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Optimality of the Delaunay Triangulation in $\mathbb{R}^d$

V. T. Rajan

Manufacturing Research Department, IBM Research Division,
T. J. Watson Research Center, Yorktown Heights, NY 10598, USA
vtrajan@watson.ibm.com

Abstract. In this paper we present new optimality results for the Delaunay triangulation of a set of points in $\mathbb{R}^d$. These new results are true in all dimensions d. In particular, we define a power function for a triangulation and show that the Delaunay triangulation minimizes the power function over all triangulations of a point set. We use this result to show that (a) the maximum min-containment radius (the radius of the smallest sphere containing the simplex) of the Delaunay triangulation of a point set in $\mathbb{R}^d$ is less than or equal to the maximum min-containment radius of any other triangulation of the point set, (b) the union of circumballs of triangles incident on an interior point in the Delaunay triangulation of a point set lies inside the union of the circumballs of triangles incident on the same point in any other triangulation of the point set, and (c) the weighted sum of squares of the edge lengths is the smallest for Delaunay triangulation, where the weight is the sum of volumes of the triangles incident on the edge. In addition we show that if a triangulation consists of only self-centered triangles (a simplex whose circumcenter falls inside the simplex), then it is the Delaunay triangulation.

1. Introduction

A **triangle** in $\mathbb{R}^d$ is a d-dimensional simplex (d-simplex), which is defined by its $(d + 1)$ vertices, and a **triangulation** of a set of points in $\mathbb{R}^d$ is a simplicial decomposition of the convex hull of the point set where the vertices of the triangles are contained in the point set. The **Delaunay triangulation** of a set of points in $\mathbb{R}^d$ is defined to be the triangulation such that the circumsphere of every triangle in the triangulation contains no point from the set in its interior. Such a triangulation exists for every point set in $\mathbb{R}^d$, and it is the dual of the Voronoi diagram [8]. The triangulation is unique if the points are in general position. (Throughout this paper we assume that, unless otherwise stated, the points are in general positions. Hence, for example, no $(d + 2)$ points are cospherical. Degeneracies can be handled using techniques such as those described in [9].)

In $\mathbb{R}^2$ the Delaunay triangulation has been studied extensively and many of its properties are known [1], [8], [17]:

(a) Among all triangulations of a set of points in $\mathbb{R}^2$, the Delaunay triangulation lexicographically maximizes the minimum angle, and also lexicographically minimizes the maximum circumradii.
(b) If every triangle in a triangulation is nonobtuse, then it is the Delaunay triangulation.
(c) A flip algorithm [13] exists which looks at the four vertices of two adjacent triangles and modifies the triangulation to ensure that it is locally Delaunay. This algorithm transforms any triangulation to the Delaunay triangulation in $O(n^2)$ time and can be used as an incremental algorithm.
(d) Optimal $O(n \log n)$ time divide-and-conquer and plane-sweep algorithms are known and elegant data structures to support their implementation exist [10], [11], [17].

In three and higher dimensions, very few results are known [8]. A "lifting" transformation (discussed below) exists that allows the Delaunay-triangulation problem in $\mathbb{R}^d$ to be transformed into a convex-hull problem in $\mathbb{R}^{d+1}$. The convex-hull algorithms can therefore be used to obtain the Delaunay triangulation. However, until recently [16], no optimality results were known in three and higher dimensions.

The Delaunay triangulation (and its dual Voronoi diagram) has been used extensively both in the design of efficient algorithms and in practical applications [8], [15]. Since the Delaunay triangulation has some optimal properties in $\mathbb{R}^2$, and efficient global and incremental algorithms exist to construct them, they have been used in finite-element mesh generation as a way of yielding "good" meshes [2], [18]. A "good" mesh is loosely defined as the one whose elements are of uniform size and shape. We have used them in $\mathbb{R}^3$ for the same purpose though no such properties were known [14].

In this paper we present optimality results for Delaunay triangulation of a set of points in $\mathbb{R}^d$. We also show that some of the well-known properties of the Delaunay triangulation in $\mathbb{R}^2$ mentioned above can, when appropriately defined, be generalized to the Delaunay triangulation in $\mathbb{R}^d$. We define a triangle to be **self-centered** if the circumcenter of the triangle lies inside or on its boundary. In $\mathbb{R}^2$ all nonobtuse triangles are self-centered and vice versa and thus it is a generalization to $\mathbb{R}^d$ of the nonobtuse triangle. We define the **Min-Containment Sphere** of a triangle to be the smallest sphere containing the triangle. In Section 2 we show that, for a self-centered triangle, the min-containment sphere is the same as the circumpshere. However, for a non-self-centered triangle, it is the circumsphere of one of the facets of the triangle, its center lies on the boundary, and its radius is less than the circumradius. (A facet of the triangle is a subsimplex of the triangle. Its circumsphere is the smallest sphere passing through its vertices.)

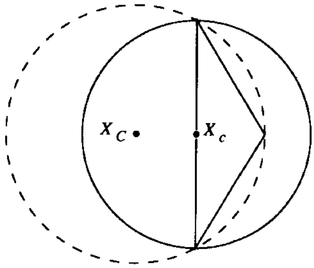

Fig. 1. The min-containment circle (solid) and the circumcircle (dashed) of a non-self-centered (obtuse) triangle in $\mathbb{R}^2$.

In $\mathbb{R}^2$ the min-containment circle of an obtuse triangle is the circle with the longest edge as the diameter. (See Fig. 1.) The **Circumball** of a triangle is the circumsphere and its interior.

In Section 2 we define a power function for a triangulation and in Section 4 we show that the Delaunay triangulation minimizes the power function over all triangulations of a point set in $\mathbb{R}^d$ and use this to prove the following results:

(a) The maximum min-containment radius of the Delaunay triangulation of a point set in $\mathbb{R}^d$ is less than the maximum min-containment radius of any other triangulation of the point set.
(b) The union of circumballs of the triangles incident on an interior point in the Delaunay triangulation of a point set lies inside the union of the circumballs of triangles incident on the same point in any other triangulation of the point set. For a point on the convex hull of the point set, this result is true provided only the portion of the union which lies in the interior cone of the convex hull at the point is considered.
(c) The weighted sum of squares of the edge lengths is the smallest for Delaunay triangulations, where the weight is proportional to the sum of volumes of the triangles incident on the edge. The sum of these weights is independent of the triangulation, therefore, the weights can be normalized.

In addition we obtain a generalization of a well-known property of the Delaunay triangulation in $\mathbb{R}^2$ to the Delaunay triangulation in $\mathbb{R}^d$. Namely, if a triangulation consists of only self-centered triangles (nonobtuse triangles in $\mathbb{R}^2$), then it is the Delaunay triangulation. In [12] we discuss the application of the power function to the visualization of data on a Delaunay triangulation in $\mathbb{R}^3$ and in [16] we discuss an incremental algorithm that can generate the Delaunay triangulation of any point set in $\mathbb{R}^d$.

We have given algebraic proofs of our results so that they can be easily seen to be valid in all dimensions without resorting to geometric intuition. However, we have also provided geometric interpretations of the results.

2. An Optimization Problem Over a Triangle in $\mathbb{R}^d$

Given a set $\{\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_{d+1}\}$ of $(d+1)$ points in $\mathbb{R}^d$ that define a triangle T, consider the function $F(\mathbf{X})$ defined at every point $\mathbf{X}$ in the space:

$$\sum_{i=1}^{d+1} \lambda_i = 1, \qquad \sum_{i=1}^{d+1} \lambda_i \mathbf{P}_i = \mathbf{X}, \tag{1}$$

$$F(\mathbf{X}) = \sum_{i=1}^{d+1} \lambda_i(\mathbf{P}_i - \mathbf{X})^2 = \sum_{i=1}^{d+1} \lambda_i \mathbf{P}_i^2 - \mathbf{X}^2. \tag{2}$$

The $(d+1)$ weights λ_i (also called the barycentric coordinates of $\mathbf{X}$) are uniquely determined by (1). Equation (2) defines $F(\mathbf{X})$ to be the weighted average of the square of the distance to each of the triangle vertices. The bold symbols denote vectors and $\mathbf{V}^2 = \mathbf{V} \cdot \mathbf{V}$ denotes the square of the norm of a vector $\mathbf{V}$. For a point $\mathbf{X}$ inside the triangle, all of the barycentric coordinates λ_i are positive and hence $F(\mathbf{X})$ is also positive. At a vertex $\mathbf{P}_i$, $\lambda_i = 1$ and all other coordinates are zero, hence $F(\mathbf{P}_i) = 0$. The following lemma gives the $F(\mathbf{X})$ at any point in space.

Lemma 1. *Let $\mathbf{X}_C$ and R denote the circumcenter and the circumradii of the triangle T, then the function $F(\mathbf{X})$ at any point $\mathbf{X}$ in space is given by*

$$\begin{aligned} F(\mathbf{X}) &= R^2 - (\mathbf{X} - \mathbf{X}_C)^2 \\ &= (R^2 - \mathbf{X}_C^2) + 2\mathbf{X}_C \cdot \mathbf{X} - \mathbf{X}^2, \end{aligned} \tag{3}$$

and $F(\mathbf{X})$ is maximized at the circumcenter $\mathbf{X}_C$ with $F(\mathbf{X}_C) = R^2$.

Proof. By definition, $(\mathbf{P}_i - \mathbf{X}_C)^2 = R^2$ for all i, hence the function $F(\mathbf{X})$ at a point $\mathbf{X}$ is given by

$$\begin{aligned} F(\mathbf{X}) &= \sum_{i=1}^{d+1} \lambda_i((\mathbf{P}_i - \mathbf{X}_C) - (\mathbf{X} - \mathbf{X}_C))^2 \\ &= \left(\sum_{i=1}^{d+1} \lambda_i\right) R^2 - 2\left(\sum_{i=1}^{d+1} \lambda_i(\mathbf{P}_i - \mathbf{X}_C)\right) \cdot (\mathbf{X} - \mathbf{X}_C) + \left(\sum_{i=1}^{d+1} \lambda_i\right)(\mathbf{X} - \mathbf{X}_C)^2 \\ &= R^2 - 2(\mathbf{X} - \mathbf{X}_C)^2 + (\mathbf{X} - \mathbf{X}_C)^2. \end{aligned}$$

Hence the result (3). The second part of the lemma follows immediately from this result. □

We can give several geometric interpretations for $F(\mathbf{X})$. The **power** of a point with respect to a sphere is the square of the length of the tangent from the point to the sphere. For points inside the sphere it is negative and is minus the square of half the length of the chord that has the point as the midpoint. From (3) we see that $F(\mathbf{X})$ is minus the power of the point $\mathbf{X}$ with respect to the circumsphere of the triangle. Hence we take the liberty of calling $F(\mathbf{X})$ the **power function** of the triangle. The following corollary is an immediate consequence of the above lemma.

Corollary 1. *The point* $\mathbf{X}$ *lies inside, on, or outside the circumsphere of* T *if and only if the power function* $F(\mathbf{X})$ *is positive, zero, or negative, respectively.*

Another interpretation of the power function is given by the following lemma, which uses the lifting transformation introduced by Edelsbrunner and Seidel [7].

Lemma 2. *Consider the transformation* $\alpha(\mathbf{P}) \mapsto (\mathbf{P}, \mathbf{P}^2)$ *which lifts points in* $\mathbb{R}^d$ *onto the paraboloid* $z = \mathbf{P}^2$ *in* $\mathbb{R}^{d+1}$. *A triangle in* $\mathbb{R}^d$ *is lifted to a triangle lying on a hyperplane in* $\mathbb{R}^{d+1}$. *Then* $F(\mathbf{X})$ *is the vertical distance of the hyperplane above the paraboloid at the point* $\mathbf{X}$.

Proof. Let the equation of the plane containing the triangle in $\mathbb{R}^{d+1}$ be given by $z = A + \mathbf{B}\cdot\mathbf{X}$ where A is a constant and $\mathbf{B}$ is a constant d-dimensional vector. The vertical distance (i.e., the distance along the z-axis) of this plane above the paraboloid $z = \mathbf{X}^2$ at a point $\mathbf{X}$ in $\mathbb{R}^d$ is $z(\mathbf{X}) = A + \mathbf{B}\cdot\mathbf{X} - \mathbf{X}^2$. The $d + 1$ constants A, $\mathbf{B}$ can be uniquely determined by the fact that both the plane and the paraboloid pass through the set of $d + 1$ points in $\mathbb{R}^{d+1}$: $\{(\mathbf{P}_i, \mathbf{P}_i^2), i = 1, \ldots, d + 1\}$, hence $z(\mathbf{P}_i) = 0$, $i = 1, \ldots, d + 1$. Since $F(\mathbf{X})$ has the same form and $F(\mathbf{P}_i) = 0$, it follows that $z(\mathbf{X}) \equiv F(\mathbf{X})$. □

If the triangle is self-centered, then the weights $\lambda_i \geq 0$ at the circumcenter. For a non-self-centered triangle one or more of the values of λ_i at the circumcenter will be negative.

Lemma 3. *If the domain of* $\mathbf{X}$ *is the triangle* T *(i.e.,* $\lambda_i \geq 0$*), then* $F(\mathbf{X})$ *is maximized at the center of the min-containment sphere of the triangle* ($\mathbf{X} = \mathbf{X}_c$) *and the value of* $F(\mathbf{X})$ *at that point is the square of the min-containment-radius* (r^2).

Proof. The problem of finding the smallest sphere containing a set of points is given by: Minimize (r^2) subject to $(\mathbf{P}_i - \mathbf{X})^2 \leq r^2$. By setting $S = r^2 - \mathbf{X}^2$ we get the quadratic programming problem: Minimize $(S + \mathbf{X}^2)$ subject to the linear constraints $S + 2\mathbf{P}_i\cdot\mathbf{X} - \mathbf{P}_i^2 \geq 0$. The dual [4], [6] of this quadratic programming problem is: Maximize $F(\mathbf{X})$ (equation (2)) subject to (1) and an additional constraint $\lambda_i \geq 0$. Thus the min-containment sphere problem and the constrained optimization problem are duals of each other and hence the result. □

Thus, for a self-centered triangle, the circumsphere is the smallest sphere spanning the triangle. If we move the center of the sphere away from any of the vertices, the radius of the minimum spanning sphere will increase. For a non-self-centered triangle, the constrained maximum of (3) is the point where the smallest sphere centered at the circumcenter touches the triangle. Let this sphere touch the facet f of the triangle at point $\mathbf{X}_c$. The sphere centered at $\mathbf{X}_c$ with radius square $F(\mathbf{X}_c)$ is the circumsphere of f. It is the smallest sphere spanning f. It also contains the remaining vertices of the triangle in its interior. (See Fig. 1.)

Lemma 4. *Let $\hat{\mathbf{u}}$ be a unit vector in the direction of a ray starting at a vertex $\mathbf{P}_i$ of the triangle T and directed toward its interior. Then the gradient of the power function $F(\mathbf{X})$ at the vertex in the direction $\hat{\mathbf{u}}$ is equal to the length of the chord of the circumsphere of T in the direction $\hat{\mathbf{u}}$.*

Proof. The gradient of the power function $F(\mathbf{X})$ at $\mathbf{P}_i$ in the direction $\hat{\mathbf{u}}$ is $\hat{\mathbf{u}} \cdot \nabla F(\mathbf{X}) = 2\hat{\mathbf{u}} \cdot (\mathbf{X}_C - \mathbf{P}_i)$. A point at a distance σ from the point $\mathbf{P}_i$ in the direction $\hat{\mathbf{u}}$ is given by $\mathbf{P}_i + \sigma\hat{\mathbf{u}}$. Substituting this into the equation for the circumsphere of the triangle yields the two roots $\sigma = 0$ and $\sigma = 2\hat{\mathbf{u}} \cdot (\mathbf{X}_C - \mathbf{P}_i)$ corresponding to the two intersection points of the chord through $\mathbf{P}_i$ in the direction $\hat{\mathbf{u}}$ with the circumsphere of T. □

Lemma 5. *The integral of the power function $F(\mathbf{X})$ over the triangle T is equal to the sum of squares of the lengths of the edges of the triangle times its volume times a constant.*

Proof. From (1) we get $\mathbf{X} = \mathbf{P}_{d+1} + \sum_{i=1}^{d} \lambda_i(\mathbf{P}_i - \mathbf{P}_{d+1})$. Therefore the integral of a function $f(\mathbf{X})$ over the triangle is given by

$$\int_{\Delta} f(\mathbf{X})\, d\mathbf{X} = J \int_0^1 d\lambda_1 \int_0^{1-\lambda_1} d\lambda_2 \cdots \int_0^{1-\sum_{i=1}^{d-1}\lambda_i} d\lambda_d\, f(\mathbf{X}), \tag{4}$$

where the Jacobian is a $d \times d$ determinant:

$$J = |\partial\mathbf{X}/\partial\boldsymbol{\lambda}| = \begin{vmatrix} \mathbf{P}_1 - \mathbf{P}_{d+1} \\ \mathbf{P}_2 - \mathbf{P}_{d+1} \\ \cdots \\ \mathbf{P}_d - \mathbf{P}_{d+1} \end{vmatrix} \tag{5}$$

and is a constant. Setting $f(\mathbf{X}) = 1$ we obtain the volume of the triangle to be $V = J/d!$. Setting

$$f(\mathbf{X}) = F(\mathbf{X}) = \sum_{i=1}^{d+1} \lambda_i \mathbf{P}_i^2 - \sum_{i=1}^{d+1} \lambda_i^2 \mathbf{P}_i^2 - 2\sum_{i=1}^{d+1}\sum_{j=1}^{i-1} \lambda_i\lambda_j \mathbf{P}_i \cdot \mathbf{P}_j$$

we obtain

$$\int_\Delta F(\mathbf{X})\,d\mathbf{X} = J\left(\sum_{i=1}^{d+1} \mathbf{P}_i^2 \int_0^1 d\lambda_i \lambda_i (1-\lambda_i)^d/(d-1)! \right.$$
$$\left. -2\sum_{i=1}^{d+1}\sum_{j=1}^{i-1} \mathbf{P}_i\cdot\mathbf{P}_j \int_0^1 d\lambda_i\lambda_i \int_0^{1-\lambda_i} d\lambda_j\lambda_j(1-\lambda_i-\lambda_j)^{d-2}/(d-2)!\right)$$
$$= (J/(d+2)!)\left(d\sum_{i=1}^{d+1}\mathbf{P}_i^2 - 2\sum_{i=1}^{d+1}\sum_{j=1}^{i-1}\mathbf{P}_i\cdot\mathbf{P}_j\right)$$
$$= (V/(d+1)(d+2))\left(\sum_{i=1}^{d+1}\sum_{j=1}^{i-1}(\mathbf{P}_i-\mathbf{P}_j)^2\right).$$

Hence the lemma. □

3. Delaunay Triangulation

A triangle T is defined to be **Delaunay valid** with respect to a point $\mathbf{P}$ if $\mathbf{P}$ does not lie inside the circumsphere of T. From Corollary 1 it follows that a triangle T is Delaunay valid with respect to a point $\mathbf{P}$ if and only if the power function of the triangle satisfies the relation $F_T(\mathbf{P}) \le 0$. The inequality is strict if the points are in general position. A triangle T in a triangulation of a set of points is called a **Delaunay triangle** if T is Delaunay valid with respect to every point in the set. A triangulation of a set of points is called the **Delaunay triangulation** of the point set if every triangle in the triangulation is a Delaunay triangle. In this section we consider the properties of two adjacent triangles that are Delaunay valid with respect to each other's vertices. We define these triangles to be Delaunay valid with respect to each other. In the next section we consider the properties of the Delaunay triangulation of a general set of points.

Let $T_1 = [\mathbf{P}_1, \ldots, \mathbf{P}_d, \mathbf{P}_{d+1}]$ and $T_2 = [\mathbf{P}_1, \ldots, \mathbf{P}_d, \mathbf{P}_{d+2}]$ be two adjacent triangles in $\mathbb{R}^d$ that lie on the opposite sides of their common face $f = [\mathbf{P}_1, \ldots, \mathbf{P}_d]$. The circumspheres of the triangles T_1 and T_2 pass through the vertices of the face f and intersect on the circumsphere of facet f lying in the hyperplane H_f containing f. Let $\mathbf{X}_f$ and R_f be the circumcenter and the circumradii of the circumsphere of f. Choose a coordinate system so that the origin is located on H_f and one of the coordinate axes, say the z-axis, is perpendicular to H_f. Let the coordinates of a point be $\mathbf{X} = (\mathbf{x}, z)$ where $\mathbf{x}$ denotes the $d-1$ coordinates in H_f. Hence $\mathbf{X}_f = (\mathbf{x}_f, 0)$. Let $\mathbf{X}_{T_i} = (\mathbf{x}_{T_i}, z_{T_i})$ and let R_{T_i} be the circumcenter and the circumradii of the triangle T_i $(i = 1, 2)$. The following lemma gives a relation between these quantities.

Lemma 6. *$\mathbf{x}_{T_i} = \mathbf{x}_f$ and $R_{T_i}^2 = R_f^2 + z_{T_i}^2$, for both $i = 1, 2$. Hence $F_i(\mathbf{X}) = F_i(\mathbf{x}, z) = (R_f^2 - \mathbf{x}_f^2) + 2\mathbf{x}\cdot\mathbf{x}_f + 2zz_{T_i} - \mathbf{X}^2$.*

Proof. For a point $\mathbf{X} = (\mathbf{x}, 0) \in H_f$ the power function $F_i(\mathbf{X})$ of the triangle T_i is identical to the power function $f(\mathbf{x})$ of the facet f since they both represent the solution to (1) and (2) in the subspace spanned by the vertices of the facet f. Hence $F_i(\mathbf{x}, 0) = R_{T_i}^2 - (\mathbf{x}_{T_i}^2 + z_{T_i}^2) + 2\mathbf{x} \cdot \mathbf{x}_{T_i} - \mathbf{x}^2 \equiv R_f^2 - \mathbf{x}_f^2 + 2\mathbf{x} \cdot \mathbf{x}_f - \mathbf{x}^2$. Making term-by-term comparison yields the results. The second part is an immediate consequence of the first part. □

The result can be interpreted as saying that the circumcenter $\mathbf{X}_{T_i}$ lies on the normal to H_f passing through $\mathbf{X}_f$. Let T_1 lie in the positive half-space $z \geq 0$ and hence T_2 lies in the negative half-space. We say that T_1 and T_2 lie above and below H_f, respectively. Let the coordinates of the two points not lying in H_f be $\mathbf{P}_{d+1} = (\mathbf{x}_{d+1}, z_{d+1})$ and $\mathbf{P}_{d+2} = (\mathbf{x}_{d+2}, z_{d+2})$, respectively, then it follows that $z_{d+1} > 0$ and $z_{d+2} < 0$.

Lemma 7. *The two adjacent triangles T_1 and T_2 are Delaunay valid with respect to each other if and only if $z_{T_1} \geq z_{T_2}$.*

Proof. The two triangles are Delaunay valid with respect to each other if and only if T_1 is Delaunay valid with respect to $\mathbf{P}_{d+2}$, and T_2 is Delaunay valid with respect to $\mathbf{P}_{d+1}$. Since $\mathbf{P}_{d+2}$ is a vertex of T_2, $F_2(\mathbf{P}_{d+2}) = 0$. T_1 is Delaunay valid with respect to $\mathbf{P}_{d+2}$ if and only if $F_1(\mathbf{P}_{d+2}) \leq 0$. That is, $F_1(\mathbf{P}_{d+2}) - F_2(\mathbf{P}_{d+2}) = 2(z_{T_1} - z_{T_2})z_{d+2} \leq 0$. Since $z_{d+2} < 0$, this is true if and only if $z_{T_1} \geq z_{T_2}$. Similarly, we can show that T_2 is Delaunay valid with respect to $\mathbf{P}_{d+1}$ if and only if $z_{T_1} \geq z_{T_2}$. □

Lemma 8. *If T_1 and T_2 are two adjacent self-centered triangles, then they are Delaunay valid with respect to each other.*

Proof. Since the triangles are self-centered, $z_{T_1} \geq 0$ and $z_{T_2} \leq 0$. Therefore $z_{T_1} \geq z_{T_2}$, and the result follows from Lemma 7. □

Lemma 9. *Let T_1 and T_2 be two adjacent Delaunay valid triangles lying above and below, respectively, their common face f. If $\mathbf{X}$ is a point above the plane H_f of f, then $F_1(\mathbf{X}) \geq F_2(\mathbf{X})$. In addition, if the point $\mathbf{X}$ is in the circumsphere of T_2, then it lies in the circumsphere of T_1.*

Proof. The coordinates of the point $\mathbf{X} = (\mathbf{x}, z)$ with $z > 0$. $F_1(\mathbf{X}) - F_2(\mathbf{X}) = 2z(z_{T_1} - z_{T_2}) \geq 0$ from Lemma 6 and 7. The equality occurs only in case of degeneracy when the $d + 2$ vertices of the two trinagles are cospherical. If the point $\mathbf{X}$ lies in the circumsphere of T_2, then $F_2(\mathbf{X}) > 0$. Hence, $F_1(\mathbf{X}) > 0$ and therefore $\mathbf{X}$ lies in the circumsphere of T_1. □

4. An Optimization Problem Over a Set of Points

In this section we discuss the properties of the Delaunay triangulation of a general set of points. In particular we show that the Delaunay triangulation minimizes the power function and use this result to prove several properties of the Delaunay triangulation.

Let $S_p = \{\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n\}$ be a set of n points in $\mathbb{R}^d$ with a convex hull denoted by $\mathrm{CH}S_p$. Consider the function $f(\mathbf{X})$ defined over the domain $\mathrm{CH}S_p$:

$$\lambda_i \geq 0, \qquad \sum_{i=1}^{n} \lambda_i = 1, \qquad \sum_{i=1}^{n} \lambda_i \mathbf{P}_i = \mathbf{X}, \tag{6}$$

$$F(\mathbf{X}, \boldsymbol{\lambda}) = \sum_{i=1}^{n} \lambda_i(\mathbf{P}_i - \mathbf{X})^2 = \sum_{i=1}^{n} \lambda_i \mathbf{P}_i^2 - \mathbf{X}^2, \tag{7}$$

$$f(\mathbf{X}) = \mathrm{Min}_{\boldsymbol{\lambda}}\, F(\mathbf{X}, \boldsymbol{\lambda}). \tag{8}$$

$F(\mathbf{X}, \boldsymbol{\lambda})$ is the weighted average of the distance square to the points with weights λ_i. For a fixed point $\mathbf{X}$, (6) provides $(d + 1)$ constraints, hence, $(n - (d + 1))$ weights λ_i can be varied independently. $f(\mathbf{X})$ is the minimum for a fixed $\mathbf{X}$ over this choice of weights. One (not necessarily optimal) choice of weights would be to give nonzero weights to only some set of $(d + 1)$ points (whose convex hull contains $\mathbf{X}$, cf. (6)) and set the remaining $(n - (d + 1))$ weights to zero. In this case the function $F(\mathbf{X}, \boldsymbol{\lambda}) = F(\mathbf{X})$ where $F(\mathbf{X})$ is the power function of the triangle defined by the chosen $(d + 1)$ points.

Lemma 10. *At* $\mathrm{Min}_\lambda\, F(\mathbf{X}, \boldsymbol{\lambda})$ *for a fixed point* $\mathbf{X}$ *the only nonzero values of* λ_i *occur for the vertices of the Delaunay triangle containing the point* $\mathbf{X}$. *Thus* $f(\mathbf{X})$ *is given by Lemma* 1:

$$f(\mathbf{X}) = F(\mathbf{X}) = R^2 - (\mathbf{X} - \mathbf{X}_C)^2, \tag{9}$$

where $\mathbf{X}_C$ *and* R *are the circumcenter and circumradius of the Delaunay triangle containing* $\mathbf{X}$.

Proof. To prove this result we use the following well-known result [3], [7] which is a consequence of Corollary 1 and Lemma 2. Consider the transformation $\alpha(\mathbf{P}) \mapsto (\mathbf{P}, \mathbf{P}^2)$ which lifts the points in $\mathbb{R}^d$ onto the paraboloid $z = \mathbf{P}^2$ in $\mathbb{R}^{d+1}$. The point S_p is lifted to a set of points $\Sigma_p = \{(\mathbf{P}_1, \mathbf{P}_1^2), (\mathbf{P}_2, \mathbf{P}_2^2), \ldots, (\mathbf{P}_n, \mathbf{P}_n^2)\}$ in $\mathbb{R}^{d+1}$. Take the lower part of the convex hull of Σ_p. Project this back into $\mathbb{R}^d$ and we get the Delaunay triangulation, $\mathscr{D}$, of S_p.

Now consider a point $(\sum_{i=1}^n \lambda_i \mathbf{P}_i, \sum_{i=1}^n \lambda_i \mathbf{P}_i^2) = (\mathbf{X}, F(\mathbf{X}, \boldsymbol{\lambda}) + \mathbf{X}^2)$, subject to the conditions in (6). This is a general point inside the convex hull of Σ_p. The minimum of F for a fixed $\mathbf{X}$ is given by the point with the lowest z coordinate, hence the point on the lower convex hull of Σ_p. This triangle (d-facet in $\mathbb{R}^{d+1}$) is the Delaunay triangle containing $\mathbf{X}$. □

Theorem 1. *Among all the triangles with vertices in* S_p *and containing the point* $\mathbf{X}$, *the Delaunay triangle minimizes the power function* $F(\mathbf{X})$ *of the triangle at the point.*

Proof. Every triangle containing the point $\mathbf{X}$ defines a particular (unique) choice of λ_i in (6) and (7), namely, set λ_i to be nonzero only at the vertices of the triangle. By Lemma 10, the choice that minimizes $F(\mathbf{X}, \boldsymbol{\lambda})$ and hence the power function $F(\mathbf{X})$ is the Delaunay triangle. □

Given any triangulation $\mathscr{T}$ of S_p we can define the function $F_{\mathscr{T}}(\mathbf{X})$ at each point $\mathbf{X}$ in $\mathrm{CH}S_p$ as the power function $F(\mathbf{X})$ of the point $\mathbf{X}$ with respect to the triangle $T \in \mathscr{T}$ that contains the point $\mathbf{X}$. Let $F_{\mathscr{D}}(\mathbf{X})$ define the corresponding function for the Delaunay triangulation $\mathscr{D}$. Then it follows from Theorem 1 that, at every point $\mathbf{X}$,

$$F_{\mathscr{D}}(\mathbf{X}) \leq F_{\mathscr{T}}(\mathbf{X}). \tag{10}$$

The equality holds only when the element (the triangle or the facet of a triangle) in the $\mathscr{D}$ that contains the point $\mathbf{X}$ is also present in $\mathscr{T}$. We can use this result to derive several optimality properties of the Delaunay triangulation.

Theorem 2. *The maximum min-containment radius of the Delaunay triangulation is less than or equal to the maximum min-containment radius of any other triangulation of the point set.*

Proof. Let $\mathbf{X}_T$ and $\mathbf{X}_D$ respectively represent the points in $\mathrm{CH}S_p$ where $F_{\mathscr{T}}(\mathbf{X})$ and $F_{\mathscr{D}}(\mathbf{X})$ attain their respective maxima. Let r_T and r_D be respectively the maximum min-containment radii of the triangulations $\mathscr{T}$ and $\mathscr{D}$, respectively. From Lemma 3, the square of the min-containment radius of a triangle T is the maximum of the power function over the triangle and hence its maxima for the triangulation $\mathscr{T}$ is equal to the maxima of $F_{\mathscr{T}}(\mathbf{X})$. The same is true for the Delaunay triangulation. Hence,

$$r_T^2 = F_{\mathscr{T}}(\mathbf{X}_T) \geq F_{\mathscr{T}}(\mathbf{X}_D) \geq F_{\mathscr{D}}(\mathbf{X}_D) = r_D^2. \tag{11}$$

Hence the result. Notice that (in the nondegenerate situation) the equality holds only if $\mathbf{X}_T \equiv \mathbf{X}_D$ and the element (triangle or the facet of a triangle) containing $\mathbf{X}_D$ in $\mathscr{D}$ is also present in $\mathscr{T}$. □

While Theorem 2 is true in all dimensions, the lexicographical version of the theorem is not true even in $\mathbb{R}^2$. A counterexample to this was given by Edelsbrunner. In Fig. 2 both the Delaunay triangulation and the non-Delaunay triangulation of this set of four points have the same maximum min-containment diameter, given by the bottom edge in the figure. However, the non-Delaunay triangulation has a smaller second-largest value for the min-containment diameter (given by the dashed diagonal) than does the Delaunay triangulation (given by the solid diagonal).

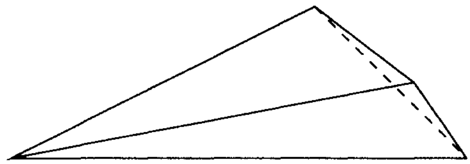

Fig. 2. The Delaunay triangulation (solid line) and the triangulation that lexicographically minimizes the maximum min-containment radius (dashed line) for a set of four points in $\mathbb{R}^2$.

Theorem 3. *The union of circumballs of triangles incident on an interior point in the Delaunay triangulation of a point set lies inside the union of the circumballs of triangles incident on the same point in any other triangulation of the point set. For a point on the convex hull of the point set, this result is true provided only the portion of the union which lies in the interior cone of the convex hull at the point is considered.*

Proof. $\mathbf{P}_i \in S_p$ is a vertex of both triangulations, therefore $F_{\mathscr{D}}(\mathbf{P}_i) = F_{\mathscr{T}}(\mathbf{P}_i) = 0$. However, at a point $\mathbf{X} \in \mathrm{CH}S_p$ in the immediate neighborhood on $\mathbf{P}_i$, $F_{\mathscr{D}}(\mathbf{X}) \leq F_{\mathscr{T}}(\mathbf{X})$. Let $\hat{\mathbf{u}}$ be a unit vector in the direction of a ray from the vertex $\mathbf{P}_i$. If $\mathbf{P}_i$ is on the convex hull of the point set $\mathrm{CH}S_p$, then $\hat{\mathbf{u}}$ is toward the interior of $\mathrm{CH}S_p$. Then the gradient of the power function in the direction $\hat{\mathbf{u}}$ satisfies the inequality $\hat{\mathbf{u}} \cdot \nabla F_{\mathscr{D}}(\mathbf{P}_i) \leq \hat{\mathbf{u}} \cdot \nabla F_{\mathscr{T}}(\mathbf{P}_i)$. The equality is true only if the element containing the initial part of $\hat{\mathbf{u}}$ is common to $\mathscr{D}$ and $\mathscr{T}$. Hence, from Lemma 4, the circumsphere of the triangle in $\mathscr{D}$ incident on $\mathbf{P}_i$ and containing $\hat{\mathbf{u}}$ does not extend beyond the corresponding circumsphere in $\mathscr{T}$ in the direction $\hat{\mathbf{u}}$. By Lemma 9, the Delaunay spheres of the adjacent triangles in $\mathscr{D}$ that are incident on $\mathbf{P}_i$ also do not extend beyond this. By choosing $\hat{\mathbf{u}}$ in every direction from a point $\mathbf{P}_i$ in the interior of $\mathrm{CH}S_p$ we get the first part of the theorem. If $\mathbf{P}_i$ is on $\mathrm{CH}S_p$, then we choose $\hat{\mathbf{u}}$ to be toward the interior, and we get the second part of the result. Notice that in both cases the equality of the two unions occurs only if every triangle in $\mathscr{T}$ incident on $\mathbf{P}_i$ is a Delaunay triangle. □

If we take a triangle in $\mathbb{R}^2$ with rays emanating from its vertices toward the interior of the triangle, then every point in the interior of the circumdisk of the triangle will be covered by one or more of these rays. Therefore, in $\mathbb{R}^2$, a corollary of Theorem 3 is that the union of circumdisks of all triangles in a Delaunay triangulation is contained in the union of circumdisks of all triangles in any other triangulation. However, in three and higher dimensions, this corollary is not true. If we take a tetrahedron in $\mathbb{R}^3$ with rays emanating from its vertices toward its interior, then there are portions of the circumball of the tetrahedron (along each edge of the tetrahedron and just outside the tetrahedron) that are not covered by these rays. Therefore, Theorem 3 cannot be used to generalize this corollary to higher dimensions. In fact, it is possible to construct a counterexample to this corollary in $\mathbb{R}^3$ using five points in three dimensions. Place three points on the xy-coordinate plane at unit distances from the origin, and place two points on the z-axis, above and below the xy-plane, each at less than unit distances from the

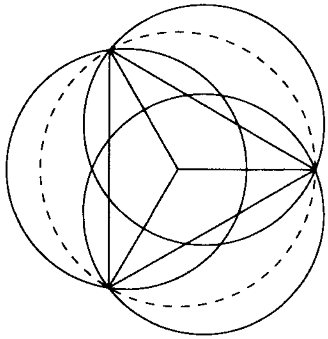

Fig. 3. The $z = 0$ section of the triangulation of a set of five points in $\mathbb{R}^3$ showing the circumspheres of the Delaunay (solid circles) and non-Delaunay (dashed circle) triangulations of the five points. Three of the points lie equidistant from the origin in the $z = 0$ plane and the two other points lie closer, on the z-axis above and below the plane.

origin. Then the Delaunay triangulation of these five points consists of three tetrahedra. Figure 3 shows the $z = 0$ section of the tetrahedra and the circumballs and shows that the circumballs of the Delaunay tetrahedra can extend beyond the circumballs of the two non-Delaunay tetrahedra. Hence the condition on the second part of Theorem 3 is necessary for three and higher dimensions.

Theorem 4. *The weighted sum of squares of the edge lengths, where the weight is proportional to the sum of volumes of the triangles incident on the edge, is the smallest for Delaunay triangulations.*

Proof. Integrate inequality (10) over the $\mathrm{CH}S_p$. By Lemma 5, each triangle in each triangulation contributes to the integral an amount equal to the sum of the squares of edge lengths of its edges times the volume of the triangle divided by the constant $(d + 1)(d + 2)$. Thus the total integral is the weighted sum of squares of the edge lengths of each triangulation, with the weight equal to the sum of the volumes of the triangle incident on each edge. Notice that since each triangle has $\binom{d+1}{2}$ edges, the sum of the weights is equal to $\binom{d+1}{2}$ times the sum of the volumes of the triangles. The sum of the volumes of the triangles is equal to the volume of $\mathrm{CH}S_p$. Thus, the sum of these weights is independent of the triangulation, therefore the weights can be normalized. Notice that $\mathscr{T}$ and $\mathscr{D}$ may in general have a different number of triangles and a different number of edges. Also that the inequality is strict, unless both triangulations are Delaunay. $\square$

Theorem 5. *If a triangulation consists of only self-centered triangles, then it is the Delaunay triangulation of that point set.*

Proof. Suppose we are given a triangulation of S_p consisting of only self-centered triangles. Then, from Lemma 8, every pair of adjacent triangles satisfies the

Delaunay condition. If we project the triangles using $\alpha(\mathbf{P})$ defined above, then this states that the resulting surface in $\mathbb{R}^{d+1}$ is convex locally. Since a surface that is locally convex everywhere is globally convex it follows that we have generated the convex hull. $\square$

This theorem shows that if a point set has a self-centered triangulation, it is unique and can be found by constructing the Delaunay triangulation of the point set.

5. Discussion

This paper provides several optimality results for Delaunay triangulation in $\mathbb{R}^d$. It suggests that the Delaunay triangulation is the most compact one in the sense of

(1) having the smallest min-containment circle,
(2) having the circumspheres of triangles incident on an interior point being closest to the point, and
(3) having the smallest weighted average of the squares of edge lengths with appropriate weights.

These optimality results are new and were not known even in $\mathbb{R}^2$. (Although an $\mathbb{R}^2$ version of Theorem 2 is implicit in Lemma 2 of [5].) Even the methods used to prove them are different from the ones used to prove other optimality results in $\mathbb{R}^2$. Those results are proven using the flip algorithm [8]. The optimality results shown in this paper provide a justification for the use of the Delaunay triangulation for the construction of meshes in $\mathbb{R}^3$.

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