \langle\nabla R, \nabla\phi\rangle)\rho\, \mathrm{dvol}_M \tag{6.22}$$

and

$$\frac{d}{d\tau}\int_M \langle\nabla\rho, \nabla\phi\rangle\, \mathrm{dvol}_M = \int_M \left(|\mathrm{Hess}\ \phi|^2 + \mathrm{Ric}(\nabla\phi, \nabla\phi) + 2\langle\mathrm{Ric}, \mathrm{Hess}\ \phi\rangle - \frac{1}{2}\nabla^2 R\right)\rho\, \mathrm{dvol}_M - \frac{1}{2\tau}\int_M \langle\nabla\rho, \nabla\phi\rangle\, \mathrm{dvol}_M\,. \tag{6.23}$$

Proof The proof is similar to that of Proposition 10. We omit the details. □

Corollary 7 *Under the hypotheses of Proposition* 15,

$$\left(\tau^{\frac{1}{2}}\frac{d}{d\tau}\right)^2\int_M \rho\log(\rho)\, \mathrm{dvol}_M = \tau\int_M \left(|\mathrm{Ric} + \mathrm{Hess}\ \phi|^2 + \frac{1}{2}H(\nabla\phi)\right)\rho\, \mathrm{dvol}_M, \tag{6.24}$$

where

$$H(X) = -R_\tau - 2\langle\nabla R, X\rangle + 2\,\mathrm{Ric}(X, X) - \frac{R}{\tau} \tag{6.25}$$

is Hamilton's trace Harnack expression. Also,

$$\left(\tau^{\frac{3}{2}}\frac{d}{d\tau}\right)^2\left(\int_M (\rho\log(\rho) + \phi\rho)\, \mathrm{dvol}_M + \frac{n}{2}\log(\tau)\right) = \tau^3\int_M \left|\mathrm{Ric} + \mathrm{Hess}\ \phi - \frac{g}{2\tau}\right|^2\rho\, \mathrm{dvol}_M\,. \tag{6.26}$$

In particular, $\int_M (\rho\log(\rho) + \phi\rho)\, \mathrm{dvol}_M + \frac{n}{2}\log(\tau)$ *is convex in* $\tau^{-\frac{1}{2}}$.

Proof This follows from Proposition 15, along with the equation

$$R_\tau = -\nabla^2 R - 2|\operatorname{Ric}|^2, \tag{6.27}$$

after some calculations. □

Remark 7 In [26] it is shown, for transport in $P(M)$ between two elements of $P^\infty(M)$, that

$$\left(\tau^{\frac{1}{2}}\frac{d}{d\tau}\right)^2 \int\limits_M \rho\log(\rho)\,\mathrm{dvol}_M \geq \frac{1}{2}\tau \int\limits_M H(\nabla\phi)\rho\,\mathrm{dvol}_M \tag{6.28}$$

and

$$\left(\tau^{\frac{3}{2}}\frac{d}{d\tau}\right)^2 \int\limits_M \rho\log(\rho)\,\mathrm{dvol}_M \geq \frac{1}{2}\tau^3 \int\limits_M H(\nabla\phi)\rho\,\mathrm{dvol}_M - \frac{n}{4}\tau. \tag{6.29}$$

7 Monotonicity of the reduced volume

In this section we give the extension of Corollary 7 to $P(M)$. We then reprove the monotonicity of Perelman's reduced volume [21].

Let $c : [\tau_0, \tau_1] \to P(M)$ be a minimizing curve for $\mathcal{A}_-$ relative to its endpoints. We assume that $c(\tau_0)$ are $c(\tau_1)$ are absolutely continuous with respect to a Riemannian volume density on M. Then $c(\tau) = (F_{\tau_0,\tau})_* c(\tau_0)$, where there is a semiconvex function $\phi_0 \in C(M)$ so that $F_{\tau_0,\tau}(m_0) = \mathcal{L}_- \exp_{m_0}^{\tau_0,\tau}(\nabla_{m_0}\phi_0)$ [2], [28, Chaps. 10, 13]. Define $\phi(\tau) \in C(M)$ by

$$2\sqrt{\tau}\phi(\tau)(m) = \inf_{m_0\in M}\left(2\sqrt{\tau_o}\phi_0(m_0) + L_-^{\tau_0,\tau}(m_0, m)\right). \tag{7.1}$$

Define $\mathcal{E} : P(M) \to \mathbb{R}\cup\{\infty\}$ as in (2.38).

Proposition 16 $\mathcal{E}(c(\tau)) + \int_M \phi(\tau)dc(\tau) + \frac{n}{2}\log(\tau)$ *is convex in* $s = \tau^{-\frac{1}{2}}$.

Proof From [26], $\mathcal{E}(c(\tau))$ is semiconvex in τ and its second derivative in the Alexandrov sense satisfies

$$\left(\tau^{\frac{3}{2}}\frac{d}{d\tau}\right)^2 \int\limits_M \rho\log(\rho)\,\mathrm{dvol}_M \geq \frac{1}{2}\tau^3 \int\limits_M H(\nabla\phi(\tau))c(\tau) - \frac{n}{4}\tau. \tag{7.2}$$

(Strictly speaking, the paper [26] assumes that $c(\tau_0), c(\tau_1) \in P^\infty(M)$, but the proof works when $c(\tau_0)$ and $c(\tau_1)$ are just absolutely continuous probability measures). Now

$$\int\limits_M \phi(\tau)dc(\tau) = \int\limits_M (\phi(\tau)\circ F_{\tau_0,\tau})dc(\tau_0). \tag{7.3}$$

From [28, Theorem 7.36], for $c(\tau_0)$-almost all $m_0 \in M$ one has

$$(\phi(\tau)\circ F_{\tau_0,\tau})(m_0) - (\phi(\tau_0))(m) = L_-^{\tau_0,\tau}(m_0, F_{\tau_0,\tau}(m_0)), \tag{7.4}$$

with $F_{\tau_0,\tau}(m_0)$ describing an $\mathcal{L}_-$-geodesic parametrized by τ.

Given such an $m_0 \in M$, put $\gamma(\tau) = F_{\tau_0,\tau}(m_0)$ and write $X = \frac{d\gamma}{d\tau}$. We evaluate $\left(\tau^{\frac{3}{2}}\frac{d}{d\tau}\right)^2 \phi(\gamma(\tau))$ using formulas from [21, Sect. 7]; see also [7, Section 18]. Write

$X(\tau) = \frac{d\gamma}{d\tau}$. Then

$$\frac{d}{d\tau}\left(2\sqrt{\tau}\phi(\gamma(\tau))\right) = \frac{d}{d\tau} L_-^{\tau_0,\tau}(m_0, \gamma(\tau)) = \sqrt{\tau}\left(R(\gamma(\tau),\tau) + |X(\tau)|^2\right), \tag{7.5}$$

so

$$\tau^{\frac{3}{2}}\frac{d}{d\tau}\phi(\gamma(\tau)) = -\frac{1}{2}\sqrt{\tau}\phi(\gamma(\tau)) + \frac{1}{2}\tau^{\frac{3}{2}}\left(R(\gamma(\tau),\tau) + |X(\tau)|^2\right). \tag{7.6}$$

From [21, (7.3)],

$$\frac{d}{d\tau}\left(R(\gamma(\tau),\tau) + |X(\tau)|^2\right) = -H(X) - \frac{1}{\tau}\left(R(\gamma(\tau),\tau) + |X(\tau)|^2\right). \tag{7.7}$$

Using (7.6) and (7.7), one obtains

$$\left(\tau^{\frac{3}{2}}\frac{d}{d\tau}\right)^2 \phi(\gamma(\tau)) = -\frac{1}{2}\tau^3 H(X). \tag{7.8}$$

For $c(\tau_0)$-almost all $m_0 \in M$, we have [28, Chapter 13]

$$X(\tau) = (\nabla\phi(\tau))(\gamma(\tau)). \tag{7.9}$$

Equations (7.3) and (7.8) give

$$\left(\tau^{\frac{3}{2}}\frac{d}{d\tau}\right)^2 \int_M \phi(\tau)dc(\tau) = \int_M \left(H(\nabla\phi(\tau)) \circ F_{\tau_0,\tau}\right) dc_0(\tau) = \int_M H(\nabla\phi(\tau))dc(\tau). \tag{7.10}$$

As

$$\left(\tau^{\frac{3}{2}}\frac{d}{d\tau}\right)^2 \log(\tau) = \frac{1}{2}\tau, \tag{7.11}$$

the proposition follows. □

Remark 8 We expect that one can prove Proposition 16 using the Eulerian approach and a density argument, along the lines of [5], but we do not pursue this here.

We now consider the limiting case when $\tau_0 = 0$ and $c(0) = \delta_p$. We remark that the preceding results of this section are valid if we just assume that only $c(\tau_1)$ is absolutely continuous with respect to a Riemannian volume density [28, Chapter 13]. Fix $p \in M$ and, following the notation of [21, Section 7], put $L(m,\tau) = L_-^{0,\tau}(p,m)$. Choose $c(\tau_1) \in P(M)$ to be absolutely continuous with respect to a Riemannian measure. For each $m_1 \in M$, choose a (minimizing) $\mathcal{L}_-$-geodesic $\gamma_{m_1} : [0,\tau_1] \to M$ with $\gamma_{m_1}(0) = p$ and $\gamma_{m_1}(\tau_1) = m_1$. It is uniquely defined for almost all $m_1 \in M$ [7, Section 17]. Let $\mathcal{R}_\tau : M \to M$ be the map given by $\mathcal{R}_\tau(m_1) = \gamma_{m_1}(\tau)$. Then as τ ranges in $[0,\tau_1]$, $c(\tau) = (\mathcal{R}_\tau)_* c(\tau_1)$ describes a minimizing curve for $\mathcal{A}_-$ relative to its endpoints. If $\tau > 0$ then $c(\tau)$ is absolutely continuous with respect to a Riemannian volume density [28, Chap. 13].

From (7.5),

$$\phi(\tau) = l(\cdot,\tau) = \frac{L(\cdot,\tau)}{2\sqrt{\tau}}. \tag{7.12}$$

Proposition 17 $\mathcal{E}(c(\tau)) + \int_M \phi(\tau)dc(\tau) + \frac{n}{2}\log(\tau)$ *is nondecreasing in* τ.

Proof Put $s = \tau^{-\frac{1}{2}}$. If we can show that $\mathcal{E}(c(\tau)) + \int_M \phi(\tau)\, dc(\tau) + \frac{n}{2}\log(\tau)$ approaches a constant as $s \to \infty$, i.e. as $\tau \to 0$, then the convexity in s will imply that $\mathcal{E}(c(\tau)) + \int_M \phi(\tau)\, dc(\tau) + \frac{n}{2}\log(\tau)$ is nonincreasing in s, i.e. nondecreasing in τ.

Let $\mathcal{L}\exp(\overline{\tau}) : T_pM \to M$ be the $\mathcal{L}$-exponential map of [21, Section 7]. That is, for $V \in T_pM$, $(\mathcal{L}\exp(\overline{\tau}))\,(V) = \gamma(\overline{\tau})$ where $\gamma : [0, \overline{\tau}] \to M$ is the $\mathcal{L}_-$-geodesic with $\gamma(0) = p$ and $\lim_{\tau\to 0} \sqrt{\tau}\gamma'(\tau) = V$.

Let Ω_{τ_1} be the set of vectors $V \in T_pM$ for which $\{\mathcal{L}\exp(\tau')(V)\}_{\tau'\in[0,\tau_1]}$ is $\mathcal{L}_-$-minimizing relative to its endpoints. Put $\widehat{c}(\tau_1) = \left(\mathcal{L}\exp(\tau_1)^{-1}\right)_* c(\tau_1)$, a measure on Ω_{τ_1}. Then $c(\tau) = \mathcal{L}\exp(\tau)_*\widehat{c}(\tau_1)$. Computing

$$\mathcal{E}(c(\tau)) + \int_M \phi(\tau) dc(\tau) + \frac{n}{2}\log(\tau) \tag{7.13}$$

with respect to the metric $g(\tau)$ on M is the same as computing

$$\mathcal{E}(\widehat{c}(\tau_1)) + \int_{\Omega_{\tau_1}} (\phi(\tau) \circ \mathcal{L}\exp(\tau))\, d\widehat{c}(\tau_1) + \frac{n}{2}\log(\tau) \tag{7.14}$$

with respect to the metric $\widehat{g}(\tau) = \mathcal{L}exp(\tau)^* g(\tau)$ on Ω_{τ_1}.

As $\tau \to 0$, one approaches the Euclidean situation; see [7, Section 16]. One can check that $(\phi(\tau) \circ \mathcal{L}exp(\tau))\,(V)$ approaches $|V|^2$ uniformly on the compact set $\overline{\Omega_{\tau_1}}$, where $|V|^2$ is the norm squared of $V \in T_pM$ with respect to g_{T_pM}. Thus

$$\lim_{\tau\to 0} \int_{\Omega_{\tau_1}} (\phi(\tau) \circ \mathcal{L}exp(\tau))\ d\widehat{c}(\tau_1) = \int_{\Omega_{\tau_1}} |V|^2 d\widehat{c}(\tau_1). \tag{7.15}$$

Also, $\frac{\widehat{g}(\tau)}{4\tau}$ approaches the flat Euclidean metric g_{T_pM} on $\overline{\Omega_{\tau_1}}$. Writing $\widehat{c}(\tau_1) = \rho_1\, \mathrm{dvol}(g_{T_pM})$, for small τ the density of $\widehat{c}(\tau_1)$ relative to $\mathrm{dvol}(\widehat{g}(\tau))$ is asymptotic to $(4\tau)^{-\frac{n}{2}}\, \rho_1$. Thus

$$\begin{aligned}
&\lim_{\tau\to 0} \left(\mathcal{E}(\widehat{c}(\tau_1)) + \frac{n}{2}\log(\tau)\right) \\
&= \lim_{\tau\to 0} \left(\int_{\Omega_{\tau_1}} (4\tau)^{-\frac{n}{2}} \rho_1 \cdot \log((4\tau)^{-\frac{n}{2}} \rho_1) \cdot (4\tau)^{\frac{n}{2}}\, \mathrm{dvol}_{T_pM} + \frac{n}{2}\log(\tau) \right) \\
&= \int_{\Omega_{\tau_1}} \rho_1 \log(\rho_1)\, \mathrm{dvol}_{T_pM} - \frac{n}{2}\log(4).
\end{aligned} \tag{7.16}$$

The proposition follows. □

Corollary 8 $\tau^{-\frac{n}{2}} \int_M e^{-l}\, \mathrm{dvol}_M$ *is nonincreasing in* τ.

Proof Given $0 < \tau' < \tau'' < \tau_1$, take

$$c(\tau'') = \frac{e^{-\phi(\tau'')}\, \mathrm{dvol}_M}{\int_M e^{-\phi(\tau'')}\, \mathrm{dvol}_M}. \tag{7.17}$$

Then

$$\mathcal{E}(c(\tau'')) + \int_M \phi(\tau'')dc(\tau'') + \frac{n}{2}\log(\tau'') = -\log\left((\tau'')^{-\frac{n}{2}} \int_M e^{-\phi(\tau'')}\,\mathrm{dvol}_M\right). \tag{7.18}$$

Applying Proposition 17, with τ_1 replaced by τ'', gives

$$\mathcal{E}(c(\tau')) + \int_M \phi(\tau')dc(\tau') + \frac{n}{2}\log(\tau') \leq \mathcal{E}(c(\tau'')) + \int_M \phi(\tau'')dc(\tau'') + \frac{n}{2}\log(\tau''). \tag{7.19}$$

However, $\mathcal{E}(\mu) + \int_M \phi(\tau')d\mu + \frac{n}{2}\log(\tau')$ is minimized by $-\log((\tau')^{-\frac{n}{2}}\int_M e^{-\phi(\tau')}\,\mathrm{dvol}_M)$, as μ ranges over probability measures that are absolutely continuous with respect to a Riemannian measure on M. Thus

$$-\log\left((\tau')^{-\frac{n}{2}} \int_M e^{-\phi(\tau')}\,\mathrm{dvol}_M\right) \leq -\log\left((\tau'')^{-\frac{n}{2}} \int_M e^{-\phi(\tau'')}\,\mathrm{dvol}_M\right). \tag{7.20}$$

The corollary follows. □

Remark 9 This procedure of converting a convexity statement to a monotonicity statement works for the $\mathcal{L}_-$-cost and the $\mathcal{L}_+$-cost but does not work for the $\mathcal{L}_0$-cost.

8 Ricci flow on a smooth metric-measure space

In this section we give a definition of Ricci flow on a smooth metric-measure space. Our approach is to consider the Ricci flow on a warped product manifold $\overline{M}$ and compute the induced flow on the base M. This is in analogy to what works in defining Ricci tensors for smooth metric-measure spaces [9].

It turns out that there is a 1-parameter family of such generalized Ricci flows, depending on a parameter $N \in [\dim(M), \infty]$. In the case $N = \infty$, there is the curious fact that the (smooth positive) measure can be absorbed by diffeomorphisms of M, so one just reduces to the usual Ricci flow equation on M.

Let T^q have a fixed flat metric given in local coordinates by $\sum_{i=1}^q dx_i^2$. Put $\overline{M} = M \times T^q$ with a time-dependent warped-product metric

$$\overline{g}(t) = \sum_{\alpha,\beta=1}^n g_{\alpha\beta}(t)dx^\alpha dx^\beta + u(t)^{\frac{2}{q}} \sum_{i=1}^q dx_i^2. \tag{8.1}$$

We also write $u = e^{-\Psi}$. If M is compact then the pushforward of the normalized volume density $\frac{\mathrm{dvol}_{\overline{M}}}{\mathrm{vol}(\overline{M})}$ under the projection $\overline{M} \to M$ is $\frac{u\ \mathrm{dvol}_M}{\int_M u\ \mathrm{dvol}_M}$.

The scalar curvature $\overline{R}$ of $\overline{M}$ equals

$$\begin{aligned} R_q &= R - 2u^{-1}\nabla^2 u + \left(1 - \frac{1}{q}\right) u^{-2}|\nabla u|^2 \\ &= R + 2\nabla^2\Psi - \left(1 + \frac{1}{q}\right)|\nabla\Psi|^2, \end{aligned} \tag{8.2}$$

which is the modified scalar curvature considered in [11]. From [14, Section 4], the Ricci flow equation on $\overline{M}$ becomes

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nabla^2 u, \\ \frac{\partial g_{\alpha\beta}}{\partial t} &= -2R_{\alpha\beta} + 2u^{-1}u_{;\alpha\beta} - \left(2 - \frac{2}{q}\right) u^{-2} u_{;\alpha} u_{;\beta}. \end{aligned} \tag{8.3}$$

Note that

$$\frac{\partial}{\partial t}(u \,\mathrm{dvol}_M) = -R_q u \,\mathrm{dvol}_M . \tag{8.4}$$

Equivalently,

$$\begin{aligned} \frac{\partial \Psi}{\partial t} &= \nabla^2 \Psi - |\nabla \Psi|^2, \\ \frac{\partial g_{\alpha\beta}}{\partial t} &= -2\left(R_{\alpha\beta} + \Psi_{;\alpha\beta} - \frac{1}{q}\Psi_{;\alpha}\Psi_{;\beta}\right). \end{aligned} \tag{8.5}$$

The right-hand side of (8.5) involves the modified Ricci curvature

$$\mathrm{Ric}_q = \mathrm{Ric} + \mathrm{Hess}\ \Psi - \frac{1}{q} d\Psi \otimes d\Psi \tag{8.6}$$

considered in [9] and [22]. If u dvol_M is a (smooth positive) probability measure then we consider (8.5) to be the N-Ricci flow equations for the smooth metric-measure space $(M, g, u\ \mathrm{dvol}_M)$, with $N = n + q$. This is in analogy to the N-Ricci curvature considered in [15]. (If $N = n$ then we require Ψ to be locally constant and just use the usual Ricci flow equation on M. That is, in the noncollapsing situation we take the measure to be the n-dimensional Hausdorff measure).

Taking $q = \infty$, we consider the ∞-Ricci flow equations to be

$$\begin{aligned} \frac{\partial \Psi}{\partial t} &= \nabla^2 \Psi - |\nabla \Psi|^2, \\ \frac{\partial g_{\alpha\beta}}{\partial t} &= -2\left(R_{\alpha\beta} + \Psi_{;\alpha\beta}\right). \end{aligned} \tag{8.7}$$

Remark 10 The occurrence of the Bakry-Émery tensor on the right-hand side of (8.7) is different from its occurrence in Perelman's modified Ricci flow [21]. In (8.7) the function $u = e^{-\Psi}$ satisfies a forward heat equation, whereas in Perelman's work the corresponding measure e^{-f} dvol_M satisfies a backward heat equation.

Example 1 We now give a trivial example of collapsing of Ricci flow solutions. For $u\,\mathrm{dvol}_M \in P^\infty(M)$, put $u_j = \frac{1}{j}u$. Give $\overline{M}$ the corresponding warped-product metric $\overline{g}_j$. Suppose that $g(t)$ and $u(t)$ satisfy (8.3). Consider the corresponding solution $(\overline{M}, \overline{g}_j(\cdot))$ to the Ricci flow equation. For any time t, as $j \to \infty$, the metric-measure spaces $\left(\overline{M}, \overline{g}_j(t), \frac{\mathrm{dvol}_{\overline{M}_j}}{\mathrm{vol}(\overline{M}_j)}\right)$ converge in the measured Gromov–Hausdorff topology to $(M, g(t), u(t)\,\mathrm{dvol}_M)$, which satisfies (8.3) by construction.

Example 2 To give another example, consider the most general T^q-invariant Ricci flow on $\overline{M}$. We can write

$$\overline{g}(t) = \sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}(t) dx^\alpha dx^\beta + \sum_{i,j=1}^{q} G_{ij}(t)\left(dx^i + A^i(t)\right)\left(dx^j + A^i(t)\right). \tag{8.8}$$

Put $u = \sqrt{\det(G_{ij})}$, $X^i_{j,\alpha} = \sum_{k=1}^{q} G^{ik} \partial_\alpha G_{kj} - \frac{2}{q} u^{-1} \partial_\alpha u \delta^i_j$ and $F^i_{\alpha\beta} = \partial_\alpha A^i_\beta - \partial_\beta A^i_\alpha$. From [14, Section 4], the Ricci flow equation on $\overline{M}$ implies that the evolution of u and $g_{\alpha\beta}$ is given by

$$\frac{\partial u}{\partial t} = \nabla^2 u - \frac{u}{4} \sum g^{\alpha\gamma} g^{\beta\delta} G_{kl} F^k_{\alpha\beta} F^l_{\gamma\delta},$$
$$\frac{\partial g_{\alpha\beta}}{\partial t} = -2R_{\alpha\beta} + 2u^{-1} u_{;\alpha\beta} - \left(2 - \frac{2}{q}\right) u^{-2} u_{;\alpha} u_{;\beta} + \sum g^{\gamma\delta} G_{ij} F^i_{\alpha\gamma} F^j_{\beta\delta} + \frac{1}{2} \operatorname{Tr}(X_\alpha X_\beta). \tag{8.9}$$

As before, by uniformly rescaling the torus fibers we can construct a sequence of Ricci flow solutions $\left(\overline{M}, \overline{g}_j(t), \frac{\operatorname{dvol}_{\overline{M}_j}}{\operatorname{vol}(\overline{M}_j)}\right)$ which, for each time, converge in the measured Gromov–Hausdorff topology to $(M, g(t), u(t) \operatorname{dvol}_M)$, satisfying (8.9). Note that instead of satisfying the N-Ricci flow equations (8.3), a solution of (8.9) satisfies the inequalities,

$$\frac{\partial u}{\partial t} - \nabla^2 u \le 0,$$
$$\frac{\partial g_{\alpha\beta}}{\partial t} + 2R_{\alpha\beta} - 2u^{-1} u_{;\alpha\beta} + \left(2 - \frac{2}{q}\right) u^{-2} u_{;\alpha} u_{;\beta} \ge 0. \tag{8.10}$$

End of example.

Returning to (8.5), adding a Lie derivative with respect to $\nabla\Psi$ to the right-hand side gives the equations

$$\frac{\partial \Psi}{\partial t} = \nabla^2 \Psi,$$
$$\frac{\partial g_{\alpha\beta}}{\partial t} = -2R_{\alpha\beta} + \frac{2}{q} \Psi_{;\alpha} \Psi_{;\beta}. \tag{8.11}$$

Note that

$$\frac{\partial}{\partial t} \left(e^{-\Psi} \operatorname{dvol}_M\right) = - \operatorname{Tr}(\operatorname{Ric}_q) e^{-\Psi} \operatorname{dvol}_M . \tag{8.12}$$

In particular, the ∞-Ricci flow equations (8.7) become

$$\frac{\partial \Psi}{\partial t} = \nabla^2 \Psi,$$
$$\frac{\partial g_{\alpha\beta}}{\partial t} = -2R_{\alpha\beta}. \tag{8.13}$$

That is, we obtain a forward heat equation coupled to an ordinary Ricci flow.

We now consider convexity of the entropy function for the system (8.3), where the entropy is computed relative to the background measure $u \operatorname{dvol}_M$. Consider the transport equations on M:

$$\frac{\partial \rho}{\partial t} = -u^{-1} \nabla^\alpha (\rho u \nabla_\alpha \phi) + \overline{R} \rho,$$
$$\frac{\partial \phi}{\partial t} = -\frac{1}{2} |\nabla \phi|^2 + \frac{1}{2} \overline{R}. \tag{8.14}$$

Note that $\int_M \rho u \operatorname{dvol}_M$ is constant in t, so we can take $\rho u \operatorname{dvol}_M$ to be a probability measure. Applying Corollary 4 to $\overline{M}$ implies that if (8.3) and (8.14) are satisfied then

$\int_M (\rho \log(\rho) - \phi\rho) u \, \mathrm{dvol}_M$ is convex in t. Equivalently, in terms of the Eqs. (8.11), if ρ and ϕ satisfy

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -\nabla^\alpha(\rho \nabla_\alpha \phi) + \langle \nabla \Psi, \nabla \rho \rangle + \langle \nabla \Psi, \nabla \phi \rangle \rho + \overline{R} \rho, \\ \frac{\partial \phi}{\partial t} &= -\frac{1}{2} |\nabla \phi|^2 + \langle \nabla \Psi, \nabla \phi \rangle + \frac{1}{2} \overline{R}\end{aligned} \tag{8.15}$$

then $\int_M (\rho \log(\rho) - \phi\rho) \, e^{-\Psi} \, \mathrm{dvol}_M$ is convex in t.

When $q \to \infty$, so that (8.13) holds, we claim that this convexity is no more than the convexity of Corollary 4 when applied to M, after a change of variables. Namely, when $q \to \infty$, if we put

$$\begin{aligned}\widetilde{\rho} &= e^{-\Psi} \rho, \\ \widetilde{\phi} &= \phi - \Psi\end{aligned} \tag{8.16}$$

then Eqs. (8.15) are equivalent to

$$\begin{aligned}\frac{\partial \widetilde{\rho}}{\partial t} &= -\nabla^\alpha(\widetilde{\rho} \nabla_\alpha \widetilde{\phi}) + R \widetilde{\rho}, \\ \frac{\partial \widetilde{\phi}}{\partial t} &= -\frac{1}{2} |\nabla \widetilde{\phi}|^2 + \frac{1}{2} R.\end{aligned} \tag{8.17}$$

From Corollary 4, we know that $\int_M \left(\widetilde{\rho} \, \log(\widetilde{\rho}) - \widetilde{\phi} \, \widetilde{\rho}\right) \, \mathrm{dvol}_M$ is convex in t. This is the same as saying that $\int_M (\rho \, \log(\rho) - \phi \, \rho) \, e^{-\Psi} \, \mathrm{dvol}_M$ is convex in t.

To summarize, for each $N \in [n, \infty]$ there is a N-Ricci flow (8.3). Its right-hand side involves the N-Ricci curvature tensor. A background solution of the N-Ricci flow equation implies a convexity result for the transport equations (8.14). In the special case when $N = \infty$, one can decouple the (smooth positive) measure within the metric flow by performing diffeomorphisms, to recover a forward heat equation coupled to the usual Ricci flow (8.7).

Appendix A: The $\mathcal{L}_+$-entropy

In this section, we give the analogs of Sects. 6 and 7 for the $\mathcal{L}_+$-functional that was considered in [6]. This is for possible future reference. We reprove the monotonicity of the Ilmanen–Feldman–Ni forward reduced volume.

Let M be a connected closed manifold and let $g(\cdot)$ be a Ricci flow solution on M, i.e. (3.1) is satisfied.

Definition 5 If $\gamma : [t', t''] \to M$ is a smooth curve with $t' > 0$ then its $\mathcal{L}_+$-length is

$$\mathcal{L}_+(\gamma) = \frac{1}{2} \int_{t'}^{t''} \sqrt{t} \left(g\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right) + R(\gamma(t), t) \right) dt, \tag{A.1}$$

where the time-t metric $g(t)$ is used to define the integrand.

Let $L_+^{t',t''}(m', m'')$ be the infimum of $\mathcal{L}_+$ over curves γ with $\gamma(t') = m'$ and $\gamma(t'') = m''$.

The Euler–Lagrange equation for the $\mathcal{L}_+$-functional is easily derived to be

$$\nabla_{\frac{d\gamma}{dt}} \left(\frac{d\gamma}{dt}\right) - \frac{1}{2} \nabla R + \frac{1}{2t} \frac{d\gamma}{dt} - 2 \,\mathrm{Ric}\left(\frac{d\gamma}{dt}, \cdot\right) = 0. \tag{A.2}$$

The $\mathcal{L}_+$-exponential map is defined by saying that for $V \in T_{m'}M$, one has

$$\mathcal{L}_+ \exp_{m'}^{t',t''}(V) = \gamma(t'') \tag{A.3}$$

where γ is the solution to (A.2) with $\gamma(t') = m'$ and $\left.\frac{d\gamma}{dt}\right|_{t=t'} = V$.

Definition 6 Given $\mu', \mu'' \in P(M)$, put

$$C_+^{t',t''}(\mu', \mu'') = \inf_{\Pi} \int_{M\times M} L_+^{t',t''}(m', m'')\, d\Pi(m', m''), \tag{A.4}$$

where Π ranges over the elements of $P(M \times M)$ whose pushforward to M under projection onto the first (resp. second) factor is μ' (resp. μ''). Given a continuous curve $c : [t', t''] \to P(M)$, put

$$\mathcal{A}_+(c) = \sup_{J\in\mathbb{Z}^+} \sup_{t'=t_0\le t_1\le\cdots\le t_J=t''} \sum_{j=1}^{J} C_+^{t_{j-1},t_j}(c(t_{j-1}), c(t_j)). \tag{A.5}$$

We can think of $\mathcal{A}_+$ as a generalized length functional associated to the generalized metric C_+. By [28, Theorem 7.21], $\mathcal{A}_+$ is a coercive action on $P(M)$ in the sense of [28, Definition 7.13]. In particular,

$$C_+^{t',t''}\left(\mu', \mu''\right) = \inf_{c} \mathcal{A}_+(c), \tag{A.6}$$

where c ranges over continuous curves with $c(t') = \mu'$ and $c(t'') = \mu''$.

If $c : [t_0, t_1] \to P^\infty(M)$ is a smooth curve in $P^\infty(M)$ with $t_0 > 0$ then we write $c(t) = \rho(t)\, \mathrm{dvol}_M$ and let $\phi(t)$ satisfy

$$\frac{\partial \rho}{dt} = -\nabla^i\left(\rho \nabla_i \phi\right) + R\rho. \tag{A.7}$$

Note that $\phi(t)$ is uniquely defined up to an additive constant. The scalar curvature term in (A.7) ensures that

$$\frac{d}{dt} \int_M \rho\, \mathrm{dvol}_M = 0. \tag{A.8}$$

Consider the Lagrangian

$$E_+(c) = \int_{t_0}^{t_1} \int_M \sqrt{t}\left(|\nabla\phi|^2 + R\right) \rho\, \mathrm{dvol}_M\, dt, \tag{A.9}$$

where the integrand at time t is computed using $g(t)$.

Proposition 18 *Let*

$$\rho\, \mathrm{dvol}_M : [t_0, t_1] \times [-\epsilon, \epsilon] \to P^\infty(M) \tag{A.10}$$

be a smooth map, with $\rho \equiv \rho(t, u)$. Let

$$\phi : [t_0, t_1] \times [-\epsilon, \epsilon] \to C^\infty(M) \tag{A.11}$$

be a smooth map that satisfies (A.7), *with* $\phi = \phi(t, u)$. *Then*

$$\left.\frac{dE_+}{du}\right|_{u=0} = 2\sqrt{t}\int_M \phi \frac{\partial \rho}{\partial u}\, \mathrm{dvol}_M \Big|_{t=t_0}^{t_1} - 2\int_{t_0}^{t_1}\int_M \sqrt{t}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 - \frac{1}{2}R + \frac{1}{2t}\phi\right)\frac{\partial \rho}{\partial u}\,\mathrm{dvol}_M\, dt, \quad \text{(A.12)}$$

where the right-hand side is evaluated at $u = 0$.

Proof The proof is similar to that of Proposition 14. We omit the details. □

From (A.12), the Euler–Lagrange equation for E_+ is

$$\frac{\partial \phi}{\partial t} = -\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}R - \frac{1}{2t}\phi + \alpha(t), \quad \text{(A.13)}$$

where $\alpha \in C^\infty([t_0, t_1])$. Changing ϕ by a spatially constant function, we can assume that $\alpha = 0$, so

$$\frac{\partial \phi}{\partial t} = -\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}R - \frac{1}{2t}\phi. \quad \text{(A.14)}$$

If a smooth curve in $P^\infty(M)$ minimizes E_+, relative to its endpoints, then it will satisfy (A.14). Given $t_0 \le t' < t'' \le t_1$, the viscosity solution of (A.14) satisfies

$$2\sqrt{t''}\phi(t'')(m'') = \inf_{m' \in M}\left(2\sqrt{t'}\phi(t')(m') + L_+^{t',t''}(m', m'')\right). \quad \text{(A.15)}$$

Then the solution of (A.7) satisfies

$$\rho(t'')\,\mathrm{dvol}_M = (F_{t',t''})_*(\rho(t')\,\mathrm{dvol}_M), \quad \text{(A.16)}$$

where the transport map $F_{t',t''} : M \to M$ is given by

$$F_{t',t''}(m') = \mathcal{L}_+ \exp_{m'}^{t',t''}\left(\nabla_{m'}\phi(t')\right). \quad \text{(A.17)}$$

Proposition 19 *Suppose that* (A.7) *and* (A.14) *are satisfied. Then*

$$\frac{d}{dt}\int_M \phi\rho\,\mathrm{dvol}_M = \frac{1}{2}\int_M \left(|\nabla\phi|^2 + R\right)\rho\,\mathrm{dvol}_M - \frac{1}{2t}\int_M \phi\rho\,\mathrm{dvol}_M, \quad \text{(A.18)}$$

$$\frac{1}{2}\frac{d}{dt}\int_M |\nabla\phi|^2\rho\,\mathrm{dvol}_M = \int_M \left(\mathrm{Ric}(\nabla\phi, \nabla\phi) + \frac{1}{2}\langle\nabla R, \nabla\phi\rangle\right)\rho\,\mathrm{dvol}_M - \frac{1}{2t}\int_M |\nabla\phi|^2\rho\,\mathrm{dvol}_M, \quad \text{(A.19)}$$

$$\frac{d}{dt}\int_M \rho\log(\rho)\,\mathrm{dvol}_M = \int_M (\langle\nabla\rho, \nabla\phi\rangle + R\rho)\,\mathrm{dvol}_M, \quad \text{(A.20)}$$

$$\frac{d}{dt}\int_M R\rho\,\mathrm{dvol}_M = \int_M (R_t + \langle\nabla R, \nabla\phi\rangle)\,\rho\,\mathrm{dvol}_M \quad \text{(A.21)}$$

and

$$\frac{d}{dt}\int_M \langle \nabla \rho, \nabla \phi \rangle \, \mathrm{dvol}_M$$
$$= \int_M \left(|\operatorname{Hess} \phi|^2 + \operatorname{Ric}(\nabla \phi, \nabla \phi) - 2\langle \operatorname{Ric}, \operatorname{Hess} \phi \rangle - \frac{1}{2}\nabla^2 R \right) \rho \, \mathrm{dvol}_M$$
$$-\frac{1}{2t}\int_M \langle \nabla \rho, \nabla \phi \rangle \, \mathrm{dvol}_M . \tag{A.22}$$

Proof The proof is similar to that of Proposition 15. We omit the details. □

Corollary 9 *Under the hypotheses of Proposition 19,*

$$\left(t^{\frac{1}{2}}\frac{d}{dt}\right)^2 \int_M \rho \log(\rho) \, \mathrm{dvol}_M = t \int_M \left(|\operatorname{Ric} - \operatorname{Hess} \phi|^2 + \frac{1}{2} H(\nabla \phi) \right) \rho \, \mathrm{dvol}_M, \tag{A.23}$$

where

$$H(X) = R_t + 2\langle \nabla R, X \rangle + 2 \operatorname{Ric}(X, X) + \frac{R}{t} \tag{A.24}$$

is Hamilton's trace Harnack expression. Also,

$$\left(t^{\frac{3}{2}}\frac{d}{dt}\right)^2 \left(\int_M (\rho \log(\rho) - \phi \rho) \, \mathrm{dvol}_M + \frac{n}{2}\log(t) \right)$$
$$= t^3 \int_M \left| \operatorname{Ric} - \operatorname{Hess} \phi + \frac{g}{2t} \right|^2 \rho \, \mathrm{dvol}_M . \tag{A.25}$$

In particular, $\int_M (\rho \log(\rho) - \phi \rho) \, \mathrm{dvol}_M + \frac{n}{2}\log(t)$ *is convex in* $t^{-\frac{1}{2}}$.

Proof This follows from Proposition 19, along with the curvature evolution equation (4.34). □

Let $c : [t_0, t_1] \to P(M)$ be a minimizing curve for $\mathcal{A}_+$ relative to its endpoints. We assume that $c(t_0)$ are $c(t_1)$ are absolutely continuous with respect to a Riemannian volume density on M. Then $c(t) = (F_{t_0,t})_* c(t_0)$, where there is a semiconvex function $\phi_0 \in C(M)$ so that $F_{t_0,t}(m_0) = \mathcal{L}_+ \exp_{m_0}^{t_0,t}(\nabla_{m_0}\phi_0)$ [2], [28, Chapters 10,13]. Define $\phi(t) \in C(M)$ by

$$2\sqrt{t}\phi(t)(m) = \inf_{m_0 \in M} (2\sqrt{t_o}\phi_0(m_0) + L_+^{t_0,t}(m_0, m)). \tag{A.26}$$

Define $\mathcal{E} : P(M) \to \mathbb{R} \cup \{\infty\}$ as in (2.38).

Proposition 20 $\mathcal{E}(c(t)) - \int_M \phi(t) dc(t) + \frac{n}{2}\log(t)$ *is convex in* $s = t^{-\frac{1}{2}}$.

Proof The proof is similar to that of Proposition 16. We omit the details. □

We now consider the limiting case when $t_0 = 0$ and $c(0) = \delta_p$. Fix $p \in M$. Choose $c(t_1) \in P(M)$ to be absolutely continuous with respect to a Riemannian measure. For each $m_1 \in M$, choose a (minimizing) $\mathcal{L}_+$-geodesic $\gamma_{m_1} : [0, t_1] \to M$ with $\gamma_{m_1}(0) = p$ and $\gamma_{m_1}(t_1) = m_1$. It is uniquely defined for almost all $m_1 \in M$. Let $\mathcal{R}_t : M \to M$ be the map given by $\mathcal{R}_t(m_1) = \gamma_{m_1}(t)$. Then as t ranges in $[0, t_1]$, $c(t) = (\mathcal{R}_t)_* c(t_1)$ describes a minimizing curve for $\mathcal{A}_+$ relative to its endpoints.

Take

$$\phi(t) = l_+(\cdot, t) = \frac{L_+^{0,t}(p, \cdot)}{2\sqrt{t}}. \tag{A.27}$$

Proposition 21 $\mathcal{E}(c(t)) - \int_M \phi(t) dc(t) + \frac{n}{2} \log(t)$ *is nondecreasing in* t.

Proof The proof is similar to that of Proposition 17. We omit the details. □

Corollary 10 $t^{-\frac{n}{2}} \int_M e^{l_+} \operatorname{dvol}_M$ *is nonincreasing in* t.

Proof The proof is similar to that of Corollary 8. We omit the details. □

Remark 11 In the Euclidean case, $l_+(x, t) = \frac{|x|^2}{4t}$. Because l_+ occurs with a positive sign in the exponential in Corollary 10, we cannot expect $t^{-\frac{n}{2}} \int_M e^{l_+} \operatorname{dvol}_M$ to make sense if M is noncompact. This is in contrast to what happens for Perelman's reduced volume $\tau^{-\frac{n}{2}} \int_M e^{-l} \operatorname{dvol}_M$, which makes sense if the Ricci flow has bounded sectional curvature on compact time intervals.

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