Optimality of Gaussian discord

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In this Letter we exploit the recently-solved conjecture on the bosonic minimum output entropy to show the optimality of Gaussian discord, so that the computation of quantum discord for bipartite Gaussian states can be restricted to local Gaussian measurements. We prove such optimality for a large family of Gaussian states, including all two-mode squeezed thermal states, which are the most typical Gaussian states realized in experiments. Our family also includes other types of Gaussian states and spans their entire set in a suitable limit where they become Choi-matrices of Gaussian channels. As a result, we completely characterize the quantum correlations possessed by some of the most important bosonic states in quantum optics and quantum information.

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Quantum correlations represent a fundamental resource in quantum information and computation [1, 2]. If we restrict the description of a quantum system to pure states, then quantum entanglement is synonymous with quantum correlations. However, this is not exactly the case when general mixed states are considered: Separable mixed states can still have residual correlations which cannot be simulated by any classical probability distribution [3, 4]. These residual quantum correlations are today quantified by quantum discord [5].

Quantum discord is defined as the difference between the total correlations within a quantum state, as measured by the quantum mutual information, and its classical correlations, corresponding to the maximal randomness which can be shared by two parties by means of local measurements and one-way classical communication [6]. This definition not only provides a more precise characterization of quantum correlations but also has direct application in various protocols, including quantum state merging [7], remote state preparation [8], discrimination of unitaries [9], quantum channel discrimination [10], quantum metrology [11] and quantum cryptography [12].

For bosonic systems, like the optical modes of the electromagnetic field, it is therefore crucial to compute the quantum discord of Gaussian states [13]. Despite these states being the most common in experimental quantum optics and the most studied in continuous-variable quantum information [14], no closed formula is yet known for their quantum discord. What is computed is an upper-bound, known as Gaussian discord [15, 16], which is a simplified version based on Gaussian detections only. Gaussian discord has been conjectured to be the actual discord for Gaussian states, as also supported by recent numerical studies [17, 18].

In this Letter, we connect this conjecture on Gaussian discord with the recently-solved conjecture on the bosonic minimum output entropy [19], according to which the von Neumann entropy at the output of a single-mode Gaussian channel is minimized by a pure Gaussian state at the input [13]. In particular, this optimal input state is the vacuum, or any other coherent state, when we consider Gaussian channels whose action is symmetric in the quadratures (phase-insensitive), as for instance is the case of lossy or amplifier channels (besides [19] see also [20] for an alternate proof).

We show that the minimization of the bosonic output entropy implies the optimality of Gaussian discord for a large family of Gaussian states. This family includes the important class of squeezed thermal states, which are all those states realized by applying two-mode squeezing to a pair of single-mode thermal states [15]. A key point in our analysis is showing that these states can always be decomposed into an Einstein-Podolsky-Rosen (EPR) state plus the local action of a phase-insensitive Gaussian channel. Given such decomposition, we can easily show that heterodyne detection represents the optimal local measurement for computing their quantum discord.

More generally, by extending the previous decomposition to include other forms of local Gaussian channels, we show that we can generate many other types of bipartite Gaussian states, for which the optimal local measurement is Gaussian, given by a (quasi-)projection on single-mode pure squeezed states. Furthermore, our decomposition spans the entire set of Gaussian states in a suitable (and fastly-converging) limit where they become Choi-matrices of local Gaussian channels.

As a result of our study, we are now able to compute the actual unrestricted discord of a large portion of Gaussian states, paving the way for a complete and precise characterization of the most fundamental quantum correlations possessed by bosonic systems.

Quantum discord and its Gaussian formulation. In classical information theory, the mutual information between two random variables, X and Y, can be written as $I(X,Y) = H(X) - H(X|Y)$, where $H(X)$ is the Shannon entropy of variable X, and $H(X|Y) = H(X,Y) - H(Y)$ is its conditional Shannon entropy. This notion has several inequivalent generalizations in quantum information

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theory [5], where the two variables are replaced by two quantum systems, A and B, in a joint quantum state ρ_{AB}.

A first generalization is given by the quantum mutual information [2], defined as $I(A,B) = S(A) - S(A|B)$, where $S(A) = -\mathrm{Tr}(\rho_A \log_2 \rho_A)$ is the von Neumann entropy of system A, in the reduced state $\rho_A = \mathrm{Tr}_B(\rho_{AB})$, and $S(A|B) = S(A,B) - S(B)$ is the conditional von Neumann entropy (to be computed from the joint state ρ_{AB} and the other reduced state ρ_B). The quantum mutual information is a measure of the total correlations between the two quantum systems.

A second generalization is given by

$$C(A|B) = S(A) - S_{\min}(A|B), \tag{1}$$

where the conditional term $S_{\min}(A|B)$ is the von Neumann entropy of system A minimized over all possible measurements on system B, generally described as positive operator valued measures (POVMs) $\mathcal{M}_B = \{M_k\}$ (in particular, it is sufficient to consider rank-1 POVMs [5]). Mathematically, this conditional term is written as

$$S_{\min}(A|B) := \inf_{\mathcal{M}_B} S(A|\mathcal{M}_B), \tag{2}$$

where [21]

$$S(A|\mathcal{M}_B) := \sum_k p_k S(\rho_{A|k}), \tag{3}$$

with $p_k = \mathrm{Tr}(\rho_{AB} M_k)$ the probability of outcome k, and $\rho_{A|k} = p_k^{-1}\mathrm{Tr}_B(\rho_{AB} M_k)$ the conditional state of A.

The entropic quantity $C(A|B)$ quantifies the classical correlations in the joint state ρ_{AB}, being the maximum amount of common randomness which can be extracted by local measurements and one-way classical communication [6]. Quantum discord is then defined as the difference between the total and these classical correlations [3–5]

$$D(A|B) := I(A,B) - C(A|B) = S_{\min}(A|B) - S(A|B). \tag{4}$$

When the two systems A and B are bosonic modes, we can consider their Gaussian discord $D_G(A|B) \geq D(A|B)$, where the minimization of the conditional term $S_{\min}(A|B)$ in Eq. (2) is restricted to Gaussian POVMs. Thanks to this restriction, Gaussian discord is easy to compute for two-mode Gaussian states [15, 16]. Furthermore, as we have already mentioned, Gaussian discord is conjectured to be optimal for these states, in the sense that it would represent their unrestricted quantum discord, i.e., $D_G(A|B) = D(A|B)$. This conjecture is supported by numerical studies [17, 18] and known to be true for a very limited set of Gaussian states ρ_{AB}, namely those which can be purified into a three-mode Gaussian state Φ_{ABE} symmetric in the AE subsystem [16, 22].

Normal forms, two-mode squeezed thermal states, and their decomposition. Since quantum discord and classical correlations are entropic quantities, they are invariant under local unitaries. This means that we may apply displacements to yield a zero mean value, and local Gaussian unitaries to reduce the covariance matrix (CM) into normal form [13]. Thus, without loss of generality, quantum discord can be studied for zero-mean Gaussian states ρ_{AB} with CM

$$\mathbf{V}_{AB} = \begin{pmatrix} a\mathbf{I} & \mathrm{diag}(c,c') \\ \mathrm{diag}(c,c') & b\mathbf{I} \end{pmatrix} := \mathbf{V}(a,b,c,c'), \tag{5}$$

where $\mathbf{I} = \mathrm{diag}(1,1)$ and the parameters satisfy bona-fide conditions imposed by the uncertainty principle [25–27].

For simplicity, we start from the most typical zero-mean Gaussian states, i.e., two-mode squeezed thermal states. These states have CMs of the form $\mathbf{V}_{AB} = \mathbf{V}(a,b,c,-c)$ with bona-fide conditions $a,b \geq 1$ and $c^2 \leq ab - 1 - |a-b|$. As proven in the Supplemental Material and depicted in Fig. 1, these states can always be decomposed as $\rho_{AB} = (\mathcal{E} \otimes \mathcal{I})(\sigma_{aB})$, where $\mathcal{E}$ is a phase-insensitive Gaussian channel (details below), $\mathcal{I}$ is the identity channel, and σ_{aB} is an EPR state with CM

$$\mathbf{V}_{aB} := \begin{pmatrix} b\mathbf{I} & \sqrt{b^2-1}\mathbf{C} \\ \sqrt{b^2-1}\mathbf{C} & b\mathbf{I} \end{pmatrix}, \tag{6}$$

where

$$\mathbf{C} := \begin{pmatrix} \mathrm{sign}(c) & 0 \\ 0 & -\mathrm{sign}(c) \end{pmatrix}. \tag{7}$$

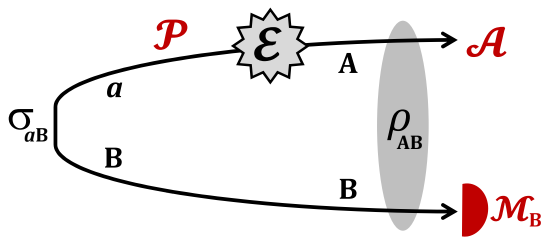

FIG. 1: *State decomposition.* A two-mode squeezed thermal state ρ_{AB} can always be decomposed into an EPR state σ_{aB} plus the local application of a phase-insensitive Gaussian channel $\mathcal{E}$. *Remote preparation* (red elements, see next section). A local measurement $\mathcal{M}_B$ on mode B generates a remote ensemble $\mathcal{A}$ on the output mode A. There will be another ensemble $\mathcal{P}$ generated on the input mode a before the channel. This input ensemble will be made by Gaussianly-modulated coherent states if $\mathcal{M}_B$ is heterodyne detection.

At the level of the second order moments, a phase-insensitive Gaussian channel $\mathcal{E}$ performs the transformation $\mathbf{V}_{aB} \to \mathbf{V}_{AB} = (\mathbf{K}\oplus\mathbf{I})\mathbf{V}_{aB}(\mathbf{K}^T\oplus\mathbf{I}) + (\mathbf{N}\oplus\mathbf{0})$, with transmission matrix $\mathbf{K} = \sqrt{\tau}\mathbf{I}$ and noise matrix $\mathbf{N} = \eta\mathbf{I}$, where $\tau \geq 0$ and $\eta \geq |1-\tau|$. In particular, this is a 'lossy channel' for transmissivity $\tau \in [0,1]$ and thermal noise $\eta \geq 1-\tau$, an 'additive-noise channel' for $\tau = 1$ and $\eta \geq 0$, or an 'amplifier channel' for $\tau > 1$ and $\eta \geq \tau - 1$. (These are the most important canonical forms of a single-mode

Gaussian channel, see [13] and Supplemental Material for this classification.) As a result, the CM of a two-mode squeezed thermal state $\mathbf{V}_{AB} = \mathbf{V}(a, b, c, -c)$ can be expressed in the equivalent form

$$\mathbf{V}_{AB} = \begin{pmatrix} (\tau b + \eta)\mathbf{I} & \sqrt{\tau(b^2-1)}\mathbf{C} \\ \sqrt{\tau(b^2-1)}\mathbf{C} & b\mathbf{I} \end{pmatrix}, \qquad (8)$$

for some choice of $\tau \geq 0$ and $\eta \geq |1-\tau|$.

Remote state preparation and connection with the bosonic minimum output entropy. As depicted in Fig. 1, the action of a local POVM $\mathcal{M}_B = \{M_k\}$ on mode B generates a remote ensemble of states $\mathcal{P}$ on the input mode a, and a corresponding ensemble $\mathcal{A}$ on the output mode A. With probability $p_k = \mathrm{Tr}(\sigma_{aB} M_k)$, we have a conditional input state $\sigma_{a|k} = p_k^{-1}\mathrm{Tr}_B(\sigma_{aB} M_k) \in \mathcal{P}$ and its corresponding channel output $\rho_{A|k} = \mathcal{E}(\sigma_{a|k}) \in \mathcal{A}$.

Assuming heterodyne detection $\mathcal{M}_B = \mathrm{het}_B$, the input ensemble $\mathcal{P}$ consists of coherent states $\sigma_{a|k} = |\alpha_k\rangle\langle\alpha_k|$ whose complex amplitudes α_k are Gaussianly-distributed (see Supplemental Material for more details on the remote preparation of Gaussian states). As a result, the output ensemble $\mathcal{A}$ will be composed of Gaussian states $\rho_{A|k} = \mathcal{E}(|\alpha_k\rangle\langle\alpha_k|)$ with Gaussianly-modulated first moments and CM equal to $(\tau+\eta)\mathbf{I}$.

The output entropy associated with the heterodyne detection is equal to the average entropy of the output ensemble $\mathcal{A}$, i.e.,

$$S(A|\mathrm{het}_B) = \int d^2k \; p_k \; S[\mathcal{E}(|\alpha_k\rangle\langle\alpha_k|)]. \qquad (9)$$

Since entropy is invariant under displacements, we may write [28] $S[\mathcal{E}(|\alpha_k\rangle\langle\alpha_k|)] = S[\mathcal{E}(|0\rangle\langle 0|)]$ and therefore

$$S(A|\mathrm{het}_B) = S[\mathcal{E}(|0\rangle\langle 0|)] = h(\tau+\eta), \qquad (10)$$

where

$$h(x) := \frac{x+1}{2}\log_2\frac{x+1}{2} - \frac{x-1}{2}\log_2\frac{x-1}{2}. \qquad (11)$$

At this point we can exploit the solved conjecture on the bosonic minimum output entropy, which states that the vacuum (or any other coherent state) minimizes the output entropy of the phase-insensitive Gaussian channel $\mathcal{E}$ among all possible input states [19, 20]

$$S[\mathcal{E}(|0\rangle\langle 0|)] = \inf_\rho S[\mathcal{E}(\rho)]. \qquad (12)$$

As a result, we may write

$$\begin{aligned} S(A|\mathrm{het}_B) &= \inf_\rho S[\mathcal{E}(\rho)] \\ &\leq \inf_{\mathcal{M}_B} S(A|\mathcal{M}_B) = S_{\min}(A|B), \end{aligned} \qquad (13)$$

where the inequality comes from the fact that any $\mathcal{M}_B$, with input ensemble $\mathcal{P} = \{p_k, \sigma_{a|k}\}$ and output ensemble $\mathcal{A} = \{p_k, \rho_{A|k}\}$, must satisfy

$$\begin{aligned} S(A|\mathcal{M}_B) &= \sum_k p_k S(\rho_{A|k}) \geq \inf_{\mathcal{A}} S(\rho_{A|k}) \\ &= \inf_{\mathcal{P}} S[\mathcal{E}(\sigma_{a|k})] \geq \inf_\rho S[\mathcal{E}(\rho)]. \end{aligned} \qquad (14)$$

Thus, there is a Gaussian POVM (heterodyne detection) which is optimal for the minimization of the output entropy $S(A|\mathcal{M}_B)$. This is equivalent to saying that the Gaussian discord of the Gaussian state ρ_{AB} is optimal, i.e., equal to its actual discord. Its calculation is therefore easy, since $S_{\min}(A|B) = h(\tau+\eta)$, leading to

$$D(A|B) = h(b) - h(\nu_-) - h(\nu_+) + h(\tau+\eta), \qquad (15)$$

where $\{\nu_\pm\}$ is the symplectic spectrum of $\mathbf{V}_{AB}$, which can be easily computed [13] from Eq. (8).

Extending the family of Gaussian states. Here we extend the previous derivation to include other Gaussian states. We first generalize the local Gaussian POVM $\mathcal{M}_B$, whose element M_k becomes a quasi-projector on the squeezed state $|k, u\rangle$ [13] with variable amplitude k but fixed CM $\mathbf{V}_{\mathrm{sq}}(u) = \mathrm{diag}(u, u^{-1})$, where $u > 0$. This measurement $\mathcal{M}_B(u)$ corresponds to a heterodyne detection for $u = 1$, and becomes a homodyne detection for $u \to 0$ or $u \to +\infty$. By applying $\mathcal{M}_B(u)$ to an EPR state σ_{aB} with variance b as in Eq. (6), we generate an ensemble $\mathcal{P}$ of amplitude-modulated squeezed states with CM $\mathbf{V}_{\mathrm{sq}}(r)$, where $r = (1+ub)(u+b)^{-1}$. The value of this squeezing ranges from $r = b^{-1}$ to $r = b$, extremes which are achieved by the two homodyne detectors (see Supplemental Material for more details).

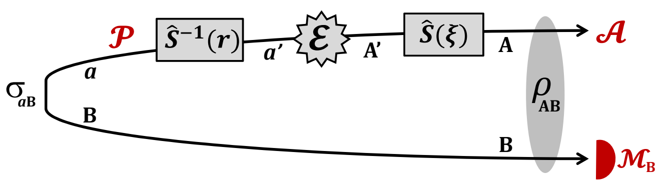

FIG. 2: *General decomposition.* We consider Gaussian states ρ_{AB} which can be decomposed into an EPR state σ_{aB} by locally applying a Gaussian channel $\mathcal{E}$ plus input-output squeezing operators. Here the local detection $\mathcal{M}_B$ is a quasi-projection onto squeezed states, so that the input ensemble $\mathcal{P}$ is composed of amplitude-modulated squeezed states.

We may now "rectify" the ensemble $\mathcal{P}$ by applying the anti-squeezing operator [13] $\hat{S}^{-1}(r)$ which transforms its states into coherent states (see Fig. 2). In this way we are sure that optimal states are fed into the Gaussian channel $\mathcal{E}$, whose output entropy is therefore minimized. Furthermore, this output entropy does not change if we apply another squeezing operator $\hat{S}(\xi)$, with a suitable ξ putting the Gaussian state ρ_{AB} into normal form. Thus the optimality of Gaussian discord is proven for any Gaussian state which is decomposable into an EPR state as

$$\rho_{AB} = (\mathcal{S}_\xi \mathcal{E} \mathcal{S}_r^{-1} \otimes \mathcal{I})(\sigma_{aB}), \;\; \mathcal{S}_x(\rho) := \hat{S}(x)\rho\hat{S}^\dagger(x), \quad (16)$$

with optimal detection given by the Gaussian $\mathcal{M}_B(u)$.

The second generalization consists of extending the Gaussian channel $\mathcal{E}$ to include negative transmissivities $\tau \leq 0$, where the channel describes the conjugate of an amplifier, which is another Gaussian channel (phase-sensitive) whose output entropy is minimized by coherent states at the input [19, 20]. We can then consider an extended Gaussian channel $\mathcal{E}$ with arbitrary $\tau \in \mathbb{R}$ and $\eta \geq |1-\tau|$, which is described by the matrices $\mathbf{K} = \sqrt{|\tau|}\mathrm{diag}[1, \mathrm{sign}(\tau)]$ and $\mathbf{N} = \eta\mathbf{I}$. Thus, the optimality of Gaussian discord is proven for all Gaussian states ρ_{AB} decomposable as in Eq. (16) with $\mathcal{E}$ being such an extended channel. As we show in the Supplemental Material, these states have CMs in the normal-form $\mathbf{V}(a, b, c, c')$, where

$$a = \theta(r)\theta(r^{-1}),\ \theta(r) := \sqrt{\eta r + |\tau|\, b}, \tag{17}$$

$$c = \pm\sqrt{|\tau|\,(b^2-1)\theta(r^{-1})/\theta(r)}, \tag{18}$$

$$c' = \mp\mathrm{sign}[\tau]\sqrt{|\tau|\,(b^2-1)\theta(r)/\theta(r^{-1})}, \tag{19}$$

with $\tau \in \mathbb{R}$, $\eta \geq |1-\tau|$ and $r \in [b^{-1}, b]$. Here the sign ambiguity comes from the type of EPR state considered in the decomposition, whose CM in Eq. (6) is generally defined with $\mathbf{C} = \pm\mathrm{diag}(1, -1)$. Also note that negative transmissivities allow us to include states with $cc' \geq 0$.

For arbitrary fixed values of $a \geq 1$ and $b \geq 1$, we generate all accessible values of the correlation parameters c and c' by exploiting the remaining degrees of freedom in the parameters τ, η and r. As shown in the numerical investigation of Fig. 3, our family represents a wide portion of all Gaussian states. It includes all states whose CMs have the form $\mathbf{V}(a, b, c, c')$ with $|c| = |c'|$, corresponding to the bisectors of the correlation plane (c, c'). Furthermore, for increasing b, our family tends to invade the entire Gaussian set very quickly. This set is completely filled for $b \to +\infty$, where the EPR state σ_{aB} becomes maximally entangled and the Gaussian state ρ_{AB} becomes the Choi matrix of the Gaussian channel $\mathcal{S}_{\xi}\mathcal{E}\mathcal{S}_r^{-1}$.

By increasing the entanglement in the EPR state, we increase the amount of squeezing r that we can generate in the input ensemble $\mathcal{P}$. The effect of this squeezing is to include output states where $|c|$ and $|c'|$ are very different, as also evident from the r-dependence in Eqs. (18) and (19). In particular, in the limit of $b \to +\infty$, homodyne detectors generate infinite squeezing in $\mathcal{P}$ and we approach all the exotic states on the axes of the plane, for which one of the correlation parameters is zero ($cc' = 0$).

Conclusion and discussion. In this Letter, we have shown that the validity of the bosonic minimum output entropy conjecture implies the optimality of Gaussian discord for a large family of Gaussian states, decomposable into EPR states subject to local Gaussian channels. In particular, this family includes the class of two-mode squeezed thermal states [15]. From this point of view, our work completely characterizes the quantum correlations possessed by the most typical states in continuous variable quantum information and experimental quantum optics. The exact size of these quantum correlations, i.e., their unconstrained quantum discord, can now be computed efficiently (note that computing discord is NP-complete in the general case [29]).

We have shown that we can rapidly fill the entire set of Gaussian states by increasing their parameter b, i.e., the thermal variance in the mode under detection. The reason is because this parameter corresponds to the variance of the EPR state involved in the decomposition. By increasing this parameter, we increase the EPR entanglement and therefore the amount of squeezing that we can remotely generate at the input of the Gaussian channel.

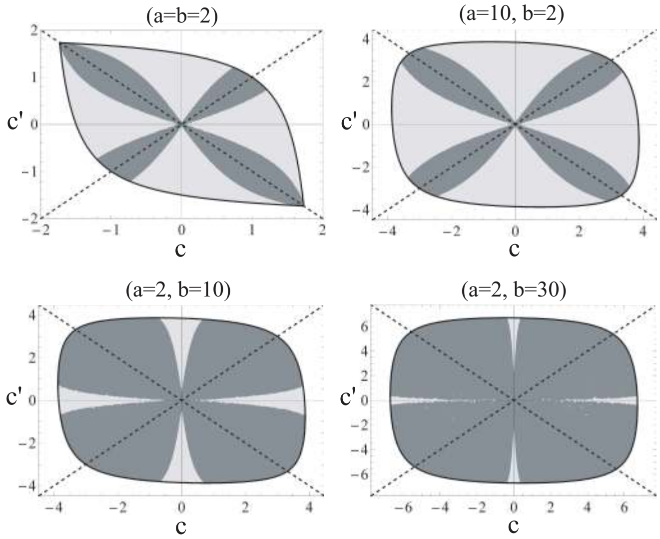

FIG. 3: For given values of a and b, the correlation parameters c and c' can only take a restricted range of physical values corresponding to the delimited regions in the panels. Within these regions the darker points are members of our Gaussian family. Plots are created by randomly testing 5×10^5 points.

We could further improve our results if an energy-constrained version of the entropy conjecture were proven for the 'pathological' canonical forms [13, 30, 31]. In fact, there exist highly phase-sensitive Gaussian channels which completely destroy the correlations in only one of the quadratures, therefore being particularly suitable to decompose exotic Gaussian states with $cc' = 0$. Unfortunately, no finite-energy state is known to be optimal for the minimization of the output entropy of these channels. If such a state were Gaussian, then it could be prepared with a limited amount of squeezing and we would span the entire set of Gaussian states more straightforwardly.

Finally, we remark that the complete characterization of the quantum correlations possessed by Gaussian states is important not only in quantum information and quantum optics (e.g., for problems of quantum metrology), but also in other fields. These include condensed matter physics (e.g., Bose-Einstein condensates), solid-state physics, relativistic quantum field theory (where Gaussian states arise from Bogoliubov transformations, e.g., in the Unruh effect or the Hawking radiation), statistical mechanics and foundations of quantum mechanics.

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[28] Any single-mode Gaussian channel can be dilated into a Gaussian unitary involving the input mode i plus two environmental modes 1 and 2. Globally the transformation of the quadratures $\hat{\mathbf{x}}^T = (\hat{q}_i, \hat{p}_i, \hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2)$ is an affine map $\hat{\mathbf{x}} \to \hat{\mathbf{x}}' = \mathbf{S}\hat{\mathbf{x}} + \mathbf{D}$, where $\mathbf{S}$ is a symplectic matrix and $\mathbf{D}$ is a displacement vector associated with the channel. Now any displacement of the input mode $\hat{\mathbf{x}} \to \hat{\mathbf{x}} + \mathbf{d}$, with $\mathbf{d}^T = (d_q, d_p, 0, 0, 0, 0)$ will be mapped into a corresponding (different) displacement of the output mode $\hat{\mathbf{x}}' \to \hat{\mathbf{x}}' + \mathbf{S}\mathbf{d}$. Since entropy is invariant under unitaries, it is not affected by these output displacements and, therefore, by the input ones.
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Supplemental Material

Contents of the document. In this Supplemental Material, we start by giving a brief review on single-mode Gaussian channels and their canonical forms, also discussing the minimization of their output entropy (Sec. I). In Sec. II we prove the decomposition of two-mode squeezed thermal states. In Sec. III we prove the parametrization of our general family of Gaussian states and we describe the random sampling of this family in more detail. In Sec. IV, we provide the mathematical tools for computing the remote preparation of Gaussian states by applying a local Gaussian measurement to an arbitrary two-mode Gaussian state. Formulas will be then specified to the case of the EPR state, subject to the detections considered in the main article. Because some of these tools are only partially known in the literature, we provide a comprehensive proof in the final Sec. V.

I. CANONICAL FORMS OF SINGLE-MODE GAUSSIAN CHANNELS

Any single-mode Gaussian channel $\mathcal{E}$ can be reduced to a simpler 'canonical form' up to Gaussian unitaries, applied at the input and the output of the channel. The canonical forms are classified as A_1, A_2, B_1, B_2, C and D (see Ref. [S1, S2] for this formalism). They transform the CM $\mathbf{V}$ of an input state as $\mathbf{V} \to \mathbf{K}\mathbf{V}\mathbf{K}^T + \mathbf{N}$, with channel matrices $\mathbf{K}$ and $\mathbf{N}$ which are diagonal.

The most typical and important canonical forms are phase-insensitive, with $\mathbf{K}$ and $\mathbf{N}$ both proportional to the identity, of the form

$$\mathbf{K} = \sqrt{\tau}\mathbf{I}, \ \ \mathbf{N} = \eta\mathbf{I}, \tag{20}$$

where τ is the transmissivity and η is thermal noise. The first example is the lossy channel (form C) for which $\tau \in$

$[0,1]$ and $\eta = (1-\tau)\omega$ with $\omega \geq 1$. In terms of the quadrature operators $\hat{\mathbf{x}} = (\hat{q},\hat{p})^T$ its action is expressed by the input-output transformations

$$\hat{\mathbf{x}} \to \sqrt{\tau}\hat{\mathbf{x}} + \sqrt{1-\tau}\hat{\mathbf{x}}_{th}, \tag{21}$$

where $\hat{\mathbf{x}}_{th}$ are the quadratures of an environmental mode in a thermal state with variance $\omega = 2\bar{n}+1$, with $\bar{n}$ being the mean number of thermal photons. At the border, for $\tau = 0$, this channel coincides with the form A_1 which is a completely depolarizing channel. For $\tau = 1$, it just becomes an identity channel (a particular instance of the form B_2).

The second example is the additive-noise channel (form B_2), which has $\tau = 1$ and $\eta \geq 0$. This channel corresponds to the transformation

$$\hat{\mathbf{x}} \to \hat{\mathbf{x}} + \boldsymbol{\xi}, \tag{22}$$

where $\boldsymbol{\xi}$ is a classical random variable with CM $\eta\mathbf{I}$. The final example of phase-insensitive canonical form is the amplifier channel (another kind of form C) which has $\tau \geq 1$ and $\eta = (\tau-1)\omega$. This is realized by

$$\hat{\mathbf{x}} \to \sqrt{\tau}\hat{\mathbf{x}} + \sqrt{\tau-1}\hat{\mathbf{x}}_{th}, \tag{23}$$

where $\hat{\mathbf{x}}_{th}$ are the quadratures of a thermal mode.

Then, we have canonical forms which are sensitive to phase. The most important is the form D which is the conjugate of the amplifier channel, i.e., the environmental output of a two-mode squeezer which dilates the amplifier channel. This form can be associated with negative transmissivities $\tau \leq 0$ and is described by the matrices

$$\mathbf{K} = \sqrt{-\tau}\mathbf{Z},\ \mathbf{N} = \eta\mathbf{I},$$

with $\mathbf{Z} := \mathrm{diag}(1,-1)$ and $\eta = (1-\tau)\omega = (1-\tau)(2\bar{n}+1)$. Its sensitivity to phase is clear from the fact that $\mathbf{K} \propto \mathrm{diag}(1,-1)$. It corresponds to the transformation

$$\hat{\mathbf{x}} \to \sqrt{-\tau}\mathbf{Z}\hat{\mathbf{x}} + \sqrt{1-\tau}\hat{\mathbf{x}}_{th}. \tag{24}$$

In the main paper, our phase-insensitive Gaussian channel is given by the forms C and B_2, while our extended Gaussian channel is given by the forms C, B_2 and D. Such an extended channel can be compactly characterized by the matrices

$$\mathbf{K} = \sqrt{|\tau|}\mathrm{diag}[1, \mathrm{sign}(\tau)],\ \mathbf{N} = \eta\mathbf{I}, \tag{25}$$

with $\tau \in \mathbb{R}$ and $\eta \geq |1-\tau|$. The corresponding input-output relations can be written in the form

$$\hat{\mathbf{x}} \to \begin{pmatrix} \sqrt{|\tau|} & 0 \\ 0 & [\mathrm{sign}(\tau)]\sqrt{|\tau|} \end{pmatrix} \hat{\mathbf{x}} + \zeta, \tag{26}$$

where ζ is a noise-variable with CM equal to $\eta\mathbf{I}$. The output entropy of this extended Gaussian channel is minimized by taking a coherent state at the input [S3].

Finally, we have the 'pathological' canonical forms A_2 and B_1, which are highly phase-sensitive. In particular, form A_2 is described by

$$\mathbf{K} = \mathrm{diag}(1,0),\ \mathbf{N} = (2\bar{n}+1)\mathbf{I}, \tag{27}$$

while form B_1 is described by

$$\mathbf{K} = \mathbf{I},\ \mathbf{N} = \mathrm{diag}(0,1). \tag{28}$$

Form A_2 is highly symmetric in the transmission matrix, and represents a sort of 'half' depolarizing channel which replaces one of the input quadratures with thermal noise. When this form is locally applied to a bipartite state, it completely destroys the correlations in one of the quadratures. By contrast, B_1 is highly asymmetric in the noise matrix and its effect is to add a unit of vacuum noise to only one of the quadratures.

For these pathological forms, the optimal input state is expected to be a squeezed vacuum. Indeed, this can easily be proven for B_1, for which the optimal input state is an infinitely-squeezed vacuum, with CM $\mathbf{V} = \mathrm{diag}(r^{-1}, r)$ where $r \to +\infty$. In fact, at the output we have a Gaussian state with covariance matrix $\mathbf{V}+\mathbf{N} = \mathrm{diag}(r^{-1}, r+1)$ whose determinant goes to 1, so that it is asympotically pure. This implies that the output entropy goes to zero, i.e., to its minimum value. However, if we enforce an energy constraint at the input of this form, then we do not know the optimal input state which minimizes the output entropy.

The situation is more involved for the other form A_2. When $\bar{n} = 0$, this is the conjugate channel of B_1 in a unitary dilation with a pure environment (Stinespring dilation). Since the global output of this dilation is pure for an input pure state (we can always restrict the search to pure states due to the concavity of the entropy), we have that the output entropy of the conjugate channel A_2 is equal to the output entropy of B_1. This trivially implies that the infinitely-squeezed vacuum is again the optimal input state. However, for $\bar{n} > 0$, the dilation involves a thermal environment, and the optimality of this state is only conjectured for A_2. Furthermore, also for this form, the optimal input is not known if we enforce an energy constraint.

II. DECOMPOSITION OF TWO-MODE SQUEEZED THERMAL STATES

Here we prove that a CM in the special normal-form $\mathbf{V}(a,b,c,-c)$ can equivalently be expressed as in Eq. (8), i.e., using the parametrization

$$a = \tau b + \eta,\ c = \sqrt{\tau(b^2-1)}, \tag{29}$$

for some choice of $\tau \geq 0$ and $\eta \geq |1-\tau|$.

Let us start by re-writing the bona-fide conditions for the parameters a, b, and c. From the uncertainty principle, we derive [S4] $a, b \geq 1$ and $c^2 \leq c^2_{\max}$, where

$$c^2_{\max} := \min\{(a-1)(b+1), (a+1)(b-1)\} \quad (30)$$
$$= ab - 1 - |a - b|. \quad (31)$$

For $a = 1$ or $b = 1$ we must have $c = 0$. This CM is just realized by taking $\tau = 0$. For any $a, b > 1$, it is sufficient to prove that we can generate all CMs with maximal correlations $c^2 = c^2_{\max}$ by means of our parametrization. If this is possible for maximal correlations, then it will also be possible for intermediate correlations $c^2 \leq c^2_{\max}$, where conditions are less stringent (since this extension is merely technical, it has been omitted).

First consider the case $a \geq b > 1$, so that $c^2_{\max} = (a+1)(b-1)$. Imposing $c^2 = (a+1)(b-1)$ and using Eq. (29), we derive $\eta = \tau - 1$ (possible for $\tau \geq 1$). Using $\tau = \eta + 1$ in the parametrization of a, we find $a = b + \eta(1 + b)$, which means that we can generate all possible values of $a \geq b$ by freely varying $\eta \geq 0$.

Consider now $b \geq a > 1$, so that $c^2_{\max} = (a-1)(b+1)$. By imposing $c^2 = (a-1)(b+1)$ and using Eq. (29), we derive $\eta = 1 - \tau$, which implies $0 \leq \eta \leq 1$. By replacing $\tau = 1 - \eta$ in the parametrization of a, we derive $a = b(1 - \eta) + \eta$. It is clear that we can generate all possible values of $a \leq b$ by freely varying $0 \leq \eta < 1$.

III. PARAMETRIZATION OF OUR FAMILY OF GAUSSIAN STATES

In order to derive the parametrization given in Eqs. (17)-(19), we work in the Heisenberg picture, considering the input-output transformations for the quadrature operators $\hat{\mathbf{x}} = (\hat{q}, \hat{p})^T$. The action of the first squeezer in Fig. 2 is given by

$$\hat{\mathbf{x}}_a \to \hat{\mathbf{x}}_{a'} = S^{-1}(r)\hat{\mathbf{x}}_a, \quad (32)$$

where r is positive and

$$S(r) := \begin{pmatrix} r^{1/2} & 0 \\ 0 & r^{-1/2} \end{pmatrix}. \quad (33)$$

The role of this squeezer is to rectify the input ensemble created by the local Gaussian POVM $\mathcal{M}_B(u)$ with $u > 0$, in such a way to have coherent states on mode a'. In fact, applying this local detection to the B mode of an EPR state σ_{aB} with CM

$$\mathbf{V}_{aB} = \begin{pmatrix} b\mathbf{I} & \sqrt{b^2-1}\mathbf{C} \\ \sqrt{b^2-1}\mathbf{C} & b\mathbf{I} \end{pmatrix}, \ \mathbf{C} = \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (34)$$

the other mode a is projected onto amplitude-modulated squeezed states with squeezing $r = (1 + ub)(u + b)^{-1}$ which ranges between b^{-1} and b (see Sec. IV for more details on the remote preparation of Gaussian states).

After the first squeezer, the action of the extended Gaussian channel is expressed by

$$\hat{\mathbf{x}}_{a'} \to \hat{\mathbf{x}}_{A'} = \begin{pmatrix} \sqrt{|\tau|} & 0 \\ 0 & [\mathrm{sign}(\tau)]\sqrt{|\tau|} \end{pmatrix} \hat{\mathbf{x}}_{a'} + \zeta \ , \quad (35)$$

where ζ is a noise-variable with CM $\eta\mathbf{I}$, where $\eta \geq |1-\tau|$ and $\tau \in \mathbb{R}$ (see Eq. (26) in Sec. I). Finally, the second squeezer realizes

$$\hat{\mathbf{x}}_{A'} \to \hat{\mathbf{x}}_A = S(\xi)\hat{\mathbf{x}}_{A'}, \quad (36)$$

where parameter ξ is chosen in such a way to put the output CM $\mathbf{V}_{AB}$ in normal-form. After simple algebra we find

$$\xi = r\frac{\theta(r^{-1})}{\theta(r)}, \quad \theta(r) := \sqrt{\eta r + |\tau|\, b}. \quad (37)$$

Using these equations and the fact that mode B is subject to the identity (before detection), we have that the input EPR state is mapped into the output Gaussian state ρ_{AB} with normal-form CM $\mathbf{V}_{AB} = \mathbf{V}(a, b, c, c')$, where

$$a = \theta(r)\theta(r^{-1}) \quad (38)$$

$$c = \pm\sqrt{|\tau|\,(b^2-1)\frac{\theta(r^{-1})}{\theta(r)}}, \quad (39)$$

$$c' = \mp\mathrm{sign}[\tau]\sqrt{|\tau|\,(b^2-1)\frac{\theta(r)}{\theta(r^{-1})}}, \quad (40)$$

which correspond to Eqs. (17)-(19).

In order to generate elements of the family, we randomly pick values for $b \geq 1$, $r \in [b^{-1}, b]$, $\tau \in \mathbb{R}$ and $\eta \geq |1 - \tau|$. Then, we compute a, c, and c' according to Eqs. (38)-(40). Alternatively, we can pick values for $a, b \geq 1$ and randomly generate values for r, τ, and η which are compatible with Eq. (38). Using these values, we finally compute c and c' according to Eqs. (39) and (40). We adopt this second procedure since it allows us to represent the elements of the family on the correlation plane (c, c') for given values of a and b.

Fixing a value for $a \geq 1$, the condition in Eq. (38) restricts the range of variability of the transmissivity. In fact, we can express the parameter η as function of a by inverting Eq. (38) and finding

$$\eta = \frac{\sqrt{4a^2r^2 + (r^2-1)^2\tau^2b^2} - (1+r^2)|\tau|b}{2r}. \quad (41)$$

Then, imposing $\eta \geq |1-\tau|$ implies $\tau \in [\tau_{\min}, \tau_{\max}]$, where

$$\tau_{\min} = \frac{b + (br+2)r - \sqrt{(r^2-1)^2b^2 + 4a^2r\gamma_+}}{2\gamma_+}, \quad (42)$$

$$\tau_{\max} = \begin{cases} \frac{b+(br-2)r-\sqrt{(r^2-1)^2b^2-4a^2r\gamma_-}}{2\gamma_-} & \text{for } a \leq b, \\ \frac{b+(br+2)r+\sqrt{(r^2-1)^2b^2+4a^2r\gamma_+}}{2\gamma_+} & \text{for } b \leq a, \end{cases} \quad (43)$$

and $\gamma_\pm := (r \pm b)(rb \pm 1)$. Thus, for any values of $a, b \geq 1$, we randomly generate the parameters $r \in [b^{-1}, b]$ and $\tau \in [\tau_{\min}, \tau_{\max}]$. Then, we compute η according to Eq. (41) to be replaced in θ of Eq. (37). We finally compute the values of c and c' according to Eqs. (39) and (40).

IV. REMOTE PREPARATION OF GAUSSIAN STATES

Let us consider an arbitrary two-mode Gaussian state ρ_{AB} with mean value $\bar{\mathbf{x}}_{AB}^T = (\bar{\mathbf{x}}_A^T, \bar{\mathbf{x}}_B^T) \in \mathbb{R}^4$ and CM

$$\mathbf{V}_{AB} = \begin{pmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{B} \end{pmatrix}, \tag{44}$$

where $\mathbf{A} = \mathbf{A}^T$, $\mathbf{B} = \mathbf{B}^T$ and $\mathbf{C}$ are 2×2 real blocks. Mode B is detected by an arbitrary Gaussian measurement, which is defined as a POVM with measurement operator

$$M_k := \pi^{-1} \hat{D}(k) \rho_0 \hat{D}(-k), \tag{45}$$

where $k = (q + ip)/2$ is the complex outcome, $\hat{D}(k) = \exp(k^* \hat{a} - k \hat{a}^\dagger)$ is the displacement operator [S5] with annihilation operator $\hat{a} = (\hat{q} + i\hat{p})/2$, and ρ_0 is a zero-mean Gaussian state with CM $\mathbf{V}_0$. This operator displaces the state by $-k$ and projects it onto the Gaussian state ρ_0. The Gaussian measurement is rank-one when ρ_0 is pure. The simplest choice is the vacuum state $\rho_0 = |0\rangle\langle 0|$, in which case the Gaussian measurement corresponds to heterodyne detection.

The Gaussian POVM can also be expressed in terms of the quadrature operators $\hat{\mathbf{x}}^T := (\hat{q}, \hat{p})$ and the real vector $\mathbf{k}^T = (q, p)$. In fact, from the decomposition of the identity

$$\hat{I} = \int_{\mathbb{C}} d^2k \; M_k = \int_{\mathbb{R}^2} d^2\mathbf{k} \; M_{\mathbf{k}},$$

we derive the equivalent measurement operator

$$M_{\mathbf{k}} := \frac{1}{4\pi} \rho_0(\mathbf{k}),$$

where $\rho_0(\mathbf{k}) := \hat{D}(\mathbf{k}) \rho_0 \hat{D}(-\mathbf{k})$ and

$$\hat{D}(\mathbf{k}) = \exp\left(\frac{i}{2} \, \hat{\mathbf{x}}^T \mathbf{\Omega} \, \mathbf{k}\right), \; \mathbf{\Omega} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{46}$$

By detecting mode B, the outcome $\mathbf{k}$ is achieved with probability

$$p(\mathbf{k}) = \frac{\exp\left[-\frac{1}{2} \mathbf{d}^T (\mathbf{B} + \mathbf{V}_0)^{-1} \mathbf{d}\right]}{2\pi \sqrt{\det(\mathbf{B} + \mathbf{V}_0)}}, \; \mathbf{d} := \bar{\mathbf{x}}_B - \mathbf{k}, \tag{47}$$

which is Gaussian with classical CM $\mathbf{V}_{\mathbf{k}} = \mathbf{B} + \mathbf{V}_0$. Correspondingly, the other mode A is projected into a conditional Gaussian state $\rho_{A|\mathbf{k}}$ with mean value

$$\bar{\mathbf{x}}_{A|\mathbf{k}} = \bar{\mathbf{x}}_A - \mathbf{C}(\mathbf{B} + \mathbf{V}_0)^{-1} \mathbf{d}, \tag{48}$$

and CM given by

$$\mathbf{V}_{A|\mathbf{k}} = \mathbf{A} - \mathbf{C}(\mathbf{B} + \mathbf{V}_0)^{-1} \mathbf{C}^T. \tag{49}$$

If we detect mode A with outcome $\mathbf{k}$, then we have to permute A and B in the formulas. This means that $\bar{\mathbf{x}}_{A|\mathbf{k}}$ and $\mathbf{V}_{A|\mathbf{k}}$ are replaced by $\bar{\mathbf{x}}_{B|\mathbf{k}}$ and $\mathbf{V}_{B|\mathbf{k}}$. Then, Eqs. (47), (48) and (49), are subject to the replacements $\bar{\mathbf{x}}_A \to \bar{\mathbf{x}}_B$, $\bar{\mathbf{x}}_B \to \bar{\mathbf{x}}_A$, $\mathbf{A} \to \mathbf{B}$, $\mathbf{B} \to \mathbf{A}$ and $\mathbf{C} \to \mathbf{C}^T$.

Despite the fact that Eq. (49) is well-known in the literature [S1, S6, S7], the other two Eqs. (47) and (48) are rarely-used and almost unknown. For the sake of completeness and pedagogical reasons we prove all three formulas in Sec. V.

A. Coherent state preparation with EPR states

For its relevance to our work, we consider the specific case where the two-mode Gaussian state ρ_{AB} is an EPR state, and mode B is heterodyned, so that coherent states are prepared on mode A. Let us start with one type of EPR state, whose CM $\mathbf{V}_{AB}$ in Eq. (44) has blocks

$$\mathbf{A} = \mathbf{B} = \mu \mathbf{I}, \; \mathbf{C} = \sqrt{\mu^2 - 1} \mathbf{Z},$$

where $\mu \geq 1$ and $\mathbf{Z} = \mathrm{diag}(1, -1)$. Afterwards we deal with the other EPR type ($\mathbf{Z} \to -\mathbf{Z}$).

Since mode B is heterodyned, we have $\mathbf{V}_0 = \mathbf{I}$ in the previous formulas. According to Eq. (47), the outcome $\mathbf{k}$ follows a Gaussian probability $p(\mathbf{k})$ with classical CM $\mathbf{V}_{\mathbf{k}} = (\mu + 1)\mathbf{I}$, as expected since the reduced state ρ_B is thermal with CM $\mu \mathbf{I}$. Correspondingly, the other EPR mode A is projected into a conditional Gaussian state $\rho_{A|\mathbf{k}}$ with mean-value

$$\bar{\mathbf{x}}_{A|\mathbf{k}} := \mathbf{a} = \frac{\sqrt{\mu^2 - 1}}{\mu + 1} \mathbf{Z} \mathbf{k}, \tag{50}$$

from Eq. (48), and CM $\mathbf{V}_{A|\mathbf{k}} = \mathbf{I}$, from Eq. (49). It is therefore a coherent state with a Gaussianly-modulated mean-value $\mathbf{a} = \mathbf{a}(\mathbf{k})$. In particular, the Gaussian distribution $p(\mathbf{a})$ has CM $(\mu - 1)\mathbf{I}$, as evident from the decomposition of the average (thermal) state

$$\rho_A = \int d^2\mathbf{k} \; p(\mathbf{k}) \, |\mathbf{a}(\mathbf{k})\rangle \langle \mathbf{a}(\mathbf{k})| = \int d^2\mathbf{a} \; p(\mathbf{a}) \, |\mathbf{a}\rangle \langle \mathbf{a}| \,,$$

where

$$p(\mathbf{a}) = \frac{\mu + 1}{\mu - 1} p[\mathbf{k}(\mathbf{a})] = \frac{\exp\left(\frac{-\mathbf{a}^T \mathbf{a}}{2(\mu - 1)}\right)}{2\pi(\mu - 1)}.$$

In the complex notation, the amplitude of the remote coherent state is given by

$$\alpha = \frac{\sqrt{\mu^2 - 1}}{\mu + 1} k^*, \tag{51}$$

where k is the complex outcome of the heterodyne detection. For large μ, we have $\mathbf{a} \to \mathbf{Z}\mathbf{k}$ and $\alpha \to k^*$.

Consider now the other type of EPR state, i.e., the one having $\mathbf{C} = -\sqrt{\mu^2 - 1} \mathbf{Z}$. From Eqs. (47)-(49), we see that the sign change in $\mathbf{C}$ only affects the first moments in Eq. (48). This means that mode A is projected on coherent states as before, the only difference being the relation between their amplitude and the outcome of the heterodyne detection, which is now

$$\bar{\mathbf{x}}_{A|\mathbf{k}} = \frac{-\sqrt{\mu^2-1}}{\mu+1}\mathbf{Zk}. \quad (52)$$

Thus, heterodyning mode B of an arbitrary EPR state with $\mathbf{C} = \pm\sqrt{\mu^2-1}\mathbf{Z}$ projects mode A into an ensemble of coherent states with coherent amplitudes given by Eq. (50) or Eq. (52). In terms of entropy there is no difference, since the entropy of a Gaussian state depends only on the second-order statistical moments.

B. Squeezed-state preparation with EPR states

Here we show how we can create an ensemble of squeezed states by detecting one mode of an EPR state. To achieve this, the Gaussian state ρ_0 in the measurement operator M_k of Eq. (45) must be the squeezed vacuum, with CM $\mathbf{V}_{\mathrm{sq}}(u) = \mathrm{diag}(u, u^{-1})$ where $u > 0$. Let us apply this local detection to mode B of an EPR state with blocks $\mathbf{A} = \mathbf{B} = \mu\mathbf{I}$ and $\mathbf{C} = \pm\sqrt{\mu^2-1}\mathbf{Z}$. Using Eq. (49) we derive

$$\mathbf{V}_{A|\mathbf{k}} = \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix},\ r := \frac{1+u\mu}{u+\mu}. \quad (53)$$

This is clearly the CM of a squeezed state. In particular, it is easy to check that for $u \to 0$ (corresponding to homodyne q-detection), we get $r \to \mu^{-1}$, while for $u \to +\infty$ (which is homodyne p-detection) we get $r \to \mu$. The amount of squeezing we can generate on mode A is limited by the amount of correlations in the EPR state (quantified by the parameter μ). Finally, note that for $u = 1$, we have heterodyne detection and we get $r = 1$ as expected (coherent states).

Finally, using Eqs. (47) and (48) we can derive both the Gaussian probability of the outcomes and the modulated mean value of the states in terms of these outcomes. However, we omit here this calculation, being irrelevant for the computation of the entropy.

V. GAUSSIAN PREPARATION: PROOFS

For the sake of completeness, we prove here the formulas (47)-(49) for the remote preparation of Gaussian states. We start by introducing the partial-trace rule, which is a central tool in our demonstration.

A. Partial-trace rule

Consider a two-mode state ρ_{AB} with characteristic function [S5] $\chi(\alpha,\beta) := \mathrm{Tr}[\rho_{AB}\hat{D}(\alpha)\otimes\hat{D}(\beta)]$, and a state ρ_0 of mode B with characteristic function $\chi_0(\beta) := \mathrm{Tr}[\rho_0\hat{D}(\beta)]$. We consider the partial trace of their product, i.e., the operator $\hat{O} := \mathrm{Tr}_B[\rho_{AB}(I\otimes\rho_0)]$, with associated characteristic function $\chi[\hat{O}](\alpha) := \mathrm{Tr}[\hat{O}\hat{D}(\alpha)]$. This function can be computed via the partial-trace rule [S8]

$$\chi[\hat{O}](\alpha) = \int_{\mathbb{C}} \frac{d^2\beta}{\pi}\chi(\alpha,-\beta)\chi_0(\beta). \quad (54)$$

This is a simple extension of the trace rule for single-mode states ρ and ρ_0, which reads

$$\mathrm{Tr}\,(\rho\rho_0) = \int_{\mathbb{C}} \frac{d^2\beta}{\pi}\chi(-\beta)\chi_0(\beta). \quad (55)$$

In particular, when ρ and ρ_0 are Gaussian with statistical moments $\{\bar{\mathbf{x}},\mathbf{V}\}$ and $\{\bar{\mathbf{x}}_0,\mathbf{V}_0\}$, it is straightforward to check that

$$\mathrm{Tr}\,(\rho\rho_0) = \frac{2\exp\left[-\frac{1}{2}(\bar{\mathbf{x}}-\bar{\mathbf{x}}_0)^T(\mathbf{V}+\mathbf{V}_0)^{-1}(\bar{\mathbf{x}}-\bar{\mathbf{x}}_0)\right]}{\sqrt{\det(\mathbf{V}+\mathbf{V}_0)}}. \quad (56)$$

B. Proof of Eqs. (47)-(49)

Let us start with the proof of Eq. (47). We write

$$p(\mathbf{k}) = \mathrm{Tr}(\rho_B M_{\mathbf{k}}) = \frac{1}{4\pi}\mathrm{Tr}\left[\rho_B\rho_0(\mathbf{k})\right], \quad (57)$$

where $\rho_B = \mathrm{Tr}_A(\rho_{AB})$ is the reduced state of mode B. Since ρ_B and $\rho_0(\mathbf{k})$ are single-mode Gaussian states with statistical moments $\{\bar{\mathbf{x}}_B,\mathbf{B}\}$ and $\{\mathbf{k},\mathbf{V}_0\}$, we can directly apply Eq. (56) to obtain Eq. (47).

To compute the statistical moments $\bar{\mathbf{x}}_{A|\mathbf{k}}$ and $\mathbf{V}_{A|\mathbf{k}}$ of the conditional Gaussian state

$$\rho_{A|\mathbf{k}} = p(\mathbf{k})^{-1}\mathrm{Tr}_B(\rho_{AB}M_{\mathbf{k}}) = \frac{1}{4\pi p(\mathbf{k})}\mathrm{Tr}_B\left[\rho_{AB}\rho_0(\mathbf{k})\right],$$

we apply the partial-trace rule to $\hat{O}_{\mathbf{k}} := \mathrm{Tr}_B\left[\rho_{AB}\rho_0(\mathbf{k})\right]$. Here it is convenient to adopt the Cartesian decomposition $\hat{a} = (\hat{q}+i\hat{p})/2$ and $\alpha = (q+ip)/2$, and use the following real variables

$$\hat{\mathbf{x}} := \begin{pmatrix} \hat{q} \\ \hat{p} \end{pmatrix},\ \tilde{\mathbf{x}} := \begin{pmatrix} \mathrm{Im}\,\alpha \\ -\mathrm{Re}\,\alpha \end{pmatrix},\ \tilde{\mathbf{y}} := \begin{pmatrix} \mathrm{Im}\,\beta \\ -\mathrm{Re}\,\beta \end{pmatrix}.$$

In this notation the displacement becomes $\hat{D}(\tilde{\mathbf{x}}) = \exp\left(i\hat{\mathbf{x}}^T\tilde{\mathbf{x}}\right)$ and the characteristic function of $\rho_{A|\mathbf{k}}$ reads

$$\chi_{A|\mathbf{k}}(\tilde{\mathbf{x}}) := \mathrm{Tr}[\rho_{A|\mathbf{k}}\hat{D}(\tilde{\mathbf{x}})] = \frac{1}{4\pi p(\mathbf{k})}\chi[\hat{O}_{\mathbf{k}}](\tilde{\mathbf{x}}), \quad (58)$$

where $\chi[\hat{O}_{\mathbf{k}}](\tilde{\mathbf{x}})$ is the characteristic function of $\hat{O}_{\mathbf{k}}$. Using the partial-trace rule we get

$$\chi[\hat{O}_{\mathbf{k}}](\tilde{\mathbf{x}}) = \int_{\mathbb{R}^2} \frac{d^2\tilde{\mathbf{y}}}{\pi}\chi(\tilde{\mathbf{x}},-\tilde{\mathbf{y}})\chi_0(\tilde{\mathbf{y}}), \quad (59)$$

where $\chi_0(\tilde{\mathbf{y}}) := \mathrm{Tr}[\rho_0(\mathbf{k})\hat{D}(\tilde{\mathbf{y}})] = e^{-\frac{1}{2}\tilde{\mathbf{y}}^T\mathbf{V}_0\tilde{\mathbf{y}}-i\mathbf{k}^T\tilde{\mathbf{y}}}$ and

$$\chi(\tilde{\mathbf{x}},\tilde{\mathbf{y}}) := \mathrm{Tr}[\rho_{AB}\hat{D}(\tilde{\mathbf{x}})\otimes\hat{D}(\tilde{\mathbf{y}})] \\ = e^{-\frac{1}{2}\left(\tilde{\mathbf{x}}^T\mathbf{A}\tilde{\mathbf{x}}+\tilde{\mathbf{y}}^T\mathbf{B}\tilde{\mathbf{y}}+2\tilde{\mathbf{y}}^T\mathbf{C}^T\tilde{\mathbf{x}}\right)-i\bar{\mathbf{x}}_A^T\tilde{\mathbf{x}}-i\bar{\mathbf{x}}_B^T\tilde{\mathbf{y}}}.$$

By solving the Gaussian integral [S9] in Eq. (59) and replacing in Eq. (58), we obtain

$$\chi_{A|\mathbf{k}}(\tilde{\mathbf{x}}) = \exp\left(-\frac{1}{2}\tilde{\mathbf{x}}^T\mathbf{V}_{A|\mathbf{k}}\tilde{\mathbf{x}}-i\bar{\mathbf{x}}_{A|\mathbf{k}}^T\tilde{\mathbf{x}}\right),$$

where the statistical moments, $\bar{\mathbf{x}}_{A|\mathbf{k}}$ and $\mathbf{V}_{A|\mathbf{k}}$, are those in Eqs. (48) and (49).

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[S8] In order to prove Eq. (54), we first express the states in terms of their characteristic functions

$$\rho_{AB} = \int_{\mathbb{C}^2}\frac{d^2\alpha d^2\beta}{\pi^2}\chi(\alpha,\beta)\hat{D}(-\alpha)\otimes\hat{D}(-\beta).$$

$$\rho_0 = \int_{\mathbb{C}}\frac{d^2\beta}{\pi}\chi_0(\beta)\hat{D}(-\beta),$$

Then, we may write

$$\hat{O} = \int_{\mathbb{C}^3}\frac{d^2\alpha d^2\beta' d^2\beta}{\pi^3}\mathrm{Tr}\left[\hat{D}(-\beta')\hat{D}(-\beta)\right] \\ \times \chi(\alpha,\beta')\chi_0(\beta)\hat{D}(-\alpha).$$

Since $\mathrm{Tr}[\hat{D}(-\beta')\hat{D}(-\beta)] = \pi\delta^{(2)}(-\beta'-\beta)$, we get

$$\hat{O} = \int_{\mathbb{C}}\frac{d^2\alpha}{\pi}\left[\int_{\mathbb{C}}\frac{d^2\beta}{\pi}\chi(\alpha,-\beta)\chi_0(\beta)\right]\hat{D}(-\alpha),$$

where the term inside the square brackets is $\chi[\hat{O}](\alpha)$.

[S9] For any $n\times n$ real symmetric and positive-definite matrix $\mathbf{Q}$, we may write

$$\int_{\mathbb{R}^n} d^n\mathbf{x}\; e^{-\mathbf{x}^T\mathbf{Q}\mathbf{x}+i\mathbf{d}^T\mathbf{x}} = \sqrt{\frac{\pi^n}{\det\mathbf{Q}}}\; e^{-\frac{\mathbf{d}^T\mathbf{Q}^{-1}\mathbf{d}}{4}}.$$