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OPTIMALITY CONDITIONS FOR SYSTEMS DRIVEN BY NONLINEAR EVOLUTION EQUATIONS

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Using the Dubovitskii-Milyutin theory we derive necessary and sufficient conditions for optimality for a class of Lagrange optimal control problems monitored by a nonlinear evolution equation and involving initial and/or terminal constraints. An example of a parabolic control system is also included.

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1. INTRODUCTION

The problem of necessary conditions for infinite dimensional control systems has been considered by several authors starting with the influential paper of Friedman [2] and the book of Lions [8], who concentrated on linear systems. Later Lions [9] considered systems with an initial condition which is not determined by an a priori given function, but instead is assumed to belong to a specified set (Lions calls these systems "systems with insufficient data"). The work of Lions was extended by Papageorgiou [10] to a class of nonlinear systems using the Dubovitskii-Milyutin formalism. His results were carried further by Ledzewicz [5], [6] who considered systems with terminal data (see [5]) as well as abnormal problems and problems with equality constraints (see [6]). Ledzewicz's approach was also based on the Dubovitskii-Milyutin formalism.

This paper continues in the same direction, improves some of the results of the above mentioned papers and establishes optimality conditions for a fairly broad class of nonlinear systems with state and control constraints. Our approach uses both the Dubovitskii-Milyutin theory for constrained optimization problems [1]. A very comprehensive presentation of the Dubovitskii-Milyutin theory can be found in the monograph of Girsanov [3], which also presents applications to finite dimensional control systems. Also extensions of the Dubovitskii-Milyutin method can be found in Ledzewicz [5], [6] and in the references quoted therein.

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2. PRELIMINARIES

Our mathematical setting is the following. Let $T = [0, b]$ and H a separable Hilbert space. Also let $X \subseteq H$ be a subspace carrying the structure of a separable reflexive Banach space which embeds continuously and densely into H. Identifying H with its dual (pivot space) we have $X \subseteq H \subseteq X^*$ with all the embeddings being continuous and dense. Such a triple (X, H, X^*) is usually called an "evolution triple" of spaces (the name "Gelfand triple" can also be found in the literature). By $\|\cdot\|$ (resp. $|\cdot|$, $\|\cdot\|_*$) we will denote the norm of X (resp. of H. X^*). Also by $(\cdot, \cdot)$ we will denote the inner product of H and by $\langle\cdot,\cdot\rangle$ the duality brackets for the dual pair (X, X^*). The two are compatible in the sense that $\langle\cdot,\cdot\rangle|H \times X = (\cdot,\cdot)$. Let $W(T) = \{x \in L^2(X) : \dot{x} \in L^2(X^*)\}$ (here the derivative is understood in the sense of vector valued distributions). Equipped with the norm $\| x \|_{W(T)} = (\| x \|^2_{L^2(X)} + \| \dot{x} \|^2_{L^2(X^*)})^{1/2}$ $W(T)$ becomes a Banach space (a Hilbert space if X is a Hilbert space). We know that $W(T)$ embeds continuously in $C(T, H)$. So every element in $W(T)$ has a representative which is continuous as a function from T into H. Moreover if X embeds into H compactly (which is often the case in applications), then $W(T)$ embeds into $L^2(H)$ compactly. For details we refer to Zeidler [12]. Finally let Y be a separable Banach space modelling the control space.

3. OPTIMALITY CONDITIONS

We start this section by examining the following optimal control problem:

$$\left\{\begin{array}{c} J(x, u) = \int_0^b L(t, x, (t), u(t))\, dt \rightarrow inf \\ s.t.\ \dot{x}(t) + A(t, x(t), u(t)) = 0 \ a.e. on\ T \\ (x(0), x(b)) \in C,\ u(t) \in U(t)\ a.e.,\ u(.)\ is\ measurable. \end{array}\right\} \tag{1}$$

The hypotheses on the data of (1) are the following:

H(A): $A : T \times X \times Y \rightarrow X^*$ *is an operator such that*

(a) $t \rightarrow A(t, x, u)$ *is measurable,*

(b) $\langle A(t, x', u) - A(t, x, u) \geq \theta \| x' - x \|^2$ *for every* $t \in T$, $x' \in X$, $u \in U(t)$ *and with* $\theta > 0$,

(c) $(x, u) \rightarrow A(t, x, u)$ *is continuously Frechet differentiable,*

(d) $\| A(t, x, u) \|_* \leq a_1(t) + c_1\|x\|$ *a.e. for every* $u \in U(t)$ *with* $a_1 \in L^2_+$, $c_1 > 0$,

(e) $\langle A(t, x, u), x\rangle \geq c_2\| x \|^2$ *for every* $u \in U(t)$ *with* $c_2 > 0$.

H(L): $L : T \times H \times Y \rightarrow \mathbb{R}$ is an integrand such that

(a) $t \rightarrow L(t, x, u)$ *is measurable,*

(b) $(x, u) \rightarrow L(t, x, u)$ *is continuously Frechet differentiable,*

(c) *for every* $x \in C(T, H)$ *and every* $u(\cdot) \in L^2(Y)$ $J(x, u)$ *is finite.*

H(U): $U : T \rightarrow 2^Y \setminus \{\emptyset\}$ *is a multifunction with closed and convex values such that* $GrU = \{(t,u) \in T \times Y : u \in U(t)\} \in \mathcal{L} \times B(Y)$ *with* $\mathcal{L}$ *being the Lebesgue* σ*-field of* T *and* $B(Y)$

the Borel σ-field of Y, there is $\epsilon_0 > 0$ *for which for almost all* $t \in T$ *we can find* $u \in U(t)$ *such that* $B_{\epsilon_0}(u) = \{v \in Y : \| u = v\| Y_{\epsilon 0}\} \subseteq U(t)$ *and* $t \rightarrow |U(t)| = sup\{\| u \|_Y : u \in U(t)\} \in L_+^\infty$.

H(C): $C \subseteq H \times H$ *is nonempty closed convex and has nonempty interior.*

In what follows by $A_x(t,x(t),u(t))$, $A_u(t,x(t),u(t))$, $L_x(t,x(t),u(t))$ *and* $L_u(t,x(t),u(t))$, we will denote the Frechet derivatives of $A(t,\cdot,u)$, $A(t,x,\cdot)$, $L(t,\cdot,u)$ and $L(t,x,\cdot)$, respectively at the point $(t,x(t),u(t))$. Also by S_U we will denote the set of measurable selections of $U(\cdot)$; i.e., $S_U = \{u : T \rightarrow Y$ measurable such that $u(t) \in U(t)$ a.e.$\}$. Evidently $S_U \subseteq L^\infty(Y)$ (check hypothesis H(U)).

We start with an auxiliary result that guarantees the solvability of the adjoint equation in the system of necessary and sufficient conditions that we will establish (compare with proposition 2.1 in [10]).

PROPOSITION 3.1 *If hypotheses* H(A), H(L), H(U), H(C), *hold, for some admissible state-control pair* (x, u) *of* (1) $t \rightarrow L_x(t, x(t), u(t))$ *belongs in* $L^2(H)$ *and* $z(\cdot) \in L^2(H)$, *then there exists* $p \in W(T)$ *such that*– $\dot{p}(t) + A_x(t, x(t), u(t))^* p(t) + L_x(t, x(t), u(t)) = z(t)$ *a.e.* $p(b) \in H$.

Proof From hypothesis H(A)(b) we know that $\langle A(t, x', u(t)) - A(t, x(t), u(t))\rangle \geq \theta \| x' - x(t) \|^2$, hence $\langle A_x(t, x(t), u(t)) (x' - x(t)) + 0(\| x - x(t) \|), x' - x(t)\rangle \geq \theta \| x' - x(t)\|^2$. Let $x' - x(t) = \epsilon p$. Then we see that $\langle A_x(t, x(t), u(t))\epsilon p + o(\epsilon p), \epsilon p\rangle \geq \theta\epsilon^2\| p\|^2$. Divide by ϵ^2 and let $\epsilon \downarrow 0$ to get that $\langle A_x(t, x(t), u(t))p, p \rangle \geq \theta \| p \|^2$. Finally, apply theorem 30.A of Zeidler [12] to establish the existence of a solution $p(\cdot) \in W(T)$ for the adjoint equation. *Q.E.D.*

Now we are adequately equipped to establish necessary and sufficient conditions for an admissible pair (x, u) of problem (1) to be optimal.

THEOREM 3.2 *If hypotheses H(A), H(L), H(U), H(C) hold and for the admissible state-control pair* $(x, u) \in W(T) \times S_U$ *we have that* $\| A_x(t, x(t), u(t))\|_{\mathcal{L}(X,X^*)} \leq \eta_1$, $\| A_u(t, x(t), u(t)) \|_{\mathcal{L}(Y,X^*)} \leq \eta_2$, $t \rightarrow L_x(t, x(t), u(t))$ *belongs in* $L^2(H)$ *and* $t \rightarrow L_u(t, x(t), u(t))$ *belongs in* $L^1(Y^*)$, *then* (x,u) *is solution of* (1) *if and only if there exists* $p \in W(T)$ *solving the "adjoint" equation* $-\dot{p}(t) + A'_x(t, x(t), u(t))^* p(t) + L_x(t,x(t), u(t)) = 0$ *a.e. on T and for which the following two "minimum principles" hold.*

$$(L_u(t, x, (t), u(t)) - A_u(t, x(t), u(t))^* p(t), v - u(t))_{Y^*, Y} \geq 0 \text{ a.e. for all } v \in U(t) \tag{2}$$

$$(p(b), c_2 - x(b)) - (p(0), c_1 - x(0)) \geq 0 \text{ for all } (c_1, c_2) \in C \tag{3}$$

Proof As we already mentioned we will implement the Dubovitskii-Milyutin formalism. We start by analyzing the cost criterion. Since $J(\cdot,\cdot)$ is convex, via the monotone convergence theorem, we get (with D denoting the gradient of $L(t, x, u)$ in both (x, u)):

$$J'(x, u)(h, v) = \int_0^b D L(t, x(t), u(t)) (h(t), v(t))\, dt, \; h \in W(T), \; v \in L^\infty(Y).$$

Recall that $DL(t, x(t), u(t))(h(t),v(t)) = L_x(t, x(t), u(t))h(t) + L_u(t, x(t), u(t))v(t)$. Applying theorem 7.4 of Girsanov [3] we get that the cone of directions of decrease of the cost criterion $J(\cdot,\cdot)$ at (x, u) is given by

$$K_d = \{(h,v) \in W\ (T) \times L^\infty(Y): J'(x, u)(h, v) < 0\}.$$

For the moment, assume that $K_d \neq \emptyset$ (eventually we will remove this assumption). Then dual cone of K_d is given by $K_d^* = \{-\lambda\, J'(x, u) : \lambda \geq 0\}$.

Next we pass to the analysis of the equality constraint determined by the evolution equation. Define $P : W(T) \times L^\infty(Y) \to L^2(X^*) \times H$

$$P(x,' x')(t) = (\dot{x}'(t) + A\ (t, x'(t), u'(t)),\ x'(0)).$$

Because of hypothesis H(A) the map $\hat{A} : W(T) \times L^\infty(Y) \to L^2(X^*)$ defined by $\hat{A}(x',u')(\cdot) = A(\cdot,\ x'(\cdot),\ u'(\cdot))$ is continuously Frechet differentiable at $(x(0), x(b),\ x,\ u)$. So P is continuously Frechet differentiable at the same point and we have

$$P'(x, u)\ (h, v)(t) = (\dot{h}\ (t) + A_x\ (t, x(t), u(t))\ h(t) + A_u(t, x(t), u(t))\ v\ (t), h(0)).$$

We will show that $P'(x,u)(\cdot,\cdot)$ is surjective. To this end, let $(g,v,h_0) \in L^2(X^*) \times L^\infty(Y) \times H$ be arbitrary and consider the Cauchy problem

$$\left\{\begin{array}{c} \dot{h}\ (t) + A_x\ (t, x(t), u(t))\ h(t) + A_u\ (t, x(t), u(t))\ v(t) = z(t)\ a.\ e. \\ h(0) = h_0 \end{array}\right\}$$

Apply theorem 30.A of Zeidler [12] to deduce that this problem has a unique solution $h \in W(T)$. Hence $P'(x,u)$ is surjective as claimed. Therefore Lyusternik's theorem (see Girsanov [3], theorem 9.1) tells us that if $Q_1 = \{(x',u') \in W(T) \times L^\infty(Y) : P(x',u') = 0\}$ (the equality constraint set), the tangent cone to Q_1 at (x,u) is given by

$$T(Q_1) = \{(h,v) \in W(T)\ \times\ L^\infty\ (Y): P'(x, u)\ (h, v) = \ 0\}\ =\ ker\, P'(x, u).$$

Thus $T(Q_1)^* = \{w^* \in W(T)^* \times (L^\infty(Y))^* : w^*(h,v) = 0$ for all $(h,v) \in T(Q_1)\}$.

Finally we analyze the inequality constraints. So set $Q_2 = C \times S_U$. We claim that $intS_U \neq \emptyset$ in $L^\infty(Y)$. Indeed let $L(t) = \{u \in U(t) : B_{\epsilon_0}(u) \subseteq U(t)\}$. By hypothesis H(U), $L(t) \neq \emptyset$ for almost all $t \in T$ and $GrL = \{(t,u) : d(u,bdU(t)) \geq \epsilon_0\}$. But $(t,u) \to d(u,bdB(t))$ is a Caratheodory function (*cf.* hypothesis H(U) and theorem 4.6 (iii) of Himmelberg [4]) and so is jointly measurable. Hence $GrL \in \mathcal{L} \times B(Y)$ and so via Aumann's selection theorem (see [4], theorem 5.2) we get $u : T \to Y$ measurable such that $u(t) \in L(t)$ a.e. on T. Evidently $u \in intS_U$ (the interior considered in $L^\infty(Y)$). So $C \times S_U$ is convex with nonempty interior in $H \times H \times L^\infty(Y)$. By theorem 10.5 of [3], the dual of the cone of feasible directions of Q_2 at the point $(x(0,x(b),u)$ is given by $K_f(Q_2)^* = (C \times S_U)^* = C^* \times S^*_U$. Hence $(c^*,u^*) \in K_f(Q)^* \subseteq H \times H \times L^\infty(Y)^*$ if and only if the functional $-c^* = (-c_1^*,-c_2^*) \in H \times H$ supports the set C at the point $(x(0), x(b))$ and $u^* \in L^\infty(Y)^*$ supports S_U

at u (although we will eventually show that $u^* \in L^1(Y^*_{w^*}) = \{z^* : T \to Y^*$ which are w^*-measurable, that is, for every $x \in X$ $t \to (z^*(t),x)_{Y^*,Y}$ is measurable and $\| z^*(\cdot) \| \in L^1(T)\}$, this can be assumed from the beginning without any loss of generality as we illustrate in remark (3) at the end of this section).

Now that we have a concrete description of all the relevant dual cones, we can apply the Dubovitskii-Milyutin theorem [1] (see also [3], theorem 6.1) and obtain the Euler equation. So we can find $y^* \in K^*_d$, $w^* \in T(Q_1)^*$ and $(c^*, u^*) \in K_f(Q_2)^*$ not all of them simultaneously zero such that

$$(0,y^*) + (0,w^*) + (c^*,u^*) = 0.$$

This means that for every $(h_0,h_1,h,v) \in H \times H \times W(T) \times L^\infty(Y)$ we have

$$y^*(h,v) + w^*(h,v) + (c^*,(h_0,h_1)) + u^*(v) = 0.$$

Recall from the analysis of the equality constraint that if $(h,v) \in T(Q_1)$, then $w^*(h,v)= 0$. This means that if for a given $v \in L^\infty(Y)$ we choose $h \in W(T)$ so that it solves

$$\left\{ \begin{array}{c} \dot{h}(t) + A_x(t, x(t), u(t))\, h(t) + A_u(t, x(t), u(t))\, v(t) = 0 \text{ a. e.} \\ h(0) = h_0 \end{array} \right\} \tag{4}$$

(we have already established the existence of such an $h \in W(T)$), then $w^*(h,v) = 0$ and thus the Euler equation becomes

$$y^*(h,v) + (c^*,h_0, h_1)) + u^*(v) = 0$$

hence

$$-\lambda J'(x,u)(h,v) + (c^*, (h_0,h_1)) + u^*(v) = 0. \tag{5}$$

If $\lambda = 0$, then since $(h_0, h_1, v) \in H \times H \times L^\infty(Y)$ was arbitrary, we get that $c^* = 0$, $u^* = 0$, and so $w^* = 0$, a contradiction. Therefore $\lambda \neq 0$ and without any loss of generality, we can have $\lambda = 1$.

Consider the following adjoint evolution equation

$$\left\{ \begin{array}{c} -\dot{p}(t) + A_x(t, x(t), u(t))^* p(t) = -L_x(t, x(t), u(t)) \text{ a. e.} \\ p(b) \in H. \end{array} \right\}$$

From Proposition 3.1 we know that this problem has a solution (in fact, unique) $p \in W(T)$. Making use of this adjoint state we have

$$\int_0^b (L_x(t, x(t), u(t)), h(t))\, dt = \int_0^b \langle \dot{p}(t) - A_x(t, x(t), u(t))^* p(t), h(t) \rangle\, dt$$
$$= \int_0^b \langle \dot{p}(t), h(t) \rangle\, dt - \int_0^b \langle A_x(t, x(t), u(t))^* p(t), h(t) \rangle\, dt.$$

But from the integration by parts formula for functions in $W(T)$ (see Zeidler [12], proposition 23.23, p. 422), we have

$$\int_0^b = \langle \dot{p}(t), \mathrm{h}(t)\rangle = (p(b), h(0)) - (p(0), h_0) - \int_0^b \langle p(t), \dot{h}(t)\rangle\, dt.$$

Also we have $\int_0^b \langle A_x(t, x(t), u(t))^* p(t), h(t)\rangle dt = \int_0^b \langle p(t), A_x(t, x(t), u(t))h(t)\rangle\, dt$. So finally we can write

$$\int_0^b (L_x(t,x(t), u(t)), h(t))dt$$

$$= \int_0^b \langle p(t), -\dot{h}(t) - A_x(t,x(t), u(t))h(t)\rangle dt + (p(b), h(b)) - (p(0), h_0)$$

$$= \int_0^b \langle p(t), A_u(t,x(t), u(t))v(t)\rangle\, dt + (p(b),h(b)) - (p(0),h_0)\ (cf. (4)).$$

Using this fact, into (5) we get

$$(c^*,(h_0,h_1)) + u^*(v) = \int_0^b \langle p(t), A_u(t,x(t),u(t))v(t)\rangle dt + \int_0^b (L_u(t,x(t), u(t)), v(t))_{Y^*Y}dt$$

$$+ (p(b), h(b)) - (p(0),h_0)$$

for every $(h_0,h_1,h,v) \in H \times H \times W(T) \times L^\infty(Y)$.

Evidently $(c^*,(h_0, p(b))) = (p(b), h(b)) - (p(0), h(0))$ and $u^*(v) = \int_0^b (L_u(t, x(t),u(t)) - A_u(t,x(t),u(t))^* p(t),v(t))_{Y^*,Y}dt$.

Recall that $-c^* = (-c^*_1,-c^*_2)$ supports C at $(x(0),x(b))$, while $-u^*$ supports S_U at u. So we get

$$(p(b),c_2 - x(b)) - (p(0),c_1 - x(0)) \geq 0 \textit{ for all } (c_1,c_2) \in C$$

and $\int_0^b (L_u(t,x(t), u(t)) - A_u(t,x(t), u(t))^* p(t), v(t) - u(t))_{Y^*,Y} \geq 0$ for all $v \in S_U$.

Suppose that for some $D \subseteq T$ with $\lambda(D) > 0$ we have

$$\inf [(L_u(t,x(t), u(t)) - A_u(t,x(t), u(t))^* p(t),v - u(t))_{Y^*,Y} : v \in U(t)] < 0 \text{ for } t \in D.$$

Set $\Gamma(t) = \{u \in U(t) : (L_u(t,x(t),u(t)) - A_u(t,x(t),u(t))^* p(t),v - u(t))_{Y^*,Y} < 0\}$. Then $\Gamma(t) \neq \emptyset$ for $t \in D$ and clearly $Gr\Gamma \in \mathcal{L} \times B(Y)$. So we can apply once again Aumann's selection theorem and get $\hat{v} : T \to Y$ measurable such that $\hat{v}(t) \in \Gamma(t)$ a.e. Then let $v = \chi_{D^c}u + \chi_D\hat{v}$. Note that $v \in S_U$ and

$$\int_0^b (L_u(t, x(t), u(t)) - A_u(t,x(t), u(t))^* p(t), v(t) - u(t))_{Y^*,Y}\, dt < 0$$

a contradiction. So we have established the two minimum principles (2) and (3).

It remains to remove the hypothesis that $K_d \neq \emptyset$ made earlier in the proof. If $K_d = \emptyset$, then $\int_0^b (L_x(t, x(t), u(t)), h(t))dt + \int_0^b (L_u(t,x(t), u(t)), v(t))_{Y^*,Y}dt = 0$ *for every* $(h,v) \in W(T) \times L^2(Y)$. *First let* $v = 0$ *to get*

$$\int_0^b (L_x(t, x(t), u(t)), h(t))\, dt = 0 \text{ for every } h \in W(T).$$

Set $h(t) = \varphi(t)\, z\ \varphi\ C_0^\infty(0,b),\ z \in X$. Then

$$\int_0^b \varphi(t)(L_x(t, x(t), u(t)), z)\, dt = 0.$$

Because $\varphi \in C_0^\infty\ (0,b)$ was arbitrary we deduce that

$$(L_x(t, x(t), u(t)), z) = 0 \text{ a.e.}$$

for every $z \in X$ and the exceptional Lebesgue null set is independent of $z \in X$ since the latter is separable. Recalling the X embeds densely into H we get that $L_x(t,x(t),u(t)) = 0$ a.e. Next let h = 0 to get

$$\int_0^b (L_u(t, x(t), u\,(t)), v(t))_{Y^*,Y}\ dt = 0 \text{ for every } v\ L^\infty(Y).$$

Then automatically we have the $L_u(t,x(t)) = 0$ *a.e.* So by letting p(b) = 0, we get that p $\equiv 0$ and then the minimum principles become trivial. This proves the necessity part.

For the sufficiency part we will apply theorem 15.2 of Girsanov [3]. Note that $J(\cdot,\cdot)$ is a convex functional which is lower semicontinuous (this can be easily verified using Fatou's lemma) and finite everywhere (*cf.* hypothesis H(L)(c)); Then $J(\cdot,\cdot)$ is continuous (in fact, locally Lipschitz). Because $intQ_2 = int(C \times S_u) \neq \emptyset$ we can apply theorem 15.2 of [3] and get the sufficiency part.

Q.E.D.

A careful reading of the proof reveals that if instead of (1) we consider

$$\left\{\begin{array}{c} J(x, u) = \int_0^b L\,(t, x(t), u(t))\, dt \to inf \\ \text{s.t. } \dot{x}\,(t) + A(t, x(t), u(t)) = 0 \text{ a.e. on } T \\ (x(0), x(b)) \in C, u \in V \subseteq L^2\,(Y). \end{array}\right\} \tag{6}$$

with the hypothesis.

H(V): $V \subseteq L^2\,(Y)$ *is a nonempty, closed and convex set with a nonempty interior, then the minimum principle* (2) *takes an integral form. More precisely we have:*

THEOREM 3.3 *If hypotheses H(A), H(L), H(V), H(C) hold and for the admissible state-control pair* $(x,u) \in W\,(T) \times V$ *we have that* $\| A(t,x(t), u(t))\|_{\mathcal{L}\,(X,X^*)} \leq \eta_1$,

$\| A_u(t,x(t), u(t))\|_{\mathcal{L}(Y,X^*)} \leq \eta_2$, $t \to L_x(t,x(t),u(t))$ belongs in $L^2(H)$ and $t \to L_u(t,x(t), u(t))$ belongs in $L^1(Y^*)$, then (x,u) is a solution of (6) if and only if there exists $p \in W(T)$ solving the "adjoint" equation $-\dot{p}(t) + A_x(t,x(t), u(t))^* p(t) + L_x(t,x(t), u(t)) = 0$ a.e. on T and for which the following two "minimum principles" hold

$$\int_0^b L_u(t, x(t), u(t)) - A_u(t,x(t), u(t))^* p(t), v(t) - u(t))_{Y^*,Y} dt \geq 0 \text{ for all } v \in V$$

$$(p(b), c_2 - x(b)) - (p(0), c_1 - x(0)) \geq 0 \text{ for all } (c_1, c_2) \in C.$$

Moreover, using the techniques of Ledzewicz [5], [6] (appropriately modified to take into account the full nonlinearity of the system), we get a slight extension of her work (she has the control variable appearing linearly in the dynamics). More specifically consider the problem

$$\left\{\begin{array}{l} J(x,u) = \int_0^b L(t,x(t), u(t))\, dt \to inf \\ \text{s.t. } \dot{x}(t) + A(t,x(t), u(t)) = 0 \text{ a.e. em on } T \\ x(0) = x_0, x(b) = x_1, u(t) \in U(T) \text{ a.e.} \end{array}\right\} \tag{7}$$

For this problem we have the following result

THEOREM 3.4 *If hypotheses* H(A), H(L), H(U) *hold and for the admissible state-control pair $(x, u) \in W(T) \times S_U$ we have $\| A_x(t,x(t),u(t))\|_{\mathcal{L}(X,X^*)} \leq \eta_1$, $\| A_u(t,x(t), u(t))\|_{\mathcal{L}(Y,X^*)} \leq \eta_2$, $t \to L_x(t,x(t), u(t))$ belongs in $L^2(H)$ and $t \to L_u(t,x(t), u(t))$ belong in $L^1(Y^*)$, then (x,u) is a solution of (7) if and only if there exists $p \in W(T)$ solving the "adjoint" equation $-\dot{p}(t) + A_x(t,x(t), u(t))^* p(t) + L_x(t,x(t), u(t)) = 0$ a.e. on T and for which the following minimum principle holds*

$$inf[(L_u(t, x(t), u(t)) - A_u(t, x(t), u(t))^* p(t), v - u(t))_{Y^*Y} : v \in V(t)] \geq 0 \text{ a.e.}$$

(1) If in theorem 3.3 $V = L^2(Y)$, then the minimum principle becomes

$$L_u(t, x(t), u(t)) = A_u(t, x(t) u(t))^* p(t) \text{ a.e.}$$

(2) If from hypothesis H(L), we drop H(L)(c), then the sufficiency part is no longer true but the necessity part remains in tact.

(3) As we mentioned in the course of the proof of Theorem 3.2, without any loss of generality we can take the functional u^* in the Euler equation to be in $L^*(Y^*_{w^*})$. Let us show why this is so. Using Levin's decomposition [7], we can write $u^* = u_a^* + u_s^*$ with u_a^* being the "absolutely continuous" part of u^* (*i.e.*, $u^*_a \in L^1(Y^*_{w^*})$) and u_s^* the "singular" part of u^* (*i.e.*, there exists a decreasing sequence $\{C_m\}_{m\geq 1}$ of Lebesgue measurable subsets of T such that $\lambda(C_m) \downarrow 0$ as $m \to \infty$ and $u_s^*(v) = u_x^*(x_{C_m^c} v + x_{C_m} u \in SU$. Evidently $v_m \xrightarrow{\lambda} v$ (*i.e.*, convergence in measure) and so from Papageorgiou [11], we have $v_m \xrightarrow{m} v$ (*i.e.*, convergence in the Mackey topology $m(L^\infty(Y), L^1(Y^*_{w^*}))$. Then $0 \leq u^*(v_m - u) = u_a^*(v_m - u) + u_s^*(v_m - u) = u_a^*(v_m - u) \to u_a^*(v - u)$. So from the beginning we can consider $u_a^* \in L^1(Y^*_{w^*})$.

4. AN EXAMPLE

In this section we work out an example of a distributed parameter optimal control problem to illustrate the applicability of the results in section 3.

The problem under consideration is the following Lagrange optimal control problem monitored by a nonlinear parabolic equation. Let $T = [0,b]$, $Z \subseteq \mathbb{R}^N$ a bounded domain with smooth boundary Γ and let $D = grad = (\partial/\partial_{zi})^N_{i=1}$

$$\left\{\begin{array}{ll} J(x,u) = \int_0^b \int_Z l\,(t, z, x(t, z), u(t, z))\, dzdt \to inf & \\ \text{s.t. } \frac{\partial x}{\partial t} - diva\,(t, z, Dx(t, z), u(t, z)) = 0 \text{ a.e.} & \text{on } T \times Z \\ x\,|_\Gamma = 0, x(0,\cdot) \in C_1, x(h,\cdot) \in C_2, u(t,\cdot) \in U(t) \text{ a.e.} & \text{on } T. \end{array}\right\} \tag{8}$$

We make the following hypothesis on the data of (8).

H(a): $a : T \times Z \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ is a function such that

(a) $(t,z) \to a(t,z,y,u)$ *is measurable,*

(b) $(a(t,z,y,u) - a(t,z,y',u), y - y;)_{\mathbb{R}^N} \geq \theta \| y - y' \|^2$ *for all* $t \in T$, $(z, u) \in GrU$ with $\theta > 0$,

(c) $(y,u) \to a(t,z,y,u)$ *is continuously differentiable,*

(d) $\| a(t, z,y,u(z)) \| \leq a_1(t,z) + c_1(z)\| y\|$ *a.e. on* $T \times Z$ *for every* $u \in U(t)$ *and with* $a_2(\cdot,\cdot) \in L^2(T \times Z)$, $c_1(\cdot) \in L^2(Z)$ *and* $\| a_y(t,z,y,u(z)) \|$, $\| a_u(t,z,y,u(z))\| \leq a_2(t,z) + c_2(z) \| y \|$ *a.e. on* $T \times Z$ *for every* $u \in U(t)$ *and with* $a_2(\cdot,\cdot) \in L^\infty(T, L^2(Z))$ and $c^2(\cdot) \in L^2(Z)$,

(e) $(a(t,z,y,u(z)),y)_{\mathbb{R}^N} \geq c_3(z)\| y \|^2$ *a.e. on* Z *for all* $(t,y) \in T \times \mathbb{R}^N$ *and* $u \in U(t)$ *and with* $c_3(\cdot) \in L^\infty(Z)$, $0< \hat{c}_3 \leq c_3(z)$ *for every* $z \in Z$.

H(l): $l : T \times Z \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an integrand such that

(a) $(t,z) \to l\,(t,z,x,u)$ is measurable,

(b) $(x,u) \to l\,(t,z,x,u)$ is continuously differentiable and convex

(c) $J(x,u)$ is finite for every $x \in C(T, L^2(Z))$ and every $u \in L^\infty(T \times Z)$ and $|l_x(t,z,x,u(z))|$, $|l_u(t,z,x,u(z))| \leq \gamma(t,z) + c(t)|x|$ a.e. on $T \times Z$ for every $u \in U(t)$ with $\gamma(\cdot,\cdot) \in L^2(T \times Z)$ and $c(\cdot) \in L^1(T)$.

$H(U)_1$: $U((t) = \{u \in L^1(Z) : \| u \|_1 \leq r(t)\}$ with $r(\cdot) \in L^\infty(T)$, $r(t) \geq \beta > 0$ for every $t \in T$.

$H(C)_1$: $C_1 = \{x \in L^2(Z) : \| x \|_2 \leq r_1\}$ and $C_2 = \{x \in L^2(Z) : \| x \|_2 \leq r_2\}$ with r_0, $r_1 > 0$.

In this case, $X = H^1_0(Z)$, $H = L^2(Z)$, $X^* = H^{-1}(Z)$ and $Y = L^1(Z)$. By the Sobolev embedding theorem (X, H, X^*) is an evolution triple. Let $A : T \times X \to X^*$ be defined by $\langle A(t,x),y\rangle = \int_Z (a(t,z,Dx(z)), Dy(z))\mathbb{R}^N dz$ for every $y \in X = H^1_0(Z)$. Also let $L : T \times H \times Y \to \mathbb{R}$ be defined by $L(t,x,u) = \int_Z l\,(t,z,x(z))dz$. Then we can rewrite (8) in its equivalent abstract formulation (1). Also it is routine to check that by virtue of hypotheses H(a), H(l), $H(U)_1$, and $H(C)_1$, hypotheses H(A), H(L), H(U), H(C) hold and $\| A_x(t,x(t),u(t)) \|_{\mathcal{L}(X,X^*)} \leq \eta_1$, $\| A_u(t,x(t), u(t)) \|_{\mathcal{L}(Y,X^*)} \leq \eta_2$, $t \to L_x(t, x(t), u(t))$ belongs in $L^2(H)$ and $t \to L_u(t,x(t),u(t))$ belongs in $L^1(Y^*)$ for every admissible pair (x,u). So we can apply Theorem 3.2 and get:

THEOREM 4.1 *An admissible state-control pair* $(x, u) \in [W^{1,2}(T, H^{-1}(Z)) \cap L^2(T, H_0^1(Z))] \times L^\infty(T, L^1(Z))$ *is optimal if and only if there exists* $p \in W^{1,2}(T, H^{-1}(Z)) \cap L^2(T, H^1{}_0(Z))$ *such that*

$$\frac{\delta p}{\delta t} + diva_y(t, z, Dx(t, z), u(t, z))\, Dp(t, z) - l_x(t, z, x(t, z), u(t, z)) = 0 \ a.e. \ on \ T \times Z$$

$$\int_Z (l_u(t, z, x(t, z), u(t, z)) - diva_u(t, z, Dx(t, z), u(t, z))\, Dp(t, z))(v(z) - u(t, z))dz \geq 0$$

for every v in $U(t)$,

$$\int_Z p(b, z)(c_2(z) - x(b, z))\, dz - \int_Z p(0, z)(c_1(z) - x(0, z))\, dz \geq 0$$

for every $c_2 \in C_2$ *and* $c_1 \in C_1$.

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