Optimal value range in interval linear programming

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Introduction

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- An interval matrix

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Consider the interval linear program

$$f(A, b, c) \equiv \inf c^T x \ \text{ subject to } \ Ax = b, x \ge 0,$$

where $A \in A^I$, $b \in b^I$, $c \in c^I$.

Lower and upper bounds of the optimal value

$$\underline{f} \equiv \inf f(A, b, c) \quad \text{subject to} \quad A \in A^I, b \in b^I, c \in c^I,$$
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Theorem (Rohn 2006)

We have

$$\underline{f} = \inf \underline{c}^T x \;\; \textit{subject to} \;\; \underline{A}x \le \overline{b}, \; \overline{A}x \ge \underline{b}, \; x \ge 0.$$

Let $\underline{f}$ be finite or let the right-hand side of (1) be positively infinite. Then

$$\overline{f} = \sup b_c^T y + b_\Delta^T |y| \;\; \textit{subject to} \;\; A_c^T y - A_\Delta^T |y| \le \overline{c}. \qquad (1)$$

Unified approach

Consider the general interval linear program

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Dual problem

$$\max b^T x \;\text{ subject to }\; x \in N(A,c).$$

Solution sets of $M(A, b)$ and $N(A, c)$, respectively, are

$$M \equiv \{x \in M(A, b) \mid A \in A^I, b \in b^I\},$$
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Theorem

We have

$$\underline{f} = \inf c_c^T x - c_\Delta^T |x| \ \ \textit{subject to} \ \ x \in M.$$

Let $\overline{f} < \infty$ *or let the right-hand side of* (2) *be positively infinite. Then*

$$\overline{f} = \sup b_c^T y + b_\Delta^T |y| \ \ \textit{subject to} \ \ x \in N. \qquad (2)$$

Algorithm

Definition (Strong solvability)

The system describing $M(A, b)$, $A \in A^I$, $b \in b^I$, is strongly solvable if $M(A, b)$ is nonempty for all $A \in A^I$, $b \in b^I$.

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1. Compute

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$$\overline{\varphi} := \sup b_c^T y + b_\Delta^T |y| \text{ subject to } y \in \mathcal{N}.$$
3. If $M(A, b)$, $A \in \mathbf{A}^I$, $b \in \mathbf{b}^I$, is strongly solvable, then set $\overline{f} := \overline{\varphi}$, otherwise set $\overline{f} := \infty$.

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Example (simple case)

Let $M(A, b) = \{x \mid Ax \leq b,\ x \geq 0\}$.

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$$\begin{aligned} N(A,c) &= \{y \mid A^T y \ge c,\ y \ge 0\}, \\ M &= \{x \mid \underline{A}x \le \overline{b},\ x \ge 0\}, \\ N &= \{y \mid \overline{A}^T y \ge \underline{c},\ y \ge 0\}. \end{aligned}$$

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Strong solvability of $M(A,b)$ is equivalent to solvability to

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Optimal value bounds

$$\underline{f} = \inf \underline{c}^T x \quad \text{subject to} \quad x \in M,$$
$$\overline{f} = \sup \overline{b}^T y \quad \text{subject to} \quad y \in N.$$

Example (with dependences)

Let the feasible set be described as follows

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$$M = \Big\{x, y \mid \underline{A}x \leq \overline{b}^1,\ \overline{A}x \geq \underline{b}^1,\ \underline{A}y \leq \overline{b}^2,\ \overline{A}y \geq \underline{b}^2,\ x, y \geq 0,$$

$$b^1_\Delta y^T + b^2_\Delta x^T + A_\Delta |xy^T - yx^T| \geq |(A_c x - b^1_c)y^T - (A_c y - b^2_c)x^T|\Big\},$$

$$N = \Big\{u, v \mid A_c^T u - A_\Delta^T |u| \leq \overline{c}^1,\ A_c^T v - A_\Delta^T |v| \leq \overline{c}^2,$$

$$(\overline{c}^1 - A_c^T u)|v_k| + (\overline{c}^2 - A_c^T v)|u_k| + A_\Delta^T |v_k u - u_k v| \geq 0\ \forall k : u_k v_k < 0\Big\}.$$

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Theorem

If C is fixed and $A \in A^I$, $b \in b^I$, $d \in d^I$, then

$$\underline{f} = \inf x^T Cx + \underline{d}^T x \ \textit{ subject to } \ \underline{A}x \le \overline{b}, \ x \ge 0,$$

$$\overline{f} = \sup -x^T Cx - \underline{b}^T u \ \textit{ subject to } \ 2Cx + \overline{A}^T u + \overline{d} \ge 0, u \ge 0.$$

The End.