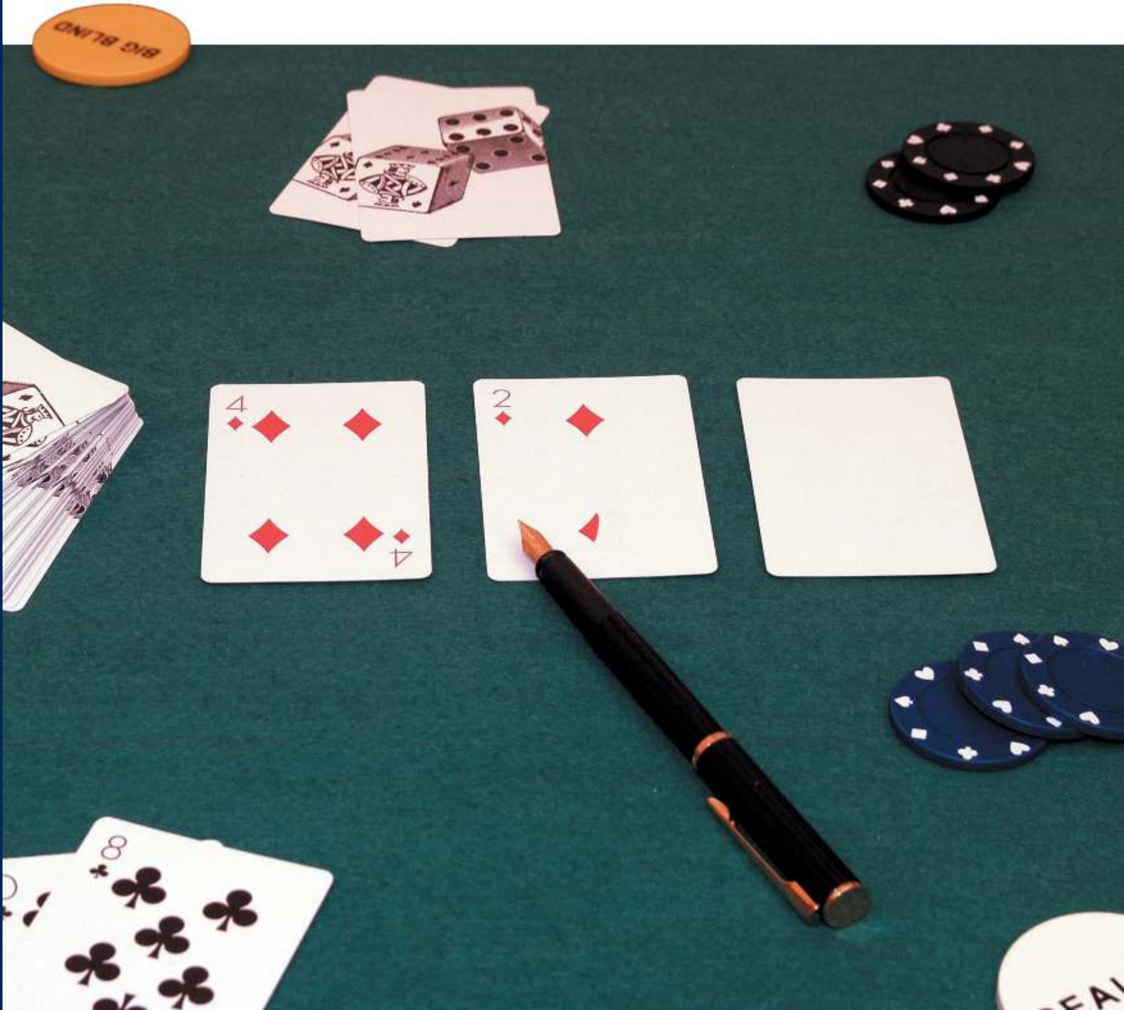
RUTGER KERKKAMP

Optimisation Models for Supply Chain Coordination under Information Asymmetry



Optimisation Models for Supply Chain Coordination under Information Asymmetry

Optimisation Models for Supply Chain Coordination under Information Asymmetry

Optimalisatiemodellen voor coördinatie in productieketens met asymmetrische informatie

Thesis

to obtain the degree of Doctor from the
Erasmus University Rotterdam
by command of the
Rector Magnificus

Prof.dr. R.C.M.E. Engels

and in accordance with the decision of the Doctorate Board.

The public defence shall be held on

Friday 5 October 2018 at 09:30 hours

by

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Erasmus Research Institute of Management - ERIM
The joint research institute of the Rotterdam School of Management (RSM)
and the Erasmus School of Economics (ESE) at the Erasmus University Rotterdam
Internet: www.erim.eur.nl

ERIM Electronic Series Portal: repub.eur.nl

ERIM PhD Series in Research in Management, 462
ERIM reference number: EPS-2018-462-LIS
ISBN 978-90-5892-523-7

Cover image: R.B.O. Kerkkamp
Cover design: PanArt, www.panart.nl

This publication (cover and interior) is printed by Tuijtel on recycled paper, BalanceSilk®.
The ink used is produced from renewable resources and alcohol free fountain solution.
Certifications for the paper and the printing production process: Recycle, EU Ecolabel, FSC®, ISO14001.
More info: www.tuijtel.com

Acknowledgements

First and foremost, I would like to thank my supervisors, Wilco van den Heuvel and Albert Wagelmans, for enabling me to do a PhD and assisting me throughout these four years. Albert, I have always greatly appreciated your insights and opinions at the key moments of each research project. The same holds for our conversations on non-research-related topics, which often formed the start and end of our meetings. Wilco, as my daily supervisor, I do not want to know how many e-mails I have sent you. Your guidance, your feedback on all preliminary results, and our plethora of discussions have been vital during these years. I enjoyed assisting you in teaching Combinatorial Optimisation, especially since you allowed me to take on more responsibilities each time. Knowing the experiences of many other PhD candidates, I am fortunate that we have a compatible working style and ethos, based on mutual trust and respect, and that we have maintained it throughout these years.

A doctoral thesis defence cannot take place without a suitable opposition. I am grateful to Prof.dr.ir. Rommert Dekker, Prof.dr.ir. Karen Aardal, and Prof.dr. Safia Kedad-Sidhoum for being part of my inner doctoral committee. Furthermore, I greatly appreciate that Prof.dr. Edwin Romeijn and Prof.dr. Otto Swank join the opposition in the thesis defence. I would like to thank the entire doctoral committee for their time and effort.

My paranymphs, Judith Mulder and Kevin Dalmeijer, deserve my gratitude as well. Thank you for all the proofreading and provided assistance, and for being awesome colleagues.

In the first months of my PhD, Charlie started a great initiative, supported by Kevin in the execution, which has grown to a tradition worth preserving. Sha, Weina, Xiao, and Xuan, I enjoyed our language sessions and our cultural exchange. Harwin, Thomas, Thomas, and Indy, thank you for all the conversations and research discussions we have had.

Last but not least, my fellow colleagues also deserve to be mentioned here. I will attempt to list all of you, excluding those mentioned above, in pseudo-random order: Chiel, Dennis, Eric, Evelot, Jan, Kristiaan, Marieke, Mathijs, Naut, Nemanja, Nick, Remy, Rolf, Rowan, Twan, and Willem. It would be ironic to forget Paul.

You have all contributed in some way to this conclusion.

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Chapter 1

Introduction

Contents

This chapter is based on Kerkkamp (2017).

1.1 The challenge of information asymmetry

Suppose you are a seller of a certain product and are aware of a potential buyer. You have a single opportunity to offer this buyer a menu of several contracts to choose from, where each contract prescribes the buyer's order quantity and his total price. However, the buyer does not blindly accept any contract: the proposed order quantity and the corresponding price must be satisfactory to him. Any unfavourable contract will be declined. That is, after offering the menu of contracts to the buyer, he will either reject all contracts or accept the contract that is most beneficial to himself. In the first case, you lose this potential buyer, since we assume that renegotiations are not possible. Hence, whether the buyer accepts a contract needs to be taken into account when designing the menu of contracts.

A complicating factor is that the buyer does not share all his information with you for strategic bargaining reasons, which includes his acceptable combinations of order quantities and prices. Facing this information asymmetry, how can you design a menu of contracts such that your expected profit is maximised?

The described problem is called a *principal-agent problem*. The principal (the seller) wants to persuade the agent (the buyer) to take a certain action, without having all relevant information on the buyer. There is information asymmetry between both parties: the buyer has private information, e.g., his maximum budget or his maximum price per unit of products. To achieve his goal, the seller can design an *incentive mechanism* to persuade the buyer to act in the seller's interest and change his default behaviour. We consider a mechanism consisting of a menu of contracts where a side payment (a financial compensation) is used as incentive. In order to make optimal use of a menu, these contracts must be carefully designed to be in line with each other. Typically, an optimisation problem has to be solved to determine a menu that maximises the seller's profit.

The modelling of the principal-agent problem is essential for the resulting incentive mechanism. Small variations in the model can already change the buyer's observable behaviour in his choice of contracts. Two important modelling aspects are the buyer's private information and the maximum allowed number of contracts in the menu. For example, if the private information can take on only two possible values, then a natural approach is to offer a menu with two contracts, namely one intended for each possibility. This idea can of course be extended to a general but finite number of possibilities, referred to as the *discrete* case. If the private information lies in a continuous range, the *continuous* case, we can interpret it as the limit of the discrete case. However, we can question how you would communicate the resulting menu with infinitely many contracts to the buyer. Perhaps we should restrict the number of contracts, hopefully without losing too much expected profit.

We continue with a formal definition of the considered problem in order to elaborate on the mentioned design choices and to position Chapters 2-5 in this framework.

1.2 The principal-agent problem

We describe the principal-agent problem in a setting with a seller and a potential buyer, where the buyer has private information. The seller has a single type of product, which we assume is infinitely divisible. Selling $x \in \mathbb{R}_{\geq 0}$ products to the buyer results in a seller's utility $\psi(x)$ expressed in monetary value, e.g., the utility $\psi(x)$ is the seller's profit. The buyer's utility for obtaining x products depends on his private information. We restrict the problem to the case where the buyer's private information can be encoded in a single-dimensional parameter $\theta \in \mathbb{R}$. This parameter is called the buyer's *type* and models, for example, the buyer's marginal value for a unit of products. Thus, the buyer with type θ gains a utility $\phi(x|\theta)$ for an order quantity x.

From the seller's perspective, we assume that the buyer's type θ follows a strictly positive distribution ω on a set $\Theta \subseteq \mathbb{R}$. The seller constructs a menu of contracts to maximise his expected net utility, relative to the distribution ω, where each contract specifies an order quantity $x \in \mathbb{R}_{\geq 0}$ and a side payment $z \in \mathbb{R}$ from the buyer to the seller. These side payments form the incentive mechanism and allow the seller to affect the buyer's behaviour. If the buyer accepts a contract (x, z), then he obtains a net utility of $\phi(x|\theta) - z$ and the seller gains a net utility of $\psi(x) + z$. We assume that the buyer is willing to accept a contract if his net utility meets a certain threshold $\phi^*(\theta)$, which might depend on his type. The threshold $\phi^*(\theta)$ is called the buyer's *reservation level* or *default option*.

Given the described setting, the seller designs a number of contracts and assigns each type $\theta \in \Theta$ one of these contracts, denoted by $(x(\theta), z(\theta))$. The design and assignment of contracts must satisfy two conditions. First, each assigned contract must be acceptable for the corresponding buyer type by taking his reservation level into account:

$$\phi(x(\theta)|\theta) - z(\theta) \geq \phi^*(\theta), \qquad \forall\, \theta \in \Theta. \tag{1.1}$$

These are known as the *Individual Rationality* (IR) constraints. Second, the assigned contract must be the buyer's most preferred contract of the menu in terms of net utility:

$$\phi(x(\theta)|\theta) - z(\theta) \geq \phi(x(\hat{\theta})|\theta) - z(\hat{\theta}), \qquad \forall\, \theta, \hat{\theta} \in \Theta. \tag{1.2}$$

Constraints (1.2) are the *Incentive Compatibility* (IC) constraints and align the side payments across the contracts. The IR and IC constraints are typical mechanism-design constraints.

Since the buyer does not share his private information, he can lie about his type θ, and will either reject all contracts or choose the most beneficial contract for him. However, by designing the menu such that constraints (1.1) and (1.2) are satisfied, it is always optimal for buyer type $\theta \in \Theta$ to accept his intended contract $(x(\theta), z(\theta))$. In other words, the mechanism prevents any (financial) incentive to lie. Consequently, the buyer's choice is directly related to his type and we can express the seller's expected net utility by $\mathrm{E}_\theta \left(\psi(x(\theta)) + z(\theta)\right)$.

Hence, the seller's optimisation problem is to determine a menu satisfying (1.1) and (1.2) that maximises $\mathrm{E}_\theta\left(\psi(x(\theta)) + z(\theta)\right)$, referred to as the *contracting problem.* We emphasise that with the described mechanism the buyer will always accept a contract from the menu, regardless of his type. We call such a mechanism *robust.*

In the above description of the principal-agent problem and the incentive mechanism, we have not specified whether the type distribution ω is discrete or continuous, and how many contracts the menu can contain. These are important modelling decisions and depend on, for example, the information on the buyer available to the seller and the communication of the menu to the buyer. We will discuss three modelling approaches in the next sections. These models differ in the distribution ω of the buyer's type (discrete or continuous) and in the number of contracts in the menu (finite or infinite). Table 1.1 provides an overview of the contracting models.

Model type		**Number of contracts** **Finite**	**Infinite**
Probability distribution	**Discrete**	Discrete (Chapter 2)	-
	Continuous	Pooling (Chapters 3-4)	Continuous (Chapter 5)

Table 1.1: Variants of the contracting model.

1.2.1 The continuous model

The first contracting model we discuss is the *continuous model*, where the buyer's type θ is continuously distributed on an interval $\Theta = [\underline{\theta}, \bar{\theta}] \subseteq \mathbb{R}$. Furthermore, the menu may contain infinitely many contracts. The continuous model is given by

$$\begin{aligned}
&\max_{x,z} && \int_{\underline{\theta}}^{\bar{\theta}} \omega(\theta)\Big(\psi(x(\theta)) + z(\theta)\Big)\mathrm{d}\theta \\
&\text{s.t.} && \phi(x(\theta)|\theta) - z(\theta) \geq \phi^*(\theta), && \forall\, \theta \in [\underline{\theta}, \bar{\theta}], \\
& && \phi(x(\theta)|\theta) - z(\theta) \geq \phi(x(\hat{\theta})|\theta) - z(\hat{\theta}), && \forall\, \theta, \hat{\theta} \in [\underline{\theta}, \bar{\theta}], \\
& && x(\theta) \geq 0, && \forall\, \theta \in [\underline{\theta}, \bar{\theta}].
\end{aligned}$$

The objective is to maximise the seller's expected net utility. The constraints are the IR constraints (1.1), the IC constraints (1.2), and the domain constraints.

To give an example of the continuous model, suppose the private information θ is the buyer's monetary value of a unit of products. A corresponding utility function could be $\phi(x|\theta) = \theta x$, which directly expresses the monetary value of the order x. The private information can also be more abstract and, for example, be related to market saturation by having $\phi(x|\theta) = rx - \theta x^2$. Here, $r \in \mathbb{R}_{\geq 0}$ is a given marginal value of a unit of products. This concave utility function models a situation where an excess of products results in additional costs for the buyer, implying that for each type there is an order quantity maximising the buyer's utility.

1.2.2 The discrete model

The second contracting model is the *discrete model*. Here, the buyer's type has a discrete distribution on a finite set $\Theta = \{\theta_1, \ldots, \theta_K\}$ with corresponding strictly positive probabilities $\omega_1, \ldots, \omega_K$, for some $K \in \mathbb{N}_{\geq 1}$. In the discrete model the seller offers a menu consisting of K contracts, denoted by (x_k, z_k) for $k \in \mathcal{K} = \{1, \ldots, K\}$. The formulation for the discrete model is

$$\begin{aligned}
\max_{x,z} \quad & \sum_{k \in \mathcal{K}} \omega_k \Big(\psi(x_k) + z_k \Big) \\
\text{s.t.} \quad & \phi(x_k|\theta_k) - z_k \geq \phi^*(\theta_k), && \forall\, k \in \mathcal{K}, \\
& \phi(x_k|\theta_k) - z_k \geq \phi(x_l|\theta_k) - z_l, && \forall\, k, l \in \mathcal{K}, \\
& x_k \geq 0, && \forall\, k \in \mathcal{K}.
\end{aligned}$$

For example, the discrete model can be used when the buyer outsources part of his operations to a subcontractor, such as a warehouse owner where the buyer will store his inventory. The available subcontractors are public knowledge and only finitely many possibilities exist. The buyer's private information is the used subcontractor, determining the buyer's holding cost for example, which affects his utility of an order quantity.

1.2.3 The pooling model

Suppose that the buyer's type is continuously distributed on $\Theta = [\underline{\theta}, \bar{\theta}]$. Applying the continuous model typically results in a complex menu of contracts, where each buyer type is assigned a different contract. This effectively means that the seller needs to communicate an infinite number of contracts. There are situations where such a menu is undesirable, for example, due to the difficulty in communicating the menu to the buyer. It would be more manageable if only a limited number of contracts are offered.

If we discretise the interval $[\underline{\theta}, \bar{\theta}]$ into K type representatives $\{\theta_1, \ldots, \theta_K\}$, we can apply the discrete model to obtain a menu with K contracts. However, all non-represented types do not exist in the discrete model and their choice of contracts is not taken into account. In general, this implies that the resulting menu is not robust, i.e., in some cases the buyer will reject all contracts.

To prevent this issue, we should use the *pooling model*, which we also call the *robust pooling model* to emphasise its robustness property. It is a combination of the discrete and continuous models. First, the seller decides the number of contracts $K \in \mathbb{N}_{\geq 1}$ in the menu. Second, he partitions $[\underline{\theta}, \bar{\theta}]$ into K subintervals, denoted by $[\underline{\theta}_k, \bar{\theta}_k]$ for $k \in \mathcal{K} = \{1, \ldots, K\}$. Finally, the seller uses a mechanism as seen before to construct a menu of K contracts, where the k-th contract (x_k, z_k) will be assigned to all types in $[\underline{\theta}_k, \bar{\theta}_k]$. In other words, all buyer types in a subinterval are pooled and are incentivised to accept the same contract. The pooling model is given by

$$\max_{x,z} \quad \sum_{k \in \mathcal{K}} \Big(\int_{\underline{\theta}_k}^{\bar{\theta}_k} \omega(\theta) \mathrm{d}\theta \Big) \Big(\psi(x_k) + z_k \Big)$$

$$\begin{aligned}
\text{s.t.} \qquad & \phi(x_k|\theta_k) - z_k \geq \phi^*(\theta_k), && \forall\, \theta_k \in [\underline{\theta}_k, \bar{\theta}_k], k \in \mathcal{K}, \\
& \phi(x_k|\theta_k) - z_k \geq \phi(x_l|\theta_k) - z_l, && \forall\, \theta_k \in [\underline{\theta}_k, \bar{\theta}_k], k, l \in \mathcal{K}, \\
& x_k \geq 0, && \forall\, k \in \mathcal{K}.
\end{aligned}$$

In contrast to the discrete model, the pooling model takes all buyer types into account and guarantees a robust menu. Compared to the continuous model, the pooling approach offers a controllable number of contracts in the menu. Furthermore, the buyer types are pooled a priori by the partition in the pooling model. The optimisation of the partition can be included into the optimisation model, eliminating the fixed pooling, but this significantly increases the complexity of solving the model.

1.3 Outline

In Chapters 2-5 we analyse contracting models in various supply chain coordination problem settings with two parties. These models are either as shown in the previous sections or variations thereof. In all chapters the downstream party of the supply chain has single-dimensional private information. We refer to these chapters for references to the literature on the introduced contracting concepts. Given the mathematical setting as specified in each chapter, the main research goals are to determine whether the associated contracting model can be solved efficiently and to derive structural properties of the optimal menus of contracts.

Below, we will position the remaining chapters in the framework of mechanism design by referring to the contracting models introduced in Section 1.2. Furthermore, we will outline the topic of each chapter and state on which publication or report it is based.

Tables 1.1 and 1.2 provide a classification of the models analysed in Chapters 2-5. The differences in the model types, the probability distributions, and the number of contracts have been discussed in the previous sections. Table 1.2 also shows if the setting of the underlying supply chain coordination problem is a continuous or combinatorial optimisation problem. In addition, the table lists whether the model uses a single- or multi-objective approach. That is, the single-objective approach maximises the seller's expected net utility, whereas the multi-objective approach balances expected and worst-case net utility. Finally, we indicate whether the buyer's reservation level ϕ^* depends on his type. The type dependency leads to different structures in the optimal menus and affects the solution approach.

Chapter	**Coordination problem**	**Objective approach**	**Probability distribution**	**Number of contracts**	**Type-dependent reservation level**
Chapter 2	Continuous	Single	Discrete	Finite	Yes
Chapter 3	Continuous	Single	Continuous	Finite	No
Chapter 4	Continuous	Multi	Continuous	Finite	No
Chapter 5	Combinatorial	Single	Continuous	Infinite	Yes

Table 1.2: Overview of the analysed contracting models.

In Chapter 2 we analyse a contracting model in the context of a supplier and a retailer, where the supplier designs a menu to minimise his expected costs. We determine a solution approach for a general number of retailer types and derive structural properties of the optimal menus. It is based on Kerkkamp et al. (2018c) of which Kerkkamp et al. (2016) is an earlier report version.

Chapter 3 considers the pooling model for certain utility maximisation and cost minimisation problems. In addition to solving the pooling models, we focus on optimising the partition of the buyer types. This chapter is based on the report Kerkkamp et al. (2017).

A similar analysis is performed in Chapter 4 for a specific utility maximisation problem where the seller wants to balance his expected and worst-case net utility. We apply a constraint-wise multi-objective approach and determine the optimal partition for the resulting pooling model. The research is based on the report Kerkkamp et al. (2018a).

In Chapter 5 we again consider a cost minimisation problem with a supplier and a retailer. However, the costs follow from combinatorial optimisation problems and do not have manageable closed-form formulas. We present a two-stage solution approach and identify cases for which this approach has polynomial running time. The chapter is based on the report Kerkkamp et al. (2018b).

Finally, we conclude our main findings in Chapter 6.

All chapters can be read independently as they will (re)introduce the necessary concepts and notation. The research has been conducted independently under supervision of the promotors who provided guidance in the research directions, verified the mathematical results, and assisted in finalising the writing.

For completeness, this introductory chapter is an adaptation of Kerkkamp (2017).

Chapter 2

Two-echelon supply chain coordination under information asymmetry with multiple types

Abstract

In this chapter, we analyse a principal-agent contracting model with asymmetric information between a supplier and a retailer. Both the supplier and the retailer have the classical non-linear economic ordering cost functions consisting of ordering and holding costs. We assume that the retailer has the market power to enforce any order quantity. Furthermore, the retailer has private holding costs. The supplier wants to minimise his expected costs by offering a menu of contracts with side payments as an incentive mechanism. We consider a general number of discrete single-dimensional retailer types with type-dependent default options.

A natural and common model formulation is non-convex, but we present an equivalent convex formulation. Hence, the contracting model can be solved efficiently for a general number of retailer types. We also derive structural properties of the optimal menu of contracts. In particular, we completely characterise the optimum for two retailer types and provide a minimal list of candidate contracts for three types. We show that the retailer's lying behaviour is more complex than simply lying to have higher costs. Finally, we prove a sufficient condition to guarantee unique contracts in the optimal solution for a general number of retailer types.

This chapter is based on Kerkkamp et al. (2018c).

Contents

2.1 Introduction

We consider the classical two-echelon Economic Order Quantity (EOQ) setting with a supplier and a retailer. Both the supplier and the retailer act as fully rational individualistic entities that want to minimise their own costs. It is well known that such individualistic viewpoints are suboptimal for the entire supply chain. This loss of efficiency is often called the price of anarchy, see for example Perakis and Roels (2007). We assume that the supply chain uses a pull ordering strategy, i.e., the retailer places orders at the supplier. Therefore, the retailer's default ordering policy is optimal for himself. The supplier can decrease his costs by somehow persuading the retailer to change to a different ordering policy.

One way the supplier can do so is by offering a contract to the retailer that typically includes a side payment or discounts. If the contract is accepted by the retailer, the costs for the entire supply chain decrease and the resulting profit is divided between the two parties as agreed upon in the contract. Being selfish, the supplier wants the largest possible share of this profit. Depending on the type of contract, it is non-trivial to determine a contract that maximises the supplier's profit and that is accepted by the retailer.

The complexity of the matter is increased significantly if the retailer has private information that is not shared with the supplier. For example, the retailer's cost structure can be undisclosed. Furthermore, private information typically leads to inefficiencies for the supply chain, see for example Inderfurth et al. (2013). This partial cooperation between the supplier and the retailer leads to a principal-agent optimisation problem with asymmetric information.

In the case that the retailer holds private information, the supplier can use mechanism design or incentive theory to improve his situation, see Laffont and Martimort (2002). That is, he presents a specially designed menu of contracts for the retailer to choose from. We focus on constructing the optimal menu of contracts that minimises the supplier's expected costs, provided that the retailer is not worse off by choosing one of these contracts.

Our setting fits in the active broader research on supply chain coordination, see for example Lambert and Cooper (2000), Leng and Parlar (2005), and Stadtler (2008). Ideally, all parties in a supply chain should cooperate fully for maximum efficiency. Such (centralised) cooperation is often difficult to achieve in practice, as parties do not want to share their private information or become too dependent on each other. However, even under information asymmetry, cooperation to improve efficiency is essential in order to be part of the increasingly competitive market.

To further specify the considered optimisation problem and our contribution to the literature, we need to introduce the economical setting.

2.1.1 Contracting model

The retailer faces external demand for a particular product with constant rate $d \in \mathbb{R}_{>0}$, which must be satisfied immediately, i.e., there is no backlogging. Placing an order at the supplier has an ordering cost of $f \in \mathbb{R}_{>0}$ for the retailer. Delivery of the order is assumed to be instantaneous (no lead times). Furthermore, the retailer has inventory holding cost of $h \in \mathbb{R}_{>0}$ per product unit and time unit.

Since we assume that the retailer minimises his own costs and can place any order, he places an order if and only if his inventory is depleted (the zero-inventory property). An order quantity of $x \in \mathbb{R}_{>0}$ units leads to an average holding cost per time unit of $\frac{1}{2}hx$ and an average ordering cost of $df\frac{1}{x}$. In total, the average costs per time unit for the retailer is given by

$$\phi_R(x) = df\tfrac{1}{x} + \tfrac{1}{2}hx,$$

which is minimised by ordering the well-known economic order quantity $x_R^* = \sqrt{2df/h}$ (see Banerjee (1986)). The minimal costs are $\phi_R^* = \phi_R(x_R^*) = \sqrt{2dfh}$.

The cost structure of the supplier is similar: the supplier has an order handling cost $F \in \mathbb{R}_{>0}$ and inventory holding cost $H \in \mathbb{R}_{>0}$. Procurement by the supplier takes place with constant rate $p \in \mathbb{R}_{\geq d}$. To minimise his own costs, the supplier follows a just-in-time lot-for-lot policy. That is, the supplier does not batch the retailer's orders and completes procurement of an order exactly on time. Note that F can be interpreted as a production setup cost provided that $p > d$ and batching is not allowed.

Per time unit the supplier has average holding costs of $\frac{1}{2}H\frac{d}{p}x$ and average order handling costs of $dF\frac{1}{x}$. This leads to a total cost for the supplier of

$$\phi_S(x) = dF\tfrac{1}{x} + \tfrac{1}{2}H\tfrac{d}{p}x,$$

which is minimised if the order quantity is $x_S^* = \sqrt{2Fp/H}$.

The supplier and retailer both have their own optimal order quantity and either policy is suboptimal for the entire supply chain (unless $x_R^* = x_S^*$), see Banerjee (1986). From the perspective of the supply chain, the supplier and retailer can cooperate to lower the total joint costs. The joint costs are given by

$$\phi_J(x) = d(f+F)\tfrac{1}{x} + \tfrac{1}{2}\big(h + H\tfrac{d}{p}\big)x,$$

with optimal joint order quantity $x_J^* = \sqrt{2d(f+F)/(h+H\frac{d}{p})}$. It is not difficult to verify that x_J^* always lies between x_R^* and x_S^* (see Lemma 2.17 on page 44). Therefore, lower joint costs can be achieved by deviating from the individually optimal order quantities. Whether such coordination takes place depends on further assumptions on power relations and market options.

As mentioned before, we assume that both the supplier and the retailer behave rationally and want to minimise their own costs. Furthermore, we assume that the retailer has the market power to enforce any order quantity on the supplier. Consequently, the retailer chooses his own optimal order quantity x_R^* by default, called the *default ordering policy* or *default option*. By using incentive mechanisms, the supplier can persuade the retailer to deviate from the default policy. We analyse using a side payment $z \in \mathbb{R}$ to the retailer as an incentive mechanism for cooperation. Note that side payments can be realised, for example, via contract-dependent quantity discounts. The pair (x, z) of an order quantity x and a side payment z is called a contract.

The presented contract (x, z) must be constructed such that the retailer is not worse off than with his default option: $\phi_R(x) - z \leq \phi_R^*$. This condition is called the Individual Rationality (IR) constraint or participation constraint. If the offered contract leads to the same costs for the retailer as his default option, we assume that the retailer is indifferent and that the supplier can convince the retailer to choose the contract preferred by the supplier. By assumption, the supplier can do so without any additional costs. Hence, the retailer always accepts the presented contract if it satisfies the IR constraint.

If the supplier has complete information of the supply chain, it is straightforward to determine that the optimal contract offers the joint order quantity $x = x_J^*$ and minimal side payment $z = \phi_R(x_J^*) - \phi_R^*$. The resulting ordering policy leads to perfect supply chain coordination: it is optimal for the entire supply chain, as if there is a central decision maker.

However, we study the case that the retailer has private information on his cost structure: either the ordering cost f or the holding cost h is private (but not both). We consider the case that the supplier is uncertain about the retailer's holding cost, which is without loss of generality as will be shown in Section 2.2.1. The supplier has narrowed the retailer's real holding cost down to $K \in \mathbb{N}$ possible scenarios. Each scenario corresponds to a so-called *retailer type*. Type $k \in \mathcal{K} = \{1, \ldots, K\}$ has cost function

$$\phi_R^k(x) = df\tfrac{1}{x} + \tfrac{1}{2}h_k x,$$

where $0 < h_1 < h_2 < \cdots < h_{K-1} < h_K$ are the possible holding costs. This affects the retailer's individually optimal order quantity, which now depends on the retailer's type. Consequently, the retailer's default option is type dependent, since it is his own optimal order quantity by our assumptions. As such, we add the index $k \in \mathcal{K}$ to our notation to discern between retailer types. For example, for type $k \in \mathcal{K}$ the default order quantity is $x_R^{k*} = \sqrt{2df/h_k}$ with corresponding costs $\phi_R^{k*} = \phi_R^k(x_R^{k*})$. Note that type-independent default options can be used if, for example, logistical operations can be outsourced to a third party for a fixed fee. We do not consider this option.

The supplier designs a menu of K contracts for the retailer to choose from, one for each retailer type. For each type $k \in \mathcal{K}$ the supplier constructs a contract (x_k, z_k) that is individually rational for that specific type, similar to before. However, the retailer can lie about his type and choose any of the presented contracts if it is beneficial for him to do so. This situation is also called a contracting or screening game in the literature, see Laffont and Martimort (2002).

Furthermore, the supplier assigns an objective weight $\omega_k \in \mathbb{R}_{>0}$ to each type $k \in \mathcal{K}$, indicating its likelihood, and minimises his expected costs. Without loss of generality, ω is a probability distribution ($\sum_{k\in\mathcal{K}} \omega_k = 1$), but this is not required for the model and our results.

This leads to the following non-linear optimisation problem:

$$\min_{x,\tilde{x},z,\tilde{z}} \quad \sum_{k\in\mathcal{K}} \omega_k \left(\phi_S(\tilde{x}_k) + \tilde{z}_k\right) \tag{2.1}$$

$$\text{s.t.} \quad \phi_R^k(x_k) - z_k \le \phi_R^{k*}, \qquad \forall\, k \in \mathcal{K}, \tag{2.2}$$

$$(\tilde{x}_k, \tilde{z}_k) \in \{(x_1, z_1), \dots, (x_K, z_K)\}, \qquad \forall\, k \in \mathcal{K}, \tag{2.3}$$

$$\phi_R^k(\tilde{x}_k) - \tilde{z}_k \le \phi_R^k(x_l) - z_l, \qquad \forall\, k, l \in \mathcal{K}, \tag{2.4}$$

$$x_k > 0, \qquad \forall\, k \in \mathcal{K}.$$

The designed contracts (x_k, z_k) must satisfy the IR constraints (2.2). The pair $(\tilde{x}_k, \tilde{z}_k)$ denotes the chosen contract by retailer type $k \in \mathcal{K}$, which must be one of the presented contracts, see constraints (2.3). The retailer chooses the most beneficial contract for himself by possibly lying, which is enforced by constraints (2.4). The supplier's objective is to minimise his expected costs including side payments, see (2.1).

Consider an optimal solution to the non-linear problem and suppose that the retailer lies about his true type. By relabelling the presented contracts, we can construct another optimal solution for which the retailer will never lie about his type, i.e., $(\tilde{x}_k, \tilde{z}_k) = (x_k, z_k)$ for all $k \in \mathcal{K}$. This is also known as the revelation principle (see Laffont and Martimort (2002) and Myerson (1982)), which states that without loss of optimality the supplier can restrict his design to incentive-compatible direct coordination mechanisms and obtain a truthful choice of contract by the retailer.

For example, suppose the retailer type $k \in \mathcal{K}$ lies being type $l \in \mathcal{K}$. This implies that $(\tilde{x}_k, \tilde{z}_k) = (x_l, z_l)$ and in particular

$$\phi_R^k(x_l) - z_l = \phi_R^k(\tilde{x}_k) - \tilde{z}_k \overset{(2.4)}{\le} \phi_R^k(x_k) - z_k \overset{(2.2)}{\le} \phi_R^{k*}.$$

So, contract (x_l, z_l) is individually rational for type k. Relabelling or redefining (x_k, z_k) to be equal to (x_l, z_l) leads to an equivalent feasible solution where type k does not lie.

A direct consequence is that we can use the following equivalent simpler non-linear model:

$$\min_{x,z} \quad \sum_{k\in\mathcal{K}} \omega_k \left(\phi_S(x_k) + z_k\right) \tag{2.5}$$

$$\text{s.t.} \quad \phi_R^k(x_k) - z_k \le \phi_R^{k*}, \qquad \forall\, k \in \mathcal{K}, \tag{2.6}$$

$$\phi_R^k(x_k) - z_k \le \phi_R^k(x_l) - z_l, \qquad \forall\, k, l \in \mathcal{K}, \tag{2.7}$$

$$x_k > 0, \qquad \forall\, k \in \mathcal{K}.$$

We call this simpler model the *default contracting model.* Here, (2.7) are the Incentive Compatibility (IC) constraints that prevent types from lying, provided that we make the following conventional assumption. If the IC constraint (2.7) where type k compares to the contract for type l is tight, then type k is indifferent between contracts (x_k, z_k) and (x_l, z_l). In this case, we assume that the supplier can convince the retailer to choose contract (x_k, z_k) without any additional cost. We address this assumption in Section 2.5.2. Consequently, we can implicitly set $(\tilde{x}_k, \tilde{z}_k) = (x_k, z_k)$ and drop the choice of contracts completely from the model. Note that the menu of contracts with $(x_k, z_k) = (x_R^{k*}, 0)$ for all $k \in \mathcal{K}$ is a feasible solution.

From this point onwards, we denote a menu of contracts by (x, z), where $x = (x_1, \ldots, x_K)$ and $z = (z_1, \ldots, z_K)$. A single contract is denoted by (x_k, z_k) for $k \in \mathcal{K}$.

2.1.2 Connection to the literature

Similar models have been studied in the literature and there are many variations. One variation is to consider a continuous range of retailer types such as in Corbett and de Groote (2000) and Corbett et al. (2004). Kerschbamer and Maderner (1998) and Pinar (2015) analyse contracting models with structurally different cost functions and Mussa and Rosen (1978) consider a quality-differentiated spectrum of products. Another variation is to analyse single-period contracting such as newsvendor problems (see Burnetas et al. (2007) and Cachon (2003)). In Cakanyildirim et al. (2012) the roles of the supplier and retailer are swapped: the supplier has private information and the retailer designs a menu of contracts. We focus on literature that closely relates to our model, see also Table 2.1 for a comparison.

In this chapter, we assume that only one cost parameter of the retailer is private, which leads to so-called single-dimensional types. Pishchulov and Richter (2016) analyse the same setting, but with two-dimensional retailer types. That is, both the ordering cost and the holding cost are private. Their research provides a complete analysis of the model in Sucky (2006), who considers the same problem. Both use optimality conditions to determine a list of candidates for the optimal solution. However, the analysis is restricted to only two retailer types, whereas we consider a general number of types, albeit single-dimensional types. From our results we see different qualitative properties of the optimal solution for two types versus more than two types.

Li et al. (2012) incorporate a controllable lead time into the contracting model. The retailer has additional safety stock proportional to the square root of the lead-time demand. Only two retailer types are considered. The two types are two-dimensional, but the type with low costs has lower ordering and holding costs than the type with high costs.

In Voigt and Inderfurth (2011) the supplier's setup cost (or order handling cost) is an additional decision variable in the contracting model. The supplier has to decide whether to lower his setup cost at the cost of lost investment opportunities. Furthermore, the supplier has no holding costs and the retailer no ordering costs. Besides the differences in cost functions, their model assumes the same default option for all retailer types. To our knowledge, their work is the only paper with a related model that considers a general number of retailer types, although the authors do make assumptions on the distribution of the retailer types. Our results show that having type-dependent default options increases the complexity of the retailer's lying behaviour and the optimal menus, compared to having a type-independent default option.

Another model similar to ours is discussed in Zissis et al. (2015), but there are only two retailer types. Furthermore, the supplier has no holding costs, which reduces the number of optimal menus of contracts that can occur. Since we analyse the case for two types in detail, our results generalise their derived structural properties of the optimal menu of contracts.

In light of the previous references, we emphasise that the inclusion of both ordering/setup costs and holding costs for the retailer and supplier results in structurally different optimal menus of contracts. This is because both involved parties have a finite individually optimal order quantity. Deviating from that quantity leads to higher costs. This is not true if only one type of cost (ordering or holding) is included, since then the individually optimal order quantity is either zero or infinity. Furthermore, in the literature it is common to assume that the supplier prefers a larger order quantity than the retailer. We do not make this assumption and therefore also provide insight into contracts when the supplier prefers smaller order quantities.

Paper	**Supplier's costs:**		**Retailer's costs:**		**Number of types:**		**Type-dependent default option**	**Dimension of type:**	
	Setup	**Holding**	**Ordering**	**Holding**	**Two**	**Multiple**		**One**	**Two**
Sucky (2006)	✓	✓	✓	✓	✓		✓	✓	✓
Voigt and Inderfurth (2011)	✓			✓	✓	✓		✓	
Li et al. (2012)	✓	✓	✓	✓	✓		✓	✓	✓
Zissis et al. (2015)	✓		✓	✓	✓		✓	✓	
Pishchulov and Richter (2016)	✓	✓	✓	✓	✓		✓	✓	✓
This chapter	✓	✓	✓	✓	✓	✓	✓	✓	

Table 2.1: Comparison of related literature.

2.1.3 Contribution

We consider a principal-agent contracting model with asymmetric information under the EOQ setting. Our model distinguishes itself from the literature by having a general number of retailer types with type-dependent default options. Furthermore, the supplier and the retailer have both ordering/setup costs and holding costs. Consequently, a typical analysis using optimality conditions is complex and does not appear to lead to a generalisable solution method.

Our main contributions are as follows. First, we show that our non-convex model has a hidden convexity, which is achieved by a change of decision variables. Hence, in practice we can numerically solve our model to optimality for a general number of retailer types using various efficient techniques. Second, we determine structural properties of the optimal solution for a general number of retailer types. The analysis shows significant differences in the structure of optimal menus of contracts for two types compared to more than two types. Third, we prove a sufficient condition to guarantee unique contracts in the optimal solution. We provide counterexamples when this condition is omitted.

In particular, we use the structural properties to analyse the difference between two and three retailer types. To do so, we analytically solve the model for these two cases. We provide a complete characterisation of the optimal solution for the case with two retailer types. The derived closed-form formulas of the optimal solution are not only simpler than those found in related literature, they also show additional structure of the solution. For the specific case of three retailer types we did not find any results in the literature. We give a minimal list of candidate contracts for the optimal solution of the problem with three retailer types. The analysis shows that the retailer's lying behaviour is more complex than simply lying to have higher costs.

To conclude, our results show that having type-dependent default options increases the complexity of the retailer's lying behaviour and the possible structures of the optimal menus. In particular, certain properties and behaviour are only observed for more than two retailer types.

The remainder is organised as follows. In Section 2.2 we present an alternative model which shows the hidden convexity and leads to an efficient solution method. We continue with structural properties of the contracting model in Section 2.3. In Section 2.4 we discuss the optimal menus of contracts for two and three retailer types, where we give examples of each occurring optimal menu. The derivations of the optimal contracts are given in Appendix 2.D. We end with a general discussion of our results in Section 2.5.

2.2 Efficient solution method

In this section we show that the contracting problem can be solved efficiently. This insight becomes apparent after a change of decision variables of the contracting model. Before we give the details, we prove that for single-dimensional retailer types we can assume without loss of generality that the retailer's holding cost is private. Consequently, we can efficiently solve two kinds of contracting models.

2.2.1 Equivalence when one cost parameter is private

Consider a contracting problem where all retailer types instead have the same holding cost h, but different ordering costs f_k. We can transform any such problem to an equivalent contracting problem where all types have the same ordering cost $\hat{f}$, but different holding costs $\hat{h}_k$.

The transformation is as follows. For arbitrary $\hat{d} \in \mathbb{R}_{>0}$ and $\hat{p} \in \mathbb{R}_{\geq \hat{d}}$, define the following parameters:

$$\hat{\omega}_k = \omega_k, \qquad \hat{H} = 2(dF)\frac{\hat{p}}{\hat{d}}, \qquad \hat{F} = (\tfrac{1}{2}H\tfrac{d}{p})\frac{1}{\hat{d}}, \qquad \hat{f} = (\tfrac{1}{2}h)\frac{1}{\hat{d}}, \qquad \hat{h}_k = 2(df_k).$$

These parameters are well defined and result in a contracting problem instance where all retailer types have the same ordering cost, instead of the same holding cost. To distinguish the instances, let $\hat{S}$ be the supplier and $\hat{R}$ the retailer for the newly constructed problem. We claim that both instances are equivalent, i.e., both have the same optimal objective value and there is a bijection between the optimal solutions.

To show any equivalence between instances, the important expressions of the contracting model are: ϕ_S, ϕ_R^k, and ϕ_R^{k*}. Consider any order quantity $x_k \in \mathbb{R}_{>0}$ and set $\hat{x}_k = 1/x_k$, leading to the expressions:

$$\begin{aligned}
\phi_S(x_k) &= dF\frac{1}{x_k} + \frac{1}{2}H\frac{d}{p}x_k = \frac{1}{2}H\frac{d}{p}\frac{1}{\hat{x}_k} + dF\hat{x}_k = \hat{d}\hat{F}\frac{1}{\hat{x}_k} + \frac{1}{2}\hat{H}\frac{\hat{d}}{\hat{p}}\hat{x}_k = \phi_{\hat{S}}(\hat{x}_k),\\
\phi_R^k(x_k) &= df_k\frac{1}{x_k} + \frac{1}{2}hx_k = \frac{1}{2}h\frac{1}{\hat{x}_k} + df_k\hat{x}_k = \hat{d}\hat{f}\frac{1}{\hat{x}_k} + \frac{1}{2}\hat{h}_k\hat{x}_k = \phi_{\hat{R}}^k(\hat{x}_k),\\
\phi_R^{k*} &= \sqrt{2df_kh} = \sqrt{2\hat{d}\hat{f}\hat{h}_k} = \phi_{\hat{R}}^{k*},
\end{aligned}$$

where the equalities follow by definition. Thus, any menu (x, z) is a feasible solution for the original instance if and only if $(\hat{x}, z)$ is feasible for the newly constructed instance. Moreover, the objective values of the two instances are equal.

To conclude, the qualitative properties of the contracting model with one private cost parameter are irrespective of which cost parameter (ordering or holding cost) is private.

2.2.2 Alternative convex model

The default contracting model is not convex, since the IC constraints (2.7) state

$$df\left(\tfrac{1}{x_k} - \tfrac{1}{x_l}\right) + \tfrac{1}{2}h_k(x_k - x_l) + z_l - z_k \leq 0, \qquad \forall\, k, l \in \mathcal{K}.$$

Here, the term $-1/x_l$ is not convex in the decision variables. Non-convex optimisation problems are generally difficult to solve, but we show that this is not the case for our problem. We reveal a hidden convexity of our problem by changing the perspective from side payments to so-called *information rents.*

An alternative contracting model can be obtained by rescaling the side payments as follows. The individual rationality constraints imply that $z_k \geq \phi_R^k(x_k) - \phi_R^{k*} \geq 0$. As such, it is natural to interpret the value $\phi_R^k(x_k) - \phi_R^{k*}$ as the minimum side payment that always has to be paid to satisfy the IR constraint. We introduce a new variable y_k which denotes the additional side payment required by the IC constraints:

$$y_k = z_k - (\phi_R^k(x_k) - \phi_R^{k*}) \geq 0.$$

This variable is also known as the information rent for type k. Substituting $z_k = y_k + \phi_R^k(x_k) - \phi_R^{k*}$ in the default contracting model leads to:

$$\begin{aligned}
\min_{x,y} \quad & \sum_{k\in\mathcal{K}} \omega_k \left(\phi_S(x_k) + \phi_R^k(x_k) + y_k - \phi_R^{k*}\right) && && (2.8)\\
\text{s.t.} \quad & y_k \geq 0, && \forall\, k \in \mathcal{K}, && (2.9)\\
& y_l - y_k + \phi_R^l(x_l) - \phi_R^k(x_l) \leq \phi_R^{l*} - \phi_R^{k*}, && \forall\, k, l \in \mathcal{K}, && (2.10)\\
& x_k > 0, && \forall\, k \in \mathcal{K}.
\end{aligned}$$

So, (2.9) are the IR constraints and (2.10) are the IC constraints. The new objective function (2.8) exposes that the joint costs $\phi_S + \phi_R^k$ for the entire supply chain have to be minimised, together with the information rents y_k. Hence, the aspect of supply chain coordination is more visible than in the default model. We call the new model the *alternative contracting model* to differentiate it from the earlier defined default model.

By definition of y_k, there is a bijection between the feasible region of the alternative model and that of the default model. Furthermore, the corresponding objective values are the same. Hence, we can solve the default model by solving the alternative model and vice versa.

Although both models are equivalent in the sense mentioned above, there is one significant difference. Notice that the non-linear terms in (2.10) cancel out if we expand the cost functions:

$$y_l - y_k + \tfrac{1}{2}(h_l - h_k)x_l = y_l - y_k + \phi_R^l(x_l) - \phi_R^k(x_l) \leq \phi_R^{l*} - \phi_R^{k*}.$$

Thus, all constraints of the alternative model are linear in the decision variables. Since the objective function is convex, we conclude that the alternative model is convex. Moreover, the feasible solution $x_k = x_R^{k*}$ and $y_k = \epsilon \in \mathbb{R}_{>0}$ for all $k \in \mathcal{K}$ is a Slater point, i.e., strictly feasible. It is well known that a convex model with differentiable functions and Slater points can be solved efficiently using scalable methods such as interior-point or cutting-plane methods (see Bertsekas (2015) and Boyd and Vandenberghe (2004)). This conclusion is stated in Theorem 2.1.

Theorem 2.1. *The contracting model can be solved efficiently via the alternative model.*

Proof. The proof is given in the above discussion. □

Remark 2.1. Recalling the results from Section 2.2.1, we note that the contracting model with single-dimensional types can be solved efficiently. If both the ordering cost f and the holding cost h are private information, we have two-dimensional retailer types specified by cost parameters (f_k, h_k). In this case, both the default model and the alternative model fall in the category of Difference of Convex functions (DC) programming. In the literature, there exist good numerical methods to find local optima of DC models, see Horst et al. (1991) and Pham Dinh and Le Thi (2014). However, to guarantee global optimality such methods need to be incorporated into, for example, a branch-and-bound procedure. ⧫

To conclude, in practice we can determine optimal solutions of our problem numerically. We have implemented a cutting-plane algorithm using Gurobi as linear-programming solver. Typical computational times are less than a second for one hundred types on a standard desktop computer. However, it is worthwhile to further analyse the model theoretically. In the following sections we determine qualitative properties of the optimal menu of contracts and in some cases even provide closed-form solutions. The used model (default or alternative) has no significant effect on the results. Hence, we present all results using the default model and place remarks where needed for the alternative model.

2.3 Structural properties

We continue with additional properties of the contracting model and its optimal solutions. These results hold for a general number of retailer types. In particular, the model is connected to a one-to-all shortest path problem in a certain directed graph. This allows us to use the theory of the shortest path problem and have a different view of the contracting model. Furthermore, we use the well-known Karush-Kuhn-Tucker conditions to determine structures in the optimal solution. In the end, we derive a sufficient condition to guarantee unique contracts in the optimal solution. Moreover, the analysis leads to a minimal list of menus of contracts for two and three retailer types which contains the optimal solution. These are discussed in Section 2.4. All proofs of the results in this section are given in Appendix 2.B.

2.3.1 Shortest path interpretation

A closer look into the structure of the IR and IC constraints shows a connection with a dual shortest path interpretation. For given fixed quantities x_k, constraints (2.6) and (2.7) can be seen as the dual constraints of a shortest path problem. To be precise, for given x_k the contracting model is equivalent to the dual of a specific minimum cost flow formulation for the one-to-all shortest path problem. A similar connection to shortest paths has been described in Rochet and Stole (2003) and Vohra (2012).

The related flow problem is as follows. Consider the directed graph $G = (\mathcal{V}, \mathcal{A})$ with nodes $\mathcal{V} = \{s\} \cup \mathcal{K}$ and arcs $\mathcal{A} = \{(s,k) : k \in \mathcal{K}\} \cup \{(k,l) : k,l \in \mathcal{K}, k \neq l\}$. That is, G is the complete graph of K retailer nodes with a source added. See Figure 2.1 for an example. We call such a graph an *IRIC graph*, which stands for Individual Rationality and Incentive Compatibility graph for reasons to become apparent. The lengths (or costs) of the arcs are:

- arc (s,k) with $k \in \mathcal{K}$ has length $\phi_R^{k*} - \phi_R^k(x_k)$,
- arc (k,l) with $k,l \in \mathcal{K}$, $k \neq l$, has length $\phi_R^l(x_k) - \phi_R^l(x_l)$.

Finally, node s has supply $\sum_{k \in \mathcal{K}} \omega_k$ and each retailer node $k \in \mathcal{K}$ has demand ω_k. There are no capacity restrictions on the arcs. Consequently, flow will be sent along shortest paths in the optimal solution of the flow formulation. Hence, we see this flow formulation as a one-to-all shortest path representation.

For fixed order quantities x_k, the contracting problem needs to determine the optimal side payments z_k by solving (2.5)-(2.7). This is the dual of the flow problem in the corresponding IRIC graph, see Appendix 2.A for the details on the (dual) flow formulation.

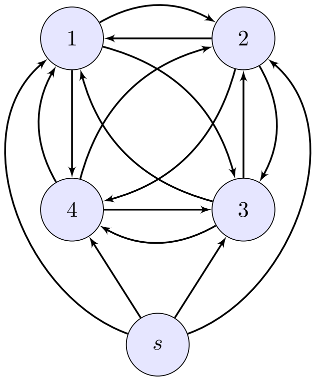


Figure 2.1: IRIC graph for $K = 4$ retailer types.

It is useful to mention some well-known properties of (dual) flow formulations, see also Ahuja et al. (1993). Consider the optimal solution (x, z) of the contracting model. The value $-z_k$ is equal to the length of the shortest (s, k)-path in the IRIC graph corresponding to x. Moreover, strong duality implies that the IRIC graph contains a negative cycle if and only if its dual flow problem is infeasible. In such cases there exist no side payments that will satisfy the IC constraints for the considered order quantities x_k. Thus, the IC constraints can be satisfied if and only if the corresponding IRIC graph has no negative cycles.

In the non-degenerate case, where the shortest (s, k)-path is unique for all $k \in \mathcal{K}$, the set of all used arcs in the optimal shortest paths from s to the other nodes forms a spanning tree in the IRIC graph. In the degenerate case, this does not hold, but the optimal shortest paths can be modified such that the used arcs form a spanning tree again. In particular, if the set of all used arcs in the optimal shortest paths contains cycles, these cycles must have length zero.

From the complementary slackness conditions it follows that if arc (i, j) is in the spanning tree, then the corresponding constraint in the dual is satisfied with equality. For example, if arc (s, k) is part of the shortest path tree, then the IR constraint for type k is tight. If arc (k, l) is used, with $k, l \in \mathcal{K}$, then the IC constraint $\phi_R^l(x_l) - z_l \leq \phi_R^l(x_k) - z_k$ is satisfied with equality.

Due to the bijection between retailer types and retailer type nodes, and the bijection between arcs and the IR and IC constraints, we often interchange interpretation and terminology. For example, we can refer to outgoing arcs out of a retailer type, referring to the outgoing arcs of the corresponding node in the graph. These insights explain why we use the name 'IRIC graph'.

2.3.2 Adjacent retailer types

Since the types are ordered such that $h_1 < \cdots < h_K$, there is a sense of adjacent or neighbouring types. We define the neighbours of type $k \in \mathcal{K}$ to be the types $k - 1$ and $k + 1$, where types 1 and K have only one neighbour. The adjacency of types plays an important role as we will see.

Intuitively, one would expect that in an optimal solution a type with higher holding cost gets offered a lower order quantity (i.e., more frequent orderings) to prevent too high inventory costs. Lemma 2.2 shows that this intuition is mathematically correct.

Lemma 2.2. *Any feasible menu of contracts satisfies* $x_1 \geq \cdots \geq x_K$.

The ordering (or monotonicity) in the order quantities often holds for contracting models with single-dimensional types and well-behaved cost (or utility) functions. However, this does not hold in general, even for single-dimensional types. See for examples and further discussion Araujo and Moreira (2010), Laffont and Martimort (2002), Schottmüller (2015), and Vohra (2012). There is no guaranteed monotonicity in the side payments (see Section 2.4 for examples).

A consequence of Lemma 2.2 is that adjacent retailer types follow both from the holding costs and from the (feasible) order quantities. In fact, using this result we can restrict the incentive compatibility constraints to take only the neighbouring types into account, without changing the feasible region. See Lemma 2.3 for the result. We call these constraints the adjacent IC constraints.

Lemma 2.3. *The adjacent incentive compatibility constraints are sufficient to ensure general incentive compatibility.*

We can use Lemma 2.3 to prove that order quantities satisfying $x_1 \geq \cdots \geq x_K > 0$ can always be extended to a feasible menu of contracts (x, z), see Corollary 2.4. Therefore, we call such order quantities feasible for the contracting model.

Corollary 2.4. *For given order quantities satisfying* $x_1 \geq \cdots \geq x_K > 0$*, it is feasible and optimal to determine the side payments via the shortest path interpretation.*

2.3.3 KKT conditions

Since the contracting model consists of continuously differentiable functions with a continuous domain, there are well-known necessary conditions for optimality and even sufficient optimality conditions in certain cases. Using these conditions we can design candidate solutions for further inspection. This allows us to analytically investigate properties of the optimal menu of contracts. In the following sections we use the Karush-Kuhn-Tucker (KKT) optimality conditions to do so (see Karush (1939) and Kuhn and Tucker (1951)).

The default contracting model is non-convex, but with a slight detour we can show that the KKT conditions are necessary and sufficient. Recall that we have an equivalent convex model with a Slater point, namely the alternative contracting model of Section 2.2.2. Thus, the KKT conditions are necessary and sufficient for the alternative model (see for example Boyd and Vandenberghe (2004)). It turns out that both models lead to the same KKT conditions, from which we conclude that the KKT conditions are also necessary and sufficient for the default model.

With the above mentioned remarks in mind, we determine the KKT conditions for the contracting model. Using Lemma 2.3 we only incorporate the adjacent IC constraints in our model. The Lagrangian function with Lagrange multipliers $\lambda \in \mathbb{R}^K_{\geq 0}$ and $\mu \in \mathbb{R}^{2K-2}_{\geq 0}$ is given by:

$$\begin{aligned} L(x, z, \lambda, \mu) = &\sum_{k \in \mathcal{K}} \omega_k \left(\phi_S(x_k) + z_k\right) + \sum_{k \in \mathcal{K}} \lambda_k \left(\phi_R^k(x_k) - z_k - \phi_R^{k*}\right) \\ &+ \sum_{k \in \mathcal{K} \setminus \{1\}} \mu_{k-1,k} \left(\phi_R^k(x_k) - z_k - \phi_R^k(x_{k-1}) + z_{k-1}\right) \\ &+ \sum_{k \in \mathcal{K} \setminus \{K\}} \mu_{k+1,k} \left(\phi_R^k(x_k) - z_k - \phi_R^k(x_{k+1}) + z_{k+1}\right). \end{aligned}$$

We deliberately choose this order of the indices of μ and will explain in Section 2.3.3.1 why this notation is useful.

The KKT conditions consist of primal and dual feasibility, complementary slackness, and stationarity constraints (see Boyd and Vandenberghe (2004)). The dual feasibility constraints require all multipliers to be non-negative. The complementary slackness constraints are:

$$\begin{aligned} \lambda_k \left(\phi_R^k(x_k) - z_k - \phi_R^{k*}\right) &= 0, && \forall\, k \in \mathcal{K}, \\ \mu_{k-1,k} \left(\phi_R^k(x_k) - z_k - \phi_R^k(x_{k-1}) + z_{k-1}\right) &= 0, && \forall\, k \in \mathcal{K} \setminus \{1\}, \\ \mu_{k+1,k} \left(\phi_R^k(x_k) - z_k - \phi_R^k(x_{k+1}) + z_{k+1}\right) &= 0, && \forall\, k \in \mathcal{K} \setminus \{K\}. \end{aligned}$$

For each $k \in \mathcal{K}$, the stationarity constraints with respect to x_k are:

$$\begin{aligned} \omega_k \frac{\mathrm{d}\phi_S}{\mathrm{d}x}(x_k) + \lambda_k \frac{\mathrm{d}\phi_R^k}{\mathrm{d}x}(x_k) + (\mu_{k-1,k} + \mu_{k+1,k}) \frac{\mathrm{d}\phi_R^k}{\mathrm{d}x}(x_k) \\ - \mu_{k,k-1} \frac{\mathrm{d}\phi_R^{k-1}}{\mathrm{d}x}(x_k) - \mu_{k,k+1} \frac{\mathrm{d}\phi_R^{k+1}}{\mathrm{d}x}(x_k) = 0, \end{aligned} \tag{2.11}$$

and with respect to z_k:

$$\omega_k - \lambda_k - (\mu_{k-1,k} + \mu_{k+1,k}) + (\mu_{k,k-1} + \mu_{k,k+1}) = 0, \tag{2.12}$$

where all ill-defined multipliers with out-of-bound indices are set to zero. We can simplify the stationarity constraints by substituting (2.12) in (2.11):

$$\omega_k \Big(- \frac{d(f+F)}{x_k^2} + \tfrac{1}{2}\big(h_k + H\tfrac{d}{p}\big)\Big) + \tfrac{1}{2}\mu_{k,k-1}(h_k - h_{k-1}) + \tfrac{1}{2}\mu_{k,k+1}(h_k - h_{k+1}) = 0. \tag{2.13}$$

To conclude, the KKT conditions consist of the primal and dual feasibility constraints, complementary slackness constraints, and stationarity constraints (2.12) and (2.13).

Remark 2.2. The KKT conditions for the alternative model directly give (2.12) and (2.13). ♦

We use the KKT conditions to determine candidate solutions to analyse properties of the optimal menu of contracts. When solving the KKT conditions, only the non-zero Lagrange multipliers are relevant. However, we do not know a priori which multipliers are non-zero. Therefore, we initially consider all 2^{3K-2} possible cases. For each case, we *partly* solve the corresponding KKT conditions in order to completely specify the corresponding menu (x, z) for this case. We do not solve for the Lagrange multipliers. See Appendix 2.D for the details. This leads to candidate solutions, which can be infeasible or suboptimal. We call these candidate solutions *KKT menus* and their contracts *KKT contracts*. The optimal solution of our model satisfies all KKT conditions and is equal to the best feasible KKT menu. KKT menus that are optimal for some instance are called *valid*.

Thus, the KKT conditions lead to a (large) set of KKT menus. Without further analysis, determining this set in general will be intractable due to its size. Hence, we analyse our problem to exclude certain KKT menus and provide additional insight into optimal menus of contracts. As we will see, we can express many structural properties intuitively in terms of a graph closely related to the Lagrange multipliers and the IRIC graph. Therefore, we first introduce this graph before continuing to the analysis of KKT menus.

2.3.3.1 KKT graph

The shortest path interpretation of Section 2.3.1 still holds if we only use adjacent IC constraints (Lemma 2.3). The corresponding *Adjacent IRIC graph* is shown in Figure 2.2. Now notice that the order of indices of μ corresponds nicely to the Adjacent IRIC graph. If $\mu_{lk} > 0$, then the equality $\phi_R^k(x_k) - z_k = \phi_R^k(x_l) - z_l$ must hold by the KKT complementary slackness conditions. Hence, arc (l, k) is used by the shortest paths, as discussed in Section 2.3.1. The same holds for λ_k, constraint $\phi_R^k(x_k) - z_k \leq \phi_R^{k*}$, and arc (s, k). Consequently, we have bijections between multipliers λ (or μ), the IR (or IC) constraints, and certain arcs in the Adjacent IRIC graph. As such, we can refer to the multiplier of an arc in the Adjacent IRIC graph.

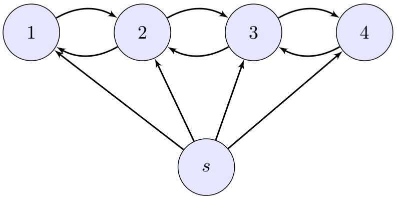


Figure 2.2: Adjacent IRIC graph for $K = 4$ types.

We can visualise a KKT menu in the Adjacent IRIC graph by only considering the arcs for which the corresponding multipliers are strictly positive. That is, we have a directed graph $\hat{G} = (\mathcal{V}, \hat{\mathcal{A}})$ with $\mathcal{V} = \{s\} \cup \mathcal{K}$ and arcs

- (s, k) with $k \in \mathcal{K}$ if $\lambda_k > 0$,

- $(k, k-1)$ with $k \in \mathcal{K} \setminus \{1\}$ if $\mu_{k,k-1} > 0$,
- $(k, k+1)$ with $k \in \mathcal{K} \setminus \{K\}$ if $\mu_{k,k+1} > 0$.

We call this graph the *KKT graph*. The arcs of the KKT graph indicate which arcs are for certain part of shortest paths in the IRIC graph. Unfortunately, there could be arcs in a shortest path for which the multiplier is zero, as degenerate cases may occur.

The KKT graph allows for easy-to-draw names of KKT menus. We call arc (s, k) the Up arc for retailer type $k \in \mathcal{K}$, arc $(k, k+1)$ the Right arc, and arc $(k, k-1)$ the Left arc. The name of a KKT menu is simply a list of the Up, Left, and Right arcs for each retailer type from 1 to K in the corresponding KKT graph. If a retailer type has no Up, Left, and Right arcs, we denote it by 'x'. For example, KKT menu 1Right2UpLeft3UpLeftRight4x is shown in Figure 2.3.

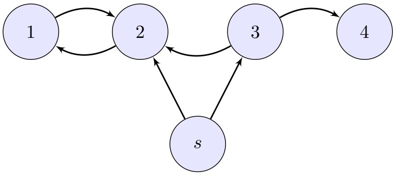


Figure 2.3: KKT graph for 1Right2UpLeft3UpLeftRight4x.

In the results to come, we often use the term 'connected component' of the KKT graph. To avoid confusion, a subset $\mathcal{S} \subseteq \mathcal{V}$ is a connected component if between each pair of nodes in $\mathcal{S}$ there exists an *undirected* path in the graph. Also, a 'maximal' set according to some condition is maximal by inclusion.

2.3.4 Properties of optimal contracts

The result that only adjacent IC constraints need to be taken into account greatly reduces the number of possible KKT menus to consider. We continue to analyse which cases can also be excluded from consideration, i.e., which combinations of strictly positive multipliers (or which KKT graphs) can occur. We will express the results in terms of intuitive structures of the KKT graph.

2.3.4.1 Reachable from source node

We start with Lemma 2.5, which shows an explicit connection to shortest paths and spanning trees.

Lemma 2.5. *Every retailer node $k \in \mathcal{K}$ must be reachable from source node s in the KKT graph.*

Notice that this is a stronger property than the fact that the side payments follow from shortest paths. Shortest paths imply that each node is reachable from s using only arcs for which the corresponding constraint is tight. As weak complementary slackness may hold, tightness does not automatically imply that the corresponding multiplier is strictly positive. However, a strictly positive multiplier does imply tightness of the constraint. This result allows us to discard certain combinations of multipliers, significantly reducing the number of options.

The next lemma describes a general pattern (a 'T-pattern') that will never occur in the optimal solution.

Lemma 2.6. *There exist no $k \in \mathcal{K} \setminus \{1, K\}$ such that the constraints corresponding to arcs (s, k), $(k, k-1)$, and $(k, k+1)$ are simultaneously satisfied with equality.*

Corollary 2.7. *A retailer node directly connected to node s in the KKT graph has at most one outgoing arc.*

Corollary 2.7 implies that the graph in Figure 2.3 is never a valid KKT graph, since node 3 has Up, Left, and Right arcs (violating the corollary).

2.3.4.2 Cycles are restrictive

As a result of Lemma 2.3, the only cycles of interest are 2-cycles between adjacent nodes. A node can be part of two 2-cycles, which leads to a so called *2-cycle chain*: nodes $\mathcal{S} = \{i, i+1, \ldots, j-1, j\} \subseteq \mathcal{K}$ are part of a 2-cycle chain if $\mu_{k,k+1}, \mu_{k+1,k} > 0$ for $k \in \{i, \ldots, j-1\}$. We show that the retailer types of a 2-cycle chain in the KKT graph have the same contract, also called *bunching* in the literature. This is formalised in the next lemma and corollary.

Lemma 2.8. *Both incentive compatibility constraints between retailer types i and j are tight if and only if $x_i = x_j$. Furthermore, if $x_i = x_j$ then $z_i = z_j$ must hold.*

Corollary 2.9. *The retailer types of a 2-cycle chain in the KKT graph are offered the same contract.*

The KKT conditions become more restrictive if certain types have the same order quantity, as it introduces additional dependency between the decision variables. Using this fact, we can exclude more cases from consideration, see Lemma 2.10.

Lemma 2.10. *In the KKT graph, a maximal 2-cycle chain must have at least one ingoing arc (possibly from node s) and exactly one outgoing arc.*

For example, the graph in Figure 2.3 is not a valid KKT graph, since nodes 1 and 2 form a 2-cycle but do not have an outgoing arc.

A direct consequence is that for two retailer types KKT graphs with 2-cycles are not valid KKT graphs. For more than two retailer types 2-cycles in the optimal solution can actually occur, see Section 2.4. This implies that types can get the same contract in the optimal solution. We return to this issue in Section 2.3.5.

2.3.4.3 The joint order quantity

Recall that if a type $k \in \mathcal{K}$ is assigned its joint order quantity, $x_k = x_J^{k*}$, then perfect supply chain coordination occurs for that type. If a retailer node $k \in \mathcal{K}$ has no outgoing arcs in the KKT graph it is straightforward to determine that $x_k = x_J^{k*}$ must hold. The next lemma shows that this is an if-and-only-if relation.

Lemma 2.11. *In the optimal solution, $x_k = x_J^{k*}$ if and only if node $k \in \mathcal{K}$ has no outgoing arcs in the KKT graph.*

In fact, we can strengthen Lemma 2.11 to also relate the joint order quantity to retailer nodes with outgoing arcs in the KKT graph, see Lemma 2.12.

Lemma 2.12. *Let retailer node $k \in \mathcal{K}$ be part of a maximal 2-cycle chain $\mathcal{S} = \{i, i+1, \dots, k, \dots, j-1, j\} \subseteq \mathcal{K}$ in the KKT graph. If node k is not part of a 2-cycle we set $i = j = k$. The optimal solution satisfies the following properties:*

- *$x_k > x_J^{k*}$ if and only if $\mu_{i,i-1} = 0$ and $\mu_{j,j+1} > 0$,*
- *$x_k = x_J^{k*}$ if and only if $\mu_{k,k-1} = 0$ and $\mu_{k,k+1} = 0$ (so $i = j = k$),*
- *$x_k < x_J^{k*}$ if and only if $\mu_{i,i-1} > 0$ and $\mu_{j,j+1} = 0$.*

In particular, Lemma 2.12 implies that $x_1 \geq x_J^{1*}$ and $x_K \leq x_J^{K*}$ must hold in the optimal solution.

Our last result for this section, Lemma 2.13, states that at least one type is assigned the joint order quantity in the optimal solution. Hence, perfect supply chain coordination always occurs for at least one type.

Lemma 2.13. *In the optimal solution the order quantity for at least one retailer type is the joint order quantity. Moreover, the total costs for at least one retailer type equals its default costs.*

2.3.5 Uniqueness of contracts

Additional assumptions are needed in order to guarantee that each contract in the optimal menu is unique. In this section, we consider the special case where we have equidistant holding costs and uniformity on the retailer types. That is, we assume that $\omega_k = 1/K$ for all $k \in \mathcal{K}$ and $h_{k+1} = h_k + \delta$ for all $k \in \mathcal{K}$ for some $\delta \in \mathbb{R}_{>0}$.

The assumptions lead to the following KKT stationarity conditions for $k \in \mathcal{K}$:

$$\Big(- \frac{2d(f+F)}{x_k^2} + H\tfrac{d}{p} \Big) + h_k + \delta(\mu_{k,k-1} - \mu_{k,k+1}) = 0, \tag{2.14}$$

$$1 - \lambda_k - \mu_{k-1,k} - \mu_{k+1,k} + \mu_{k,k-1} + \mu_{k,k+1} = 0, \tag{2.15}$$

where all ill-defined multipliers with out-of-bound indices are set to zero. Notice that without loss of generality we set $\omega_k = 1$ in the KKT conditions by uniformly rescaling all multipliers.

It turns out that uniformity on types and equidistant holding costs is sufficient to guarantee *a priori* to obtain an optimal menu with unique contracts, see Theorem 2.14. Be aware that the exclusion of 2-cycles in the KKT graph does not automatically imply that all contracts are unique, at least not without improving the result of Corollary 2.9.

Theorem 2.14. *Assume uniformity on types and equidistant holding costs. In the optimal solution all contracts are unique.*

Corollary 2.15. *If we assume uniformity on types and equidistant holding costs, the KKT graph has no cycles.*

If we remove one of the two assumptions, there are instances where the optimal menu does not have unique contracts. Table 2.2 provides such examples where the optimal solution has non-unique contracts. Notice that there are no examples with only two retailer types. In the next section, we will prove in Theorem 2.16 that the optimal menu for two types always has unique contracts.

Optimal menu			F	H	f	h_1	h_2	h_3	Objective	x_1	z_1	x_2	z_2	x_3	z_3
1x	2LeftRight	3UpLeft	1	1	1	5	9	10	2.059 836 692	0.816 497	0.154 678	0.544 331	0.086 637	0.544 331	0.086 637
1Up	2LeftRight	3UpLeft	1	1	1	4	9	10	2.008 977 459	0.894 427	0.078 461	0.547 903	0.092 520	0.547 903	0.092 520
1UpRight	2LeftRight	3x	1	5	6	1	2	6	5.339 122 934	2	0.535 898	2	0.535 898	1.128 152	0.238 786
1UpRight	2LeftRight	3Up	1	4	9	1	2	6	5.231 467 672	2.459 866	0.646 029	2.459 866	0.646 029	1.414 214	0.214 297

(a) Examples for $K = 3$ with unevenly spaced holding costs, unit rates, and uniform types.

Optimal menu			F	H	f	h_1	h_2	h_3	Objective	x_1	z_1	x_2	z_2	x_3	z_3
1x	2LeftRight	3UpLeft	1	1	1	3	4	5	1.671 733 102	1	0.053 004	0.715 282	0.023 977	0.715 282	0.023 977
1Up	2LeftRight	3UpLeft	1	1	1	3	5	7	1.753 612 795	1	0.050 510	0.646 084	0.067 423	0.646 084	0.067 423
1UpRight	2LeftRight	3x	1	3	7	1	2	3	4.018 909 135	2.708 013	0.197 270	2.708 013	0.197 270	1.632 993	0.286 427
1UpRight	2LeftRight	3Up	1	3	1	1	2	3	2.522 142 495	1.035 276	0.069 350	1.035 276	0.069 350	0.816 497	0

(b) Examples for $K = 3$ with equidistant holding costs, unit rates, and type probabilities $\omega_1 = 10/21$, $\omega_2 = 1/21$, $\omega_3 = 10/21$.

Table 2.2: Examples of non-unique contracts in the optimal solution for three retailer types.

2.4 Optimal menus of contracts

The KKT conditions lead to a list of KKT menus (candidate solutions), one of which is the optimal solution. The analysis of the KKT conditions in Section 2.3.4 excludes certain KKT menus from consideration, which allows us to focus on the remaining cases. Furthermore, when determining formulas for these KKT menus, we can often reuse parts of the solution of subproblems or symmetric cases.

We have determined the formulas for all valid KKT menus for two and three retailer types. These lists of KKT menus are *minimal*, i.e., if we omit any menu there are instances for which we would fail to determine the optimum.

In principle, we can use the same techniques to solve the problem for more retailer types. However, it seems that the number of KKT menus increases rapidly and that we need to solve a few completely new cases when increasing the number of types. Unfortunately, this implies that we do not end up with a practical analytical solution method for a general number of retailer types. For a general solution method, we resort to the procedure described in Section 2.2.2.

In the next sections, we provide and discuss example instances and their optimal KKT menu for two and three retailer types. These instances have been solved by determining the best feasible KKT menu, which are derived in Appendix 2.D. As a verification step, all instances have also been solved using a cutting-plane procedure (see Section 2.2.2). Furthermore, to check the minimality of the lists of KKT menus, we have verified that exactly one KKT menu is optimal.

When interpreting the KKT graphs, recall the theoretical results in Section 2.3. For example, an arc (s, k) implies that the total costs for type k in the corresponding menu is equal to its default costs. Furthermore, perfect supply chain coordination occurs for types without any outgoing arcs. A cycle in the KKT graph implies that those two types get the same contract, but not vice versa (see Section 2.3.5). Finally, for $k \in \mathcal{K}$ and $l \in \{k-1, k+1\}$ an arc (l, k) indicates that retailer type k conceptually equally prefers contracts k and l, but chooses contract k by assumption. This assumption is further discussed in Section 2.5.2. We refer to this situation by saying that type k is *prevented from lying* to be type l. Thus, an arc between retailer types points to the type that has been prevented from lying.

2.4.1 Two retailer types

From our analysis we can reduce the number of KKT menus significantly if there are only two retailer types ($K = 2$). From the $2^4 = 16$ cases, there remain 5 cases that can occur, see Figure 2.4. For example, in Figure 2.4a there is perfect supply chain coordination for both types, even with information asymmetry. The details of the derivation of the corresponding KKT menus are given in Appendix 2.D.2. All 5 KKT menus can be optimal, see Table 2.3 for example instances and their optimal solution. We conclude that our analysis is tight for $K = 2$, i.e., we cannot exclude any of these KKT menus.

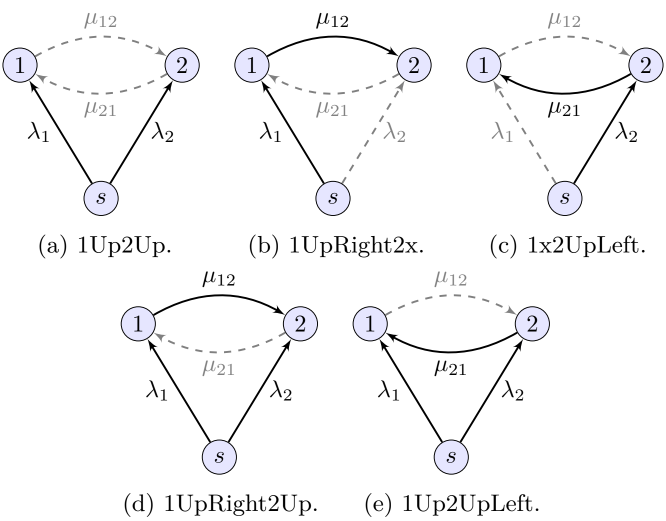


Figure 2.4: All KKT graphs for two retailer types.

Optimal menu		F	H	f	h_1	h_2	Objective	x_1	z_1	x_2	z_2
1x	2UpLeft	2	1	1	1	2	2.181 540 551	1.732 051	0.055 748	1.224 745	0.041 241
1Up	2Up	1	1	1	1	2	1.439 157 589	1.414 214	0	1.154 701	0.020 726
1Up	2UpLeft	1	1	1	2	4	1.560 477 933	1.154 701	0.020 726	0.828 427	0.035 534
1UpRight	2x	1	2	4	1	2	2.569 918 513	2.236 068	0.078 461	1.581 139	0.164 500
1UpRight	2Up	1	1	4	1	2	1.562 913 839	2.343 146	0.050 253	1.825 742	0.016 632

Table 2.3: Example instances for two retailer types with unit rates and uniform types.

Although the KKT approach is viable for two retailer types, there is a faster and easier solution approach. Theorem 2.16 provides closed-form formulas for the optimal menu of contracts in this case. The theorem shows that the contracts for the types are unique (i.e., not the same) and that each order quantity lies between the default and joint order quantity for that type. This implies that the optimal menu will always result in a better coordination between the supplier and the retailer, i.e., the joint costs for the entire supply chain are lower. These properties do not hold in general for three or more types, which will be discussed in Section 2.4.3.

Theorem 2.16. *For two retailer types ($K = 2$) the unique optimal menu of contracts is given by:*

$$x_1 = \min\left\{\sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p} + \frac{\omega_2}{\omega_1}(h_1 - h_2)}}, \max\left\{\frac{2\sqrt{2df}}{\sqrt{h_1}+\sqrt{h_2}}, \sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p}}}\right\}\right\},$$

$$x_2 = \max\left\{\sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p} + \frac{\omega_1}{\omega_2}(h_2 - h_1)}}, \min\left\{\frac{2\sqrt{2df}}{\sqrt{h_1}+\sqrt{h_2}}, \sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p}}}\right\}\right\},$$

$$z_1 = df\frac{1}{x_1} + \frac{1}{2}h_1 x_1 - \sqrt{2dfh_1} + \max\left\{0, \sqrt{2df}(\sqrt{h_1} - \sqrt{h_2}) + \frac{1}{2}(h_2 - h_1)x_2\right\},$$

$$z_2 = df\frac{1}{x_2} + \frac{1}{2}h_2 x_2 - \sqrt{2dfh_2} + \max\left\{0, \sqrt{2df}(\sqrt{h_2} - \sqrt{h_1}) + \frac{1}{2}(h_1 - h_2)x_1\right\},$$

where we ignore any ill-defined values if $h_1 + H\frac{d}{p} + \frac{\omega_2}{\omega_1}(h_1 - h_2) \leq 0$.

Furthermore, we have that $x_1 > x_2$ and that each order quantity x_k lies in the closed interval with endpoints x_R^{k*} and x_J^{k*}. Finally, $x_k = x_R^{k*}$ if and only if $x_S^* = x_R^{k*}$, which implies $x_J^{k*} = x_R^{k*}$.

Proof. The proof is given in Appendix 2.C. □

The closed-form formulas and properties in Theorem 2.16 for the optimal menu of contracts can be determined relatively easily using only calculus for differentiable convex functions. So there is no need to evaluate multiple menus or even use KKT conditions to determine the optimal solution.

We relate our results for $K = 2$ to Pishchulov and Richter (2016), who analyse the same problem for two retailer types but with two-dimensional private information. They use a KKT approach to determine the optimal solution, which consequently generalises our KKT approach for $K = 2$ (but not Theorem 2.16). Their results show that for two-dimensional types 8 KKT graphs can occur. The 3 additional cases are variations with 2-cycles in the KKT graph, where bunching ($x_1 = x_2$) occurs. However, for any instance only 5 KKT graphs need to be evaluated (determined by checking if the joint order quantities coincide). In particular, no bunching occurs if the two joint order quantities differ (which holds for our model).

The need to evaluate 5 cases also appears in the setting of Kerschbamer and Maderner (1998), which is complementary to our model and results. For two retailer types with strictly increasing convex cost functions (and a linear utility function for the supplier) only 5 cases can occur, all without bunching. This is similar to our findings for two types.

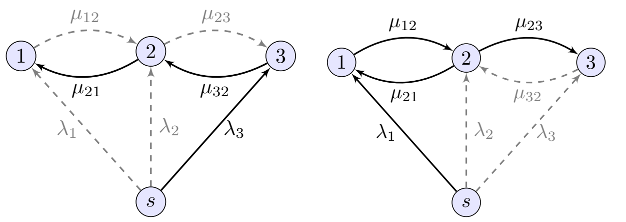
2.4.2 Three retailer types

For three retailer types ($K = 3$) our results reduce the number of KKT menus from $2^7 = 128$ to only 23. For the details we refer to Appendix 2.D.3. For each of the 23 KKT menus, we have found instances where that menu is optimal. We conclude that our analysis is tight for $K = 3$. See Table 2.4 for example instances and their optimal solution. Observe that such examples can already be found in a small integer range for the cost parameters.

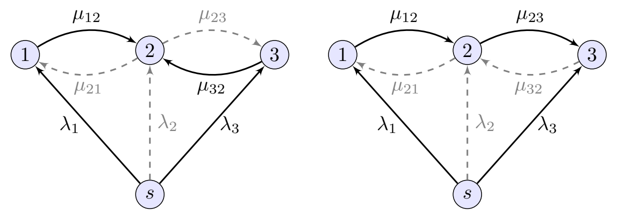
The analysis and these examples show certain structures in the optimal solution that do not occur in related literature. For example, Voigt and Inderfurth (2011) study multiple retailer types, but due to their cost functions only the KKT graph in Figure 2.5a occurs. For our model, many more possible optimal structures exist, which complicates the analysis.

Furthermore, in certain optima the same contract is offered to multiple types: see menus with cycles in their KKT graph, such as in Figure 2.5b. We also draw additional attention to the solution corresponding to Figure 2.5c. Here, type 2 is simultaneously prevented from lying to have lower holding cost h_1 and higher cost h_3. Hence, we cannot limit our analysis to cases where all retailer arcs follow the same direction, which does hold when the default option is type independent. In particular, this shows that the retailer does not necessarily have an inherent incentive to pretend to have a higher holding cost. As a final note, there is no monotonicity or general ordering in the side payments.

Similar structures have been found for a different model in Kerschbamer and Maderner (1998). They consider retailer types with strictly increasing convex cost functions and a supplier with a linear utility function. In particular, their results include (sometimes weaker) analogues to Lemmas 2.3, 2.5, and 2.12 and Corollary 2.2 for a general number of types. This allows them to reduce the number of cases to consider. For three types, they determine just a few more possible structures than our 23 cases, since they do not have an analogue to Lemma 2.6.



(a) 1x2Left3UpLeft. (b) 1UpRight2LeftRight3x.



(c) 1UpRight2x3UpLeft. (d) 1UpRight2Right3Up.

Figure 2.5: Example KKT graphs for three retailer types that occur in optimal solutions.

Optimal menu			F	H	f	h_1	h_2	h_3	Objective	x_1	z_1	x_2	z_2	x_3	z_3
1x	2UpLeft	3Up	1	1	1	3	4	20	2.027 573 769	1	0.079 821	0.816 497	0.029 311	0.436 436	0.331 090
1x	2UpLeft	3UpLeft	1	1	1	3	4	6	1.716 382 902	1	0.079 821	0.816 497	0.029 311	0.635 674	0.016 054
1x	2Left	3UpLeft	1	1	1	3	4	5	1.689 666 918	1	0.099 524	0.816 497	0.049 014	0.707 107	0.019 703
1x	2LeftRight	3UpLeft	1	1	1	5	9	10	2.059 836 692	0.816 497	0.154 678	0.544 331	0.086 637	0.544 331	0.086 637
1Up	2x	3UpLeft	1	1	1	1	3	4	1.552 112 933	1.414 214	0	1	0.079 821	0.816 497	0.029 311
1Up	2Up	3Up	1	1	1	1	2	9	1.653 409 937	1.414 214	0	1.154 701	0.020 726	0.632 456	0.184 548
1Up	2Up	3UpLeft	1	1	1	1	2	3	1.483 843 092	1.414 214	0	1.154 701	0.020 726	0.898 979	0.011 352
1Up	2UpLeft	3Up	1	1	2	3	4	8	1.535 007 626	1.224 745	0.006 009	1.071 797	0.009 619	0.816 497	0.058 622
1Up	2UpLeft	3UpLeft	1	1	1	2	3	5	1.606 428 503	1.154 701	0.020 726	0.898 979	0.011 352	0.712 788	0.022 634
1Up	2UpRight	3x	1	3	6	1	8	9	3.072 312 060	1.870 829	0.678 448	1.183 216	0.005 830	1.080 123	0.025 909
1Up	2UpRight	3Up	1	1	7	1	3	4	1.631 813 245	2.828 427	0.147 430	2.005 148	0.017 995	1.788 854	0.007 513
1Up	2Left	3UpLeft	1	1	1	2	3	4	1.580 844 039	1.154 701	0.020 726	0.898 482	0.011 470	0.758 372	0.006 931
1Up	2LeftRight	3UpLeft	1	1	1	4	9	10	2.008 977 459	0.894 427	0.078 461	0.547 903	0.092 520	0.547 903	0.092 520
1UpRight	2x	3Up	1	2	3	1	2	5	2.254 654 654	2	0.050 510	1.414 214	0.086 044	1.069 045	0.001 630
1UpRight	2x	3UpLeft	1	6	1	1	7	8	3.612 045 267	0.775 694	0.262 802	0.554 700	0.002 932	0.517 411	0.002 344
1UpRight	2Up	3Up	1	1	3	1	2	3	1.466 674 185	2.029 224	0.043 520	1.632 993	0.006 009	1.414 214	0
1UpRight	2Up	3UpLeft	1	5	1	1	6	7	3.281 949 676	0.819 955	0.215 343	0.603 023	0.003 279	0.555 112	0.002 673
1UpRight	2UpRight	3x	1	2	5	1	2	3	2.619 772 030	2.619 717	0.056 184	2	0.027 864	1.549 193	0.079 140
1UpRight	2UpRight	3Up	1	1	5	1	2	3	1.575 560 678	2.619 717	0.056 184	2.010 179	0.025 384	1.732 051	0.007 602
1UpRight	2Right	3x	1	3	3	1	2	3	3.078 862 750	2	0.050 510	1.414 214	0.086 044	1.154 701	0.173 530
1UpRight	2Right	3Up	1	2	4	1	2	5	2.393 966 461	2.239 120	0.077 549	1.584 379	0.161 041	1.195 229	0.010 156
1UpRight	2LeftRight	3x	1	5	6	1	2	6	5.339 122 934	2	0.535 898	2	0.535 898	1.128 152	0.238 786
1UpRight	2LeftRight	3Up	1	4	9	1	2	6	5.231 467 672	2.459 866	0.646 029	2.459 866	0.646 029	1.414 214	0.214 297

Table 2.4: Example instances for three retailer types with unit rates and uniform types.

2.4.3 Differences between two or more types

In this section, we discuss observed differences in the optimal menus of contracts for two and three retailer types. First of all, the bounds on the order quantities in Theorem 2.16 are a unique property for the case with only two retailer types. To be more specific, an optimal order quantity x_k is not bounded by x_R^{k*} or x_J^{k*}

when there are more than two retailer types. For an example, see Table 2.5 where type 2 is not bounded by its default or joint order quantity. In fact, this can even occur if the retailer type and the supplier both desire the same order quantity, i.e., $x_R^{k*} = x_S^* = x_J^{k*}$, as the example in Table 2.5 shows for type $k = 2$. Thus, for $K = 2$ the optimal menu will always result in a better coordination between the supplier and the retailer, but this need not be the case for some contracts in the optimal menu for $K = 3$. In such cases, an inefficient order quantity is optimal since the side payments for the entire menu would otherwise be too high.

Furthermore, for three or more types it can be optimal to have duplicate contracts in the menu, as we have shown in Sections 2.3.5 and 2.4.2. Recall that by Theorem 2.14 we need additional assumptions to guarantee to have unique contracts. In contrast, for two types we know from Theorem 2.16 that the optimal menu never contains duplicate contracts.

Finally, related to the above points, we observe structurally different optimal menus of contracts for three or more types compared to two types.

Unfortunately, our analysis suggests that for each number of retailer types K we need to solve some cases from scratch. That is, we were unable to reuse old results for $K - 1$ types in a scalable way to solve the problem for K types. For example, cases similar to Figures 2.5c and 2.5d are troublesome. Therefore, the analytical KKT approach does not seem to be a generalisable solution approach. For a general number of retailer types, the scalable technique described in Section 2.2.2 is preferred.

To conclude, there are significant differences in the qualitative properties of the optimal menu of contracts for two types compared to more than two types.

F	H	f	h_1	h_2	h_3	Optimal menu			Objective
3	15	1	1	5	7	1UpRight	2Right	3Up	9.716 539

Type k	x_k	z_k	x_R^{k*}	x_J^{k*}	$\phi_S(x_k) + z_k$	$\phi_S(x_R^{k*})$
1	0.844 369	0.192 288	1.414 214	0.707 107	10.078 003	12.727 922
2	0.638 707	0.059 480	0.632 456	0.632 456	9.546 772	9.486 833
3	0.603 023	0.027 234	0.534 522	0.603 023	9.524 842	9.621 405

Table 2.5: Example instance for $K = 3$ with unit rates and uniform types.

2.5 Discussion and conclusion

Before we conclude with the main insights of our research, we discuss consequences of two model assumptions. First, using the expected costs as objective function is common in the literature. However, this can lead to the peculiar situation where the supplier's action to offer a menu of contracts results in an increase in the supplier's costs compared to taking no such action. See Section 2.5.1 for more details. Second, we discuss the screening capability of the contracting model in Section 2.5.2. Finally, we conclude our findings in Section 2.5.3.

2.5.1 Unfavourable realisations

The objective of the contracting model is to minimise the expected costs of the supplier based on the K scenarios, where each scenario corresponds to a retailer type. Therefore, it could be that for a certain realisation of the scenarios the supplier is worse off using the menu of contracts instead of accepting the retailer's default option. The example in Table 2.5 shows that this can indeed happen: for a realisation of the scenario of retailer type 2 the supplier would be better off with the default option. We note that examples for $K = 2$ also exist.

If this phenomenon is not allowed, then we have to add the following constraints to the model to prevent it:

$$\phi_S(x_k) + z_k \leq \phi_S(x_R^{k*}), \qquad \forall\, k \in \mathcal{K}. \tag{2.16}$$

We can interpret the new constraints (2.16) as the individual rationality constraints for the supplier. Notice that the menu with all default order quantities and zero side payments is still feasible. Furthermore, adding these constraints to the default contracting model will lead to an optimal objective value at least that of the default model.

If we consider the alternative convex model of Section 2.2.2, the equivalent constraints of (2.16) are convex. Therefore, we can still efficiently solve the alternative model after adding the IR constraints for the supplier. Of course, the theoretical analysis has to be redone after adding these constraints.

In the literature, models with and without individual rationality for the supplier are used. For example, all references in Table 2.1 do not use (2.16). Including (2.16) is out of scope for this chapter, but it would be interesting to determine its effect on the structure of optimal menus.

2.5.2 Screening capability

The contracting model is often called a screening model, i.e., it allows the supplier to correctly identify the retailer's type. The idea is as follows. We have shown that there exists an optimal solution where the retailer does not lie about his type (also known as the revelation principle). Therefore, by observing the contract chosen by the retailer, we can determine his true type. Unfortunately, this idea has some issues, which we discuss in this section.

First, we have the issue of the retailer's indifference between contracts. Consider a type $k \in \mathcal{K}$. If the IC constraint where type k compares to the contract for type $l \in \mathcal{K}$ is tight, then type k is indifferent between contracts (x_k, z_k) and (x_l, z_l). We have assumed that in this case the supplier can convince the retailer to choose contract (x_k, z_k) without any additional cost. Without this assumption, we need to model the IC constraints as strict inequalities or add a secondary objective for the retailer to determine its choice. For example, the retailer could be inequity averse, a topic analysed in Voigt (2015).

Second, it may be optimal to assign the same contract to multiple retailer types. We have showed in Section 2.4 that this phenomenon can indeed occur if and only if there are more than two types. For such an optimal solution, we cannot distinguish those types by the retailer's choice. Of course, we can modify the contract to be unique at a small cost in objective value, by the hidden convexity of the feasible region. Another possibility is to make assumptions on the retailer types to guarantee unique contracts as seen in Section 2.3.5.

To conclude, the screening capability of the contracting model should be treated with care, especially if there are more than two retailer types. These observations are still in line with the revelation principle, since not lying is always a weakly-dominant strategy for the retailer.

2.5.3 Main insights

Our model extends the current literature by having a general number of retailer types with type-dependent default options. The inclusion of type-dependent default options and both ordering/setup costs and holding costs for the retailer and supplier increases the structural complexity of optimal menus of contracts.

Our analysis shows that an optimal menu of contracts for three or more retailer types has different structural properties than an optimal menu for two types. For two retailer types the contracts in the optimal menu are unique (see Theorem 2.16), whereas for three or more retailer types it may be optimal to present the same contract to multiple types. This insight affects the screening capability of the contracting model, as discussed above. However, if the distribution of the retailer types is uniform and equidistant, then we are guaranteed to have different contracts for each type (see Theorem 2.14).

Furthermore, for two retailer types the order quantities in the optimal menu lie between their default and joint order quantities (see Theorem 2.16). Thus, all contracts in the optimal menu improve the efficiency of the entire supply chain. This does not hold for all contracts in the optimal menu for more than two types, as we have counterexamples (see Table 2.5).

Besides the monotonicity in order quantities, there are also other general properties. In any optimal menu of contracts the order quantity for at least one type is its joint order quantity (see Lemma 2.13). If the retailer's true type is that specific type, then perfect supply chain coordination takes place. Similarly, in any optimal menu the resulting costs for at least one type is the same as its default costs. An important observation for the previous two properties is that it could hold for any retailer type, so not necessarily the type with either highest or lowest holding cost. In particular, we have given examples where perfect supply chain coordination is achieved only for the middle type and not for the lowest or highest type. Also, there are examples where there is perfect coordination for all types. Therefore, we cannot consider the types with highest or lowest holding cost as 'best' and 'worst' types, as is common in the literature.

By considering more than two types we observe additional properties of the retailer's lying behaviour. For example, consider three adjacent types with different contracts, say types 1, 2, and 3 with holding costs $h_1 < h_2 < h_3$ respectively. The following situation cannot occur: types 1 and 3 are simultaneously prevented from lying to have (higher, respectively lower) holding cost h_2. However, it can happen that type 2 is simultaneously prevented from lying to have lower (h_1) and higher (h_3) holding cost. Thus, the retailer's lying behaviour is more complex than just having an inherent incentive to lie having higher costs. To our knowledge, such behaviour is uncommon in the literature.

Overall, having type-dependent default options significantly impacts the retailer's lying behaviour and the possible optimal menus, especially for more than two retailer types. Furthermore, not all properties for two retailer types generalise to more types. These phenomena have to be taken into account when using contracting models.

To conclude, we can efficiently solve our model for a general number of retailer types by a change of decision variables (see Theorem 2.1). Changing the perspective from side payments to information rents reveals the hidden convexity of our model. Remarkably, the current literature seems to focus on formulations using side payments. As our case illustrates, the change of perspective can lead to valuable insights or solution approaches. Therefore, for other contracting models we would like to promote the use or investigation of an alternative formulation using information rents.

Appendix

2.A Flow problem in the IRIC graph

In Section 2.3.1 we showed a connection between the contracting model and a shortest path interpretation. For fixed order quantities x_k, the contracting model is equivalent to the dual of the minimum cost flow problem in the corresponding IRIC graph. In this appendix we explicitly state the involved flow formulations. The minimum cost flow problem in the IRIC graph is given by:

$$\begin{aligned}
\min_{u} \quad & \sum_{k\in\mathcal{K}} \left(\phi_R^{k*} - \phi_R^k(x_k)\right) u_{sk} + \sum_{k\in\mathcal{K}}\sum_{l\in\mathcal{K}} \left(\phi_R^l(x_k) - \phi_R^l(x_l)\right) u_{kl} \\
\text{s.t.} \quad & \sum_{k\in\mathcal{K}} u_{sk} = \sum_{k\in\mathcal{K}} \omega_k, \\
& \sum_{l\in\mathcal{K}} u_{kl} - \sum_{l\in\mathcal{K}} u_{lk} - u_{sk} = -\omega_k, \qquad \forall\, k\in\mathcal{K}, \\
& u \geq 0.
\end{aligned}$$

Its dual is as follows:

$$\begin{aligned}
\max_{v} \quad & \sum_{k\in\mathcal{K}} \omega_k v_s - \sum_{k\in\mathcal{K}} \omega_k v_k \\
\text{s.t.} \quad & v_s - v_k \leq \phi_R^{k*} - \phi_R^k(x_k), \qquad \forall\, k\in\mathcal{K}, \\
& v_k - v_l \leq \phi_R^l(x_k) - \phi_R^l(x_l), \qquad \forall\, k,l\in\mathcal{K}.
\end{aligned}$$

For a feasible dual solution v, adding the same constant to each v_i results in another equivalent feasible solution with the same objective value. Thus, we can set $v_s = 0$

without loss of generality, resulting in:

$$
\begin{aligned}
\max_{v} \quad & -\sum_{k\in\mathcal{K}} \omega_k v_k \\
\text{s.t.} \quad & -v_k \leq \phi_R^{k*} - \phi_R^k(x_k), && \forall\, k \in \mathcal{K}, \\
& v_l - v_k \leq \phi_R^k(x_l) - \phi_R^k(x_k), && \forall\, k, l \in \mathcal{K}.
\end{aligned}
$$

By taking the minus sign in the objective out of the optimisation problem, we obtain a minimisation problem, which is equal to the contracting model with fixed order quantities (see (2.5)-(2.7)).

2.B Proofs of Section 2.3

This appendix states all the proofs of Section 2.3. Most proofs condition on the Lagrange multipliers and use linear combinations of the KKT conditions to derive the desired results.

2.B.1 Proofs of Section 2.3.2

Proof of Lemma 2.2. Consider a feasible menu of contracts (x, z). From the shortest path interpretation in Section 2.3.1 we know that no negative cycles exist in the corresponding IRIC graph. In particular, any 2-cycle in the IRIC graph has non-negative length. Without loss of generality, consider $i, j \in \mathcal{K}$ with $h_i < h_j$ and consider the length of the 2-cycle between nodes i and j, which must be non-negative:

$$
\begin{aligned}
& \left(\phi_R^i(x_j) - \phi_R^i(x_i)\right) + \left(\phi_R^j(x_i) - \phi_R^j(x_j)\right) \geq 0 \\
\iff \quad & \tfrac{1}{2}(h_j - h_i)(x_i - x_j) \geq 0 \\
\overset{(h_i < h_j)}{\iff} \quad & x_i - x_j \geq 0.
\end{aligned}
$$

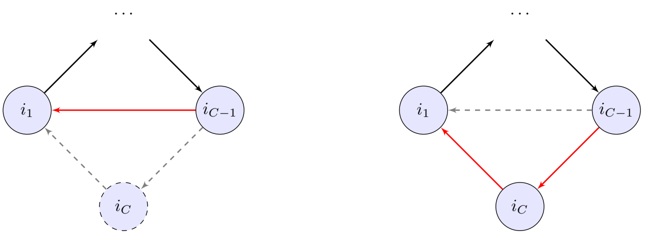
Hence, $x_i \geq x_j$ must hold in any feasible solution. □

Proof of Lemma 2.3. Let (x, z) be the optimal menu of contracts when we *only* use the adjacent IC constraints, instead of all IC constraints. Consider a cycle $\mathcal{C} = (i_1, \ldots, i_C)$ of unique retailer nodes in the IRIC graph corresponding to (x, z). We prove that any such cycle has non-negative length, implying that *all* IC constraints are satisfied. The proof is by induction on the cardinality of $\mathcal{C}$.

If $C = 2$, then the adjacent IC constraints enforce that the cycle length is non-negative. Therefore, let $C > 2$ and without loss of generality, assume that type i_C has the greatest holding cost. By induction, the cycle $(i_1, \ldots, i_{C-1})$ has non-negative length. We compare the difference in length between the two cycles, see also Figure 2.6:

$$\begin{aligned}&\Big(\big(\phi_R^{i_C}(x_{i_{C-1}}) - \phi_R^{i_C}(x_{i_C})\big) + \big(\phi_R^{i_1}(x_{i_C}) - \phi_R^{i_1}(x_{i_1})\big) \Big) - \big(\phi_R^{i_1}(x_{i_{C-1}}) - \phi_R^{i_1}(x_{i_1})\big)\\ &\qquad = \phi_R^{i_C}(x_{i_{C-1}}) - \phi_R^{i_C}(x_{i_C}) + \phi_R^{i_1}(x_{i_C}) - \phi_R^{i_1}(x_{i_{C-1}})\\ &\qquad = \tfrac{1}{2}(h_{i_C} - h_{i_1})(x_{i_{C-1}} - x_{i_C}) \geq 0.\end{aligned}$$

The inequality follows from our assumptions on the holding costs ($h_{i_C} > h_{i_1}$) and Lemma 2.2. Thus, $\mathcal{C}$ must have non-negative length as well. Consequently, all IC constraints hold without explicitly incorporating the corresponding IC constraints in the optimisation model. To conclude, (x, z) is also optimal for the complete contracting model with all IC constraints. □



(a) Smaller cycle $(i_1, \ldots, i_{C-1})$. (b) Larger cycle $\mathcal{C} = (i_1, \ldots, i_C)$.

Figure 2.6: Relevant arcs in the induction proof of Lemma 2.3.

Proof of Corollary 2.4. From Lemma 2.3 it follows that for feasibility we only need to determine side payments such that the IR constraints and the adjacent IC constraints are satisfied. From the shortest path interpretation, we know that side payments satisfying the adjacent IC constraints exist if and only if 2-cycles in the corresponding graph have non-negative length. Now consider arbitrary $i, j \in \mathcal{K}$ with $h_i < h_j$. The proof of Lemma 2.2 shows that the 2-cycle between i and j has non-negative length if and only if $x_i \geq x_j$, which holds by assumption.

Hence, we can determine feasible side payments by solving a one-to-all shortest path problem as described in Section 2.3.1. Furthermore, this leads to the best possible feasible side payments with respect to the given order quantities. □

2.B.2 Proofs of Section 2.3.4

Proof of Lemma 2.5. First, suppose $k \in \mathcal{K}$ has no ingoing arcs, i.e., $\lambda_k = \mu_{k-1,k} = \mu_{k+1,k} = 0$. From (2.12) we have:

$$\omega_k + \mu_{k,k-1} + \mu_{k,k+1} = 0 \qquad \Longrightarrow \qquad \mu_{k,k-1} + \mu_{k,k+1} < 0.$$

This contradicts the fact that all multipliers are non-negative. Hence, any node in $\mathcal{K}$ must have an ingoing arc.

Second, let $\mathcal{S} = \{i, i+1, \ldots, j-1, j\} \subseteq \mathcal{K}$ be a maximal (by inclusion) connected component that is not reachable from s. Adding up (2.12) for all $k \in \mathcal{S}$ results in:

$$\sum_{k\in\mathcal{S}} \omega_k - \mu_{i-1,i} - \mu_{j+1,j} + \mu_{i,i-1} + \mu_{j,j+1} = 0.$$

Notice that all internal arcs of $\mathcal{S}$ cancel out and $\lambda_k = 0$ for all $k \in \mathcal{S}$. Furthermore, $\mu_{i-1,i} = 0$ and $\mu_{j+1,j} = 0$ by maximality of $\mathcal{S}$. This leads to a contradiction, since $\omega_k > 0$ for all $k \in \mathcal{K}$. To conclude, every maximal connected component is reachable from s.

Finally, by iteratively using that each node has an ingoing arc we can conclude that every node must be reachable from node s. □

Proof of Lemma 2.6. Let $i, j, k \in \mathcal{K}$, $i < k < j$, be such that the constraints corresponding to arcs (s, k), (k, i), and (k, j) are simultaneously satisfied with equality. Consequently, we have:

$$\begin{aligned}
&\phi_R^k(x_k) - z_k = \phi_R^{k*}, && \\
&\phi_R^i(x_i) - z_i = \phi_R^i(x_k) - z_k, && \phi_R^j(x_j) - z_j = \phi_R^j(x_k) - z_k,\\
&\phi_R^i(x_i) - z_i \le \phi_R^{i*}, && \phi_R^j(x_j) - z_j \le \phi_R^{j*}.
\end{aligned}$$

Combining these relations leads to the following:

$$\begin{aligned}
\phi_R^{i*} &\ge \phi_R^i(x_i) - z_i = \phi_R^i(x_k) - z_k = \phi_R^i(x_k) - \phi_R^k(x_k) + \phi_R^{k*}\\
&= \tfrac{1}{2}(h_i - h_k)x_k + \phi_R^{k*},\\
\phi_R^{j*} &\ge \phi_R^j(x_j) - z_j = \phi_R^j(x_k) - z_k = \phi_R^j(x_k) - \phi_R^k(x_k) + \phi_R^{k*}\\
&= \tfrac{1}{2}(h_j - h_k)x_k + \phi_R^{k*}.
\end{aligned}$$

Rewriting these results gives:

$$\frac{\phi_R^{i*} - \phi_R^{k*}}{h_i - h_k} \le \tfrac{1}{2}x_k \le \frac{\phi_R^{j*} - \phi_R^{k*}}{h_j - h_k}.$$

Recall that $\phi_R^{l*} = \sqrt{2dfh_l}$ for all $l \in \mathcal{K}$. Thus, we arrive at the following inequality:

$$\begin{aligned}
\frac{\sqrt{h_k} - \sqrt{h_i}}{h_k - h_i} \le \frac{\sqrt{h_j} - \sqrt{h_k}}{h_j - h_k} \qquad &\iff \qquad \frac{1}{\sqrt{h_k} + \sqrt{h_i}} \le \frac{1}{\sqrt{h_j} + \sqrt{h_k}}\\
&\iff \qquad \sqrt{h_i} \ge \sqrt{h_j}.
\end{aligned}$$

The first inequality compares two slopes between three points on the square root curve. Such an inequality never holds for $h_i < h_k < h_j$, as the equivalent inequality shows. □

Proof of Corollary 2.7. Suppose a node $k \in \mathcal{K}$ directly connected to s has more outgoing arcs. The direct connection to node s implies $\lambda_k > 0$. Furthermore, the outgoing arcs must be $(k, k-1)$ and $(k, k+1)$, so $\mu_{k,k-1}, \mu_{k,k+1} > 0$. The KKT complementary slackness conditions imply that the corresponding constraints are tight, violating Lemma 2.6. □

Proof of Lemma 2.8. First, suppose the order quantities for types $i, j \in \mathcal{K}$ are the same. The incentive compatibility constraints state:

$$\begin{aligned} \phi_R^i(x_i) - z_i \leq \phi_R^i(x_j) - z_j = \phi_R^i(x_i) - z_j \qquad &\Longleftrightarrow \qquad z_i \geq z_j, \\ \phi_R^j(x_j) - z_j \leq \phi_R^j(x_i) - z_i = \phi_R^j(x_j) - z_i \qquad &\Longleftrightarrow \qquad z_j \geq z_i. \end{aligned}$$

Thus, $z_i = z_j$ must hold and both contracts are the same. Consequently, substituting $z_i = z_j$ shows that both incentive compatibility constraints are tight.

Second, suppose both incentive compatibility constraints between i and j are tight:

$$\phi_R^i(x_i) - z_i = \phi_R^i(x_j) - z_j, \qquad \phi_R^j(x_j) - z_j = \phi_R^j(x_i) - z_i.$$

Combining both equalities leads to:

$$\begin{aligned} \phi_R^i(x_i) - \phi_R^j(x_i) = \phi_R^i(x_j) - \phi_R^j(x_j) \quad &\Longleftrightarrow \quad \tfrac{1}{2}(h_i - h_j)x_i = \tfrac{1}{2}(h_i - h_j)x_j \\ &\Longleftrightarrow \quad x_i = x_j. \end{aligned}$$

The first equivalence follows from having the same ordering cost f and the last equivalence from $h_i \neq h_j$. As proved above, $x_i = x_j$ implies that $z_i = z_j$. Thus, the contracts for types i and j are the same. □

Proof of Corollary 2.9. A 2-cycle in the KKT graph means that $\mu_{kl}, \mu_{lk} > 0$ for some adjacent $k, l \in \mathcal{K}$. From complementary slackness it follows that both incentive compatibility constraints between types k and l must be tight. Lemma 2.8 implies that $x_k = x_l$ and $z_k = z_l$. Repeating this argument for all 2-cycles in the 2-cycle chain completes the proof. □

Proof of Lemma 2.10. The statement that at least one ingoing arc exists follows directly from Lemma 2.5. We prove the statement for the outgoing arcs by contradiction.

Let $\mathcal{S} = \{i, i+1, \ldots, j-1, j\} \subseteq \mathcal{K}$ be such a maximal 2-cycle chain and suppose that $\mathcal{S}$ as a whole has no outgoings arcs. By Corollary 2.9, all $k \in \mathcal{S}$ get the same contract, say order quantity x. The stationarity conditions (2.13) state that

$$\omega_k\Big(-\frac{d(f+F)}{x^2} + \tfrac{1}{2}\big(h_k + H\tfrac{d}{p}\big)\Big) + \underbrace{\tfrac{1}{2}\mu_{k,k-1}(h_k - h_{k-1})}_{>0} + \underbrace{\tfrac{1}{2}\mu_{k,k+1}(h_k - h_{k+1})}_{<0} = 0.$$

By assumption, $\mu_{i,i-1} = 0$ or non-existent (if $i = 1$). Likewise, $\mu_{j,j+1} = 0$ or non-existent (if $j = K$). The stationarity constraint for type i requires that:

$$-d(f+F)\frac{1}{x^2} + \frac{1}{2}\big(h_i + H\tfrac{d}{p}\big) > 0.$$

This implies that the similar term in the stationarity constraint for j is also strictly positive, as $h_i < h_j$. However, the resulting constraint only contains strictly positive terms, which is infeasible. Thus, $\mathcal{S}$ as a whole has at least one outgoing arc.

Suppose $\mathcal{S}$ as a whole has two outgoing arcs, i.e., $\mu_{i,i-1}, \mu_{j,j+1} > 0$. By Lemma 2.5, all nodes in $\mathcal{S}$ must be reachable from s via arcs with strictly positive multipliers. Since all nodes in $\mathcal{S}$ have two outgoing arcs by assumption, Corollary 2.7 implies that $\lambda_i, \ldots, \lambda_j = 0$, otherwise the solution is infeasible. Therefore, $\mu_{i-1,i} > 0$ and/or $\mu_{j+1,j} > 0$ must hold, but this contradicts the maximality of $\mathcal{S}$. To conclude, $\mathcal{S}$ has exactly one outgoing arc. □

Proof of Lemma 2.11. First, suppose node $k \in \mathcal{K}$ has no outgoing arcs in the KKT graph, i.e., $\mu_{k,k-1} = \mu_{k,k+1} = 0$. The KKT stationarity condition (2.13) requires that:

$$\omega_k\Big(- d(f+F)\frac{1}{x_k^2} + \frac{1}{2}\big(h_k + H\tfrac{d}{p}\big)\Big) = 0 \qquad \Longrightarrow \qquad x_k = x_J^{k*},$$

which proves one direction of the lemma.

Second, suppose that $x_k = x_J^{k*}$. Again using the KKT stationarity condition (2.13), we get:

$$\mu_{k,k-1}(h_k - h_{k-1}) + \mu_{k,k+1}(h_k - h_{k+1}) = 0.$$

Since $h_{k-1} < h_k < h_{k+1}$, either $\mu_{k,k-1}, \mu_{k,k+1} > 0$ (node k has two outgoing arcs) or $\mu_{k,k-1} = \mu_{k,k+1} = 0$ (node k has no outgoing arcs). In the latter case we are done. Therefore, suppose $\mu_{k,k-1}, \mu_{k,k+1} > 0$. We discern two cases.

Case I: node k is not part of a 2-cycle, i.e., $\mu_{k-1,k} = \mu_{k+1,k} = 0$. By Lemma 2.5 node k must be reachable from node s, implying $\lambda_k > 0$. This case is infeasible, see Corollary 2.7.

Case II: node k is part of a 2-cycle. Let k be part of the maximal 2-cycle chain $\mathcal{S} = \{i, i+1, \ldots, k, \ldots, j-1, j\} \subseteq \mathcal{K}$. Recall that from Corollary 2.9 we know that all types in $\mathcal{S}$ have the same contract. Furthermore, by Lemma 2.10 either $\mu_{i,i-1} > 0$ or $\mu_{j,j+1} > 0$ (but not both).

Consider the case that $\mu_{i,i-1} > 0$, and thus $\mu_{j,j+1} = 0$ and $j > k$. The KKT stationarity conditions state:

$$\omega_j\Big(- d(f+F)\frac{1}{x_j^2} + \frac{1}{2}\big(h_j + H\tfrac{d}{p}\big)\Big) + \frac{1}{2}\mu_{j,j-1}(h_j - h_{j-1}) = 0.$$

Since $\mu_{j,j-1}(h_j - h_{j-1}) > 0$, it must hold that $x_j < x_J^{j*}$. Since $x_k = x_j$, we have the required contradiction:

$$x_J^{k*} = x_k = x_j < x_J^{j*} < x_J^{k*}.$$

The other case, $\mu_{i,i-1} = 0$ and $\mu_{j,j+1} > 0$, is similar and is omitted.

To conclude, if $x_k = x_J^{k*}$ node k must have no outgoing arcs in the KKT graph, which completes the proof. □

Proof of Lemma 2.12. First, assume that k is part of a 2-cycle chain $(i < j)$. From Corollary 2.9 we know that all types in $\mathcal{S}$ have the same contract. Furthermore, by Lemma 2.10 either $\mu_{i,i-1} > 0$ or $\mu_{j,j+1} > 0$ (but not both), so we can consider these two cases.

Consider the case that $\mu_{i,i-1} > 0$ and thus $\mu_{j,j+1} = 0$. The KKT stationarity conditions state:

$$\omega_j \left(\frac{\mathrm{d}\phi_S}{\mathrm{d}x}(x_j) + \frac{\mathrm{d}\phi_R^j}{\mathrm{d}x}(x_j) \right) + \underbrace{\frac{1}{2}\mu_{j,j-1}(h_j - h_{j-1})}_{>0} = 0 \qquad \Longrightarrow \qquad x_j < x_J^{j*}.$$

That is, the expression can only be zero if the first term is strictly negative. Since $x_k = x_j$, we have the required result: $x_k = x_j < x_J^{j*} \le x_J^{k*}$.

Next, consider the other case, $\mu_{i,i-1} = 0$ and $\mu_{j,j+1} > 0$. Again, the KKT conditions imply:

$$\omega_i \left(\frac{\mathrm{d}\phi_S}{\mathrm{d}x}(x_i) + \frac{\mathrm{d}\phi_R^i}{\mathrm{d}x}(x_i) \right) + \underbrace{\frac{1}{2}\mu_{i,i+1}(h_i - h_{i+1})}_{<0} = 0 \qquad \Longrightarrow \qquad x_i > x_J^{i*}.$$

Thus, we get $x_k = x_i > x_J^{i*} \ge x_J^{k*}$. This concludes the proof if k is part of a 2-cycle.

Second, assume that k is not part of a 2-cycle ($i = j = k$). We consider each of the four possible cases. If $\mu_{i,i-1} > 0$ and $\mu_{j,j+1} = 0$, or $\mu_{i,i-1} = 0$ and $\mu_{j,j+1} > 0$, we can reuse the above cases. If $\mu_{k,k-1} = \mu_{k,k+1} = 0$, then k has no outgoing arcs and Lemma 2.11 implies $x_k = x_J^{k*}$. Finally, consider the last case: $\mu_{k,k-1}, \mu_{k,k+1} > 0$. Lemma 2.5 implies that $\lambda_k > 0$, $\mu_{k-1,k} > 0$, or $\mu_{k+1,k} > 0$. This either leads to infeasibility by Lemma 2.6 or a 2-cycle which is excluded by assumption.

The statements also hold in the reverse direction by trivial contradiction arguments using the above derived implications. □

Proof of Lemma 2.13. The result that there exists a retailer type with the same total costs as its default option follows directly from Lemma 2.5. That is, there exists a $k \in \mathcal{K}$ such that $\lambda_k > 0$. Hence, $\phi_R^k(x_k) - z_k = \phi_R^{k*}$ by complementary slackness.

By combining Lemmas 2.10 and 2.11, we can prove the other claim as follows. Suppose each retailer node has an outgoing arc in the KKT graph. Thus, arc $(1, 2)$ from type 1 to type 2 exists in the graph. If arc $(2, 3)$ is in the graph, we continue to type 3. If type 2 has only outgoing arc $(2, 1)$, then type 2 forms a 2-cycle with type 1. By Lemma 2.10 arc $(2, 3)$ must exist as well, a contradiction. Repeat this argument until we reach type K. Since node K also has an outgoing arc, a 2-cycle with type $K - 1$ is formed. Again by Lemma 2.10, this cycle must have an outgoing arc, namely arc $(K - 1, K - 2)$. Repeat this argument until we reach type 1. Hence, all retailer nodes are part of the same 2-cycle chain which contradicts Lemma 2.10.

Thus, there exists at least one type with no outgoing arcs. Lemma 2.11 states that this retailer type is assigned the joint order quantity in the optimal solution. □

2.B.3 Proofs of Section 2.3.5

Proof of Theorem 2.14. For all types $k \in \mathcal{K}$, let $\omega_k = 1/K$ and $h_{k+1} = h_k + \delta$ for some $\delta \in \mathbb{R}_{>0}$. First, realise that if $x_k = x_l$ for some $l > k + 1$, then all intermediate types also have the same order quantity: $x_k = x_{k+1} = \cdots = x_{l-1} = x_l$. This follows from the ordering of the order quantities (Lemma 2.2). Second, if $x_k = x_l$ then automatically $z_k = z_l$ must hold to be feasible (Lemma 2.8). So, both contracts are exactly the same.

Assume that there are contracts for retailer types that are the same, else there is nothing to prove. Let $\mathcal{S} = \{i, i+1, \ldots, j-1, j\} \subseteq K$ with $i < j$ be a maximal set of types with the same contract. Note that Lemmas 2.6 and 2.8 imply that $\lambda_{i+1}, \ldots, \lambda_{j-1} = 0$. We have to distinguish two cases based on the KKT multipliers.

Case I: $\mu_{i,i-1} > 0$. We have three direct implications: $\lambda_i = 0$ (by Lemma 2.6), $\mu_{i-1,i} = 0$ (by maximality of $\mathcal{S}$ and Corollary 2.9), and thus $\mu_{i+1,i} > 0$ (by Lemma 2.5). Furthermore, since all nodes in $\mathcal{S}$ must be reachable (Lemma 2.5), we have that $\mu_{i+1,i}, \ldots, \mu_{j,j-1} > 0$. Finally, we can conclude that $\mu_{j,j+1} = 0$ with a simple argument by contradiction using the maximality of $\mathcal{S}$ or Lemmas 2.5 and 2.6.

Now we can derive two contradictory equations. The first equation is as follows. Since $\lambda_i, \ldots, \lambda_{j-1} = 0$, the sum of the corresponding KKT conditions (2.15) from i to $j-1$ is equal to

$$(j-i) + \mu_{i,i-1} - \mu_{j,j-1} + \mu_{j-1,j} = 0.$$

For the second equation we need to consider the KKT conditions (2.14) for type i and j. As $x_i = x_j$ both conditions have a common part, hence the difference must be equal:

$$h_i + \delta(\mu_{i,i-1} - \mu_{i,i+1}) = h_j + \delta\mu_{j,j-1} \quad \Longleftrightarrow \quad \mu_{i,i-1} - \mu_{i,i+1} = (j-i) + \mu_{j,j-1}.$$

Finally, both equations combined state that

$$-\mu_{i,i+1} = 2(j-i) + \mu_{j-1,j} \geq 2 + \mu_{j-1,j} > 0.$$

This contradicts that $\mu_{i+1,i}$ is non-negative.

Case II: $\mu_{i,i-1} = 0$. The KKT stationarity for type i simplifies to

$$\left(-\frac{2d(f+F)}{x_i^2} + H\tfrac{d}{p}\right) + h_i = \delta\mu_{i,i+1} \geq 0.$$

Therefore, for any $k \in \mathcal{S}$, $k > i$, it must hold that $\mu_{k,k+1} > 0$, since $x_k = x_i$ and from the above inequality:

$$\begin{aligned} 0 &= \left(-\frac{2d(f+F)}{x_k^2} + H\tfrac{d}{p}\right) + h_k + \delta(\mu_{k,k-1} - \mu_{k,k+1}) \\ &\geq (h_k - h_i) + \delta(\mu_{k,k-1} - \mu_{k,k+1}). \end{aligned}$$

Consequently, $\lambda_j = 0$ by Lemma 2.6, and $\mu_{j+1,j} = 0$ by maximality of $\mathcal{S}$ and Corollary 2.9.

As in Case I, we derive two contradictory equations. The sum of the corresponding KKT conditions (2.15) from $i+1$ to j is equal to

$$(j-i) + \mu_{i+1,i} - \mu_{i,i+1} + \mu_{j,j+1} = 0.$$

The KKT conditions (2.14) for type i and j lead to:

$$h_i - \delta\mu_{i,i+1} = h_j + \delta(\mu_{j,j-1} - \mu_{j,j+1}) \iff -\mu_{i,i+1} = (j-i) + \mu_{j,j-1} - \mu_{j,j+1}.$$

These two equations give a contradiction:

$$0 = 2(j-i) + \mu_{j,j-1} + \mu_{i+1,i} \geq 2.$$

To conclude, a menu with non-unique contracts between retailer types is never optimal, irrespective of the actual value of δ. This shows that uniformity of types and equidistant holding costs are sufficient for unique contracts. □

Proof of Corollary 2.15. Suppose the KKT graph has a 2-cycle (which are the only cycles possible). Corollary 2.9 implies that those two types have the same contract, which contradicts Theorem 2.14. □

2.C Proof of Theorem 2.16

In this appendix, we give the proof of Theorem 2.16. The proof only requires some basic calculus for differentiable convex functions and the results from Section 2.3.2. In particular, we do not use the KKT conditions in any way. First, we give two lemmas that relate certain values that appear in optimal contracts and are needed to prove the theorem.

Lemma 2.17. *For $k \in \mathcal{K}$, x_J^{k*} lies in the closed interval with endpoints x_R^{k*} and x_S^*, and is equal to either endpoint if and only if $x_R^{k*} = x_S^*$.*

Proof. This follows from simple algebraic manipulation. Let $\sim$ denote the ordering relation between two numbers, i.e., $\sim \in \{=, \geq, >, \leq, <\}$. We have:

$$\begin{aligned}
x_J^{k*} \sim x_R^{k*} \quad &\iff \quad \frac{2d(f+F)}{h_k + H\frac{d}{p}} \sim \frac{2df}{h_k} \quad &&\iff \quad h_k(f+F) \sim f(h_k + H\tfrac{d}{p}) \\
&\iff \quad h_k F \sim fH\tfrac{d}{p} \quad &&\iff \quad \frac{2dF}{H\frac{d}{p}} \sim \frac{2df}{h_k} \\
&\iff \quad x_S^* \sim x_R^{k*},
\end{aligned}$$

and a similar equivalence for the supplier: $x_J^{k*} \sim x_S^* \iff x_R^{k*} \sim x_S^*$. □

Lemma 2.18. *For $k, l \in \mathcal{K}$ with $k < l$, we have:*

$$x_R^{k*} > \frac{2(\phi_R^{l*} - \phi_R^{k*})}{h_l - h_k} > x_R^{l*}.$$

Proof. The square root function is concave, so we can relate its gradient as follows, using $h_k < h_l$:

$$\frac{1}{2\sqrt{h_k}} > \frac{\sqrt{h_l} - \sqrt{h_k}}{h_l - h_k} > \frac{1}{2\sqrt{h_l}}.$$

Therefore, we have

$$
\begin{aligned}
x_R^{k*} &= \sqrt{\frac{2df}{h_k}} > 2\sqrt{2df}\left(\frac{\sqrt{h_l}-\sqrt{h_k}}{h_l-h_k}\right) = \frac{2(\phi_R^{l*}-\phi_R^{k*})}{h_l-h_k},\\
x_R^{l*} &= \sqrt{\frac{2df}{h_l}} < 2\sqrt{2df}\left(\frac{\sqrt{h_l}-\sqrt{h_k}}{h_l-h_k}\right) = \frac{2(\phi_R^{l*}-\phi_R^{k*})}{h_l-h_k}.
\end{aligned}
$$

This proves the lemma. □

We continue with the proof of Theorem 2.16.

Proof of Theorem 2.16. By Lemma 2.2 any feasible solution must satisfy $x_1 \geq x_2$. Assuming $x_1 \geq x_2$ the optimal side payments are determined by the shortest paths in the corresponding IRIC graph (see also Corollary 2.4). There are only two paths possible: directly from node s or via the other retailer node. Thus, the side payments are:

$$
\begin{aligned}
z_1 &= -\min\left\{\phi_R^{1*}-\phi_R^1(x_1), \left(\phi_R^{2*}-\phi_R^2(x_2)\right)+\left(\phi_R^1(x_2)-\phi_R^1(x_1)\right)\right\}\\
&= \max\left\{\phi_R^1(x_1)-\phi_R^{1*}, \phi_R^1(x_1)-\phi_R^{2*}+\frac{1}{2}(h_2-h_1)x_2\right\}\\
&= \phi_R^1(x_1)-\phi_R^{1*}+\max\left\{0, \phi_R^{1*}-\phi_R^{2*}+\frac{1}{2}(h_2-h_1)x_2\right\},
\end{aligned}
$$

and likewise

$$
z_2 = \phi_R^2(x_2)-\phi_R^{2*}+\max\left\{0, \phi_R^{2*}-\phi_R^{1*}+\frac{1}{2}(h_1-h_2)x_1\right\}.
$$

Therefore, the contribution of x_1 to the objective function is

$$
\omega_1\left(\phi_S(x_1)+\phi_R^1(x_1)\right)+\omega_2\max\left\{0, \phi_R^{2*}-\phi_R^{1*}+\frac{1}{2}(h_1-h_2)x_1\right\}. \tag{2.17}
$$

The expression (2.17) is a continuous convex function in x_1 with one non-differentiable point. Consequently, its minimiser is the value of x_1 such that the derivative is zero or changes from negative to positive. The derivative of (2.17) is given by

$$
\begin{cases}
\omega_1\left(-d(f+F)\frac{1}{x_1^2}+\frac{1}{2}\left(h_1+H\frac{d}{p}\right)\right)+\frac{1}{2}\omega_2(h_1-h_2) & \text{if } x_1 < \frac{2(\phi_R^{2*}-\phi_R^{1*})}{h_2-h_1}\\
\omega_1\left(-d(f+F)\frac{1}{x_1^2}+\frac{1}{2}\left(h_1+H\frac{d}{p}\right)\right) & \text{otherwise}
\end{cases}.
$$

This implies that there are three critical values for x_1:

$$
x_1 = x_J^{1*}, \qquad x_1 = \sqrt{\frac{2d(f+F)}{h_1+H\frac{d}{p}+\frac{\omega_2}{\omega_1}(h_1-h_2)}}, \qquad x_1 = \frac{2(\phi_R^{2*}-\phi_R^{1*})}{h_2-h_1}.
$$

Notice that we have the following relation indicated by $\sim\in\{=,\geq,>,\leq,<\}$:

$$\begin{aligned}
&\omega_1\left(-d(f+F)\frac{1}{x_1^2}+\frac{1}{2}\left(h_1+H\frac{d}{p}\right)\right)+\frac{1}{2}\omega_2(h_1-h_2)\sim 0\\
\iff\quad &\left(h_1+H\frac{d}{p}+\frac{\omega_2}{\omega_1}(h_1-h_2)\right)x_1^2\sim 2d(f+F).
\end{aligned}$$

So if $h_1+H\frac{d}{p}+\frac{\omega_2}{\omega_1}(h_1-h_2)\leq 0$, the gradient is strictly negative for all $x_1<2(\phi_R^{2*}-\phi_R^{1*})/(h_2-h_1)$. In this case, the minimiser is given by

$$x_1=\max\left\{\frac{2(\phi_R^{2*}-\phi_R^{1*})}{h_2-h_1},x_J^{1*}\right\}.$$

Otherwise, $h_1+H\frac{d}{p}+\frac{\omega_2}{\omega_1}(h_1-h_2)>0$ and all critical values are well-defined. Using the fact that

$$x_J^{1*}=\sqrt{\frac{2d(f+F)}{h_1+H\frac{d}{p}}}<\sqrt{\frac{2d(f+F)}{h_1+H\frac{d}{p}+\frac{\omega_2}{\omega_1}(h_1-h_2)}},$$

we end up with three cases (orderings of the critical values) for x_1, see Figure 2.7. Determining the minimiser for these cases is straightforward and leads to the optimal value of x_1:

$$x_1=\min\left\{\sqrt{\frac{2d(f+F)}{h_1+H\frac{d}{p}+\frac{\omega_2}{\omega_1}(h_1-h_2)}},\max\left\{\frac{2(\phi_R^{2*}-\phi_R^{1*})}{h_2-h_1},x_J^{1*}\right\}\right\}.$$

The proof for x_2 is similar. Its contribution to the objective value is

$$\omega_2\left(\phi_S(x_2)+\phi_R^2(x_2)\right)+\omega_1\max\left\{0,\phi_R^{1*}-\phi_R^{2*}+\frac{1}{2}(h_2-h_1)x_2\right\}.$$

The corresponding derivative is

$$\begin{cases}\omega_2\left(-d(f+F)\frac{1}{x_2^2}+\frac{1}{2}\left(h_2+H\frac{d}{p}\right)\right)+\frac{1}{2}\omega_1(h_2-h_1) & \text{if } x_2>\frac{2(\phi_R^{2*}-\phi_R^{1*})}{h_2-h_1}\\ \omega_2\left(-d(f+F)\frac{1}{x_2^2}+\frac{1}{2}\left(h_2+H\frac{d}{p}\right)\right) & \text{otherwise}\end{cases},$$

with three critical values for x_2:

$$x_2=x_J^{2*},\qquad x_2=\sqrt{\frac{2d(f+F)}{h_2+H\frac{d}{p}+\frac{\omega_1}{\omega_2}(h_2-h_1)}},\qquad x_2=\frac{2(\phi_R^{2*}-\phi_R^{1*})}{h_2-h_1}.$$

In contrast to the case for x_1, these critical values are always well-defined. See also Figure 2.7 for the minimisers, given by the formula

$$x_2=\max\left\{\sqrt{\frac{2d(f+F)}{h_2+H\frac{d}{p}+\frac{\omega_1}{\omega_2}(h_2-h_1)}},\min\left\{\frac{2(\phi_R^{2*}-\phi_R^{1*})}{h_2-h_1},x_J^{2*}\right\}\right\}.$$

It remains to verify the final claims on these optimal values for x_1 and x_2. The fact that $x_1 > x_2$ follows from the formulas for the optimal order quantities and

$$\begin{aligned}\sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p} + \frac{\omega_1}{\omega_2}(h_2 - h_1)}} < \sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p}}} < \sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p}}} \\ < \sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p} + \frac{\omega_2}{\omega_1}(h_1 - h_2)}}.\end{aligned}$$

Thus, the formulas do indeed give feasible order quantities (as $x_1 \geq x_2$ is feasible by Corollary 2.4).

Finally, the statement that the optimal order quantities lie between the default and joint order quantities follows from the formulas and

$$x_R^{1*} > \frac{2(\phi_R^{2*} - \phi_R^{1*})}{h_2 - h_1} > x_R^{2*},$$

which has been proved in Lemma 2.18. The details are as follows. We have:

$$\begin{aligned}
x_R^{1*} > x_J^{1*} \quad &\Longrightarrow \quad \begin{cases} x_1 & \leq \max\left\{\frac{2(\phi_R^{2*}-\phi_R^{1*})}{h_2-h_1}, x_J^{1*}\right\} < x_R^{1*} \\ x_1 & \geq \min\left\{\sqrt{\frac{2d(f+F)}{h_1+H\frac{d}{p}+\frac{\omega_2}{\omega_1}(h_1-h_2)}}, x_J^{1*}\right\} = x_J^{1*} \end{cases}, \\
x_R^{1*} \leq x_J^{1*} \quad &\Longrightarrow \quad \left\{ x_1 \; = \min\left\{\sqrt{\frac{2d(f+F)}{h_1+H\frac{d}{p}+\frac{\omega_2}{\omega_1}(h_1-h_2)}}, x_J^{1*}\right\} = x_J^{1*} \right. , \\
x_R^{2*} \geq x_J^{2*} \quad &\Longrightarrow \quad \left\{ x_2 \; = \max\left\{\sqrt{\frac{2d(f+F)}{h_2+H\frac{d}{p}+\frac{\omega_1}{\omega_2}(h_2-h_1)}}, x_J^{2*}\right\} = x_J^{2*} \right. , \\
x_R^{2*} < x_J^{2*} \quad &\Longrightarrow \quad \begin{cases} x_2 & \geq \min\left\{\frac{2(\phi_R^{2*}-\phi_R^{1*})}{h_2-h_1}, x_J^{2*}\right\} > x_R^{2*} \\ x_2 & \leq \max\left\{\sqrt{\frac{2d(f+F)}{h_2+H\frac{d}{p}+\frac{\omega_1}{\omega_2}(h_2-h_1)}}, x_J^{2*}\right\} = x_J^{2*} \end{cases}.
\end{aligned}$$

Moreover, $x_k = x_R^{k*}$ if and only if it equals the joint order quantity x_J^{k*} and thus corresponds with the supplier's own optimal order quantity x_S^* (see also Lemma 2.17).

□

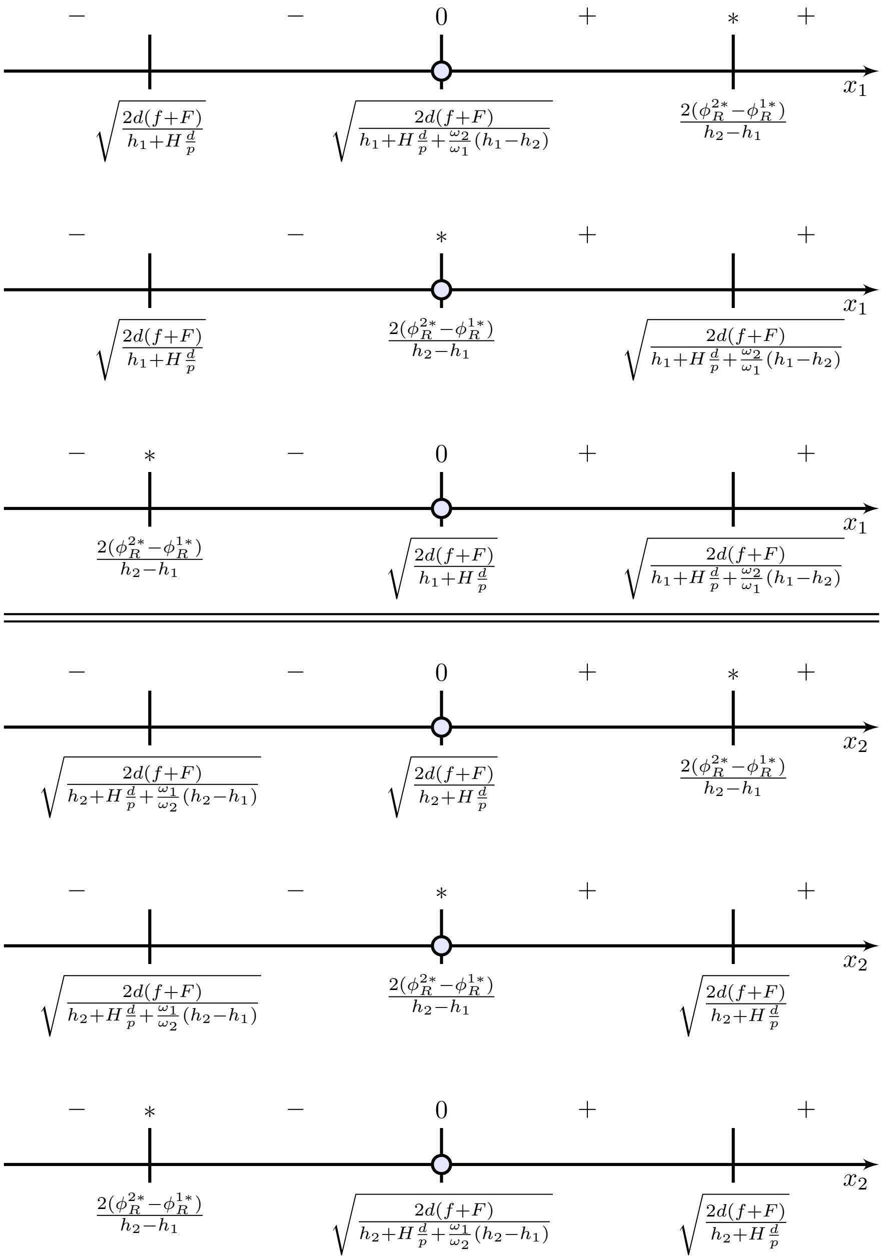


Figure 2.7: The sign of the derivative of the contribution of x_1 and x_2 to the objective value. An asterisk denotes that the point is non-differentiable. The circle indicates the minimiser.

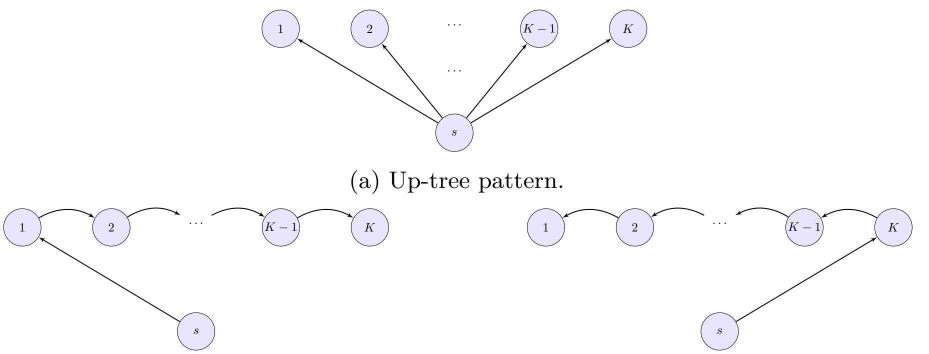
2.D Derivation of KKT menus

In this appendix, we derive the menus of contracts that follow from the KKT conditions. We only consider the cases with two or three retailer types. First, in Section 2.D.1 we give the KKT menus for certain generalisable patterns that are used for both two and three types. In Section 2.D.2 we derive all KKT menus for two types. For three types, we only show the analysis for certain cases from which all results can be reproduced, see Section 2.D.3.

2.D.1 Simple KKT menus

We can distinguish two types of KKT menus based on the corresponding KKT graph. If the KKT graph is a spanning tree, we call the corresponding menu a *simple* KKT menu. The other cases give so-called *complex* KKT menus. Recall that by Lemma 2.5 the KKT graph is either a spanning tree or a strict superset of a spanning tree.

There are three fundamental patterns for simple KKT menus: the Up-tree, Right-tree, and Left-tree, see Figure 2.8. That is, for a simple menu each connected component is one of these patterns. Note that by Lemma 2.6 we have no 'T-pattern'. In the next sections, we derive the corresponding KKT menus for each of these simple patterns.



(a) Up-tree pattern.

(b) Right-tree pattern.

(c) Left-tree pattern.

Figure 2.8: The fundamental patterns for simple KKT menus.

2.D.1.1 Up-tree pattern

For the Up-tree pattern no retailer node has outgoing arcs. Therefore, we can apply Lemma 2.11 to determine the order quantities, i.e., we have $x_k = x_J^{k*}$ for all $k \in \mathcal{K}$. The side payments follow from complementary slackness: since $\lambda_k > 0$ for all $k \in \mathcal{K}$, it must hold that $\phi_R^k(x_k) - z_k = \phi_R^{k*}$. This leads to the KKT contract for all $k \in \mathcal{K}$:

$$x_k = \sqrt{\frac{2d(f+F)}{h_k + H\frac{d}{p}}}, \qquad z_k = \phi_R^k(x_k) - \phi_R^{k*}.$$

2.D.1.2 Right-tree pattern

Based on the spanning tree in the KKT graph, we can determine formulas for the side payments. Complementary slackness with respect to $\lambda_1 > 0$ implies that $z_1 = \phi_R^1(x_1) - \phi_R^{1*}$. Likewise, $\phi_R^2(x_2) - z_2 = \phi_R^2(x_1) - z_1$ must hold, since $\mu_{1,2} > 0$. After substituting the known value for z_1, we obtain: $z_2 = \phi_R^2(x_2) + \phi_R^1(x_1) - \phi_R^2(x_1) - \phi_R^{1*}$. In general, for retailer type $k \in \mathcal{K}$ we have:

$$z_k = \phi_R^k(x_k) + \sum_{i=1}^{k-1} \left(\phi_R^i(x_i) - \phi_R^{i+1}(x_i)\right) - \phi_R^{1*}.$$

For the order quantities, first notice that adding up all KKT stationarity conditions (2.12) leads to $\sum_{k\in\mathcal{K}} \lambda_k = \sum_{k\in\mathcal{K}} \omega_k$. For the Right-tree pattern, we have $\lambda_1 = \sum_{k\in\mathcal{K}} \omega_k$. Consequently, the conditions (2.12) imply:

$$\begin{aligned}
\omega_1 - \lambda_1 + \mu_{1,2} = 0 &\quad\Longrightarrow\quad \mu_{1,2} = \lambda_1 - \omega_1 = \sum_{i=2}^{K} \omega_i, \\
\omega_k - \mu_{k-1,k} + \mu_{k,k+1} = 0 &\quad\Longrightarrow\quad \mu_{k,k+1} = \mu_{k-1,k} - \omega_k = \sum_{i=k+1}^{K} \omega_i, \\
\omega_K - \mu_{K-1,K} = 0 &\quad\Longrightarrow\quad \mu_{K-1,K} = \omega_K.
\end{aligned}$$

Thus, KKT stationarity conditions state

$$\omega_k \left(-\frac{2d(f+F)}{x_k^2} + h_k + H\tfrac{d}{p}\right) + (h_k - h_{k+1}) \sum_{i=k+1}^{K} \omega_i = 0,$$

which for $k \in \mathcal{K}$ implies the order quantity:

$$x_k = \sqrt{\frac{2d(f+F)}{h_k + H\frac{d}{p} + (h_k - h_{k+1})\frac{1}{\omega_k}\sum_{i=k+1}^{K}\omega_i}}.$$

Note that the order quantities can be complex numbers (infeasible).

2.D.1.3 Left-tree pattern

For the Left-tree pattern, the analysis is by symmetry similar to the Right-tree pattern. We have

$$x_k = \sqrt{\frac{2d(f+F)}{h_k + H\frac{d}{p} + (h_k - h_{k-1})\frac{1}{\omega_k}\sum_{i=1}^{k-1}\omega_i}} > 0,$$

$$z_k = \phi_R^k(x_k) + \sum_{i=k+1}^{K} \left(\phi_R^i(x_i) - \phi_R^{i-1}(x_i)\right) - \phi_R^{K*}.$$

Here, the order quantities are always well-defined.

2.D.2 KKT menus for two types

For two retailer types ($K = 2$) we can reduce the number of KKT menus to consider from $2^4 = 16$ to 5 cases. Table 2.6 and Figure 2.4 provide the details of these 5 cases. In the following sections we derive formulas for the KKT menus. As expected, we see the same possible optimal order quantities as derived in Theorem 2.16. Furthermore, there is a bijection between the optimal order quantity x and the KKT graph. See Section 2.4.1 for numerical examples.

Case	Menu		$\boldsymbol{\lambda_1}$	$\boldsymbol{\lambda_2}$	$\boldsymbol{\mu_{12}}$	$\boldsymbol{\mu_{21}}$
1	1Up	2Up	> 0	> 0	$= 0$	$= 0$
2	1UpRight	2x	> 0	$= 0$	> 0	$= 0$
3	1x	2UpLeft	$= 0$	> 0	$= 0$	> 0
4	1UpRight	2Up	> 0	> 0	> 0	$= 0$
5	1Up	2UpLeft	> 0	> 0	$= 0$	> 0

Table 2.6: All cases of Lagrange multipliers for two retailer types.

2.D.2.1 Case 1Up2Up

Since this is a simple KKT menu, the derivation is already given in Section 2.D.1.1. The menu of contracts is as follows:

$$x_1 = \sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p}}} > 0, \qquad x_2 = \sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p}}} > 0,$$
$$z_1 = \phi_R^1(x_1) - \phi_R^{1*} \geq 0, \qquad z_2 = \phi_R^2(x_2) - \phi_R^{2*} \geq 0.$$

2.D.2.2 Case 1UpRight2x

This menu is a Right-tree pattern, see Section 2.D.1.2. The KKT menu is given by:

$$x_1 = \sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p} + \frac{\omega_2}{\omega_1}(h_1 - h_2)}}, \qquad x_2 = \sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p}}} > 0,$$
$$z_1 = \phi_R^1(x_1) - \phi_R^{1*} \geq 0, \qquad z_2 = \phi_R^2(x_2) - \phi_R^2(x_1) + \phi_R^1(x_1) - \phi_R^{1*}.$$

Note that x_1 might be infeasible if h_2 is large. Likewise, z_2 could be negative (infeasible) for certain cost parameters.

2.D.2.3 Case 1x2UpLeft

By symmetry, this case is similar to the 1UpRight2x contract, with the roles of types 1 and 2 interchanged:

$$x_1 = \sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p}}} > 0, \qquad x_2 = \sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p} + \frac{\omega_1}{\omega_2}(h_2 - h_1)}} > 0,$$
$$z_1 = \phi_R^1(x_1) - \phi_R^1(x_2) + \phi_R^2(x_2) - \phi_R^{2*}, \qquad z_2 = \phi_R^2(x_2) - \phi_R^{2*} \geq 0.$$

Note that z_1 could be negative (infeasible) for certain cost parameters.

2.D.2.4 Case 1UpRight2Up

By complementary slackness and $\lambda_1, \lambda_2 > 0$, we can directly derive the side payments:

$$z_1 = \phi_R^1(x_1) - \phi_R^{1*} \geq 0, \qquad z_2 = \phi_R^2(x_2) - \phi_R^{2*} \geq 0.$$

Furthermore, since node 2 has no outgoing arcs, x_2 is the joint order quantity:

$$x_2 = \sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p}}} > 0.$$

Finally, by complementary slackness and $\mu_{12} > 0$, we have $\phi_R^2(x_2) - z_2 = \phi_R^2(x_1) - z_1$. Substituting the formulas for the side payments results in:

$$\phi_R^{2*} = \phi_R^2(x_1) - \phi_R^1(x_1) + \phi_R^{1*} \qquad \Longleftrightarrow \qquad \frac{1}{2}(h_2 - h_1)x_1 = \phi_R^{2*} - \phi_R^{1*}.$$

Hence, type 1 has order quantity:

$$x_1 = \frac{2\left(\phi_R^{2*} - \phi_R^{1*}\right)}{h_2 - h_1} > 0.$$

2.D.2.5 Case 1Up2UpLeft

Again by symmetry, we can reuse the analysis of the 1UpRight2Up contract to obtain:

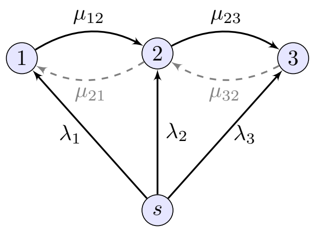
$$x_1 = \sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p}}} > 0, \qquad x_2 = \frac{2\left(\phi_R^{2*} - \phi_R^{1*}\right)}{h_2 - h_1} > 0,$$
$$z_1 = \phi_R^1(x_1) - \phi_R^{1*} \geq 0, \qquad z_2 = \phi_R^2(x_2) - \phi_R^{2*} \geq 0.$$

2.D.3 KKT menus for three types

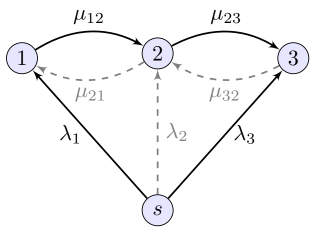
In this section we derive the menus for three retailer types ($K = 3$), see also Section 2.4.2. Table 2.7 provides an overview of all possible cases. The cases indicated as reducible can be solved by reusing KKT contracts for $K = 2$ and by using Lemma 2.11. That is, one retailer type gets offered the joint order quantity according to Lemma 2.11. For the other two types we can use the contracts for $K = 2$ derived earlier. This leaves 14 new cases to solve, but due to symmetry in the cases only 8 important cases remain. These are shown in Figure 2.9 and solved below. Compare this to the $2^7 = 128$ cases that would need to be analysed without using our results.

Case	Menu			Reducible	λ_1	λ_2	λ_3	μ_{12}	μ_{21}	μ_{23}	μ_{32}
1	1Up	2Up	3Up	Yes	>0	>0	>0	$=0$	$=0$	$=0$	$=0$
2	1UpRight	2Up	3Up	Yes	>0	>0	>0	>0	$=0$	$=0$	$=0$
3	1Up	2UpLeft	3Up	Yes	>0	>0	>0	$=0$	>0	$=0$	$=0$
4	1Up	2UpRight	3Up	Yes	>0	>0	>0	$=0$	$=0$	>0	$=0$
5	1Up	2Up	3UpLeft	Yes	>0	>0	>0	$=0$	$=0$	$=0$	>0
6	1UpRight	2UpRight	3Up	No	>0	>0	>0	>0	$=0$	>0	$=0$
7	1UpRight	2Up	3UpLeft	No	>0	>0	>0	>0	$=0$	$=0$	>0
8	1Up	2UpLeft	3UpLeft	No	>0	>0	>0	$=0$	>0	$=0$	>0
9	1Up	2UpRight	3x	Yes	>0	>0	$=0$	$=0$	$=0$	>0	$=0$
10	1UpRight	2UpRight	3x	No	>0	>0	$=0$	>0	$=0$	>0	$=0$
11	1UpRight	2x	3Up	Yes	>0	$=0$	>0	>0	$=0$	$=0$	$=0$
12	1Up	2x	3UpLeft	Yes	>0	$=0$	>0	$=0$	$=0$	$=0$	>0
13	1UpRight	2Right	3Up	No	>0	$=0$	>0	>0	$=0$	>0	$=0$
14	1Up	2Left	3UpLeft	No	>0	$=0$	>0	$=0$	>0	$=0$	>0
15	1UpRight	2x	3UpLeft	No	>0	$=0$	>0	>0	$=0$	$=0$	>0
16	1UpRight	2LeftRight	3Up	No	>0	$=0$	>0	>0	>0	>0	$=0$
17	1Up	2LeftRight	3UpLeft	No	>0	$=0$	>0	$=0$	>0	>0	>0
18	1x	2UpLeft	3Up	Yes	$=0$	>0	>0	$=0$	>0	$=0$	$=0$
19	1x	2UpLeft	3UpLeft	No	$=0$	>0	>0	$=0$	>0	$=0$	>0
20	1UpRight	2Right	3x	No	>0	$=0$	$=0$	>0	$=0$	>0	$=0$
21	1UpRight	2LeftRight	3x	No	>0	$=0$	$=0$	>0	>0	>0	$=0$
22	1x	2Left	3UpLeft	No	$=0$	$=0$	>0	$=0$	>0	$=0$	>0
23	1x	2LeftRight	3UpLeft	No	$=0$	$=0$	>0	$=0$	>0	>0	>0

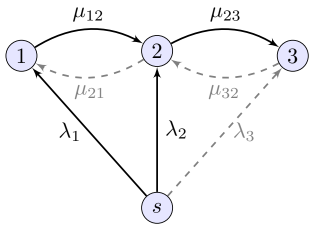
Table 2.7: All cases of Lagrange multipliers for three retailer types.



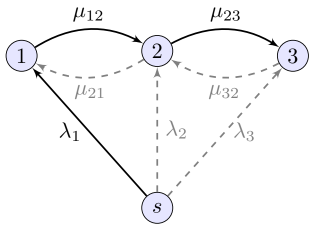
(a) 1UpRight2UpRight3Up.



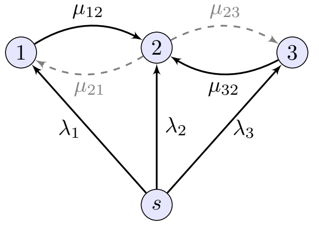
(b) 1UpRight2Right3Up.



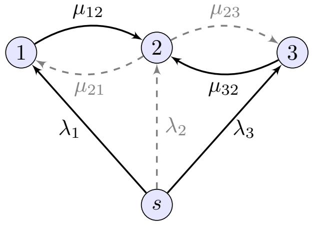
(c) 1UpRight2UpRight3x.



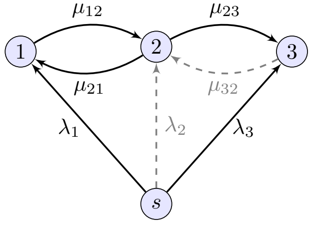
(d) 1UpRight2Right3x.



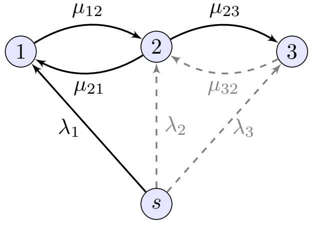
(e) 1UpRight2Up3UpLeft.



(f) 1UpRight2x3UpLeft.



(g) 1UpRight2LeftRight3Up.



(h) 1UpRight2LeftRight3x.

Figure 2.9: Relevant KKT graphs for three retailer types. Not showing reducible or symmetric cases.

2.D.3.1 Case 1UpRight2UpRight3Up

As usual, the side payments follow directly from the complementary slackness conditions:

$$z_1 = \phi_R^1(x_1) - \phi_R^{1*}, \qquad z_2 = \phi_R^2(x_2) - \phi_R^{2*}, \qquad z_3 = \phi_R^3(x_3) - \phi_R^{3*}.$$

Since node 3 has no outgoing arcs, we can apply Lemma 2.11 to obtain the order quantity:

$$x_3 = \sqrt{\frac{2d(f+F)}{h_3 + H\frac{d}{p}}}.$$

Finally, we use the complementary slackness conditions of the remaining active arcs. First, consider the following:

$$\phi_R^{2*} = \phi_R^2(x_2) - z_2 = \phi_R^2(x_1) - z_1 = \phi_R^2(x_1) - \phi_R^1(x_1) + \phi_R^{1*} = \frac{1}{2}(h_2 - h_1)x_1 + \phi_R^{1*}.$$

Likewise,

$$\phi_R^{3*} = \phi_R^3(x_3) - z_3 = \phi_R^3(x_2) - z_2 = \phi_R^3(x_2) - \phi_R^2(x_2) + \phi_R^{2*} = \frac{1}{2}(h_3 - h_2)x_2 + \phi_R^{2*}.$$

Solving both equalities leads to:

$$x_1 = \frac{2(\phi_R^{2*} - \phi_R^{1*})}{(h_2 - h_1)}, \qquad x_2 = \frac{2(\phi_R^{3*} - \phi_R^{2*})}{(h_3 - h_2)}.$$

2.D.3.2 Case 1UpRight2Right3Up

This case is more difficult to solve. The side payments are straightforward to determine:

$$z_1 = \phi_R^1(x_1) - \phi_R^{1*}, \quad z_2 = \phi_R^2(x_2) - \phi_R^2(x_1) + \phi_R^1(x_1) - \phi_R^{1*}, \quad z_3 = \phi_R^3(x_3) - \phi_R^{3*}.$$

Lemma 2.11 specifies the order quantity for retailer type 3:

$$x_3 = \sqrt{\frac{2d(f+F)}{h_3 + H\frac{d}{p}}}.$$

Furthermore, we have

$$\begin{aligned}\phi_R^{3*} &= \phi_R^3(x_3) - z_3 = \phi_R^3(x_2) - z_2 = \phi_R^3(x_2) - \phi_R^2(x_2) + \phi_R^2(x_1) - \phi_R^1(x_1) + \phi_R^{1*}\\ &= \frac{1}{2}(h_3 - h_2)x_2 + \frac{1}{2}(h_2 - h_1)x_1 + \phi_R^{1*},\end{aligned}$$

that is,

$$(h_2 - h_1)x_1 + (h_3 - h_2)x_2 = 2(\phi_R^{3*} - \phi_R^{1*}).$$

We also need to rewrite two KKT stationarity conditions, resulting in:

$$\begin{aligned}\frac{\omega_1}{(h_1 - h_2)}\left(-2d(f+F)\frac{1}{x_1^2} + h_1 + H\frac{d}{p}\right) + \mu_{12} = 0,\\ \frac{\omega_2}{(h_2 - h_3)}\left(-2d(f+F)\frac{1}{x_2^2} + h_2 + H\frac{d}{p}\right) + \mu_{23} = 0.\end{aligned}$$

Combining these two equations and using $\omega_2 - \mu_{12} + \mu_{23} = 0$ leads to:

$$\begin{aligned}&\frac{\omega_1}{(h_1 - h_2)}\left(-2d(f+F)\frac{1}{x_1^2} + h_1 + H\frac{d}{p}\right)\\ &\quad - \frac{\omega_2}{(h_2 - h_3)}\left(-2d(f+F)\frac{1}{x_2^2} + h_2 + H\frac{d}{p}\right) + \omega_2 = 0,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\frac{-2d(f+F)\omega_1}{(h_1 - h_2)}\frac{1}{x_1^2} - \frac{-2d(f+F)\omega_2}{(h_2 - h_3)}\frac{1}{x_2^2}\\ &\quad = \frac{\omega_2}{(h_2 - h_3)}\left(h_2 + H\frac{d}{p}\right) - \frac{\omega_1}{(h_1 - h_2)}\left(h_1 + H\frac{d}{p}\right) - \omega_2.\end{aligned}$$

To conclude, we have to solve a pair of equations for which we can use Lemma 2.19 in Section 2.D.3.9. As proved in the lemma, there exists a unique strictly positive and real solution. Both exact closed-form formulas and an efficient numerical solution method exist to solve these equations. Unfortunately, the formulas for x_1 and x_2 are too verbose to state here and are omitted.

2.D.3.3 Case 1UpRight2UpRight3x

The side payments are:

$$z_1 = \phi_R^1(x_1) - \phi_R^{1*}, \quad z_2 = \phi_R^2(x_2) - \phi_R^{2*}, \quad z_3 = \phi_R^3(x_3) - \phi_R^3(x_2) + \phi_R^2(x_2) - \phi_R^{2*}.$$

As seen before, the order quantity of type 3 follows from Lemma 2.11:

$$x_3 = \sqrt{\frac{2d(f+F)}{h_3 + H\frac{d}{p}}}.$$

Complementary slackness states that

$$\phi_R^{2*} = \phi_R^2(x_2) - z_2 = \phi_R^2(x_1) - z_1 = \phi_R^2(x_1) - \phi_R^1(x_1) + \phi_R^{1*},$$

that is,

$$x_1 = \frac{2(\phi_R^{2*} - \phi_R^{1*})}{h_2 - h_1}.$$

Finally, from the KKT stationarity conditions we have $\mu_{23} = \omega_3$ and

$$\omega_2\left(-2d(f+F)\frac{1}{x_2^2} + h_2 + H\frac{d}{p}\right) + \omega_3(h_2 - h_3) = 0,$$

leading to

$$x_2 = \sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p} + \frac{\omega_3}{\omega_2}(h_2 - h_3)}}.$$

2.D.3.4 Case 1UpRight2Right3x

This is a simple KKT menu, namely the Right-tree pattern (see Section 2.D.1.2). The menu of contracts is given by the order quantities

$$x_1 = \sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p} + \frac{\omega_2+\omega_3}{\omega_1}(h_1 - h_2)}}, \qquad x_2 = \sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p} + \frac{\omega_3}{\omega_2}(h_2 - h_3)}},$$
$$x_3 = \sqrt{\frac{2d(f+F)}{h_3 + H\frac{d}{p}}},$$

and side payments

$$\begin{aligned} z_1 &= \phi_R^1(x_1) - \phi_R^{1*}, \\ z_2 &= \phi_R^2(x_2) - \phi_R^2(x_1) + \phi_R^1(x_1) - \phi_R^{1*}, \\ z_3 &= \phi_R^3(x_3) - \phi_R^3(x_2) + \phi_R^2(x_2) - \phi_R^2(x_1) + \phi_R^1(x_1) - \phi_R^{1*}. \end{aligned}$$

2.D.3.5 Case 1UpRight2Up3UpLeft

We apply the general solution technique to find:

$$z_1 = \phi_R^1(x_1) - \phi_R^{1*}, \qquad z_2 = \phi_R^2(x_2) - \phi_R^{2*}, \qquad z_3 = \phi_R^3(x_3) - \phi_R^{3*},$$

and

$$x_2 = \sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p}}}.$$

Now use complementary slackness:

$$\phi_R^{2*} = \phi_R^2(x_2) - z_2 = \phi_R^2(x_1) - z_1 = \phi_R^2(x_1) - \phi_R^1(x_1) + \phi_R^{1*}.$$

We obtain a similar equation for x_3. Hence, we have

$$x_1 = \frac{2(\phi_R^{2*} - \phi_R^{1*})}{h_2 - h_1}, \qquad x_3 = \frac{2(\phi_R^{3*} - \phi_R^{2*})}{h_3 - h_2}.$$

2.D.3.6 Case 1UpRight2x3UpLeft

This case is one of the more difficult cases. The side payments are:

$$z_1 = \phi_R^1(x_1) - \phi_R^{1*}, \quad z_2 = \phi_R^2(x_2) - \phi_R^2(x_1) + \phi_R^1(x_1) - \phi_R^{1*}, \quad z_3 = \phi_R^3(x_3) - \phi_R^{3*}.$$

The order quantity for type 2 is straightforward:

$$x_2 = \sqrt{\frac{2d(f+F)}{h_2 + H\frac{d}{p}}}.$$

We use complementary slackness to find the following equation:

$$
\begin{aligned}
&& \phi_R^2(x_1) - z_1 = \phi_R^2(x_2) - z_2 = \phi_R^2(x_3) - z_3 \\
\iff && \phi_R^2(x_1) - \phi_R^1(x_1) + \phi_R^{1*} = \phi_R^2(x_3) - \phi_R^3(x_3) + \phi_R^{3*} \\
\iff && (h_2 - h_1)x_1 + (h_3 - h_2)x_3 = 2(\phi_R^{3*} - \phi_R^{1*}).
\end{aligned}
$$

The following derivation has been used before. We rewrite two KKT stationarity conditions:

$$
\begin{aligned}
\frac{\omega_1}{(h_1 - h_2)} \left(-2d(f+F)\frac{1}{x_1^2} + h_1 + H\frac{d}{p} \right) + \mu_{12} = 0, \\
\frac{\omega_3}{(h_3 - h_2)} \left(-2d(f+F)\frac{1}{x_3^2} + h_3 + H\frac{d}{p} \right) + \mu_{32} = 0.
\end{aligned}
$$

Next, combine both equations and use $\omega_2 - \mu_{12} - \mu_{32} = 0$:

$$
\begin{aligned}
&\frac{\omega_1}{(h_1 - h_2)} \left(-2d(f+F)\frac{1}{x_1^2} + h_1 + H\frac{d}{p} \right) \\
&\quad + \frac{\omega_3}{(h_3 - h_2)} \left(-2d(f+F)\frac{1}{x_3^2} + h_3 + H\frac{d}{p} \right) + \omega_2 = 0,
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&\frac{-2d(f+F)\omega_1}{(h_1 - h_2)}\frac{1}{x_1^2} - \frac{-2d(f+F)\omega_3}{(h_2 - h_3)}\frac{1}{x_3^2} \\
&= \frac{\omega_3}{(h_2 - h_3)} \left(h_3 + H\frac{d}{p} \right) - \frac{\omega_1}{(h_1 - h_2)} \left(h_1 + H\frac{d}{p} \right) - \omega_2.
\end{aligned}
$$

Thus, we solve the pair of equations using Lemma 2.19. As stated in the lemma, a unique strictly positive and real solution exists. The formulas for x_1 and x_3 are too verbose and are omitted.

2.D.3.7 Case 1UpRight2LeftRight3Up

From Corollary 2.9 we know that $x_1 = x_2$ and $z_1 = z_2$. Hence, we have side payments

$$
z_1 = z_2 = \phi_R^1(x_1) - \phi_R^{1*}, \qquad z_3 = \phi_R^3(x_3) - \phi_R^{3*}.
$$

As before, the order quantity of type 3 is:

$$
x_3 = \sqrt{\frac{2d(f+F)}{h_3 + H\frac{d}{p}}}.
$$

Using complementary slackness results in:

$$
\phi_R^{3*} = \phi_R^3(x_3) - z_3 = \phi_R^3(x_2) - z_2 = \phi_R^3(x_1) - z_1 = \phi_R^3(x_1) - \phi_R^1(x_1) + \phi_R^{1*}.
$$

Solving for x_1 gives:

$$
x_1 = x_2 = \frac{2(\phi_R^{3*} - \phi_R^{1*})}{h_3 - h_1}.
$$

2.D.3.8 Case 1UpRight2LeftRight3x

The 2-cycle implies that $x_1 = x_2$ and $z_1 = z_2$. First, we give the side payments:

$$z_1 = z_2 = \phi_R^1(x_1) - \phi_R^{1*}, \qquad z_3 = \phi_R^3(x_3) - \phi_R^3(x_2) + \phi_R^1(x_1) - \phi_R^{1*},$$

where by Lemma 2.11

$$x_3 = \sqrt{\frac{2d(f+F)}{h_3 + H\frac{d}{p}}}.$$

The KKT stationarity conditions state that $\mu_{23} = \omega_3$, hence $\mu_{12} - \mu_{21} = \omega_2 + \omega_3$. Adding up the KKT conditions

$$\omega_1 \left(-2d(f+F)\frac{1}{x_1^2} + h_1 + H\frac{d}{p} \right) + \mu_{12}(h_1 - h_2) = 0,$$
$$\omega_2 \left(-2d(f+F)\frac{1}{x_1^2} + h_2 + H\frac{d}{p} \right) + \mu_{21}(h_2 - h_1) + \omega_3(h_2 - h_3) = 0,$$

leads to:

$$(\omega_1 + \omega_2) \left(-2d(f+F)\frac{1}{x_1^2} + H\frac{d}{p} \right) + \omega_1 h_1 + \omega_2 h_2$$
$$+ (\omega_2 + \omega_3)(h_1 - h_2) + \omega_3(h_2 - h_3) = 0.$$

To conclude, the order quantity for types 1 and 2 is equal to:

$$x_1 = x_2 = \sqrt{\frac{2d(f+F)}{h_1 + H\frac{d}{p} + \frac{\omega_3}{\omega_1+\omega_2}(h_1 - h_3)}}.$$

2.D.3.9 Special system of equations

In this section we discuss a special system of equations that needs to be solved for certain KKT contracts. See Lemma 2.19 for the details.

Lemma 2.19. *Consider the pair of equations of the following form:*

$$\alpha_1 x_1 + \alpha_2 x_2 = \gamma_1, \qquad \beta_1 \frac{1}{x_1^2} - \beta_2 \frac{1}{x_2^2} = \gamma_2,$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1 \in \mathbb{R}_{>0}$ and $\gamma_2 \in \mathbb{R}$ are given parameters. These equations always have a unique strictly positive real solution, i.e., satisfying $x_1, x_2 \in \mathbb{R}_{>0}$.

Proof. First, suppose $\gamma_2 = 0$. We have

$$\beta_1 \frac{1}{x_1^2} = \beta_2 \frac{1}{x_2^2} \qquad \Longleftrightarrow \qquad x_1 = \sqrt{\frac{\beta_1}{\beta_2}} x_2.$$

Thus, the other equation implies that

$$\alpha_1 \sqrt{\frac{\beta_1}{\beta_2}} x_2 + \alpha_2 x_2 = \gamma_1 \qquad \Longleftrightarrow \qquad x_2 = \frac{\gamma_1}{\alpha_1 \sqrt{\frac{\beta_1}{\beta_2}} + \alpha_2} > 0.$$

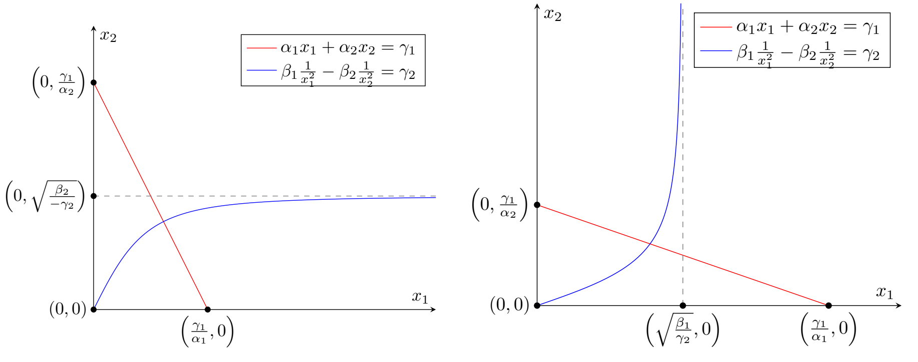
This proves the claim for $\gamma_2 = 0$.

Next, consider the case that $\gamma_2 < 0$. Notice that we can solve this case by finding the roots of a forth degree polynomial, for which an exact closed-form formula exists. This polynomial follows from substitution of one equation in the other. What remains is to show that exactly one of these four roots is strictly positive and real. To do so, solve the non-linear equation for x_2:

$$x_2 = \sqrt{\frac{\beta_2}{\beta_1 \frac{1}{x_1^2} - \gamma_2}} = \frac{\sqrt{\beta_2} x_1}{\sqrt{\beta_1 - \gamma_2 x_1^2}}.$$

This is a well-defined strictly positive solution for $x_1 > 0$. Furthermore, we have $x_2 = \gamma_1/\alpha_2 - (\alpha_1/\alpha_2)x_1$. Figure 2.10a shows the corresponding curves in the positive quadrant. Since the limits for $x_1 \to 0$ and $x_1 \to \infty$ are well-defined, this proves that a unique strictly positive real solution exists.

The case that $\gamma_2 > 0$ is similar, see Figure 2.10b, and will not be shown. □



(a) Curve if $\gamma_2 < 0$.

(b) Curve if $\gamma_2 > 0$.

Figure 2.10: Solution curve in the positive quadrant.

For completeness sake, we show how to numerically find the solution efficiently. Again, we assume $\gamma_2 < 0$ ($\gamma_2 > 0$ is similar). Consider the function θ for $x \in \mathbb{R}_{\geq 0}$:

$$\theta(x) = \frac{\sqrt{\beta_2} x}{\sqrt{\beta_1 - \gamma_2 x^2}} + \frac{\alpha_1}{\alpha_2} x - \frac{\gamma_1}{\alpha_2} \in \left[-\frac{\gamma_1}{\alpha_2}, \infty\right),$$
$$\frac{\mathrm{d}\theta}{\mathrm{d}x}(x) = \frac{\beta_1 \sqrt{\beta_2}}{(\beta_1 - \gamma_2 x^2)^{3/2}} + \frac{\alpha_1}{\alpha_2} \in \left(\frac{\alpha_1}{\alpha_2}, \sqrt{\frac{\beta_2}{\beta_1}} + \frac{\alpha_1}{\alpha_2}\right].$$

Solving $\theta(x) = 0$ is equivalent to finding the value x_1. We only need to search in the bounded domain $x \in (0, \gamma_1/\alpha_1)$, since

$$\alpha_1 x_1 = \gamma_1 - \alpha_2 x_2 < \gamma_1 \qquad \Longrightarrow \qquad x_1 < \gamma_1/\alpha_1,$$

and $x_1, x_2 > 0$. As $\alpha_1/\alpha_2 > 0$, the derivative of θ is never zero. For example, if $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, then the derivative lies between 1 and 2. This suggests that methods such as Newton-Raphson should work very well and numerical results confirm fast and accurate convergence in typically less than 10 iterations.

Chapter 3

Robust pooling for contracting models with asymmetric information

Abstract

In this chapter, we consider principal-agent contracting models between a seller and a buyer with single-dimensional private information. The buyer's type follows a continuous distribution on a bounded interval. We present a new modelling approach where the seller offers a menu of finitely many contracts to the buyer. The approach distinguishes itself from existing methods by pooling the buyer types using a partition. That is, the seller first chooses the number of contracts offered and then partitions the set of buyer types into subintervals. All types in a subinterval are pooled and offered the same contract by the design of our menu.

We call this approach robust pooling and apply it to utility maximisation and cost minimisation problems. In particular, we analyse two problems adapted from the literature. For both problems we are able to express structural results as a function of a single new parameter, which remarkably does not depend on all instance parameters. We determine the optimal partition and the corresponding optimal menu of contracts. This results in new insights into the (sub)optimality of the equidistant partition. For example, the equidistant partition is optimal for a family of instances for one of the problems. Finally, we derive performance guarantees for the equidistant and optimal partitions for a given number of contracts. For the considered problems the robust pooling approach has good performances when using only a few contracts.

This chapter is based on Kerkkamp et al. (2017).

Contents

3.1 Introduction

In principal-agent contracting problems, a principal wants to persuade an agent to perform a certain action and uses financial incentives to do so. Both parties are individually rational and only want to improve their own situation. We consider contracting problems where the principal is a seller of a certain product and the agent is a potential buyer. Thus, the seller desires either to initiate new trade with the buyer or to change the existing buyer's order quantity. In order to do so, the seller offers a contract to the buyer, describing the order quantity (the action) and a side payment (the incentive). The contract design must balance the value of the contract for both parties, since the buyer can refuse a disadvantageous contract.

The complexity of the contracting problem increases significantly when the buyer has private information on his valuation of contracts, i.e., when there is information asymmetry. In terms of mechanism design, the buyer's private information is represented by so-called *types*. That is, the buyer's identity is an element of a known set of types $\mathcal{P}$ and specified by a probability distribution on $\mathcal{P}$. The distribution of types is assumed to be common knowledge, in particular also to the seller. We consider the case where the buyer has single-dimensional private information, represented by the type $p \in \mathcal{P}$.

In case of information asymmetry, the seller offers a menu of contracts, typically one contract for each of the possible buyer types. First, the optimal menu is determined by solving a certain optimisation problem, which we will discuss in later sections. Second, this menu is offered to the buyer. Finally, the buyer either chooses to accept a contract of the menu or refuses the offer, depending on what is most beneficial for the buyer. Note that the buyer can lie about his true type and choose any contract, which complicates the seller's optimisation process.

The modelling of the buyer types $\mathcal{P}$ is crucial for the contracting problem. In the mechanism design literature there are two typical choices. First, we have the classical *discrete model*: a finite discrete set $\mathcal{P} = \{p_1, \ldots, p_K\} \subseteq \mathbb{R}$ for some $K \in \mathbb{N}_{\geq 1}$ (discrete distribution). Here, the menu consists of K contracts, one for each type. Hence, the buyer chooses from a finite number of contracts. Second, we have the classical *continuous model*: a bounded interval $\mathcal{P} = [\underline{p}, \bar{p}] \subseteq \mathbb{R}$ with $\bar{p} > \underline{p}$ (continuous distribution). Here, the menu is a function that maps every type to a contract. In other words, infinitely many contracts are offered to the buyer.

Our goal is to design and analyse a model that combines aspects of both the discrete and continuous models. For this model, the buyer's type is continuously distributed on $\mathcal{P} = [\underline{p}, \bar{p}] \subseteq \mathbb{R}$ with $\bar{p} > \underline{p}$, but only finitely many contracts are offered. The main motivation for this approach is that offering finitely many contracts is often preferred in practice, as such menus are easier to communicate and implement. The discrete and continuous approaches are not suitable for achieving this goal, which we will later discuss in more detail. This combination of the discrete and continuous approaches has received limited attention in the literature, which we will review in the next section.

We present a modelling approach which we call *robust pooling* in order to achieve the stated goal. For the robust pooling model, the buyer's type lies in a bounded interval $[\underline{p}, \bar{p}]$, but only finitely many contracts are offered. First, the seller chooses the number of contracts $K \in \mathbb{N}_{\geq 1}$ that will be offered. Second, he partitions the interval $[\underline{p}, \bar{p}]$ into K subintervals denoted by $[\underline{p}_k, \bar{p}_k]$ for $k \in \{1, \ldots, K\}$. Third, he designs a menu of K contracts with a single contract intended for each subinterval $[\underline{p}_k, \bar{p}_k]$. Finally, he offers the menu to the buyer, as usual.

Our modelling approach has two fundamental properties: pooling of types and robustness. First, the (discrete) *pooling* property refers to offering finitely many contracts, and thus offering the same contract to multiple types, *by design*. Second, the (continuous) *robustness* property means that each type $p \in \mathcal{P}$ accepts a contract from the menu and that this choice is correctly reflected in the model (for example in the objective function). In other words, the menu specifies an intended contract for each type and each type chooses its intended contract. Consequently, the buyer always accepts a contract from the menu, making the menu robust to the buyer's private information. In our case, for each $k \in \{1, \ldots, K\}$ it is for all types in $[\underline{p}_k, \bar{p}_k]$ most beneficial to choose the k-th contract.

In our approach, the seller must decide on a partition scheme, i.e., the number of contracts and an appropriately corresponding partition of $[\underline{p}, \bar{p}]$. The robust pooling model enables us to determine the effect of different partition schemes, since our model handles an arbitrary number of contracts and any partition in a natural way. Due to the robustness property, we can evaluate the use of different schemes in a fair way by directly comparing the resulting objective values of the model.

Such a fair comparison is not possible with the classical discrete approach, since varying the number of contracts also implies changing the distribution of the buyer's type, effectively changing which scenarios could happen. Moreover, if the discrete distribution is actually an approximation of a continuous distribution, then the discrete approach is generally not robust. The classical continuous approach does not pool types by design and is therefore also unsuitable.

As already hinted, there are several aspects of the robust pooling model to analyse. First, what is the complexity of the model? In particular, can we identify conditions under which the model can be solved efficiently? Second, can we quantify the performance of partition schemes? A natural choice for a partition is the equiquantile partition, where $[\underline{p}, \bar{p}]$ is partitioned into subintervals of equal probability. However, is the equiquantile partition the best possible partition and if not, how much performance is lost? Also, offering infinitely many contracts (the continuous approach) results in the best possible objective value and is partition independent. When using our approach, how many contracts should be offered to guarantee, say, 95% of this best possible value?

We continue with a literature review of related modelling techniques and contracting problems.

3.1.1 Connection to the literature

For a general reference for the classical discrete and continuous modelling approaches, see for example Laffont and Martimort (2002). To our knowledge, a combination of the discrete and continuous approaches, such as our robust pooling model, has received limited attention in the literature. Bergemann et al. (2011) consider a linear-quadratic model based on Mussa and Rosen (1978), but with limited communication between the seller and the buyer. The limited communication implies that only a menu with a limited number of contracts can be offered. Their approach is effectively a restricted form of the classical continuous approach, where the menu is restricted to have finitely many contracts. The resulting model satisfies our desired pooling and robustness properties. They are able to reformulate the problem into a mean square minimisation problem and apply quantisation theory (Lloyd-Max conditions) to determine the optimal menu of contracts and the optimal partition scheme. In particular, they show that compared to offering infinitely many contracts the loss in performance is of the order $\Theta(1/K^2)$ when using K optimal contracts.

The same modelling approach is used in Wong (2014), who analyses a more general version of the non-linear pricing problem in Bergemann et al. (2011). He determines general results on the loss of performance when offering K optimal contracts. Among other results, he proves that the loss in performance is of the order $\mathcal{O}(1/K^2)$ under more general assumptions than Bergemann et al. (2011).

We shall refer to the modelling approach used in Bergemann et al. (2011) and Wong (2014) as the *limited variety* model. In general, our robust pooling model is more restrictive than the limited variety model, since we partition (pool) types into subintervals a priori. We note that under the considered assumptions in Bergemann et al. (2011) and Wong (2014), the limited variety model effectively also partitions the types. We show in Appendix 3.A.3 that under our considered assumptions, both modelling approaches are equivalent *provided that the optimal partition scheme is used*. Nevertheless, we use our robust pooling approach for the following reasons.

First, the robust pooling model has an added benefit regarding information extraction. The seller can extract private information from the buyer by observing the buyer's chosen contract. Recall that by design the k-th contract is chosen by all types in $[\underline{p}_k, \bar{p}_k]$. Thus, after observing the buyer's choice, the seller can narrow down the buyer's type to one of the subintervals of the partition. Since the used partition is a decision made by the seller, he is able to control the accuracy of said identification in a natural and intuitive way. In general, the limited variety model cannot guarantee such structured information extraction.

Second, an implicit goal of offering a limited number of contracts is to have a simple mechanism. Partitioning $[\underline{p}, \bar{p}]$ using a certain heuristic (e.g., equidistantly or according to some 'square-root' rule) is simple and intuitive, and could have a decent performance. That is, the formulation promotes the experimentation with partition schemes. Moreover, it could be that the additional loss in performance by restricting to a partition scheme a priori is negligible.

The robust pooling model is also related to robust optimisation (see Ben-Tal et al. (2009)). That is, our model can be interpreted as a robust optimisation variant for the discrete model, where each subinterval $[\underline{p}_k, \bar{p}_k]$ is the so-called uncertainty set of type p_k. This will be further discussed when we have formalised the model.

In the recent years, there has been an increase in the application of robust optimisation to mechanism design models in the literature. For examples, see Aghassi and Bertsimas (2006), Bandi and Bertsimas (2014), Bergemann and Morris (2005), and Pinar and Kizilkale (2017). The main focus lies on making contracting models robust to the distribution of the buyer's type, i.e., it only depends on which types can occur and not on any probabilities. To our knowledge, robust optimisation has not been applied to obtain a model similar to our robust pooling model.

In the robust pooling approach, the seller must decide on a partition scheme, i.e., the number of contracts and an appropriately corresponding partition of $[\underline{p}, \bar{p}]$. Due to the robustness property, we can compare the performance of different partition schemes in a straightforward and fair way. For performance guarantees of the *optimal* partition scheme for utility maximisation problems we refer to the results of Wong (2014), provided the required assumptions are met. However, the actually attained performance of any partition scheme is difficult, if not impossible, to determine analytically in general. Therefore, we focus on two specific problems for which we are able to derive the performance of any partition scheme for any problem instance.

The first problem is based on a decreasing marginal utility setting for the buyer as his order quantity increases. It is the robust pooling variant and a generalisation of the linear-quadratic-uniform model considered in Wong (2014). The second problem uses the classical economic order quantity setting. It is the robust pooling variant of, for example, Corbett and de Groote (2000), Pishchulov and Richter (2016), and Voigt and Inderfurth (2011). Both problems will be formalised in later sections.

3.1.2 Contribution

Our contributions are as follows. We present a new modelling approach for contracting problems called robust pooling. Our approach distinguishes itself from the classical discrete and continuous models by having a continuous distribution for the buyer's type and offering a menu with finitely many contracts. Its two fundamental properties are pooling of types by design and robustness. Compared to the limited variety model, we use a partition to pool types a priori. Consequently, our modelling approach promotes the experimentation of heuristic partition schemes leading to simple mechanisms. We restrict the analysis to single-dimensional types, but the robust pooling principle can be applied to more general settings. We show that under certain assumptions robust pooling models have a simplified reformulation and can be solved efficiently.

We apply robust pooling to the Decreasing Marginal Utility (DMU) problem and the Economic Order Quantity (EOQ) problem. Both problems assume a uniform distribution of the buyer's type, which although restrictive allows for closed-form formulas. Consequently, the equiquantile partition is equidistant, which we use as a simple and intuitive benchmark partition.

For both problems, our contributions to the literature include the following. First, we derive closed-form formulas for the optimal menu and corresponding optimal objective value for any number of contracts and any partition. Second, we show that structural results and performance measures can be expressed by functions of a single new parameter based on the instance parameters. Remarkably, this parameter does not depend on all instance parameters, implying families of instances with the same structure. Third, we determine the optimal partition scheme, either analytically or numerically, depending on the problem. In particular, this leads to new insights into the (sub)optimality of the equidistant/equiquantile partition. Finally, we give performance guarantees for the equidistant and optimal partitions.

As a special case, we extend and complete the analysis of the linear-quadratic-uniform model in Wong (2014), which reveals that the equidistant partition is optimal for certain instances. Furthermore, to our knowledge, our results for the EOQ problem are new to the literature.

The remainder of this chapter is organised as follows. In Section 3.2 we consider robust pooling models in the context of utility maximisation and apply the concept to the mentioned DMU problem in Sections 3.2.3-3.2.5. In Section 3.3 we perform a similar analysis to cost minimisation models and apply it to the mentioned EOQ problem in Section 3.3.2. Finally, we conclude our findings in Section 3.4.

3.2 Contracting for maximising utility

In this section we consider principal-agent contracting models in the setting of utility maximisation. We first formalise the robust pooling model in Section 3.2.1, which we reformulate and analyse under certain assumptions in Section 3.2.2. Finally, in Sections 3.2.3-3.2.5 we consider the DMU problem and analyse the performance of partition schemes in detail. All proofs are given in Appendix 3.A.

3.2.1 The model

The principal is a seller of products and wants to initiate trade with the agent, referred to as the buyer. The seller desires to enter a contractual agreement with the buyer to provide the goods. However, the buyer does not share all his information with the seller, complicating the design of a contract. Therefore, the seller uses mechanism design to construct a menu of contracts such that the buyer can be persuaded to order at the seller.

A contract is given by an order quantity $x \in \mathbb{R}_{\geq 0}$ and a side payment $z \in \mathbb{R}$ from the buyer to the seller. That is, the contract effectively specifies how many units of product the buyer receives and for which price. The buyer can refuse any contract, but we assume he acts individually rationally and accepts an offered contract if this is most beneficial to himself.

The buyer has private information, which we assume can be represented by a single parameter $p \in \mathbb{R}_{\geq 0}$. Let $\phi_B(x|p)$ be the utility of order quantity x for the buyer with private parameter p. Likewise, $\phi_S(x)$ is the seller's utility for order quantity x. By default there is no contract (no trade) between the seller and the buyer, resulting in a default utility of zero for the buyer. Therefore, a contract (x, z) is accepted by the buyer if its net utility is non-negative: $\phi_B(x|p) - z \geq 0$. This is called the Individual Rationality (IR) constraint. The difficulty in designing a suitable contract is that the private utility parameter p is not shared with the seller. We assume that the parameter p follows a continuous distribution with strictly positive density function $\omega : [\underline{p}, \bar{p}] \to \mathbb{R}_{>0}$ on the interval $[\underline{p}, \bar{p}] \subseteq \mathbb{R}_{\geq 0}$ with $\bar{p} > \underline{p}$. This distribution is known to the seller. Each $p \in [\underline{p}, \bar{p}]$ is called a (buyer) *type*.

Instead of offering a single contract, the seller designs a menu consisting of $K \in \mathbb{N}_{\geq 1}$ contracts for the buyer to choose from. The number of contracts K is a decision made by the seller and plays a central role in the results to come. We define $\mathcal{K} = \{1, \ldots, K\}$. Next, the seller partitions $[\underline{p}, \bar{p}]$ into K subintervals $[\underline{p}_k, \bar{p}_k]$ with $\bar{p}_k > \underline{p}_k$. We call this a *proper K-partition.* Finally, the seller constructs K contracts, where contract (x_k, z_k) is designed for subinterval $[\underline{p}_k, \bar{p}_k]$ for each $k \in \mathcal{K}$. The contracts are determined by solving the following optimisation problem:

$$\max_{x,z} \quad \sum_{k \in \mathcal{K}} \left(\int_{\underline{p}_k}^{\bar{p}_k} \omega(p) \mathrm{d}p \right) (\phi_S(x_k) + z_k) \tag{3.1}$$

$$\text{s.t.} \quad \phi_B(x_k|p_k) - z_k \geq 0, \qquad \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k \in \mathcal{K}, \tag{3.2}$$

$$\phi_B(x_k|p_k) - z_k \geq \phi_B(x_l|p_k) - z_l, \qquad \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k, l \in \mathcal{K}, \tag{3.3}$$

$$x_k \geq 0, \qquad \forall\, k \in \mathcal{K}.$$

We refer to this model as the *robust pooling model.* Constraints (3.2) specify that contract (x_k, z_k) must be individually rational for the buyer with respect to all corresponding types $p_k \in [\underline{p}_k, \bar{p}_k]$. Constraints (3.3) are the Incentive Compatibility (IC) constraints. These ensure that the buyer with type $p_k \in [\underline{p}_k, \bar{p}_k]$ prefers and chooses the intended contract (x_k, z_k) over all the other contracts in the menu. Recall that the buyer chooses the most beneficial contract for himself from the menu. To be precise, we need the following assumption, which is conventional in the mechanism design literature. If the IC constraint (3.3) where type $p_k \in [\underline{p}_k, \bar{p}_k]$ compares contract (x_k, z_k) to contract (x_l, z_l) holds with equality, then type p_k is indifferent between contracts (x_k, z_k) and (x_l, z_l). In this case, we assume that the seller can convince the buyer to choose contract (x_k, z_k).

Thus, a buyer with type $p_k \in [\underline{p}_k, \bar{p}_k]$ always chooses contract (x_k, z_k) by design of the menu. This is related to the well-known revelation principle (see Laffont and Martimort (2002) and Myerson (1982)). This principle states that without loss of optimality the seller can restrict his design to incentive-compatible direct coordination mechanisms and obtain a truthful choice of contract by the buyer. In other words, for a buyer with type $p_k \in [\underline{p}_k, \bar{p}_k]$ it is a weakly-dominant strategy to choose contract (x_k, z_k).

With this insight, we return to the robust pooling model. Notice that

$$\omega_k \equiv \int_{\underline{p}_k}^{\bar{p}_k} \omega(p) \mathrm{d}p \in (0, 1] \tag{3.4}$$

for $k \in \mathcal{K}$ defines the probability ω_k that the buyer's type lies in $[\underline{p}_k, \bar{p}_k]$ and consequently that the buyer chooses contract (x_k, z_k). The seller's objective (3.1) is to maximise his own expected net utility, which is the weighted sum of his valuation $\phi_S(x_k)$ of the order quantity x_k and the received side payment z_k.

The robust pooling model has a strong connection to robust optimisation models (see Ben-Tal et al. (2009)). Our model has finitely many decision variables and infinitely many constraints. Furthermore, for $k \in \mathcal{K}$ the interval $[\underline{p}_k, \bar{p}_k]$ can be interpreted as the so-called uncertainty interval for p_k. Thus, the robust pooling model can be seen as a robust optimisation variant of the classical discrete model with K uncertain parameters $p_1, \ldots, p_K$. We will not require robust optimisation techniques in the following sections. However, these techniques can be useful to analyse more complex robust pooling models.

To conclude, the robust pooling model pools the possible buyer types p into finitely many subintervals, enabling the seller to offer finitely many contracts in the menu. Furthermore, the contracts are robust by design, meaning that the buyer will always accept a contract from the menu for any possible type $p \in [\underline{p}, \bar{p}]$ and this choice is correctly reflected in the objective function.

From this point onwards, we denote a menu of contracts by (x, z), where $x = (x_1, \ldots, x_K)$ and $z = (z_1, \ldots, z_K)$. A single contract is denoted by (x_k, z_k) for $k \in \mathcal{K}$. Also, we use ω_k defined by (3.4) in the objective instead of the integral notation.

3.2.2 Reformulation and analysis

In the robust pooling model the number of contracts K and the proper K-partition of $[\underline{p}, \bar{p}]$ are decisions made by the seller. Therefore, if solving the model is sufficiently easy, we can focus on quantifying the effect of the number of contracts and the chosen partition. For example, how many contracts should be offered to obtain 90% of the maximum possible expected net utility? Also, the equiquantile partition is a natural choice, but is it also optimal?

In order to answer such questions, we need to make assumptions and consider explicit models, as a general approach seems impossible. The first assumption is on the buyer's utility function.

Assumption 3.1. The buyer's utility function is $\phi_B(x|p) = \psi(x) + p\chi(x)$, where the functions $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $\chi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ do not depend on the type p. Moreover, χ is non-decreasing and non-negative.

Under Assumption 3.1, we make a change of variables by splitting the side payment into two parts:

$$z_k = \psi(x_k) + y_k,$$

where y_k will replace z_k as decision variable. Substitution of this definition leads to an equivalent model with simplified constraints:

$$\begin{aligned}
\max_{x,y} \quad & \sum_{k \in \mathcal{K}} \omega_k \left(\phi_S(x_k) + \psi(x_k) + y_k \right) & & \\
\text{s.t.} \quad & p_k \chi(x_k) - y_k \geq 0, & \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k \in \mathcal{K}, & \quad (3.5) \\
& p_k \chi(x_k) - y_k \geq p_k \chi(x_l) - y_l, & \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k, l \in \mathcal{K}, & \quad (3.6) \\
& x_k \geq 0, & \forall\, k \in \mathcal{K}. &
\end{aligned}$$

The benefit of this formulation is that several utility functions can be analysed as one model: different choices of ϕ_S and ψ can lead to the same function $\phi_S + \psi$. Furthermore, if $\phi_S + \psi$ is concave and χ linear, this formulation is concave, has linear constraints, and can be solved efficiently.

We continue with the first structural result for the robust pooling model. In Lemma 3.1 we essentially identify an embedded dual shortest path problem as in Rochet and Stole (2003) and Vohra (2012).

Lemma 3.1. *Under Assumption 3.1, for any feasible x it is optimal to set*

$$y_k = \underline{p}_k \chi(x_k) - \sum_{i=1}^{k-1} (\bar{p}_i - \underline{p}_i)\chi(x_i) \qquad \forall\, k \in \mathcal{K}. \tag{3.7}$$

Lemma 3.1 allows us to eliminate the variable y (or z) and obtain an optimisation problem in terms of x. However, in order to do so, we need to be able to express the feasible region in terms of x. This is shown in Lemma 3.2.

Lemma 3.2. *Under Assumption 3.1, any x is feasible if and only if x satisfies $0 \leq x_1 \leq \cdots \leq x_K$.*

In mechanism design, the buyer's type is often related to efficiency: type $p_k > p_l$ gets more utility from a fixed order quantity than type p_l, i.e., $\phi_B(x|p_k) \geq \phi_B(x|p_l)$ for all $x \geq 0$. Thus, type p_k is more efficient. Lemma 3.2 shows that the order quantities are weakly ordered in terms of the corresponding type's efficiency: a less efficient type is offered a lower or equal order quantity than a more efficient type.

We can now combine our results to get an equivalent and much simpler formulation for the robust pooling model under our assumptions, see Theorem 3.3. Notice in particular that the reformulation has finitely many linear constraints.

Theorem 3.3. *Under Assumption 3.1, the robust pooling model with infinitely many constraints is equivalent to the following problem with finitely many and linear constraints:*

$$\max_{0 \leq x_1 \leq \cdots \leq x_K} \sum_{k \in \mathcal{K}} \omega_k \left(\phi_S(x_k) + \psi(x_k) + \left(\underline{p}_k - (\bar{p}_k - \underline{p}_k) \sum_{i=k+1}^{K} \frac{\omega_i}{\omega_k} \right) \chi(x_k) \right). \tag{3.8}$$

The computational complexity of solving (3.8) depends on the shape of $\phi_S + \psi$ and χ. Furthermore, (3.8) allows for specialised (numerical) solvers, since the feasible region is independent of $\phi_S + \psi$ and χ. In Appendix 3.A.2 we derive an explicit solution for (3.8) under additional assumptions.

In conclusion, we have shown how to reformulate and solve certain robust pooling models for maximising utility. In particular, the analysis allows us to show equivalences between the robust pooling model under Assumption 3.1 and other models from the literature (see Appendix 3.A.3). We now shift to the analysis of the DMU model to quantify the effect of the chosen partition.

3.2.3 Decreasing marginal utility problem

In this section we consider a specific contracting model that fits our robust pooling setting of Section 3.2.1 and can be analysed in detail. The model is based around the concept that the marginal utility of a product decreases for the buyer as the order quantity increases.

For order quantity $x \in \mathbb{R}_{\geq 0}$ the buyer's marginal utility of an additional product is given by $p - rx^n$ for some fixed parameters $r, n \in \mathbb{R}_{>0}$. Here, $p \in [\underline{p}, \bar{p}] \subseteq \mathbb{R}_{\geq 0}$ with $\bar{p} > \underline{p}$ is the private parameter of the buyer, as introduced in Section 3.2.1. This leads to the following utility function for the buyer:

$$\phi_B(x|p) = \int_0^x (p - ru^n)\mathrm{d}u = -\tfrac{1}{n+1} r x^{n+1} + px \equiv \psi(x) + p\chi(x).$$

The buyer's utility function is strictly concave in x and is negative for large order quantities. Therefore, the buyer has a finite individually optimal order quantity. For example, this could be the case if excess products are difficult to dispose of. The case of $n = 1$, a quadratic buyer's utility function, is for example used in Chellappa and Mehra (2018) and Wong (2014).

The seller's utility function is linear in the order quantity: $\phi_S(x) = Px$, where $P \in \mathbb{R}_{>0}$ is a fixed parameter. Therefore, the seller simply wants to sell as many products as possible. Consequently, ordering no products leads to zero utility for both the seller and the buyer.

For the entire section, we assume that the distribution of p is uniform. The seller designs a menu of contracts using the robust pooling methodology described in Section 3.2.1. We refer to this problem as the *Decreasing Marginal Utility* (DMU-n) problem.

First, we derive the optimal solution and optimal objective value in Section 3.2.3.1. Second, we show how to express relative performance measures as 1-dimensional functions in Section 3.2.3.2. Finally, we discuss properties of the optimal partition in Section 3.2.3.3. These results hold for the DMU-n problem for any $n \in \mathbb{R}_{>0}$ and are applied in Sections 3.2.4 and 3.2.5 for $n = 1$ and $n = 2$, respectively. Note that the DMU-1 model is essentially the same model as in Wong (2014).

3.2.3.1 Optimal solution and objective value

The optimal solution and optimal objective value of the DMU-n model can be explicitly determined, as shown in the next theorem.

Theorem 3.4. *For given $K \in \mathbb{N}_{\geq 1}$ and proper K-partition of $[\underline{p}, \bar{p}]$, the optimal solution for the DMU-n problem is given by*

$$x_k = \begin{cases} 0 & \text{if } k < k^* \\ \sqrt[n]{\frac{P + \bar{p}_k + \underline{p}_k - \bar{p}}{r}} & \text{if } k \geq k^* \end{cases},$$

where the index of the first non-zero order quantity is $k^* = \min\{k \in \mathcal{K} : P - \bar{p} + \bar{p}_k + \underline{p}_k > 0\}$. Thus, trade occurs for types $p \in [\underline{p}_{k^*}, \bar{p}]$. The optimal objective value Γ_K is

$$\Gamma_K = \frac{n}{n+1} \frac{1}{\sqrt[n]{r}} \sum_{k=k^*}^{K} \frac{\bar{p}_k - \underline{p}_k}{\bar{p} - \underline{p}} \left(P - \bar{p} + \bar{p}_k + \underline{p}_k\right)^{\frac{n+1}{n}}. \tag{3.9}$$

The optimal objective value Γ_K is the main focus in the results to come. Notice that Γ_1 is independent of any partition, since there is no partition for a single contract ($K = 1$). For a given instance, Γ_1 is the lowest possible expected utility for the seller when using robust pooling. Furthermore, $k^* = 1$ for $K = 1$, since $P - \bar{p} + \bar{p}_K + \underline{p}_K = P + \underline{p}_K = P + \underline{p} > 0$. Thus, the optimal objective value for $K = 1$ simplifies to

$$\Gamma_1 = \frac{n}{n+1} \frac{1}{\sqrt[n]{r}} \left(P + \underline{p}\right)^{\frac{n+1}{n}}.$$

Likewise, using infinitely many contracts, i.e., letting $K \to \infty$ using sensible partitions, also leads to an objective value independent of any partition. We denote this value by Γ_∞, which is the highest possible expected utility for the seller when using robust pooling:

$$\Gamma_\infty = \frac{n}{n+1} \frac{1}{\sqrt[n]{r}} \int_{p^*}^{\bar{p}} \frac{1}{\bar{p} - \underline{p}} \left(P - \bar{p} + 2p\right)^{\frac{n+1}{n}} \mathrm{d}p,$$

where $p^* = \min\{p \in [\underline{p}, \bar{p}] : P - \bar{p} + 2p \geq 0\}$ is the continuous version of k^*. That is, p^* is the threshold for which the optimal order quantity is non-zero. To be precise, we have

$$p^* = \max\{\underline{p}, \tfrac{1}{2}(\bar{p} - P)\}.$$

Therefore, Γ_∞ can be written as

$$\Gamma_\infty = \begin{cases} \frac{n}{n+1} \frac{n}{2n+1} \frac{1}{2(\bar{p}-\underline{p})} \frac{1}{\sqrt[n]{r}} \left((P + \bar{p})^{\frac{2n+1}{n}} - \left(P - \bar{p} + 2\underline{p}\right)^{\frac{2n+1}{n}} \right) & \text{if } p^* = \underline{p} \\ \frac{n}{n+1} \frac{n}{2n+1} \frac{1}{2(\bar{p}-\underline{p})} \frac{1}{\sqrt[n]{r}} (P + \bar{p})^{\frac{2n+1}{n}} & \text{if } p^* = \frac{1}{2}(\bar{p} - P) \end{cases}.$$

Notice that Γ_K is the composite midpoint rule for numerical integration applied to the integrand of Γ_∞. In other words, determining the optimal partition for robust pooling is equal to choosing the optimal partition for the composite midpoint rule. For more details on this numerical integration, see for example Dragomir et al. (1998) and Kirmaci (2004). Therefore, we could apply results from numerical integration to obtain performance guarantees for Γ_K compared to Γ_∞. In particular, this insight implies that loss in performance (the difference between Γ_K and Γ_∞) is of the order $\mathcal{O}(1/K^2)$. This is in line with the results of Bergemann et al. (2011) and Wong (2014). However, by analysing the performance of robust pooling in more detail, we can determine the achieved performances exactly.

3.2.3.2 Performance measures

For a given partition, we would like to compare the optimal objective value Γ_K for different number of contracts K. In order to do so, it is useful to redefine the partition as follows:

$$\underline{p}_k = \underline{p} + \delta_{k-1}(\bar{p} - \underline{p}) \qquad \text{and} \qquad \bar{p}_k = \underline{p} + \delta_k(\bar{p} - \underline{p}),$$

where $\delta_0 = 0$, $\delta_k \in [0,1]$ for $k = 1, \dots, K-1$, and $\delta_K = 1$. Notice that δ_0 corresponds to $\underline{p}$ and δ_K to $\bar{p}$. Furthermore, δ_k for $k = 1, \dots, K-1$ encode the chosen points to partition $[\underline{p}, \bar{p}]$. Thus, a proper K-partition satisfies $0 = \delta_0 < \cdots < \delta_K = 1$. We denote the partition by $\Delta = \{\delta_0, \dots, \delta_K\}$. Substitution of this definition in (3.9) gives

$$\Gamma_K = \frac{n}{n+1} \frac{1}{\sqrt[n]{r}} \sum_{k=k^*}^{K} (\delta_k - \delta_{k-1}) \left(P + \underline{p} + (\delta_k + \delta_{k-1} - 1)(\bar{p} - \underline{p})\right)^{\frac{n+1}{n}}. \tag{3.10}$$

With this reformulated expression, we can for example consider the improvement of offering K contracts compared to a single contract:

$$\frac{\Gamma_K}{\Gamma_1} = \sum_{k=k^*}^{K} (\delta_k - \delta_{k-1}) \left(1 + (\delta_k + \delta_{k-1} - 1)\frac{\bar{p} - \underline{p}}{P + \underline{p}}\right)^{\frac{n+1}{n}}.$$

The parameter $(\bar{p} - \underline{p})/(P + \underline{p})$ plays a central role in all the following analysis. We call this parameter the *instance parameter* $\alpha \in \mathbb{R}_{>0}$:

$$\alpha = \frac{\bar{p} - \underline{p}}{P + \underline{p}}.$$

We will see that all structural results can be expressed in terms of α, i.e., it captures the essence of the instance. Returning to improvement Γ_K/Γ_1, we get

$$\frac{\Gamma_K}{\Gamma_1} = \sum_{k=k^*}^{K} (\delta_k - \delta_{k-1}) \Big(1 + (\delta_k + \delta_{k-1} - 1)\alpha\Big)^{\frac{n+1}{n}}.$$

In terms of α, we have $k^* = \min\left\{k \in \mathcal{K} : \delta_k + \delta_{k-1} > \frac{\alpha-1}{\alpha}\right\}$, since

$$\begin{aligned}
& & P - \bar{p} + \bar{p}_k + \underline{p}_k &> 0 \\
\iff & & P + \underline{p} + (\delta_k + \delta_{k-1} - 1)(\bar{p} - \underline{p}) &> 0 \\
\iff & & 1 + (\delta_k + \delta_{k-1} - 1)\alpha &> 0 \qquad \iff \qquad \delta_k + \delta_{k-1} > \tfrac{\alpha-1}{\alpha}.
\end{aligned}$$

Thus, if $0 < \alpha \leq 1$ any partition satisfies $k^* = 1$, i.e., all contracts instigate trade between the seller and buyer.

It is now straightforward to determine the following bounds on the relative improvement for any $K > 1$ and proper K-partition:

$$\lim_{\alpha \to 0} \frac{\Gamma_K - \Gamma_1}{\Gamma_1} = 0 \qquad \text{and} \qquad \lim_{\alpha \to \infty} \frac{\Gamma_K - \Gamma_1}{\Gamma_1} = \infty.$$

Hence, for any arbitrarily large relative improvement there exists an instance that exceeds this relative improvement. In particular, this holds for two contracts and any proper 2-partition.

It is useful to introduce a normalisation factor ν:

$$\nu = \tfrac{n+1}{n} \sqrt[n]{r} (\bar{p} - \underline{p})^{-\frac{n+1}{n}}.$$

This leads to the more manageable formula

$$\nu\Gamma_K = \sum_{k=k^*}^{K} (\delta_k - \delta_{k-1}) \left(\tfrac{1}{\alpha} + \delta_k + \delta_{k-1} - 1\right)^{\frac{n+1}{n}} . \tag{3.11}$$

The normalisation factor ν will cancel out in relative performance measures, allowing us to use (3.11) in these expressions.

Similarly, we can express Γ_∞ in terms of α. First, we focus on p^* and realise that

$$\underline{p} \geq \tfrac{1}{2}(\bar{p} - P) \qquad \Longleftrightarrow \qquad P + \underline{p} - (\bar{p} - \underline{p}) \geq 0 \qquad \Longleftrightarrow \qquad 1 - \alpha \geq 0.$$

Thus, we have

$$p^* = \begin{cases} \underline{p} & \text{if } \alpha \leq 1 \\ \frac{1}{2}(\bar{p} - P) & \text{if } \alpha > 1 \end{cases} .$$

For $0 < \alpha \leq 1$ this leads to $p^* = \underline{p}$ and

$$\nu\Gamma_\infty|_{0<\alpha\leq 1} = \frac{n}{2(2n+1)} \left((\tfrac{1}{\alpha} + 1)^{\frac{2n+1}{n}} - (\tfrac{1}{\alpha} - 1)^{\frac{2n+1}{n}} \right) . \tag{3.12}$$

Similarly, for $\alpha > 1$ we have $p^* = \frac{1}{2}(\bar{p} - P)$ and

$$\nu\Gamma_\infty|_{\alpha>1} = \frac{n}{2(2n+1)} (\tfrac{1}{\alpha} + 1)^{\frac{2n+1}{n}} . \tag{3.13}$$

Notice that for $\alpha = 1$ (3.12) and (3.13) give the same value, as expected. Furthermore, realise that $\nu\Gamma_K$ and $\nu\Gamma_\infty$ are completely determined by α. However, for a fixed α the values Γ_K and Γ_∞ can take on any value in $(0, \infty)$ by changing the parameter r.

The main benefit of robust pooling is the finite number of contracts in the menu. However, limiting the number of contracts will typically come at the cost of having a lower expected utility for the seller. Therefore, the main performance measure of interest is the *pooling performance* Γ_K/Γ_∞, which measures the fraction of expected utility achieved by offering K contracts in terms of the maximum obtainable expected utility Γ_∞.

With the above analysis, we can express relative performance measures as 1-dimensional functions of α. Hence, we are able to make graphs of performance measures in terms of α and determine performance bounds. This requires us to make $n \in \mathbb{R}_{>0}$ explicit and choose a partition scheme (see Sections 3.2.4 and 3.2.5). Before we do so, we determine general properties of an optimal partition for the DMU-n problem.

3.2.3.3 Properties of an optimal partition

A partition is equidistant if it partitions $[\underline{p}, \bar{p}]$ into equally sized subintervals. That is, we have $\delta_k^{\text{equi}} = k/K$, or equivalently $\delta_{k+1}^{\text{equi}} - \delta_k^{\text{equi}} = 1/K$ for $k = 1, \ldots, K-1$. The equidistant partition Δ^{equi} is a natural default choice, especially in the literature on numerical integration. However, is it the optimal partition for the DMU-n problem, i.e., does it maximise Γ_K?

First of all, one should realise that the optimality of partitions is not affected by the normalisation factor ν and thus only depends on the instance parameter α. Consequently, we can work with $\nu\Gamma_K$ to simplify notation. For the equidistant partition, (3.11) becomes

$$\nu\Gamma_K^{\text{equi}} = \tfrac{1}{K} \sum_{k=k^*}^{K} \left(\tfrac{1}{\alpha} + \tfrac{2k-1}{K} - 1\right)^{\frac{n+1}{n}} . \tag{3.14}$$

With the equidistant partition, the index k^* can be determined as follows:

$$\begin{aligned} \delta_k^{\text{equi}} + \delta_{k-1}^{\text{equi}} > \tfrac{\alpha-1}{\alpha} \quad &\iff \quad \frac{2k-1}{K} > \tfrac{\alpha-1}{\alpha} \quad \iff \quad k > \tfrac{1}{2}\left(1 + K\left(\tfrac{\alpha-1}{\alpha}\right)\right) \\ &\implies \quad k^* = \max\left\{1, \left\lfloor 1 + \tfrac{1}{2}\left(1 + K\left(1 - \tfrac{1}{\alpha}\right)\right)\right\rfloor\right\}. \end{aligned}$$

The range of k^* depends on the parity of K. For K odd we have $k^* \in \{1, \ldots, \frac{1}{2}(K+1)\}$ and for K even $k^* \in \{1, \ldots, \frac{1}{2}K + 1\}$.

Intuitively, if $k^* \geq 3$ for the equidistant partition, there is an inefficiency in the corresponding optimal menu of contracts, since we are offering the contract $(x, z) = (0, 0)$ multiple times. These duplicate contracts in the menu are pointless and can be used more efficiently by changing them. The next lemma confirms this intuition.

Lemma 3.5. *For any $K \in \mathbb{N}_{\geq 1}$ an optimal partition must satisfy $0 = \delta_0 < \delta_1 < \cdots < \delta_{K-1} < \delta_K = 1$ and $k^* \in \{1, 2\}$.*

Corollary 3.6. *For $K \in \mathbb{N}_{>3}$ the equidistant partition is suboptimal if $\alpha \geq \frac{K}{K-3}$.*

Corollary 3.6 does not prove or disprove whether the equidistant partition can be optimal at all. In the next sections, Sections 3.2.4 and 3.2.5, we consider the DMU-n problem for $n = 1$ and $n = 2$ and provide the answer to this question. Note that a general formula for the optimal partition seems impossible. The difficulty in finding the optimal partition becomes clearer in the next sections.

3.2.4 Application to the DMU-1 problem

In this section we specialise the results of Section 3.2.3 to the DMU-1 problem ($n = 1$). Here, the buyer has a linearly decreasing marginal utility for the products, which leads to a quadratic utility function ϕ_B. The DMU-1 problem is essentially the same as the linear-quadratic-uniform model in Wong (2014). We extend and complete his analysis by considering all possible instances, relating structural results to the instance parameter α, deriving formulas for the performance of any partition, and evaluating the performance of the equidistant partition. In particular, this reveals the new insight that the equidistant partition is optimal for a family of instances.

As we will see, the DMU-1 problem is special compared to other DMU-n problems in the sense that the optimal partition has an exceptional structure and is relatively straightforward to determine. We will first derive the optimal partition for DMU-1 in Section 3.2.4.1. In Section 3.2.4.2 we analyse the performance of the equidistant and optimal partitions in terms of the number of contracts K.

3.2.4.1 Optimal partition

Recall that for any proper K-partition of $[\underline{p}, \bar{p}]$ the normalised optimal objective value is given by (3.11), which for the DMU-1 problem is

$$\nu\Gamma_K = \sum_{k=k^*}^{K} (\delta_k - \delta_{k-1}) \left(\tfrac{1}{\alpha} + \delta_k + \delta_{k-1} - 1\right)^2,$$

where $k^* = \min\{k \in \mathcal{K} : \delta_k + \delta_{k-1} > \frac{\alpha-1}{\alpha}\}$. We will now optimise the partition to maximise Γ_K. By Lemma 3.5, we know that the optimal partition satisfies $0 = \delta_0 < \cdots < \delta_K = 1$ and $k^* \in \{1, 2\}$. The following theorem gives the optimal partition for the DMU-1 problem.

Theorem 3.7. *For $K \in \mathbb{N}_{\geq 1}$ the optimal partition Δ^{opt} for the DMU-1 problem is given by*

$$\delta_k^{\text{opt}} = \begin{cases} \frac{k}{K} & \textit{if } \alpha < \frac{K}{K-1} \\ 1 - \frac{K-k}{2K-1}\left(\frac{1}{\alpha} + 1\right) & \textit{if } \alpha \geq \frac{K}{K-1} \end{cases} \qquad \textit{for } k \in \{1, \ldots, K-1\}.$$

Hence, for $\alpha < K/(K-1)$ the equidistant partition is optimal and all contracts instigate trade ($k^* = 1$). For $\alpha \geq K/(K-1)$ the equidistant partition is suboptimal and a single contract instigates no trade ($k^* = 2$).

The result of Theorem 3.7 is quite remarkable: for $\alpha < K/(K-1)$ the equidistant partition is the optimal partition. From Corollary 3.6 we know that the equidistant partition is not always optimal. Therefore, we expected that the equidistant partition is never optimal or optimal in the limit, e.g., for $\alpha \to 0$ or $\alpha \to \infty$. However, the equidistant partition is optimal for any instance satisfying $\alpha < K/(K-1)$. For $\alpha \geq K/(K-1)$ it turns out that we effectively only have to optimise δ_1, since it is optimal to partition the remaining subinterval $[\delta_1, 1]$ equidistantly. This fact can be verified from the formula or the stationarity conditions mentioned in the proof of Theorem 3.7.

We will show that the optimal objective value Γ_K^{opt} approximates Γ_∞ with an almost correctly shaped function of α when using the optimal partition. This does not hold for the equidistant partition, which gives additional insights into why it is sometimes suboptimal. The details are as follows. For $0 < \alpha \leq 1$ the normalised objective value $\nu\Gamma_\infty$ is given by (3.12), which simplifies to

$$\nu\Gamma_\infty|_{0<\alpha\leq 1} = \tfrac{1}{6}\left((\tfrac{1}{\alpha} + 1)^3 - (\tfrac{1}{\alpha} - 1)^3\right) = \tfrac{1}{\alpha^2} + \tfrac{1}{3}. \tag{3.15}$$

For $0 < \alpha < K/(K-1)$, which implies $0 < \alpha \leq 1$, we use (3.11) for the equidistant partition:

$$\nu\Gamma_K^{\text{equi}}|_{0<\alpha<\frac{K}{K-1}} = \tfrac{1}{K}\sum_{k=1}^{K}\left(\tfrac{1}{\alpha} + \tfrac{2k-1}{K} - 1\right)^2 = \tfrac{1}{\alpha^2} + \tfrac{1}{3}(1 - \tfrac{1}{K^2}).$$

Hence, $\nu\Gamma_K^{\text{equi}}$ is of the correct order $\Theta(\alpha^{-2})$ for $0 < \alpha \leq 1$ compared to $\nu\Gamma_\infty$, but there is an error in the constant. Now consider $\alpha \geq K/(K-1)$. For $\alpha > 1$, which is implied by $\alpha \geq K/(K-1)$, (3.13) simplifies to

$$\nu\Gamma_\infty|_{\alpha>1} = \tfrac{1}{6}(\tfrac{1}{\alpha}+1)^3. \tag{3.16}$$

This is not of the same order as $\nu\Gamma_K^{\text{equi}}$, which is $\Theta(\alpha^{-2})$ for any $\alpha > 0$. The optimal partition satisfies

$$\delta_k^{\text{opt}} - \delta_{k-1}^{\text{opt}} = \frac{1}{2K-1}\left(\frac{1}{\alpha}+1\right) = \frac{1}{2K-1}\left(\frac{P+\bar{p}}{\bar{p}-\underline{p}}\right),$$

$$P+\underline{p}+(\delta_k^{\text{opt}}+\delta_{k-1}^{\text{opt}}-1)(\bar{p}-\underline{p}) = \left(1-\frac{2K-2k+1}{2K-1}\right)(P+\bar{p}) = \frac{2(k-1)}{2K-1}(P+\bar{p}).$$

Therefore, the corresponding optimal objective value (3.10) is

$$\Gamma_K^{\text{opt}}|_{\alpha\geq\frac{K}{K-1}} = \frac{1}{2r}\frac{(P+\bar{p})^3}{\bar{p}-\underline{p}}\sum_{k=2}^{K}\frac{(2(k-1))^2}{(2K-1)^3} = \frac{1}{2r}\frac{(P+\bar{p})^3}{\bar{p}-\underline{p}}\frac{2(K-1)K}{3(2K-1)^2},$$

or when normalised:

$$\nu\Gamma_K^{\text{opt}}|_{\alpha\geq\frac{K}{K-1}} = \frac{2(K-1)K}{3(2K-1)^2}\left(\tfrac{1}{\alpha}+1\right)^3. \tag{3.17}$$

Again, we see that the term $(1/\alpha+1)^3$ is correct, but there is an error in the coefficient. Thus, in both cases Γ_K^{opt} approximates Γ_∞ with an almost correctly shaped function of α. In particular, the formulas show that the approximation converges to Γ_∞ as $K \to \infty$, as should be the case.

If the equidistant partition is not optimal, the optimal partition points δ_k^{opt} deviate from the equidistant values δ_k^{equi}. Before completing the analysis, we expected that $\delta_k^{\text{opt}} < \delta_k^{\text{equi}}$ and $\delta_k^{\text{opt}} > \delta_k^{\text{equi}}$ can both occur. However, this is not the case, as explained in the next corollary.

Corollary 3.8. *For $K \in \mathbb{N}_{\geq 1}$ the optimal partition for the DMU-1 problem satisfies $\delta_0^{\text{opt}} = 0$, $\delta_K^{\text{opt}} = 1$, and*

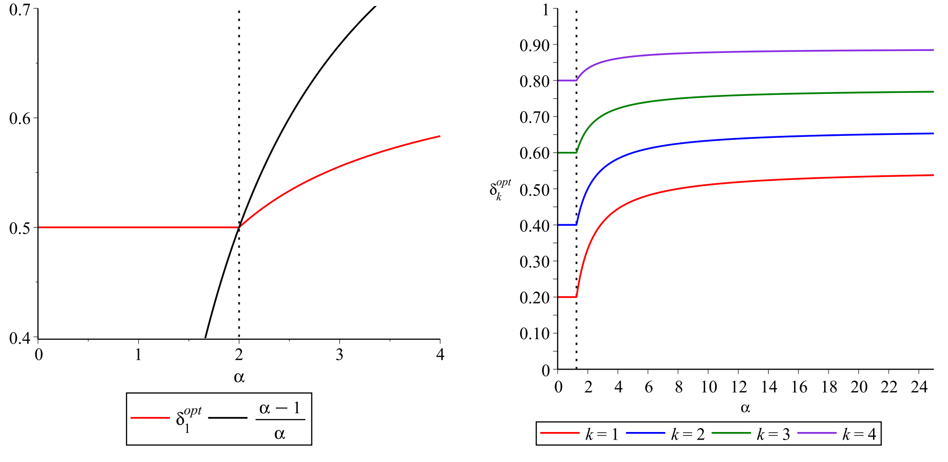
$$\delta_k^{\text{equi}} = \tfrac{k}{K} \leq \delta_k^{\text{opt}} < \tfrac{K+k-1}{2K-1} \qquad \textit{for } k \in \{1,\dots,K-1\}.$$

Thus, Corollary 3.8 shows that the optimal partition points δ_k^{opt} always deviate to the right (larger values). A possible explanation is that for $\alpha \geq K/(K-1)$ we have $k^* = 2$ for the optimal partition. In other words, one contract instigates no trade: $(x_1, z_1) = (0, 0)$. If α increases we have observed before that k^* increases for the equidistant partition. Since $k^* > 2$ is suboptimal by Lemma 3.5, we must have $\delta_1^{\text{opt}} > \delta_1^{\text{equi}}$ in order to prevent $k^* > 2$. Given δ_1^{opt}, the remaining subinterval $[\delta_1^{\text{opt}}, 1]$ is partitioned equidistantly to obtain δ_k^{opt} for $k = 2,\dots,K-1$. Thus, since $\delta_1^{\text{opt}} > \delta_1^{\text{equi}}$ we also get $\delta_k^{\text{opt}} > \delta_k^{\text{equi}}$ for $k = 2,\dots,K-1$.

Figure 3.1a shows the optimal partition for $K = 2$ in terms of α. The two curves are δ_1^{opt} in red and $(\alpha - 1)/\alpha$ in black. Left of $\alpha = 2$ (the dotted line) the optimal partition satisfies $k^* = 1$ and all contracts instigate trade. Right of $\alpha = 2$ there is no trade with the most inefficient types $p \in [\underline{p}, \underline{p}_{k^*})$. The transition in formulas of δ_1^{opt} is continuous at the breakpoint $\alpha = 2$. Furthermore, this transition occurs exactly when the equidistant partition switches from $k^* = 1$ to $k^* = 2$, i.e., when $\delta_1^{\text{equi}} = (\alpha - 1)/\alpha$, as seen in the proof of Theorem 3.7.

The optimal partition for $K = 5$ is illustrated in Figure 3.1b. For $\alpha \geq 5/4$, notice that as α increases, the seller refuses the 20% most inefficient (lowest) types p which rapidly increases to 45%, with 55% as limit.

To conclude, we have determined the optimal partition for DMU-1 and all relevant values can again be expressed in terms of α. Therefore, we can compare the performance of the equidistant and optimal partitions, which is the topic of the next section.



(a) For $K = 2$ contracts.

(b) For $K = 5$ contracts.

Figure 3.1: DMU-1: optimal partition Δ^{opt} in terms of α.

3.2.4.2 Performance of partition schemes

We compare two partition schemes: the equidistant partition Δ^{equi} and the optimal partition Δ^{opt}. As mentioned in Section 3.2.3.2 the main performance measure of interest is the pooling performance Γ_K/Γ_∞. For the DMU-1 problem, Γ_∞ is given by (3.15) and (3.16). For the equidistant partition, we have

$$\nu\Gamma_K^{\text{equi}} = \frac{1}{K} \sum_{k=k^*}^{K} \left(\frac{1}{\alpha} + \frac{2k-1}{K} - 1\right)^2,$$

where $k^* = \max\left\{1, \left\lfloor 1 + \frac{1}{2}\left(1 + K\left(1 - \frac{1}{\alpha}\right)\right)\right\rfloor\right\}$. This allows us to express $\Gamma_K^{\text{equi}}/\Gamma_\infty$ in terms of α. For $0 < \alpha < K/(K-1)$ the optimal partition is equal to the equidistant partition, but for $\alpha \geq K/(K-1)$ we have (3.17) and the pooling performance

$$\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty}\Big|_{\alpha \geq \frac{K}{K-1}} = \frac{4(K-1)K}{(2K-1)^2} = 1 - \frac{1}{(2K-1)^2}.$$

Notice that this pooling performance is constant with respect to $\alpha \geq K/(K-1)$. Figure 3.2 shows the pooling performance for $K \in \{1,2,3\}$ for the equidistant and optimal partitions. By inspection of the graphs, we conclude that the infimum of $\Gamma_K^{\text{equi}}/\Gamma_\infty$ is reached for $\alpha \to \infty$ and the infimum of $\Gamma_K^{\text{opt}}/\Gamma_\infty$ is attained for each $\alpha \geq K/(K-1)$. We have $\lim_{\alpha\to\infty} \Gamma_K^{\text{equi}}/\Gamma_\infty = 1 - 1/K^2$. This implies that the following lower bounds are tight:

$$\frac{\Gamma_K^{\text{equi}}}{\Gamma_\infty} \geq 1 - \frac{1}{K^2} \qquad \text{and} \qquad \frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty} \geq 1 - \frac{1}{(2K-1)^2}.$$

For several values of K, the performance guarantees are listed in Table 3.1.

We observe that $\Gamma_1/\Gamma_\infty \to 0$ as $\alpha \to \infty$, i.e., offering a single robustly pooled contract can perform arbitrarily bad compared to offering infinitely many contracts. However, offering two contracts with the equidistant partition always achieves at least 75% of the maximum obtainable expected utility. For the optimal partition this is 88%. The reason is as follows. A large α can be interpreted as having a large uncertainty of the buyer's efficiency, i.e., a large interval $[\underline{p}, \bar{p}]$. In order to obtain a high expected utility, the seller wants to offer different contracts to inefficient and efficient types. This is why for $k^* > 1$ the seller refuses to trade with the most inefficient types (with $p \in [\underline{p}, \underline{p}_{k^*})$). For $K = 1$, a single contract, the seller cannot make a distinction between efficient and inefficient types and always instigates trade with the buyer ($k^* = 1$). In contrast, for $K \geq 2$ the seller can refuse inefficient types ($k^* > 1$ for α large enough).

Thus, it is essential for the seller to be able to refuse the most inefficient types when there is a high uncertainty in the buyer's efficiency. This is especially noticeable for the optimal partition: for α large enough (such that $k^* = 2$) inefficient types are refused, resulting in a constant pooling performance onwards.

Finally, notice that the optimal partition greatly outperforms the equidistant partition for large values of α. In particular, Table 3.1 shows that the seller can achieve the same performance guarantee with far fewer contracts when using the optimal partition. For example, for a guarantee of 96% the seller has to offer 3 contracts with the optimal partition and 5 contracts with the equidistant partition. For either partition, good performances can be achieved with only a few contracts, which validates the robust pooling approach.

Wong (2014) restricts his analysis to instances with $\alpha \geq K/(K-1)$ (such that $k^* = 2$) and determines the corresponding optimal partition and its pooling performance. Thus, our results extend and complete the analysis of DMU-1. In particular, by considering all possible instances, we observe the remarkable optimality of the equidistant partition for each $\alpha < K/(K-1)$.

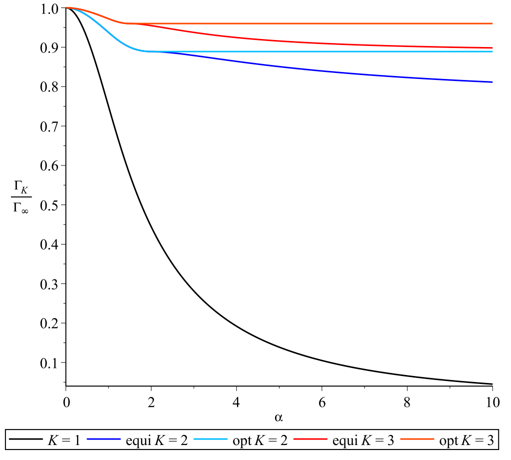


Figure 3.2: DMU-1: the pooling performance Γ_K/Γ_∞ for the equidistant and optimal partitions as functions of the instance parameter α.

K	Equidistant LB	Optimal LB
1	0	0
2	0.7500	0.8888
3	0.8888	0.9600
4	0.9375	0.9795
5	0.9600	0.9876
6	0.9722	0.9917
∞	1	1

Table 3.1: DMU-1: lower bounds for the pooling performance Γ_K/Γ_∞ for the equidistant and optimal partitions.

3.2.5 Application to the DMU-2 problem

To illustrate the special structure of the DMU-1 problem, we perform a similar analysis for the DMU-2 problem ($n = 2$). The buyer has a quadratically decreasing marginal utility for the products. Hence, for $0 < x < 1$ the marginal utility is higher than for the DMU-1 problem, but lower for $x > 1$. In Section 3.2.5.1 we show the complexity of finding closed-form formulas for the optimal partition. However, we can optimise the partition using numerical methods. In Section 3.2.5.2 we determine the performance of the equidistant and optimised partitions. Keep in mind that all values shown with four digits are truncated or rounded.

3.2.5.1 Optimal partition

For the DMU-2 problem, the normalised optimal objective value is given by

$$\nu\Gamma_K = \sum_{k=k^*}^{K} (\delta_k - \delta_{k-1}) \left(\tfrac{1}{\alpha} + \delta_k + \delta_{k-1} - 1\right)^{\frac{3}{2}},$$

where $k^* = \min\{k \in \mathcal{K} : \delta_k + \delta_{k-1} > \frac{\alpha-1}{\alpha}\}$. Again, we know by Lemma 3.5 that the optimal partition must be strictly ordered and must satisfy $k^* \in \{1, 2\}$. In Theorem 3.9 we determine the optimal partition for $K = 2$ contracts. Due to the existence of multiple local optima, determining the optimal partition is more difficult compared to the DMU-1 problem.

Theorem 3.9. *For the DMU-2 problem and $K = 2$, the optimal partition is*

$$\delta_1^{\text{opt}} = \begin{cases} \frac{1}{30}\left(\sqrt{36\frac{1}{\alpha^2} - 15} + 15 - 6\frac{1}{\alpha}\right) & \textit{if } \alpha < \alpha^{\text{trans}} \\ 1 - \frac{2}{5}(\frac{1}{\alpha} + 1) & \textit{if } \alpha \geq \alpha^{\text{trans}} \end{cases},$$

where $\alpha^{\text{trans}} \approx 1.5371$. Furthermore, δ_1^{opt} satisfies the tight bounds $0.3397 < \delta_1^{\text{opt}} < \frac{3}{5}$. Hence, for $\alpha < \alpha^{\text{trans}}$ all contracts instigate trade ($k^* = 1$). For $\alpha \geq \alpha^{\text{trans}}$ a single contract instigates no trade ($k^* = 2$).

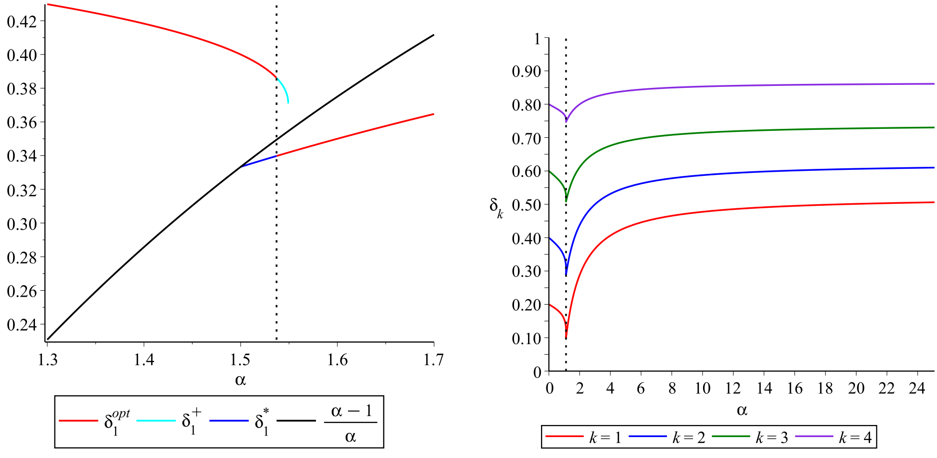
In Figure 3.3a the δ_1^{opt} for $K = 2$ is shown in red. Left of α^{trans} (the dotted line) we have $k^* = 1$ and on the right $k^* = 2$. As detailed in the proof, the difficulty of determining δ_1^{opt} is the existence of two local maxima, denoted by δ_1^+ and δ_1^*. As shown, the optimal partition jumps discontinuously from δ_1^+ to δ_1^* at α^{trans}. Figure 3.3a illustrates this jump and the coexistence of the local maxima δ_1^+ (shown in cyan) and δ_1^* (in blue) for $\alpha \in [3/2, 2/5\sqrt{15}]$. Comparing this figure with Figure 3.1a, it is clear that the properties of the optimal partition for DMU-1 are indeed exceptional.

Theorem 3.9 shows that the equidistant partition is never optimal for the DMU-2 problem (except in the limit $\alpha \to 0$). Furthermore, as α increases, δ_1^{opt} first decreases, then jumps to a lower value, and finally increases. Thus, if the uncertainty in the buyer's efficiency is large enough the optimal menu refuses trade with the most inefficient types. Moreover, as this uncertainty increases, trade is refused for more types, as is the case for the DMU-1 problem.

For a general number of contracts K, we can attempt to imitate the proof of Theorem 3.9. However, this requires to solve a complicated non-linear system of equalities, for which a general solution seems impossible. Instead, we optimise the partition numerically using the gradient-based methodology described in Appendix 3.A.7. Although the used solver can only guarantee local optimality, its performance is stable and the results correspond to our theoretical results when available. Therefore, all results indicate that the method finds the global optimum.

We see a similar structure in the optimised partition for $K \geq 2$ as observed for the optimal partition for $K = 2$: decreasing in α at first, then a discontinuous jump to a lower value, and finally increasing in α. See Figure 3.3b for the optimised partition for $K = 5$. Notice that the optimised partition points are not bounded by the equidistant partition points, as is the case for DMU-1.

To conclude, this analysis for the DMU-2 shows the special structure of the DMU-1, for which the equidistant partition can be optimal and general formulas can be determined. For the DMU-2 problem, we can numerically optimise the partition for any number of contracts. In the next section, we compare the performance of the equidistant and optimised partitions.



(a) Optimal δ_1^{opt} for $K = 2$ contracts.

(b) Optimised for $K = 5$ contracts.

Figure 3.3: DMU-2: optimised partition in terms of α.

3.2.5.2 Performance of partition schemes

As in Section 3.2.4.2, we compare the pooling performance Γ_K/Γ_∞ for the equidistant and optimised partitions. Recall that the related formulas for Γ_∞ and Γ_K^{equi} are (3.12), (3.13), and (3.14). As explained in the previous section, we only have numerical results for the optimised partition.

Figure 3.4 shows the pooling performance for $K \in \{1, 2, 3\}$ for the equidistant and optimised partitions. First of all, Γ_1/Γ_∞ is not shown completely, because it goes to zero for $\alpha \to \infty$ as seen for the DMU-1 problem. In contrast to DMU-1, the performance of the equidistant partition has local minima and maxima, and the infimum is typically attained at some finite value for α (so not for $\alpha \to \infty$). Furthermore, for a fixed instance, an equidistant $(K+1)$-partition does not always perform better than an equidistant K-partition. For example, for $\alpha = 20$ the equidistant 4-partition outperforms the equidistant 5-partition, as can be verified with (3.14). The lower bounds on the pooling performance are given in Table 3.2. Note that the lower bounds for the equidistant partition with 4 and 5 contracts are effectively the same.

For α such that $k^* = 2$ for the optimised partition, we see that the pooling performance is constant and minimal. For $K = 2$ this can be verified with Theorem 3.9. This property also holds for DMU-1. Table 3.2 also includes the lower bounds for the optimised partition.

To conclude, as for the DMU-1 problem, offering a single robust contract is not recommended. However, by offering only a few contracts, high pooling performance can be achieved of at least 88% (equidistant partition) or 92% (optimised partition). The partition can be optimised using numerical methods, which is in particular beneficial for up to five contracts.

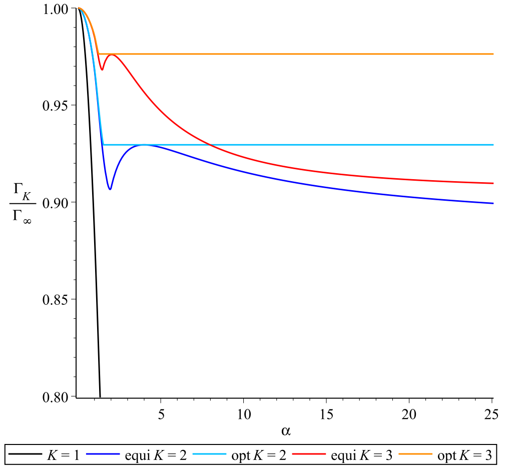


Figure 3.4: DMU-2: the pooling performance Γ_K/Γ_∞ for the equidistant and optimised partitions as functions of the instance parameter α.

K	Equidistant LB	Optimised LB
1	0	0
2	0.8838	0.9295
3	0.9065	0.9763
4	0.9681	0.9882
5	0.9681	0.9929
6	0.9842	0.9953
∞	1	1

Table 3.2: DMU-2: lower bounds for the pooling performance Γ_K/Γ_∞ for the equidistant and optimised partitions.

3.3 Contracting for minimising costs

In this section we analyse the robust pooling model in the setting of cost minimisation using the same approach as in Section 3.2. We formalise the model in Section 3.3.1. Although the models for maximising utility and for minimising cost have a similar structure, they are not equivalent under the considered assumptions. Nevertheless, the general analysis is similar and is provided in Appendix 3.B.1. When needed, we will highlight the differences between cost minimisation and utility maximisation. We apply the robust pooling model to a classical cost minimisation model based on the economic order quantity setting in Section 3.3.2. All proofs are given in Appendix 3.B.

3.3.1 The model

As in Section 3.2, the principal is a seller of products and the agent a buyer. The seller offers contracts to the buyer, which specify the order quantity $x \in \mathbb{R}_{\geq 0}$ and a side payment $z \in \mathbb{R}$. In contrast to utility maximisation, here we define z to be the side payment from the seller to the buyer to be consistent with the literature related to the model considered in Section 3.3.2.

As before, we assume that the buyer's private information can be captured by a parameter $p \in [\underline{p}, \bar{p}] \subseteq \mathbb{R}_{\geq 0}$ with $\bar{p} > \underline{p}$. The buyer's cost for order quantity x is $\phi_B(x|p)$ and the corresponding seller's cost is $\phi_S(x)$. The seller applies the same robust pooling approach as before. First, the seller decides how many contracts are offered, denoted by $K \in \mathbb{N}_{\geq 1}$. Second, he divides the interval $[\underline{p}, \bar{p}]$ into K subintervals, using a proper K-partition. Third, the seller designs a menu of K contracts by solving the following optimisation model:

$$
\begin{aligned}
\min_{x,z} \quad & \sum_{k \in \mathcal{K}} \omega_k \left(\phi_S(x_k) + z_k \right) & & \\
\text{s.t.} \quad & \phi_B(x_k|p_k) - z_k \leq \Theta, & & \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k \in \mathcal{K}, \qquad (3.18)\\
& \phi_B(x_k|p_k) - z_k \leq \phi_B(x_l|p_k) - z_l, & & \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k, l \in \mathcal{K}, \qquad (3.19)\\
& x_k \geq 0, & & \forall\, k \in \mathcal{K}.
\end{aligned}
$$

Except for (3.18), the model is essentially the same as in Section 3.2.3. Constraints (3.18) ensure individual rationality for the buyer, which need further clarification. The parameter $\Theta \in \mathbb{R}_{\geq 0}$ is the buyer's reservation level: if the buyer's net cost for a contract would exceed Θ he will not accept it. In the literature, Θ is often the cost for ordering at an outside option. Hence, Θ is also called the *outside option* or *default option*.

For utility maximisation problems, such as the DMU-n problem in Section 3.2.3, it is common that the default option is to have no trade and thus zero utility. This implies $\Theta = 0$. Therefore, we did not include Θ in the model description in Section 3.2. For cost minimisation problems this is not the case, as there is no common natural default option. For example, if the default option is to have no trade, which (virtual) cost should be assigned? If the default is to use an outside option, does the corresponding cost depend on $[\underline{p}, \bar{p}]$ or not? For the problem analysed in Section 3.3.2, and its default option, it is useful to mention Θ explicitly in the model as it affects the results.

To prepare for Section 3.3.2, we make the following assumption on the buyer's cost function to derive a simpler reformulation of the robust pooling model.

Assumption 3.2. The buyer's cost function is $\phi_B(x|p) = \psi(x) + p\chi(x)$, where the functions $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $\chi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ do not depend on the type p. Moreover, χ is non-decreasing and non-negative.

Although we can rewrite the cost minimisation problem into a utility maximisation problem, Assumption 3.2 does not fit into the framework of Section 3.2, because of the resulting negative term $-p\chi(x)$ in the buyer's utility function. Under Assumption 3.2, we can derive results equivalent to those in Section 3.2.2, which are given in Appendix 3.B.1. The proofs and results are essentially identical, with the following highlighted exceptions.

First, the change of variables by redefining the side payment includes the outside option Θ: $z_k = \psi(x_k) + y_k - \Theta$. Consequently, Θ appears as a constant in the objective function of the reformulated models.

Second, the structure of the optimal side payments and the feasible region is 'reversed' in terms of the contract indices k. For example, the feasible region is $x_1 \geq \cdots \geq x_K \geq 0$. However, in terms of buyer type's efficiency the result is not reversed. Here, a buyer with a lower parameter p is more efficient, since he has lower costs for an order quantity.

We continue to apply robust pooling to the economic order quantity model in the next section.

3.3.2 Economic order quantity problem

We consider a contracting model to which we can apply the robust pooling setting of Section 3.3.1. The considered cost functions are those of the classical economic order quantity model, which model the average cost of a trade agreement over an infinite horizon. The context of the problem is as follows.

The buyer has external demand with constant rate $d \in \mathbb{R}_{>0}$ on an infinite time horizon, which must be satisfied without backlogging. He can order products at the seller, which has an ordering cost of $f \in \mathbb{R}_{>0}$ for the buyer. Furthermore, the buyer has an inventory holding cost of $h \in \mathbb{R}_{>0}$ per product and time unit. The buyer's holding cost h is the private parameter. To minimise his own costs, the buyer orders if and only if his inventory is depleted (the zero-inventory property). Therefore, an order quantity of $x \in \mathbb{R}_{>0}$ products leads to a total cost per time unit of $\phi_B(x) = df\frac{1}{x} + \frac{1}{2}hx$.

The seller has a similar cost structure: setup cost $F \in \mathbb{R}_{>0}$ and inventory holding cost $H \in \mathbb{R}_{>0}$. Production takes place with constant rate $\rho \in \mathbb{R}_{>d}$ and according to a just-in-time lot-for-lot policy. This leads to a total costs per time unit for the seller of $\phi_S(x) = dF\frac{1}{x} + \frac{1}{2}H\frac{d}{\rho}x$.

To simplify notation, we define $R = dF$, $P = \frac{1}{2}H\frac{d}{\rho}$, $r = df$, and $p = \frac{1}{2}h$. Hence, the buyer's cost function is $\phi_B(x|p) = r\frac{1}{x} + px$, where $r \in \mathbb{R}_{>0}$ is a fixed parameter and $p \in [\underline{p}, \bar{p}] \subseteq \mathbb{R}_{\geq 0}$ the buyer's private parameter. We assume that the distribution of p is uniform. Likewise, the seller's costs are given by $\phi_S(x) = R\frac{1}{x} + Px$ for fixed parameters $R, P \in \mathbb{R}_{>0}$.

Given this setting, the seller constructs a menu of contracts using the robust pooling approach of Section 3.3.1. We refer to this problem as the *Economic Order Quantity* (EOQ) problem, which is analysed in detail in the following sections. In Section 3.3.2.1, we determine the optimal solution and corresponding optimal objective value. Section 3.3.2.2 focuses on performance measures. We show that performance measures can be expressed in terms of an instance parameter α, similar to the DMU-n problem. In Section 3.3.2.3, we analyse the optimal partition for the EOQ problem. Finally, the derived results are used in Section 3.3.2.4 to determine the performance of the equidistant partition and the optimised partition.

3.3.2.1 Optimal solution and objective value

The following theorem states the optimal solution and optimal objective value of the EOQ problem.

Theorem 3.10. *For given $K \in \mathbb{N}_{\geq 1}$ and proper K-partition of $[\underline{p}, \bar{p}]$, the optimal solution for the EOQ problem is given by*

$$x_k = \sqrt{\frac{R + r}{P - \underline{p} + \bar{p}_k + \underline{p}_k}}.$$

Hence, $x_1 > \cdots > x_K > 0$ and trade always occurs. The optimal objective value Γ_K is

$$\Gamma_K = 2\sqrt{R+r}\sum_{k=1}^{K}\frac{\bar{p}_k - \underline{p}_k}{\bar{p} - \underline{p}}\sqrt{P - \underline{p} + \bar{p}_k + \underline{p}_k} - \Theta. \tag{3.20}$$

Recall that for the DMU-n problem the optimal menu could include contracts that instigated no trade ($x_k = 0$ for some $k \in \mathcal{K}$). By Theorem 3.10 trade always occurs for the EOQ problem ($x_k > 0$ for all $k \in \mathcal{K}$).

As in Section 3.2.3 there are two extreme choices for K, namely $K = 1$ and $K = \infty$. The optimal objective value Γ_1 is the highest expected cost for the seller when using robust pooling:

$$\Gamma_1 = 2\sqrt{R+r}\sqrt{P+\bar{p}} - \Theta.$$

In contrast, Γ_∞ is the lowest expected cost for the seller:

$$\begin{aligned}\Gamma_\infty &= 2\frac{\sqrt{R+r}}{\bar{p}-\underline{p}}\int_{\underline{p}}^{\bar{p}}\sqrt{P - \underline{p} + 2p}\,\mathrm{d}p - \Theta\\ &= \frac{2}{3}\frac{\sqrt{R+r}}{\bar{p}-\underline{p}}\left(\left(P + 2\bar{p} - \underline{p}\right)^{\frac{3}{2}} - \left(P + \underline{p}\right)^{\frac{3}{2}}\right) - \Theta.\end{aligned}$$

Again, we recognise that Γ_K is the composite midpoint rule for numerical integration applied to the integrand of Γ_∞.

3.3.2.2 Performance measures

We redefine the partition into terms of δ as in Section 3.2.3:

$$\underline{p}_k = \underline{p} + \delta_{k-1}(\bar{p} - \underline{p}) \qquad \text{and} \qquad \bar{p}_k = \underline{p} + \delta_k(\bar{p} - \underline{p}),$$

where $\delta_0 = 0$, $\delta_k \in [0,1]$ for $k = 1,\ldots,K-1$, and $\delta_K = 1$. Thus, (3.20) becomes

$$\Gamma_K = 2\sqrt{R+r}\sum_{k=1}^{K}(\delta_k - \delta_{k-1})\sqrt{P + \underline{p} + (\delta_k + \delta_{k-1})(\bar{p} - \underline{p})} - \Theta. \tag{3.21}$$

We introduce the same instance parameter $\alpha \in \mathbb{R}_{>0}$ as for the DMU-n problem, but a different normalisation factor ν:

$$\alpha = \frac{\bar{p} - \underline{p}}{P + \underline{p}}, \qquad\qquad \nu = \left(2\sqrt{R+r}\sqrt{\bar{p} - \underline{p}}\right)^{-1}.$$

Consequently, the normalised optimal objective values are given by

$$\nu\Gamma_K = \sum_{k=1}^{K}(\delta_k - \delta_{k-1})\sqrt{\tfrac{1}{\alpha} + \delta_k + \delta_{k-1}} - \nu\Theta, \tag{3.22}$$

$$\nu\Gamma_\infty = \tfrac{1}{3}\left(\left(\tfrac{1}{\alpha} + 2\right)^{\frac{3}{2}} - \left(\tfrac{1}{\alpha}\right)^{\frac{3}{2}}\right) - \nu\Theta. \tag{3.23}$$

For performance measures that use differences, such as $(\Gamma_1 - \Gamma_K)/(\Gamma_1 - \Gamma_\infty)$, the outside option Θ cancels out. Therefore, these performance indicators are 1-dimensional functions in terms of α. However, the relative improvement $(\Gamma_1 - \Gamma_K)/\Gamma_1$, for example, is more difficult to analyse, since $\nu\Theta$ is in the denominator.

For other EOQ contracting problems in the literature it is common to assume that by default the buyer places orders using his own individually optimal order quantity. Hence, Θ is the corresponding minimal cost for the buyer, which implies that Θ depends on the buyer's type. However, a type-dependent outside option greatly increases the complexity of the solution structure, i.e., multiple cases would need to be considered and a general analytical approach seems impossible. The techniques and complexity are comparable to those in Chapter 2. Instead, we assume that the outside option is type independent and equal to the most restrictive minimal buyer's cost. In terms of the robust pooling model, this assumption leads to

$$\Theta^* = \inf_{p \in [\underline{p}, \bar{p}]} \inf_{x \geq 0} \phi_B(x|p) = 2\sqrt{r\underline{p}}, \tag{3.24}$$

which implies that

$$\nu\Theta^* = \sqrt{\tfrac{r}{R+r}}\sqrt{\tfrac{\underline{p}}{\bar{p}-\underline{p}}}.$$

From this point onwards, we assume that the outside option Θ is set according to (3.24).

When determining performance bounds, we take supremum or infimum of the performance measure with respect to all possible instances. This often means that $\nu\Theta^*$ must be as large or as small as possible. For example, consider $(\Gamma_1 - \Gamma_K)/\Gamma_1$. For fixed $\alpha > 0$, we want that $\nu\Theta^*$ is as small (large) as possible for the infimum (supremum). Now notice that any fixed α can be attained for any $R > 0$ and $P > 0$ by using the parameters $\underline{p}$ and $\bar{p}$. Thus, the infimum can be reached for $R \to \infty$, for which

$$\lim_{R\to\infty} \nu\Theta^* = 0.$$

Likewise, the supremum can be reached for $R \to 0$ and $P \to 0$, which implies that

$$\lim_{P\to 0}\lim_{R\to 0} \nu\Theta^* = \lim_{P\to 0} \sqrt{\tfrac{1}{\alpha} - \tfrac{P}{\bar{p}-\underline{p}}} = \tfrac{1}{\sqrt{\alpha}}.$$

To conclude, when assuming (3.24) the bounds for $(\Gamma_1 - \Gamma_K)/\Gamma_1$ and similar performance measures can still be determined by a 1-dimensional function of α.

3.3.2.3 Optimal partition

As is the case for the DMU-2 problem, a general formula for the optimal partition for the EOQ problem seems impossible. We do note that the optimal partition only depends on α and in particular not on Θ. Furthermore, the optimal partition must be a proper K-partition, see Lemma 3.11.

Lemma 3.11. *For any $K \in \mathbb{N}_{\geq 1}$ an optimal partition Δ must satisfy $0 = \delta_0 < \delta_1 < \cdots < \delta_{K-1} < \delta_K = 1$.*

We show the difficulty of finding formulas for the optimal partition by deriving the optimal partition for $K = 2$, see (the proof of) Theorem 3.12.

Theorem 3.12. *For the EOQ problem and $K = 2$, the optimal partition is*

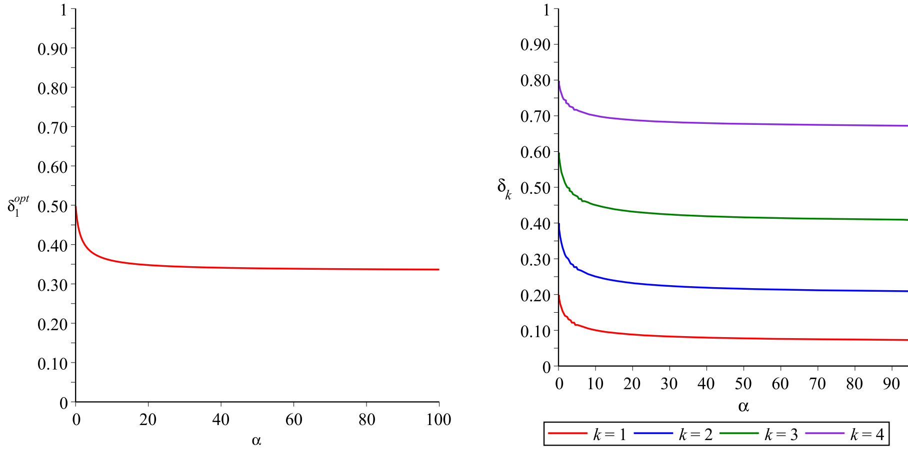
$$\delta_1^{\mathrm{opt}} = \frac{1}{6\alpha}(\sqrt{\alpha^2 + 8\alpha + 4} + \alpha - 2),$$

which satisfies the tight bounds $\frac{1}{3} < \delta_1 < \frac{1}{2}$.

For a general number of contracts, we need to solve a complicated non-linear system of equalities. However, a numerical approach is viable. We apply a similar methodology as in Section 3.2.5.1. Notice that we do not need to account for k^*, which simplifies the procedure. See Figure 3.5a for the optimal partition δ_1^{opt} for $K = 2$ and Figure 3.5b for the optimised partition for $K = 5$.

Both Theorem 3.12 for $K = 2$ and the numerical results for $K \geq 2$ show that $\delta_k^{\mathrm{opt}} \leq \delta_k^{\mathrm{equi}}$. This is also the case for DMU-1, but not DMU-2. Thus, whether δ_k^{opt} is bounded by δ_k^{equi} seems to be a problem-specific property.

To conclude, by using a numerical solution approach, we can determine optimised partitions for the EOQ problem. We continue by determining the pooling performance of the equidistant and optimised partitions.



(a) Optimal δ_1^{opt} for $K = 2$ contracts.

(b) Optimised for $K = 5$ contracts.

Figure 3.5: EOQ: optimised partition in terms of α.

3.3.2.4 Performance of partition schemes

In Section 3.3.2.2 we have shown that the infimum and supremum of relative performance measures can still be expressed as 1-dimensional functions of α. In particular, the pooling performance is calculated by rewriting Γ_K/Γ_∞ into $1 + (\Gamma_K - \Gamma_\infty)/\Gamma_\infty$. Thus, for upper bounds on the pooling performance, we use formulas (3.22) and (3.23) with $\nu\Theta^* = \alpha^{-1/2}$.

The results are shown in Figure 3.6 and Table 3.3. The performances for the equidistant and optimised partitions have roughly the same shape as Γ_1/Γ_∞, i.e., there is a global maximum for a finite α and a (lower) asymptote for $\alpha \to \infty$.

Compared to the results of Section 3.2.3, the dominant difference is that a single robust contract performs reasonably well with a pooling performance of 107%. It is not arbitrarily bad as is the case for the DMU-n problem. We believe the reason is twofold. First, when minimising costs, there is a natural lowest cost possible. In contrast, when maximising utility without budgets, there is no natural limitation. Second, the EOQ cost functions are known for being relatively insensitive to small perturbations in the order quantity or the cost parameters.

From the results, we see that the optimised partition performs only marginally better than the equidistant partition. For example, for $K = 2$ the absolute difference in pooling performance is about 0.4%.

We conclude that robust pooling obtains exceptionally good performances for the EOQ problem. Offering a single robust contract is viable, but it is recommended to offer a few more contracts for a better performance guarantee.

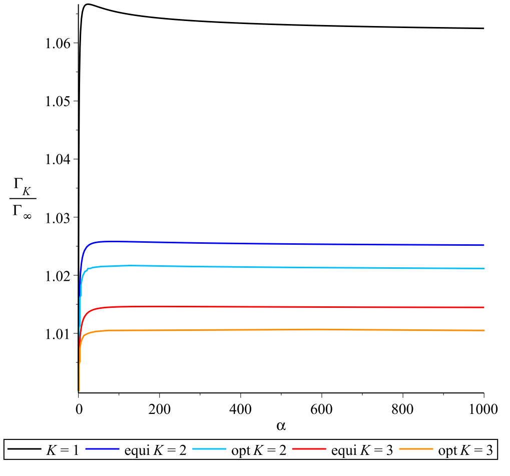


Figure 3.6: EOQ: pooling performance Γ_K/Γ_∞ for the equidistant and optimised partitions as functions of the instance parameter α.

K	Equidistant UB	Optimised UB
1	1.0667	1.0667
2	1.0259	1.0218
3	1.0147	1.0108
4	1.0098	1.0065
5	1.0071	1.0043
6	1.0055	1.0031
∞	1	1

Table 3.3: EOQ: upper bounds for the pooling performance Γ_K/Γ_∞ for the equidistant and optimised partitions.

3.4 Conclusion

We have presented and analysed a new modelling approach for principal-agent contracting models, called robust pooling. This approach considers a buyer whose type follows a continuous distribution on the interval $[\underline{p}, \bar{p}] \subseteq \mathbb{R}$. The seller wants to offer a menu with a finite number of contracts $K \in \mathbb{N}_{\geq 1}$. In our approach, the seller partitions $[\underline{p}, \bar{p}]$ into K subintervals and designs a menu with a single contract for each subinterval. The menu is constructed such that each type will choose its intended contract, making the menu robust to the buyer's private information.

With the robust pooling modelling approach we can compare offering different number of contracts in a natural and consistent way. Furthermore, we can determine performance guarantees in terms of the number of contracts offered, provided that the problem can be analysed (analytically or numerically) in sufficient detail. The existing classical continuous and discrete approaches are not suitable for this analysis. The continuous approach does not handle offering finitely many contracts. For the discrete approach such analysis requires changing the distribution of the buyer's type. This makes any comparison inconsistent, already from a modelling point of view.

Compared to the limited variety approach from the literature, we restrict the pooling of types to use a partition of $[\underline{p}, \bar{p}]$. We make this restriction to structure the buyer's choice, to obtain simple and intuitive mechanisms, to guarantee the accuracy of the extracted information on the buyer's type, and to promote the experimentation with partition schemes. For example, the seller can use the equidistant or equiquantile partitions as simple heuristics. After observing the buyer's chosen contract, the seller can narrow down the buyer's type to the corresponding subinterval. Thus, the accuracy of the extracted information is related to the width of the subintervals and is straightforwardly controlled by the seller by varying the number of contracts.

In Section 3.2 we have applied robust pooling to utility maximisation problems and in Section 3.3 to cost minimisation problems. The robust pooling model can be reformulated and simplified under certain assumptions on the buyer's utility/cost function, which are not uncommon in the literature. In particular, we have analysed two problems in detail: the DMU-n problem, based on a decreasing marginal utility, and the EOQ problem, based on the economic order quantity setting.

Our application of robust pooling to these two problems leads to new insights into the performances of partition schemes of $[\underline{p}, \bar{p}]$. A natural choice is to partition $[\underline{p}, \bar{p}]$ equidistantly. For the DMU-1 problem, the equidistant/equiquantile partition is optimal for a fully specified family of instances, but is suboptimal for other instances. The optimality seems to be a special property of DMU-1, since it is suboptimal for the DMU-2 and EOQ problems.

It is difficult to say whether the equidistant partition performs good enough for a given number of contracts, as this depends on what performance is acceptable. Naturally, the performance of the equidistant partition reaches that of the optimal partition as the number of contracts increases. However, the idea for robust pooling is to offer only a few contracts. For the DMU-1 problem, it is definitely worthwhile to optimise the partition when using up to five contracts. For example, offering 3 contracts with the optimal partition achieves at least 96% of the best possible expected utility (corresponding to infinitely many contracts). For the equidistant partition this is 88%. For the DMU-2 and EOQ problems the difference in performance is smaller. In fact, for the EOQ problem the equidistant partition performs exceptionally well.

Overall, we conclude that robust pooling with only a few contracts, say 3 to 5, leads to high performances and is a viable approach. Offering only a single contract is not advised, since being able to distinguish between inefficient and efficient types is needed for good performances. For example, the optimal menu for the DMU-n problem refuses trade with the most inefficient types in certain cases. Offering a single contract can lead to arbitrarily bad performance for the DMU-n problem.

A possible extension to our analysis is to consider partition heuristics. Based on the results of DMU-1 and DMU-2, we can design the following partition heuristic: optimise only the first partition point and partition the remaining subinterval equidistantly. This is in fact optimal for DMU-1. However, numerically optimising the entire partition is relatively straightforward. Thus, such heuristics should have a clear benefit to (numerically) optimising the entire partition. For example, a heuristic should follow a rule of thumb and not require numerical optimisation.

Appendix

3.A Addendum to Section 3.2

3.A.1 Proofs for Section 3.2.2

Proof of Lemma 3.1. Let x be feasible, i.e., there exists an y such that (x, y) is a feasible solution. The proof consists of two parts: first we show that (3.7) holds for contract $k = 1$, then we focus on the other contracts in the menu ($k > 1$).

First, realise that for an optimal y at least one IR constraint (3.5) must hold with equality. If this is not the case, we can increase all y_k by adding some $\epsilon > 0$ until at least one IR constraint is tight. This new solution is still feasible, as (3.6) only considers the difference $y_k - y_l$, which is unaffected. Moreover, the objective value of the new solution is strictly larger.

Now, suppose that $y_1 < \underline{p}_1\chi(x_1)$, then for $k \in \mathcal{K}$ we have for all $p_k \in [\underline{p}_k, \bar{p}_k]$ that

$$p_k\chi(x_k) - y_k \geq \underline{p}_k\chi(x_k) - y_k \overset{(3.6)}{\geq} \underline{p}_k\chi(x_1) - y_1 \geq \underline{p}_1\chi(x_1) - y_1 > 0.$$

Here, we use that χ is non-negative. The result implies that no IR constraint is tight, which is suboptimal as argued above. Hence, for an optimal y it must hold that $y_1 = \underline{p}_1\chi(x_1)$.

Second, fix $k \in \mathcal{K}$ with $k > 1$ and consider the following IC constraints between contracts k and $k-1$:

$$\bar{p}_{k-1}(\chi(x_k) - \chi(x_{k-1})) \leq y_k - y_{k-1} \leq \underline{p}_k(\chi(x_k) - \chi(x_{k-1})).$$

Since $\bar{p}_{k-1} = \underline{p}_k$, this implies that

$$y_k - y_{k-1} = \underline{p}_k(\chi(x_k) - \chi(x_{k-1})).$$

Using our earlier result that $y_1 = \underline{p}_1\chi(x_1)$, we obtain the following formula:

$$y_k = \sum_{i=2}^{k} \underline{p}_i(\chi(x_i) - \chi(x_{i-1})) + \underline{p}_1\chi(x_1),$$

which can be rewritten into (3.7). □

Proof of Lemma 3.2. First, we show the necessity of $x_1 \leq \cdots \leq x_K$. Let $k, k+1 \in \mathcal{K}$ and consider (3.6) for $\underline{p}_k$ and $\underline{p}_{k+1}$:

$$\begin{aligned} \underline{p}_k\chi(x_k) - y_k &\geq \underline{p}_k\chi(x_{k+1}) - y_{k+1}, \\ \underline{p}_{k+1}\chi(x_{k+1}) - y_{k+1} &\geq \underline{p}_{k+1}\chi(x_k) - y_k. \end{aligned}$$

Adding both IC constraints leads to $(\underline{p}_k - \underline{p}_{k+1})(\chi(x_k) - \chi(x_{k+1})) \geq 0$. Since $\underline{p}_k < \underline{p}_{k+1}$, this implies that $\chi(x_k) \leq \chi(x_{k+1})$ and $x_k \leq x_{k+1}$ since χ is non-decreasing.

Second, we show sufficiency of $0 \leq x_1 \leq \cdots \leq x_K$. Let $x \geq 0$ be non-decreasing and set y according to (3.7). Since χ is non-decreasing, we have $\chi(x_k) \leq \chi(x_{k+1})$ for $k \in \mathcal{K}$. It remains to check feasibility of (x, y). Fix $k \in \mathcal{K}$ and $p_k \in [\underline{p}_k, \bar{p}_k]$. For $l \in \mathcal{K}$ with $k < l$ we have

$$\begin{aligned} y_k - y_l &= \sum_{i=k}^{l-1}(y_i - y_{i+1}) \overset{(3.7)}{=} \sum_{i=k}^{l-1} \underline{p}_{i+1}(\chi(x_i) - \chi(x_{i+1})) \\ &\leq \sum_{i=k}^{l-1} \bar{p}_k(\chi(x_i) - \chi(x_{i+1})) = \bar{p}_k(\chi(x_k) - \chi(x_l)) \leq p_k(\chi(x_k) - \chi(x_l)). \end{aligned}$$

Likewise, let $l \in \mathcal{K}$ with $l < k$, then

$$\begin{aligned} y_k - y_l &= \sum_{i=l+1}^{k} (y_i - y_{i-1}) \overset{(3.7)}{=} \sum_{i=l+1}^{k} \underline{p}_i(\chi(x_i) - \chi(x_{i-1})) \\ &\leq \sum_{i=l+1}^{k} \underline{p}_k(\chi(x_i) - \chi(x_{i-1})) = \underline{p}_k(\chi(x_k) - \chi(x_l)) \leq p_k(\chi(x_k) - \chi(x_l)). \end{aligned}$$

Hence, all IC constraints (3.6) hold. Furthermore,

$$p_k\chi(x_k) - y_k \geq \underline{p}_k\chi(x_k) - y_k \overset{(3.6)}{\geq} \underline{p}_k\chi(x_1) - y_1 \geq \underline{p}_1\chi(x_1) - y_1 = 0.$$

Thus, all IR constraints (3.5) are satisfied and the solution is feasible. □

Proof of Theorem 3.3. By Lemma 3.1 we can substitute the optimal formula (3.7) for y into the optimisation model. By Lemma 3.2 we conclude that the IR and IC constraints hold if and only if $0 \leq x_1 \leq \cdots \leq x_K$. This leads to the equivalent optimisation problem

$$\max_{0 \leq x_1 \leq \cdots \leq x_K} \sum_{k \in \mathcal{K}} \omega_k \Big(\phi_S(x_k) + \psi(x_k) + \underline{p}_k\chi(x_k) - \sum_{i=1}^{k-1}(\bar{p}_i - \underline{p}_i)\chi(x_i) \Big),$$

which can be rewritten into to formulation of the theorem by collecting the terms of x_k. □

3.A.2 An explicitly solvable class of problems

In this appendix, we make additional assumptions to determine a family of explicitly solvable robust pooling models. That is, we are able to derive explicit formulas for the optimal menu of contracts. The following assumptions are a balance between the generality of the model and the brevity of the analysis, and can be weakened up to a certain extent to obtain similar results.

Assumption 3.3. The buyer has zero utility for ordering zero units of product: $\phi_B(0|p) = 0$ for all $p \in [\underline{p}, \bar{p}]$, implying $\psi(0) = \chi(0) = 0$.

Assumption 3.4. The function $\phi_S + \psi$ is strictly concave and differentiable on $\mathbb{R}_{\geq 0}$. The function χ is given by $\chi(x) = x$.

Assumption 3.5. The distribution on the private parameter p is uniform: $\omega(p) = 1/(\bar{p} - \underline{p})$, so $\omega_k = (\bar{p}_k - \underline{p}_k)/(\bar{p} - \underline{p})$ for all $k \in \mathcal{K}$.

Assuming $\phi_B(0|p) = 0$ (Assumption 3.3) ensures that there is no side payment if a contract specifies no trade, i.e., $x_k = 0$ implies $z_k = 0$. This is in line with the default situation, where the absence of trade implies zero utility for the buyer. Assumption 3.4 is needed to make (3.8) a concave maximisation problem that can be solved efficiently, for example using interior-point or cutting-plane methods (see Bertsekas (2015) and Boyd and Vandenberghe (2004)). Finally, the uniformity of p (Assumption 3.5) is significantly restrictive, but allows for a manageable exact analysis with closed-form formulas.

With the imposed additional structure, we can solve (3.8) exactly, see Theorem 3.13.

Theorem 3.13. *Under Assumptions 3.1-3.5, the robust pooling model is equivalent to the following concave problem:*

$$\max_{0 \leq x_1 \leq \cdots \leq x_K} \sum_{k \in \mathcal{K}} \frac{\bar{p}_k - \underline{p}_k}{\bar{p} - \underline{p}} \Big(\phi_S(x_k) + \psi(x_k) + (\bar{p}_k + \underline{p}_k - \bar{p}) x_k \Big).$$

The optimal order quantities are given by

$$x_k = \begin{cases} 0 & \text{if } k < k^* \\ \left(\frac{\mathrm{d}}{\mathrm{d}x} (\phi_S + \psi) \right)^{-1} (\bar{p} - \bar{p}_k - \underline{p}_k) & \text{if } k^* \leq k \leq \hat{k} \\ \infty & \text{if } k > \hat{k} \end{cases},$$

and satisfy $0 < x_{k^*} < \cdots < x_{\hat{k}} < \infty$. *Here, the index of the first non-zero order quantity is*

$$k^* = \min \left\{ K + 1, \min \left\{ k \in \mathcal{K} : \bar{p}_k + \underline{p}_k - \bar{p} > -\frac{\mathrm{d}}{\mathrm{d}x} (\phi_S + \psi)(0) \right\} \right\},$$

and the index of last finite order quantity is

$$\hat{k} = \max \left\{ 0, \max \left\{ k \in \mathcal{K} : \bar{p}_k + \underline{p}_k - \bar{p} < -\lim_{x \to \infty} \frac{\mathrm{d}}{\mathrm{d}x} (\phi_S + \psi)(x) \right\} \right\}.$$

Proof. The uniform distribution (Assumption 3.5) implies that $\omega_k = (\bar{p}_k - \underline{p}_k)/(\bar{p} - \underline{p})$. Therefore, we can simplify the summation in (3.8) as follows:

$$(\bar{p}_k - \underline{p}_k) \sum_{i=k+1}^{K} \frac{\omega_i}{\omega_k} = \sum_{i=k+1}^{K} (\bar{p}_i - \underline{p}_i) = \bar{p} - \bar{p}_k.$$

Substituting these expressions in the result of Theorem 3.3 gives the desired formulation of the optimisation problem.

For the optimal solution, we first relax the constraint $x_1 \leq \cdots \leq x_K$ to obtain a separable optimisation problem. Since the objective of the relaxed problem is strictly concave and differentiable (Assumption 3.4), its optimal solution can easily be determined. Consider contract $k \in \mathcal{K}$. We distinguish three cases.

Case I: if

$$\frac{\mathrm{d}}{\mathrm{d}x_k}(\phi_S + \psi)(0) + \bar{p}_k + \underline{p}_k - \bar{p} \leq 0,$$

then it is optimal for the relaxed problem to set $x_k = 0$. Otherwise, the optimal x_k for the relaxed problem satisfies $x_k > 0$.

Case II: if

$$\lim_{x_k \to \infty} \left(\frac{\mathrm{d}}{\mathrm{d}x_k}(\phi_S + \psi)(x_k) + \bar{p}_k + \underline{p}_k - \bar{p} \right) \geq 0,$$

then it is optimal to set $x_k = \infty$. Otherwise, a finite x is optimal. Here, we use that the above limit is zero only if it is an asymptote. This holds since $\phi_S + \psi$ is strictly concave, implying that its derivative is strictly decreasing. Furthermore, this shows that Cases I and II are indeed mutually exclusive for (fixed) k.

Case III: if the above cases do not hold the optimal x_k is found by setting the derivative to zero:

$$\frac{\mathrm{d}}{\mathrm{d}x_k}\Big(\phi_S(x_k) + \psi(x_k) + (\bar{p}_k + \underline{p}_k - \bar{p})x_k\Big) = 0$$

$$\iff \qquad \frac{\mathrm{d}}{\mathrm{d}x_k}(\phi_S + \psi)(x_k) = -(\bar{p}_k + \underline{p}_k - \bar{p}).$$

Since $\phi_S + \psi$ is strictly concave and differentiable, its derivative is continuous and invertible. Furthermore, Cases I and II are excluded, so the following value is well-defined and strictly positive:

$$\hat{x}_k = \left(\frac{\mathrm{d}}{\mathrm{d}x_k}(\phi_S + \psi) \right)^{-1} (\bar{p} - \bar{p}_k - \underline{p}_k),$$

which is the optimum for the relaxed problem. Notice that the definitions of k^* and $\hat{k}$ imply that Case III corresponds to $k \in \mathcal{K}$ such that $k^* \leq k \leq \hat{k}$. By strict concavity of $\phi_S + \psi$ we know that its derivative is strictly decreasing. Furthermore, realise that $\bar{p}_k + \underline{p}_k - \bar{p} < \bar{p}_{k+1} + \underline{p}_{k+1} - \bar{p}$ for all $k \in \mathcal{K}$. Therefore, we have $0 < \hat{x}_{k^*} < \cdots < \hat{x}_{\hat{k}}$.

Combining all cases leads to a solution satisfying $0 \leq x_1 \leq \cdots \leq x_K$, which is feasible for the non-relaxed problem. Hence, this is the optimal solution to our original problem. $\square$

Theorem 3.13 defines two indices k^* and $\hat{k}$. Typically, the index $\hat{k}$ of last finite order quantity satisfies $\hat{k} = K$ and can be omitted. If $\hat{k} < K$ then the optimal objective value is ∞, which is unrealistic and indicates that the utility functions should be reconsidered. On the other hand, the index k^* of the first non-zero order quantity can play an essential role as seen in Section 3.2.3. If $k^* > 1$ then $(x_k, z_k) = (0, 0)$ for $k < k^*$, i.e., there is no trade with types $p \in [\underline{p}, \underline{p}_{k^*})$.

3.A.3 Equivalences to other models

The structure of the reformulated robust pooling model (3.8) might be recognised by those familiar with either classical discrete contracting models or the limited variety model of Bergemann et al. (2011) and Wong (2014). In fact, under Assumption 3.1 there is an equivalence between these three models. We formalise and discuss this further in this section.

3.A.3.1 Pooling and robustness implies partitioning

In the robust pooling approach we partition $[\underline{p}, \bar{p}]$ to obtain pooling of types, i.e., a menu with finitely many contracts. As mentioned in Section 3.1, the limited variety model of Bergemann et al. (2011) and Wong (2014) achieves robustness and pooling without partitioning $[\underline{p}, \bar{p}]$ a priori. The limited variety model simply restricts the menu to include finitely many contracts. Thus, their approach is more general than our robust pooling. However, in this section we show that under Assumption 3.1 both approaches are equivalent *provided that the optimal partition scheme is used.*

Consider a menu of K contracts (x_k, z_k) that satisfies the pooling and robustness properties. Consequently, each type $p \in [\underline{p}, \bar{p}]$ chooses a contract from the menu. Without loss of generality, each contract is chosen by some types. Let $\hat{p}_k$ be the most inefficient type that chooses contract (x_k, z_k) for $k \in \mathcal{K}$. By changing the index of the contracts, we have $\underline{p} = \hat{p}_1 < \ldots < \hat{p}_K \leq \bar{p}$ without loss of generality. This implies that

$$\begin{aligned} \phi_B(x_k|\hat{p}_k) &\geq z_k, && \forall\, k \in \mathcal{K}, \\ \phi_B(x_k|\hat{p}_k) - \phi_B(x_l|\hat{p}_k) &\geq z_k - z_l, && \forall\, k, l \in \mathcal{K}. \end{aligned} \tag{3.25}$$

We will prove that, in fact, types $p \geq \hat{p}_k$ prefer contract (x_k, z_k) over contracts (x_l, z_l) with $l < k$ by verifying that the respective IR and IC constraints hold. By adding (3.25) for $k, l \in \mathcal{K}$ and by Assumption 3.1, we have

$$\begin{aligned} 0 &\leq \phi_B(x_k|\hat{p}_k) - \phi_B(x_l|\hat{p}_k) + \phi_B(x_l|\hat{p}_l) - \phi_B(x_k|\hat{p}_l) \\ &= (\hat{p}_k - \hat{p}_l)(\chi(x_k) - \chi(x_l)), && \forall\, k, l \in \mathcal{K}. \end{aligned}$$

Therefore, $\chi(x_k) \geq \chi(x_l)$ for $l < k$, since $\hat{p}_k > \hat{p}_l$ by definition. Using these results, we obtain for all $k \in \mathcal{K}$ that

$$\phi_B(x_k|p) = \psi(x_k) + p\chi(x_k) \geq \psi(x_k) + \hat{p}_k\chi(x_k) = \phi_B(x_k|\hat{p}_k) \geq z_k, \quad \forall\, p \geq \hat{p}_k,$$

and

$$\begin{aligned}\phi_B(x_k|p) - \phi_B(x_l|p) &= \psi(x_k) - \psi(x_l) + p(\chi(x_k) - \chi(x_l)) \\ &\geq \psi(x_k) - \psi(x_l) + \hat{p}_k(\chi(x_k) - \chi(x_l)) \\ &= \phi_B(x_k|\hat{p}_k) - \phi_B(x_l|\hat{p}_k) \geq z_k - z_l, \qquad \forall\, l < k, p \geq \hat{p}_k.\end{aligned}$$

These inequalities correspond to IR and IC constraints. They imply that types $p \geq \hat{p}_k$ prefer contract (x_k, z_k) over contracts (x_l, z_l) with $l < k$. Using the definition of $\hat{p}_k$, we conclude that contract (x_k, z_k) must be chosen by types $\{p \in [\underline{p}, \bar{p}] : \hat{p}_k \leq p < \hat{p}_{k+1}\}$.

Thus, under Assumption 3.1 any menu that satisfies the pooling and robustness properties effectively partitions $[\underline{p}, \bar{p}]$ and pools the respective types, exactly as our robust pooling approach.

3.A.3.2 Robustness of the discrete approach

Suppose the buyer's type p follows a continuous distribution on $[\underline{p}, \bar{p}]$ and the seller wants to offer only finitely many contracts in the menu. Of course, our robust pooling approach is designed for this task, but can we also apply the classical discrete approach? That is, can the seller select K representatives from $[\underline{p}, \bar{p}]$, assign appropriate probabilities to the representatives, apply the classical discrete approach, and achieve the same robust result as our robust pooling approach? In this section we show that the discrete approach can be robust under Assumption 3.1.

First, if a discrete model satisfies Assumption 3.1 and is robust we conclude that it must be equivalent to the robust pooling model as shown in Section 3.A.3.1. Second, we prove that under Assumption 3.1 the robust pooling model is equivalent to a specifically constructed discrete model. The proofs of Lemmas 3.1 and 3.2 show that many constraints are redundant. Of all IR constraints (3.5) only $\underline{p}_1\chi(x_1) - y_1 \geq 0$ is needed. Of all IC constraints (3.6) we need $\underline{p}_k\chi(x_k) - y_k \geq \underline{p}_k\chi(x_{k-1}) - y_{k-1}$ for all $k \in \mathcal{K}$ and the constraint $x_1 \leq \cdots \leq x_K$. The non-decreasing x can be enforced by replacing it with the IC constraints $\underline{p}_k\chi(x_k) - y_k \geq \underline{p}_k\chi(x_{k+1}) - y_{k+1}$ for all $k \in \mathcal{K}$. Adding a few more redundant IR and IC constraints, gives the following equivalent optimisation problem:

$$\begin{array}{lll}\max\limits_{x,y} & \sum\limits_{k\in\mathcal{K}} \omega_k\left(\phi_S(x_k) + \psi(x_k) + y_k\right) & \\ \text{s.t.} & \underline{p}_k\chi(x_k) - y_k \geq 0, & \forall\, k \in \mathcal{K}, \\ & \underline{p}_k\chi(x_k) - y_k \geq \underline{p}_k\chi(x_l) - y_l, & \forall\, k, l \in \mathcal{K}, \\ & x_k \geq 0, & \forall\, k \in \mathcal{K}.\end{array}$$

This is the classical discrete variant for the contracting problem, where each subinterval $[\underline{p}_k, \bar{p}_k]$ is represented by its most inefficient type $\underline{p}_k$ and this type has probability ω_k.

To conclude, the discrete model satisfying Assumption 3.1 has a hidden robustness provided that the representative of each subinterval $[\underline{p}_k, \bar{p}_k]$ is its *most inefficient* type $\underline{p}_k$ and this type has probability ω_k. Consequently, a *robust* discrete model using two types to approximate $[\underline{p}, \bar{p}]$ should *not* choose the extreme types $\underline{p}$ and $\bar{p}$ as representatives, since the contract for type $\bar{p}$ will be chosen with probability zero. Hence, effectively only a single contract is used.

3.A.4 Proofs for Section 3.2.3

Proof of Theorem 3.4. We apply Theorem 3.13 where

$$\frac{\mathrm{d}}{\mathrm{d}x}(\phi_S + \psi)(x) = P - rx^n \qquad \Longrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}x}(\phi_S + \psi)(0) = P.$$

Since $P - \bar{p} + \bar{p}_K + \underline{p}_K = P + \underline{p}_K > 0$, we get $k^* = \min\{k \in \mathcal{K} : P - \bar{p} + \bar{p}_k + \underline{p}_k > 0\}$. Furthermore, since ψ decreases super-linearly we have $\hat{k} = K$. In other words, all contracts are sensible ($x_k, z_k < \infty$) and at least one contract instigates trade ($x_K > 0$). The results now follow from Theorem 3.13. $\square$

Proof of Lemma 3.5. Consider an optimal partition Δ that does not satisfy the properties stated in the lemma. First, recall that $k^* = \min\left\{k \in \mathcal{K} : \delta_k + \delta_{k-1} > \frac{\alpha-1}{\alpha}\right\}$. Suppose $k^*(\Delta) > 2$, which requires $K > 2$. Construct a new partition $\hat{\Delta}$ with $\hat{\delta}_1 = \delta_{k^*-1}$, $\hat{\delta}_k = \delta_{k^*}$ for $1 < k < k^*$ and $\hat{\delta}_k = \delta_k$ otherwise. By construction, the partition $\hat{\Delta}$ leads to the same objective value as Δ and is therefore an optimal partition. Furthermore, we have

$$\begin{aligned} \hat{\delta}_1 + \hat{\delta}_0 = \delta_{k^*-1} + \delta_0 \leq \delta_{k^*-1} + \delta_{k^*-2} &\leq \tfrac{\alpha-1}{\alpha} \\ \hat{\delta}_k + \hat{\delta}_{k-1} \geq \delta_{k^*} + \delta_{k^*-1} &> \tfrac{\alpha-1}{\alpha}, \qquad \text{for } k = 2, \ldots, K. \end{aligned}$$

Thus, $k^*(\hat{\Delta}) = 2$. Therefore, by applying this transformation, we can assume without loss of generality that the optimal partition Δ satisfies $k^*(\Delta) \in \{1, 2\}$.

Second, we modify the partition Δ into a strictly better one, which is a contradiction. The details require two cases to be analysed.

Case I: there exists an index $i \in \{1, \ldots, K-1\}$ such that $\delta_{i-1} = \delta_i < \delta_{i+1}$. Notice that $i + 1 \geq 2 \geq k^*(\Delta) \in \{1, 2\}$. Therefore, $\delta_{i+1} + \delta_i > \frac{\alpha-1}{\alpha}$ and there exists an $0 < \epsilon < 1$ such that

$$(1 - \epsilon)\delta_{i+1} + (1 + \epsilon)\delta_i > \tfrac{\alpha-1}{\alpha}.$$

Construct a new partition $\hat{\Delta}$ by setting $\hat{\delta}_i = (1 - \epsilon)\delta_{i+1} + \epsilon\delta_i$ and $\hat{\delta}_k = \delta_k$ otherwise. By construction, we have $\hat{\delta}_{i-1} < \hat{\delta}_i < \hat{\delta}_{i+1}$, $i \geq k^*(\hat{\Delta})$, and $k^*(\hat{\Delta}) \leq k^*(\Delta)$. The normalised objective value corresponding to $\hat{\Delta}$ differs from that of Δ as follows: the terms

$$\begin{aligned} &\sum_{k=\max\{i, k^*(\Delta)\}}^{i+1} (\delta_k - \delta_{k-1}) \left(\tfrac{1}{\alpha} + \delta_k + \delta_{k-1} - 1\right)^{\frac{n+1}{n}} \\ &= (\delta_{i+1} - \delta_i) \left(\tfrac{1}{\alpha} + \delta_{i+1} + \delta_i - 1\right)^{\frac{n+1}{n}} \end{aligned}$$

are replaced by

$$
\begin{aligned}
(\hat{\delta}_i - \hat{\delta}_{i-1}) \left(\tfrac{1}{\alpha} + \hat{\delta}_i + \hat{\delta}_{i-1} - 1\right)^{\frac{n+1}{n}} &+ (\hat{\delta}_{i+1} - \hat{\delta}_i) \left(\tfrac{1}{\alpha} + \hat{\delta}_{i+1} + \hat{\delta}_i - 1\right)^{\frac{n+1}{n}} \\
&= (1-\epsilon)(\delta_{i+1} - \delta_i) \left(\tfrac{1}{\alpha} + (1-\epsilon)\delta_{i+1} + (1+\epsilon)\delta_i - 1\right)^{\frac{n+1}{n}} \\
&\quad + \epsilon(\delta_{i+1} - \delta_i) \left(\tfrac{1}{\alpha} + (2-\epsilon)\delta_{i+1} + \epsilon\delta_i - 1\right)^{\frac{n+1}{n}} \\
&> (\delta_{i+1} - \delta_i) \left(\tfrac{1}{\alpha} + \delta_{i+1} + \delta_i - 1\right)^{\frac{n+1}{n}} .
\end{aligned}
$$

The inequality follows from strict convexity of the function $(\cdot)^{\frac{n+1}{n}}$ on $\mathbb{R}_{\geq 0}$. This implies that $\hat{\Delta}$ is strictly better than Δ, which contradicts the optimality of Δ.

Case II: $\delta_{K-1} = \delta_K = 1$. Define $i = \min\{k \in \{1, \ldots, K\} : \delta_k = \delta_K\}$ to be the first partition point that coincides with δ_K. Notice that $\delta_0 < \delta_i$ and $\delta_i + \delta_{i-1} = 1 + \delta_{i-1} \geq 1 > \frac{\alpha-1}{\alpha}$, so $i \geq k^*(\Delta)$. Therefore, there exists an $0 < \epsilon < 1$ such that $(1-\epsilon)\delta_i + (1+\epsilon)\delta_{i-1} > \frac{\alpha-1}{\alpha}$.

Construct a new partition $\hat{\Delta}$ by setting $\hat{\delta}_i = (1-\epsilon)\delta_i + \epsilon\delta_{i-1}$ and $\hat{\delta}_k = \delta_k$ otherwise. By construction, we have $\hat{\delta}_{i-1} < \hat{\delta}_i < \hat{\delta}_{i+1}$ and $k^*(\hat{\Delta}) = k^*(\Delta)$. The normalised objective value corresponding to $\hat{\Delta}$ differs from that of Δ as follows: the terms

$$
\begin{aligned}
\sum_{k=\max\{i,k^*(\Delta)\}}^{i+1} & (\delta_k - \delta_{k-1}) \left(\tfrac{1}{\alpha} + \delta_k + \delta_{k-1} - 1\right)^{\frac{n+1}{n}} \\
&= (\delta_i - \delta_{i-1}) \left(\tfrac{1}{\alpha} + \delta_i + \delta_{i-1} - 1\right)^{\frac{n+1}{n}}
\end{aligned}
$$

are replaced by

$$
\begin{aligned}
(\hat{\delta}_i - \hat{\delta}_{i-1}) \left(\tfrac{1}{\alpha} + \hat{\delta}_i + \hat{\delta}_{i-1} - 1\right)^{\frac{n+1}{n}} &+ (\hat{\delta}_{i+1} - \hat{\delta}_i) \left(\tfrac{1}{\alpha} + \hat{\delta}_{i+1} + \hat{\delta}_i - 1\right)^{\frac{n+1}{n}} \\
&= (1-\epsilon)(\delta_i - \delta_{i-1}) \left(\tfrac{1}{\alpha} + (1-\epsilon)\delta_i + (1+\epsilon)\delta_{i-1} - 1\right)^{\frac{n+1}{n}} \\
&\quad + \epsilon(\delta_i - \delta_{i-1}) \left(\tfrac{1}{\alpha} + (2-\epsilon)\delta_i + \epsilon\delta_{i-1} - 1\right)^{\frac{n+1}{n}} \\
&> (\delta_i - \delta_{i-1}) \left(\tfrac{1}{\alpha} + \delta_i + \delta_{i-1} - 1\right)^{\frac{n+1}{n}} .
\end{aligned}
$$

Hence, $\hat{\Delta}$ is strictly better than Δ, which contradicts the optimality of Δ. This concludes the proof. □

Proof of Corollary 3.6. For $K > 3$ and for $\alpha \geq K/(K-3)$ we have $k^* \geq 3$, which is suboptimal by Lemma 3.5. □

3.A.5 Proofs for Section 3.2.4

Proof of Theorem 3.7. By Lemma 3.5 we know that $k^* \in \{1, 2\}$ for the optimal partition. First, we analyse the objective function Γ_K when we consider k^* as a parameter independent of the chosen partition (which it is not). To simplify notation, we use the normalised $\nu\Gamma_K$, which does not affect the optimality of a partition. Suppose $k^* = 1$, then the normalised objective function is

$$\begin{aligned}\nu\Gamma_K|_{k^*=1} = \Big(& (\delta_1 - \delta_0)(\tfrac{1}{\alpha} + \delta_1 + \delta_0 - 1)^2 + (\delta_2 - \delta_1)(\tfrac{1}{\alpha} + \delta_2 + \delta_1 - 1)^2 \\ & + (\delta_3 - \delta_2)(\tfrac{1}{\alpha} + \delta_3 + \delta_2 - 1)^2 + (\delta_4 - \delta_3)(\tfrac{1}{\alpha} + \delta_4 + \delta_3 - 1)^2 + \cdots \\ & + (\delta_{K-1} - \delta_{K-2})(\tfrac{1}{\alpha} + \delta_{K-1} + \delta_{K-2} - 1)^2 \\ & + (\delta_K - \delta_{K-1})(\tfrac{1}{\alpha} + \delta_K + \delta_{K-1} - 1)^2\Big).\end{aligned}$$

Since terms cancel out, this is a quadratic function for each δ_k, $k = 1, \ldots, K-1$. Note that $\delta_0 = 0$ and $\delta_K = 1$ are fixed for any partition. Setting the gradient to zero gives $(\delta_{k+1} - \delta_{k-1})(\delta_{k+1} + \delta_{k-1} - 2\delta_k) = 0$ for all $k \in \{1, \ldots, K-1\}$. By Lemma 3.5 we know that $\delta_{k+1} > \delta_{k-1}$ must hold for the optimal partition, so the only possibility is $\delta_k = \frac{1}{2}(\delta_{k+1} + \delta_{k-1})$ for all $k \in \{1, \ldots, K-1\}$. The solution to this linear system of equalities is

$$\delta_k = \frac{k}{K} \equiv \delta_k^{\mathrm{equi}},$$

which is the equidistant partition Δ^{equi}.

Likewise, suppose $k^* = 2$, then we have

$$\begin{aligned}\nu\Gamma_K|_{k^*=2} = \Big(& (\delta_2 - \delta_1)(\tfrac{1}{\alpha} + \delta_2 + \delta_1 - 1)^2 \\ & + (\delta_3 - \delta_2)(\tfrac{1}{\alpha} + \delta_3 + \delta_2 - 1)^2 + (\delta_4 - \delta_3)(\tfrac{1}{\alpha} + \delta_4 + \delta_3 - 1)^2 + \cdots \\ & + (\delta_{K-1} - \delta_{K-2})(\tfrac{1}{\alpha} + \delta_{K-1} + \delta_{K-2} - 1)^2 \\ & + (\delta_K - \delta_{K-1})(\tfrac{1}{\alpha} + \delta_K + \delta_{K-1} - 1)^2\Big).\end{aligned}$$

This is cubic in δ_1 and quadratic in the other δ_k ($k \in \{2, \ldots, K-1\}$). Setting the gradient to zero gives

$$(1 - \tfrac{1}{\alpha} + \delta_2 - 3\delta_1)(\tfrac{1}{\alpha} - 1 + \delta_2 + \delta_1) = 0$$

and $(\delta_{k+1} - \delta_{k-1})(\delta_{k+1} + \delta_{k-1} - 2\delta_k) = 0$ for $k \in \{2, \ldots, K-1\}$. The roots of the derivative of the cubic function are

$$\delta_1 = \tfrac{1}{3}(1 - \tfrac{1}{\alpha} + \delta_2) \qquad \text{and} \qquad \delta_1 = 1 - \tfrac{1}{\alpha} - \delta_2.$$

By closer investigation of the shape of this cubic function, the larger value of these two corresponds to the maximum. Solving the system of linear equations for both cases results in:

$$\begin{aligned}\delta_1 = \tfrac{1}{3}(1 - \tfrac{1}{\alpha} + \delta_2) \qquad & \Longrightarrow \qquad \delta_k = \frac{K+k-1}{2K-1} - \frac{K-k}{2K-1}\frac{1}{\alpha}, \\ \delta_1 = 1 - \tfrac{1}{\alpha} - \delta_2 \qquad & \Longrightarrow \qquad \delta_k = \frac{K+k-3}{2K-3} - \frac{K-k}{2K-3}\frac{1}{\alpha}.\end{aligned}$$

Since δ_1 is larger in the first case, this is the correct solution. Hence,

$$\delta_k = \frac{K+k-1}{2K-1} - \frac{K-k}{2K-1}\frac{1}{\alpha} = 1 - \frac{K-k}{2K-1}\left(\frac{1}{\alpha}+1\right) \equiv \delta_k^{\text{cubic}}.$$

We refer to this partition as Δ^{cubic}.

Finally, we check which of these partitions is feasible. First, we check correctness with k^*. We have ($\Delta^{\text{equi}} \Rightarrow k^* = 1$) if and only if $\delta_1^{\text{equi}} > \frac{\alpha-1}{\alpha}$, which is $\frac{1}{K} > \frac{\alpha-1}{\alpha}$ or equivalently $\alpha < \frac{K}{K-1}$. Likewise, ($\Delta^{\text{cubic}} \Rightarrow k^* = 2$) if and only if $\delta_1^{\text{cubic}} \le \frac{\alpha-1}{\alpha}$ and $\delta_2^{\text{cubic}} + \delta_1^{\text{cubic}} > \frac{\alpha-1}{\alpha}$. These conditions require the following:

$$\begin{aligned}
\delta_1^{\text{cubic}} \le \tfrac{\alpha-1}{\alpha} \quad &\iff \quad 1 - \tfrac{K-1}{2K-1}\left(\tfrac{1}{\alpha}+1\right) \le \tfrac{\alpha-1}{\alpha} \quad &\iff \quad \alpha \ge \tfrac{K}{K-1},\\
\delta_2^{\text{cubic}} + \delta_1^{\text{cubic}} > \tfrac{\alpha-1}{\alpha} \quad &\iff \quad 2 - \tfrac{2K-3}{2K-1}\left(\tfrac{1}{\alpha}+1\right) > \tfrac{\alpha-1}{\alpha} \quad &\iff \quad \alpha > 0.
\end{aligned}$$

Moreover, we need to check $\delta_1^{\text{cubic}} > \delta_0 = 0$:

$$\delta_1^{\text{cubic}} > 0 \quad \iff \quad 1 - \tfrac{K-1}{2K-1}\left(\tfrac{1}{\alpha}+1\right) > 0 \quad \iff \quad \alpha > \tfrac{K-1}{K}.$$

This condition is trivially satisfied for the range $\alpha \ge \frac{K}{K-1}$ corresponding to Δ^{cubic}.

To conclude, for $\alpha < K/(K-1)$ the optimal partition is Δ^{equi}, whereas for $\alpha \ge K/(K-1)$ the optimal partition is Δ^{cubic}. □

Proof of Corollary 3.8. For $\alpha < K/(K-1)$ this is trivial as the optimal partition is equidistant. For $\alpha \ge K/(K-1)$ we have

$$\delta_k^{\text{opt}} = \tfrac{K+k-1}{2K-1} - \tfrac{K-k}{2K-1}\tfrac{1}{\alpha} \in \left[\tfrac{k}{K}, \tfrac{K+k-1}{2K-1}\right).$$

Combining these properties gives the desired result. □

3.A.6 Proofs for Section 3.2.5

Proof of Theorem 3.9. First, we consider $k^* = 1$, for which

$$\nu\Gamma_2 = \delta_1\left(\tfrac{1}{\alpha} + \delta_1 - 1\right)^{\frac{3}{2}} + (1-\delta_1)\left(\tfrac{1}{\alpha}+\delta_1\right)^{\frac{3}{2}}.$$

Setting the derivative with respect to δ_1 to zero and solving the equation for δ_1 results in two solutions:

$$\delta_1^+ = \tfrac{1}{30}\left(\sqrt{36\tfrac{1}{\alpha^2} - 15} + 15 - 6\tfrac{1}{\alpha}\right) \quad \text{and} \quad \delta_1^- = \tfrac{1}{30}\left(-\sqrt{36\tfrac{1}{\alpha^2} - 15} + 15 - 6\tfrac{1}{\alpha}\right).$$

The partition point δ_1^+ is a local maximum, whereas δ_1^- does not maximise the objective. Furthermore, δ_1^+ is valid for $\alpha \in (0, \frac{2}{5}\sqrt{15}]$, i.e., it is feasible and corresponds to $k^* = 1$.

Second, consider $k^* = 2$, where the normalised optimal objective is

$$\nu\Gamma_2 = (1-\delta_1)\left(\tfrac{1}{\alpha}+\delta_1\right)^{\frac{3}{2}}.$$

Again, setting its derivative to δ_1 to zero, leads to a single feasible solution

$$\delta_1^* = 1 - \tfrac{2}{5}(\tfrac{1}{\alpha} + 1).$$

The partition point δ_1^* is valid for $\alpha \in [\frac{3}{2}, \infty)$.

In contrast to the DMU-1 problem, the valid intervals overlap: $\alpha \in (0, 1.5491]$ for $k^* = 1$ and $\alpha \in [1.5, \infty)$ for $k^* = 2$. It turns out that the optimal δ_1 switches from δ_1^+ to δ_1^* (a discontinuous jump) as α increases. The partitions δ_1^+ and δ_1^* are both optimal for $\alpha^{\text{trans}} \approx 1.5371$ (the exact expression for α^{trans} is too verbose). This is illustrated in Figure 3.7, where the maximum on the left corresponds to δ_1^* and that on the right to δ_1^+.

Finally, the lower bound on δ_1^{opt} is attained at the switch to δ_1^* (at α^{trans}) and the upper bound is reached for $\alpha \to \infty$. □

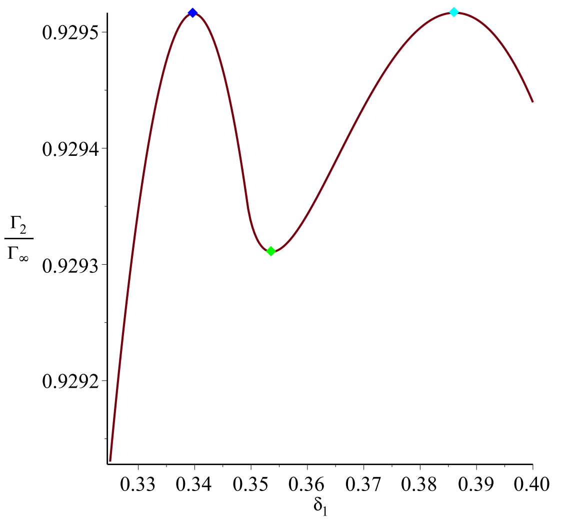


Figure 3.7: DMU-2: pooling performance Γ_2/Γ_∞ at α^{trans} in terms of the partition δ_1. The shown points are δ_1^* (left), δ_1^- (middle), and δ_1^+ (right).

3.A.7 Numerical solver

We describe the used methodology to numerically optimise the partition for the DMU-2 problem of Section 3.2.5. For each α, we have to maximise Γ_K or equivalently Γ_K/Γ_∞, so we can use the formulas of the normalised objective values $\nu\Gamma_K$ and $\nu\Gamma_\infty$. However, the formula for Γ_K contains index k^*, which depends on the partition. From Lemma 3.5 we know that $k^* \in \{1, 2\}$ for the optimal partition. Therefore, we simply optimise twice: for $k^* = 1$ in the formula with the restriction $\delta_1 > \frac{\alpha-1}{\alpha}$, and for $k^* = 2$ with the restrictions $\delta_1 \leq \frac{\alpha-1}{\alpha}$ and $\delta_1 + \delta_2 > \frac{\alpha-1}{\alpha}$. The optimal partition is the best of the resulting partitions. Note that $k^* = 1$ is always optimal for $0 < \alpha \leq 1$, since $k^* = 2$ is infeasible for this range.

For DMU-1 and DMU-2, we have inspected the shape of Γ_K as a function of the used partition for $K = 2$ and $K = 3$. From our observations, Γ_K with k^* fixed to 1 or 2 is a smooth function with a clear maximum on the respective domain that can be found using a gradient-based search. Thus, we apply a gradient-based search to maximise Γ_K with k^* fixed to either 1 or 2. To be precise, we use Maple's built-in solver 'NLPSolve' for non-linear programs with only bounds on the variables, which uses the modified-Newton method, and verify the feasibility of the obtained partition. That is, we check if $0 = \delta_1 < \delta_2 < \cdots < \delta_{K-1} < \delta_K = 1$ and if the partition indeed results in the used value for k^*.

We have also used Maple's built-in solver 'NLPSolve' for constrained non-linear programs, which uses sequential quadratic programming. With this solver we can enforce the required constraints on δ_k directly. Note that we still separately solve for k^* fixed to either 1 or 2. Both solvers give the same results, also when specifying different starting solutions.

The used methodology only guarantees to find a local maximum. However, the numerical solver always finds the same local optimum, i.e., it is stable. Furthermore, the results are consistent with our available theoretical results, such as for DMU-1 with any K and for DMU-2 with $K = 2$. Therefore, all results indicate that the numerical solver is able to find the global maximum.

3.B Addendum to Section 3.3

3.B.1 Reformulation and analysis

We follow the approach of Section 3.2 to reformulate the robust pooling model for cost minimisation, given in Section 3.3.1. First, we make Assumption 3.2, so the buyer's cost function is given by $\phi_B(x|p) = \psi(x) + p\chi(x)$. Under this assumption, we perform a change of variables by redefining the side payment as

$$z_k = \psi(x_k) + y_k - \Theta.$$

Notice that compared to Section 3.2 we include the outside option Θ in this change of variables. Substitution leads to an equivalent model with simpler constraints:

$$\begin{aligned}
\min_{x,y} \quad & \sum_{k\in\mathcal{K}} \omega_k\big(\phi_S(x_k) + \psi(x_k) + y_k\big) - \Theta \\
\text{s.t.} \quad & p_k\chi(x_k) - y_k \le 0, && \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k \in \mathcal{K}, && (3.26)\\
& p_k\chi(x_k) - y_k \le p_k\chi(x_l) - y_l, && \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k, l \in \mathcal{K}, && (3.27)\\
& x_k \ge 0, && \forall\, k \in \mathcal{K}.
\end{aligned}$$

This formulation clearly shows that Θ has no effect on the optimal order quantities and is simply a constant to be included in the side payment. We continue with the structure of the optimal solution.

Lemma 3.14. *Under Assumption 3.2, for any feasible x it is optimal to set*

$$y_k = \bar{p}_k \chi(x_k) + \sum_{i=k+1}^{K} (\bar{p}_i - \underline{p}_i)\chi(x_i) \qquad \forall\, k \in \mathcal{K}. \tag{3.28}$$

Proof. The proof is essentially the same as that of Lemma 3.1. The only difference is that the optimal y must satisfy $y_K = \bar{p}_K \chi(x_K)$. In other words, we have a tight IR constraint for $k = K$ instead of $k = 1$.

First, realise that for an optimal y at least one IR constraint (3.26) must hold with equality. If this is not the case, we can decrease all y_k by subtracting some $\epsilon > 0$ until at least one IR constraint is tight. This new solution is still feasible, as (3.27) only considers the difference $y_k - y_l$, which is unaffected. Moreover, the objective value of the new solution is strictly smaller.

Suppose that $y_K > \bar{p}_K \chi(x_K)$, then for $k \in \mathcal{K}$ we have for all $p_k \in [\underline{p}_k, \bar{p}_k]$ that

$$p_k \chi(x_k) - y_k \leq \bar{p}_k \chi(x_k) - y_k \overset{(3.27)}{\leq} \bar{p}_k \chi(x_K) - y_K \leq \bar{p}_K \chi(x_K) - y_K < 0.$$

That is, no IR constraint is tight, which is a contradiction. Hence, for an optimal y we must have $y_K = \bar{p}_K \chi(x_K)$.

Second, fix $k \in \mathcal{K}$ with $k < K$ and consider the IC constraints between contracts k and $k+1$:

$$\bar{p}_k(\chi(x_k) - \chi(x_{k+1})) \leq y_k - y_{k+1} \leq \underline{p}_{k+1}(\chi(x_k) - \chi(x_{k+1})).$$

Since $\bar{p}_k = \underline{p}_{k+1}$, this implies that

$$y_k - y_{k+1} = \bar{p}_k(\chi(x_k) - \chi(x_{k+1})).$$

Using our earlier result that $y_K = \bar{p}_K \chi(x_K)$, we obtain the following formula:

$$y_k = \sum_{i=k}^{K-1} \bar{p}_i(\chi(x_i) - \chi(x_{i+1})) + \bar{p}_K \chi(x_K),$$

which can be rewritten into (3.28). □

Lemma 3.14 shows that the side payment z_k for contract $k \in \mathcal{K}$ only depends on the order quantities of contracts with a higher index $(k+1, \ldots, K)$. In terms of indices, this dependency is reversed in Lemma 3.1. However, in terms of efficiency the result is not reversed. Thus, both lemmas state that the side payment depends on the order quantities corresponding to less efficient buyers. We observe this phenomenon also in the feasible region, see Lemma 3.15.

Lemma 3.15. *Under Assumption 3.2, any x is feasible if and only if $x_1 \geq \cdots \geq x_K \geq 0$.*

Proof. The proof is the same as that of Lemma 3.2, except that all inequalities related to the constraints are reversed.

First, we show the necessity of $x_1 \geq \cdots \geq x_K$. Let $k, k+1 \in \mathcal{K}$ and consider (3.27) for $\bar{p}_k$ and $\bar{p}_{k+1}$:

$$\begin{aligned} \bar{p}_k \chi(x_k) - y_k &\leq \bar{p}_k \chi(x_{k+1}) - y_{k+1}, \\ \bar{p}_{k+1} \chi(x_{k+1}) - y_{k+1} &\leq \bar{p}_{k+1} \chi(x_k) - y_k. \end{aligned}$$

Adding both IC constraints leads to $(\bar{p}_k - \bar{p}_{k+1})(\chi(x_k) - \chi(x_{k+1})) \leq 0$. Since we have chosen $\bar{p}_k < \bar{p}_{k+1}$, this implies that $\chi(x_k) \geq \chi(x_{k+1})$ and thus $x_k \geq x_{k+1}$.

Second, we show sufficiency of $x_1 \geq \cdots \geq x_K \geq 0$. Let $x \geq 0$ be non-increasing and set y according to (3.28). It remains to check feasibility of (x, y). Fix $k \in \mathcal{K}$ and $p_k \in [\underline{p}_k, \bar{p}_k]$. For $l \in \mathcal{K}$ with $k < l$ we have

$$\begin{aligned} y_k - y_l &= \sum_{i=k}^{l-1} (y_i - y_{i+1}) \overset{(3.28)}{=} \sum_{i=k}^{l-1} \bar{p}_i (\chi(x_i) - \chi(x_{i+1})) \\ &\geq \sum_{i=k}^{l-1} \bar{p}_k (\chi(x_i) - \chi(x_{i+1})) = \bar{p}_k (\chi(x_k) - \chi(x_l)) \geq p_k (\chi(x_k) - \chi(x_l)). \end{aligned}$$

Likewise, let $l \in \mathcal{K}$ with $l < k$, then

$$\begin{aligned} y_k - y_l &= \sum_{i=l+1}^{k} (y_i - y_{i-1}) \overset{(3.28)}{=} \sum_{i=l+1}^{k} \bar{p}_{i-1} (\chi(x_i) - \chi(x_{i-1})) \\ &\geq \sum_{i=l+1}^{k} \underline{p}_k (\chi(x_i) - \chi(x_{i-1})) = \underline{p}_k (\chi(x_k) - \chi(x_l)) \geq p_k (\chi(x_k) - \chi(x_l)). \end{aligned}$$

Hence, all IC constraints (3.27) hold. Furthermore,

$$p_k \chi(x_k) - y_k \leq \bar{p}_k \chi(x_k) - y_k \overset{(3.27)}{\leq} \bar{p}_k \chi(x_K) - y_K \leq \bar{p}_K \chi(x_K) - y_K = 0.$$

Thus, all IR constraints (3.26) are satisfied and the solution is feasible. $\square$

With these Lemmas we reformulate the robust pooling problem in terms of only the order quantities x, see Theorem 3.16. Again, notice the slight changes compared to Theorem 3.3.

Theorem 3.16. *Under Assumption 3.2, the robust pooling model with infinitely many constraints is equivalent to the following problem with finitely many and linear constraints:*

$$\min_{x_1 \geq \cdots \geq x_K \geq 0} \sum_{k \in \mathcal{K}} \omega_k \left(\phi_S(x_k) + \psi(x_k) + \Big(\bar{p}_k + (\bar{p}_k - \underline{p}_k) \sum_{i=1}^{k-1} \frac{\omega_i}{\omega_k} \Big) \chi(x_k) \right) - \Theta.$$

Proof. By Lemma 3.14 we can substitute the optimal formula (3.28) for y into the optimisation model. By Lemma 3.15 we conclude that the IR and IC constraints hold if and only if $x_1 \geq \cdots \geq x_K \geq 0$. This leads to the equivalent optimisation problem

$$\min_{x_1 \geq \cdots \geq x_K \geq 0} \sum_{k \in \mathcal{K}} \omega_k \Big(\phi_S(x_k) + \psi(x_k) + \bar{p}_k \chi(x_k) + \sum_{i=k+1}^{K} (\bar{p}_i - \underline{p}_i) \chi(x_i) \Big) - \Theta,$$

which can be rewritten into to formulation of the theorem by collecting the terms of x_k. □

We make additional assumptions to find a closed form solution, see Assumptions 3.6 and 3.7.

Assumption 3.6. The function $\phi_S + \psi$ is strictly convex and differentiable on $\mathbb{R}_{\geq 0}$. The function χ is given by $\chi(x) = x$.

Assumption 3.7. The distribution on the private parameter p is uniform: $\omega(p) = 1/(\bar{p} - \underline{p})$, so $\omega_k = (\bar{p}_k - \underline{p}_k)/(\bar{p} - \underline{p})$ for all $k \in \mathcal{K}$.

These assumptions allow us to derive Theorem 3.17, which corresponds to Theorem 3.13 for utility maximisation.

Theorem 3.17. *Under Assumptions 3.2-3.7, the robust pooling model is equivalent to the following convex problem:*

$$\min_{x_1 \geq \cdots \geq x_K \geq 0} \sum_{k \in \mathcal{K}} \frac{\bar{p}_k - \underline{p}_k}{\bar{p} - \underline{p}} \Big(\phi_S(x_k) + \psi(x_k) + (\bar{p}_k + \underline{p}_k - \underline{p}) x_k \Big) - \Theta.$$

The optimal solution is given by

$$x_k = \begin{cases} \infty & \text{if } k < \hat{k} \\ \left(\frac{\mathrm{d}}{\mathrm{d}x} (\phi_S + \psi) \right)^{-1} (\underline{p} - \bar{p}_k - \underline{p}_k) & \text{if } \hat{k} \leq k \leq k^* \\ 0 & \text{if } k > k^* \end{cases} .$$

and satisfy $\infty > x_{\hat{k}} > \cdots > x_{k^*} > 0$. *Here, the index of the last non-zero order quantity is*

$$k^* = \max \left\{ 0, \max \left\{ k \in \mathcal{K} : \bar{p}_k + \underline{p}_k - \underline{p} < -\frac{\mathrm{d}}{\mathrm{d}x} (\phi_S + \psi)(0) \right\} \right\},$$

and the index of the first finite order quantity is

$$\hat{k} = \min \left\{ K + 1, \min \left\{ k \in \mathcal{K} : \bar{p}_k + \underline{p}_k - \underline{p} > -\lim_{x \to \infty} \frac{\mathrm{d}}{\mathrm{d}x} (\phi_S + \psi)(x) \right\} \right\}.$$

Proof. The proof is similar to that of Theorem 3.13, except that we use the strict convexity instead of strict concavity of $\phi_S + \psi$.

The uniform distribution implies that $\omega_k = (\bar{p}_k - \underline{p}_k)/(\bar{p} - \underline{p})$. Therefore, we can simplify the summation as follows:

$$(\bar{p}_k - \underline{p}_k) \sum_{i=1}^{k-1} \frac{\omega_i}{\omega_k} = \sum_{i=1}^{k-1} (\bar{p}_i - \underline{p}_i) = \underline{p}_k - \underline{p}.$$

Substituting these expressions in the result of Theorem 3.16 gives the desired formulation.

For the structure of the optimal solution, we first relax the constraint $x_1 \geq \cdots \geq x_K$ to obtain a separable optimisation problem. Since the objective of the relaxed problem is strictly convex and differentiable (Assumption 3.6), its optimal solution can easily be determined. Consider contract $k \in \mathcal{K}$. We distinguish three cases.

Case I: if

$$\frac{\mathrm{d}}{\mathrm{d}x_k}(\phi_S + \psi)(0) + \bar{p}_k + \underline{p}_k - \underline{p} \geq 0,$$

then it is optimal for the relaxed problem to set $x_k = 0$. Otherwise, the optimal x_k for the relaxed problem satisfies $x_k > 0$.

Case II: if

$$\lim_{x_k \to \infty} \left(\frac{\mathrm{d}}{\mathrm{d}x_k}(\phi_S + \psi)(x_k) + \bar{p}_k + \underline{p}_k - \underline{p} \right) \leq 0,$$

then it is optimal to set $x_k = \infty$. Otherwise, a finite x is optimal. Here, we use that the above limit is zero only if it is an asymptote. This holds since $\phi_S + \psi$ is strictly convex, implying that its derivative is strictly increasing. Furthermore, this shows that Cases I and II are indeed mutually exclusive for (fixed) k.

Case III: if the above cases do not hold the optimal x_k is found by setting the derivative to zero:

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}x_k}\Big(\phi_S(x_k) + \psi(x_k) + (\bar{p}_k + \underline{p}_k - \underline{p})x_k \Big) = 0 \\ \iff \quad & \frac{\mathrm{d}}{\mathrm{d}x_k}(\phi_S + \psi)(x_k) = -(\bar{p}_k + \underline{p}_k - \underline{p}). \end{aligned}$$

Since $\phi_S + \psi$ is strictly convex and differentiable, its derivative is continuous and invertible. Furthermore, Cases I and II are excluded, so the following value is well-defined and strictly positive:

$$\hat{x}_k = \left(\frac{\mathrm{d}}{\mathrm{d}x_k}(\phi_S + \psi) \right)^{-1} (\underline{p} - \bar{p}_k - \underline{p}_k),$$

which is the optimum for the relaxed problem. Notice that the definitions of k^* and $\hat{k}$ imply that Case III corresponds to $k \in \mathcal{K}$ such that $\hat{k} \leq k \leq k^*$. By strict convexity of $\phi_S + \psi$ we know that its derivative is strictly increasing. Furthermore, realise that $\bar{p}_k + \underline{p}_k - \underline{p} < \bar{p}_{k+1} + \underline{p}_{k+1} - \underline{p}$ for all $k \in \mathcal{K}$. Therefore, we have $\hat{x}_{\hat{k}} > \cdots > \hat{x}_{k^*} > 0$.

Combining all cases leads to a solution satisfying $x_1 \geq \cdots \geq x_K \geq 0$, which is feasible for the non-relaxed problem. Hence, this is the optimal solution to our original problem. $\square$

3.B.2 Proofs for Section 3.3.2

Proof of Theorem 3.10. We apply Theorem 3.17 stated in Appendix 3.B.1 to the EOQ problem. Since $\phi_S(x)+\psi(x) = (R+r)\frac{1}{x}+Px$ it is straightforward to determine that $k^* = K$ and $\hat{k} = 1$ (note that the definitions of k^* and $\hat{k}$ differ from Section 3.2). The results now follow directly from the theorem after simplifying the expressions. □

Proof of Lemma 3.11. Consider an optimal partition that does not satisfy the properties stated in the lemma. We modify this partition into a strictly better one, which is a contradiction. The details require two cases to be analysed.

Case I: there exists a partition point $i \in \{1, \ldots, K-1\}$ such that $\delta_{i-1} = \delta_i < \delta_{i+1}$. Construct a new partition $\hat{\Delta}$ by setting $\hat{\delta}_i = \frac{1}{2}\delta_{i+1} + \frac{1}{2}\delta_i$ and $\hat{\delta}_k = \delta_k$ otherwise. By construction, we have $\hat{\delta}_{i-1} < \hat{\delta}_i < \hat{\delta}_{i+1}$. The normalised objective value corresponding to $\hat{\Delta}$ differs from that of Δ as follows: the terms

$$\sum_{k=i}^{i+1}(\delta_k - \delta_{k-1})\sqrt{\tfrac{1}{\alpha} + \delta_k + \delta_{k-1}} = (\delta_{i+1} - \delta_i)\sqrt{\tfrac{1}{\alpha} + \delta_{i+1} + \delta_i}$$

are replaced by

$$\begin{aligned}
&(\hat{\delta}_i - \hat{\delta}_{i-1})\sqrt{\tfrac{1}{\alpha} + \hat{\delta}_i + \hat{\delta}_{i-1}} + (\hat{\delta}_{i+1} - \hat{\delta}_i)\sqrt{\tfrac{1}{\alpha} + \hat{\delta}_{i+1} + \hat{\delta}_i} \\
&\quad = \tfrac{1}{2}(\delta_{i+1} - \delta_i)\sqrt{\tfrac{1}{\alpha} + \tfrac{1}{2}\delta_{i+1} + \tfrac{3}{2}\delta_i} + \tfrac{1}{2}(\delta_{i+1} - \delta_i)\sqrt{\tfrac{1}{\alpha} + \tfrac{3}{2}\delta_{i+1} + \tfrac{1}{2}\delta_i} \\
&\quad < (\delta_{i+1} - \delta_i)\sqrt{\tfrac{1}{\alpha} + \delta_{i+1} + \delta_i}.
\end{aligned}$$

The inequality follows from strict concavity of the square-root function on $\mathbb{R}_{\geq 0}$. This implies that $\hat{\Delta}$ is strictly better than Δ, which contradicts the optimality of Δ.

Case II: $\delta_{K-1} = \delta_K = 1$. Define $i = \min\{k \in \{1, \ldots, K\} : \delta_k = \delta_K\}$ to be the first partition point that coincides with δ_K. Notice that $\delta_0 < \delta_i$. Construct a new partition $\hat{\Delta}$ by setting $\hat{\delta}_i = \frac{1}{2}\delta_i + \frac{1}{2}\delta_{i-1}$ and $\hat{\delta}_k = \delta_k$ otherwise. By construction, we have $\hat{\delta}_{i-1} < \hat{\delta}_i < \hat{\delta}_{i+1}$. The normalised objective value corresponding to $\hat{\Delta}$ differs from that of Δ as follows: the terms

$$\sum_{k=i}^{i+1}(\delta_k - \delta_{k-1})\sqrt{\tfrac{1}{\alpha} + \delta_k + \delta_{k-1}} = (\delta_i - \delta_{i-1})\sqrt{\tfrac{1}{\alpha} + \delta_i + \delta_{i-1}}$$

are replaced by

$$\begin{aligned}
&(\hat{\delta}_i - \hat{\delta}_{i-1})\sqrt{\tfrac{1}{\alpha} + \hat{\delta}_i + \hat{\delta}_{i-1}} + (\hat{\delta}_{i+1} - \hat{\delta}_i)\sqrt{\tfrac{1}{\alpha} + \hat{\delta}_{i+1} + \hat{\delta}_i} \\
&\quad = \tfrac{1}{2}(\delta_i - \delta_{i-1})\sqrt{\tfrac{1}{\alpha} + \tfrac{1}{2}\delta_i + \tfrac{3}{2}\delta_{i-1}} + \tfrac{1}{2}(\delta_i - \delta_{i-1})\sqrt{\tfrac{1}{\alpha} + \tfrac{3}{2}\delta_i + \tfrac{1}{2}\delta_{i-1}} \\
&\quad < (\delta_i - \delta_{i-1})\sqrt{\tfrac{1}{\alpha} + \delta_i + \delta_{i-1}}.
\end{aligned}$$

Hence, $\hat{\Delta}$ is strictly better than Δ, which contradicts the optimality of Δ. This concludes the proof. □

Proof of Theorem 3.12. For $K = 2$ the normalised optimal objective value is given by

$$\nu\Gamma_2 = \delta_1\sqrt{\tfrac{1}{\alpha} + \delta_1} + (1-\delta_1)\sqrt{\tfrac{1}{\alpha} + 1 + \delta_1} - \nu\Theta^*.$$

The derivative with respect to δ_1 is

$$\frac{1-\delta_1}{2\sqrt{\frac{1}{\alpha} + \delta_1 + 1}} + \sqrt{\tfrac{1}{\alpha} + \delta_1} - \sqrt{\tfrac{1}{\alpha} + \delta_1 + 1} + \frac{\delta}{2\sqrt{\frac{1}{\alpha} + \delta_1}}$$

and has a single root for $\alpha > 0$, namely $\delta_1^{\mathrm{opt}} = \frac{1}{6\alpha}(\sqrt{\alpha^2 + 8\alpha + 4} + \alpha - 2)$. Furthermore, for $\delta_1 = 0$ the derivative is

$$-\sqrt{\tfrac{1}{\alpha} + 1} + \sqrt{\tfrac{1}{\alpha}} + \tfrac{1}{2}\left(\tfrac{1}{\alpha} + 1\right)^{-1/2} < 0.$$

The inequality follows from the strict concavity of the square-root function. Likewise, for $\delta_1 = 1$ the derivative is

$$-\sqrt{\tfrac{1}{\alpha} + 2} + \sqrt{\tfrac{1}{\alpha} + 1} + \tfrac{1}{2}\left(\tfrac{1}{\alpha} + 1\right)^{-1/2} > 0.$$

Hence, the root of the derivative is indeed the minimiser of the optimal objective value. The bounds follow from the following estimates:

$$\begin{aligned}\delta_1^{\mathrm{opt}} &< \frac{1}{6\alpha}(\sqrt{(2\alpha+2)^2} + \alpha - 2) = \tfrac{1}{2},\\ \delta_1^{\mathrm{opt}} &> \frac{1}{6\alpha}(\sqrt{(\alpha+2)^2} + \alpha - 2) = \tfrac{1}{3}.\end{aligned}$$

The bounds correspond to the limits $\lim_{\alpha\to 0}\delta_1(\alpha) = \frac{1}{2}$ and $\lim_{\alpha\to\infty}\delta_1(\alpha) = \frac{1}{3}$. $\square$

Chapter 4

Balancing expected and worst-case utility in contracting models with asymmetric information and pooling

Abstract

In this chapter, we consider a principal-agent contracting problem between a seller and a buyer, where the buyer has single-dimensional private information. The buyer's type is assumed to be continuously distributed on a closed interval. The seller designs a menu of finitely many contracts by pooling the buyer types a priori using a partition scheme. He maximises either his minimum utility or his expected utility, or uses a multi-objective approach. For each variation, we determine tractable reformulations and the optimal menu of contracts under certain conditions.

These results are applied to a contracting problem with quadratic utilities. We show that the optimal objective value is completely determined by the partition scheme, a single aggregate instance parameter, and a parameter encoding the seller's guaranteed obtained utility. This enables us to derive the optimal partition and exact performance guarantees. Our analysis shows that the seller should always offer at least two contracts in order to have reasonable performance guarantees, resulting in at least 88% of the expected utility compared to offering infinitely many contracts. By also optimising obtained worst-case utility, he can potentially achieve only 64% of the maximum expected utility.

This chapter is based on Kerkkamp et al. (2018a).

Contents

4.1 Introduction

We consider a principal-agent problem where the principal is a seller of products and where the agent is a potential buyer. The seller has the initiative and market power to make a one-time offer to the buyer in which he presents a menu of contracts. We assume that this is a take-it-or-leave-it offer, i.e., we do not consider repeated offers or renegotiations. Each contract specifies an order quantity $x \in \mathbb{R}_{\geq 0}$ and a side payment $z \in \mathbb{R}$ from the buyer to the seller. The buyer has the market power to accept or reject any contract from the menu. Furthermore, we assume that both the seller and the buyer act individually rationally and want to maximise their own utility. Thus, the buyer will accept an offered contract if this is most beneficial to himself.

The buyer has private information that he does not share with the seller. We consider the case where the buyer's private information can be encoded into a single-dimensional parameter p, referred to as the buyer's *type*. We assume that the buyer's type can take on values in $[\underline{p}, \bar{p}] \subseteq \mathbb{R}$ with $\bar{p} > \underline{p}$ and follows a continuous distribution with strictly positive density function $\omega : [\underline{p}, \bar{p}] \to \mathbb{R}_{>0}$. Although the buyer's type is private, this distribution is known to the seller.

The seller has utility function $\phi_S : \mathbb{R}_{\geq 0} \to \mathbb{R}$ for an order quantity and his net utility also includes the side payment. Thus, if the buyer accepts a contract (x, z), the resulting seller's net utility is $\phi_S(x) + z$. Likewise, a buyer with type p has utility function $\phi_B(\cdot|p) : \mathbb{R}_{\geq 0} \to \mathbb{R}$. His net utility for contract (x, z) is $\phi_B(x|p) - z$. If the buyer rejects all contracts, we assume that his net utility is zero, which is also known as the buyer's *reservation level*.

Due to the buyer's private information, the seller can and will use mechanism design to construct a menu of contracts for the buyer to choose from. For a general reference on mechanism design for contracting problems, see for example Laffont and Martimort (2002). We consider the case where the seller only offers a limited number of contracts, similar to the robust pooling approach in Chapter 3. That is, the seller first decides how many contracts are offered, indicated by $K \in \mathbb{N}_{\geq 1}$. For notational convenience, let $\mathcal{K} = \{1, \ldots, K\}$. Second, the seller partitions $[\underline{p}, \bar{p}]$ into K subintervals $[\underline{p}_k, \bar{p}_k]$ with $\bar{p}_k > \underline{p}_k$ for $k \in \mathcal{K}$. Such a partition is called a *proper K-partition*. Third, the seller constructs a menu of K contracts by solving a specific optimisation model, given below, which depends on the chosen partition. Finally, this menu is offered to the buyer.

From this point onwards, we refer to a contract by (x_k, z_k) with $k \in \mathcal{K}$ and to a menu of contracts by (x, z), where $x = (x_1, \ldots, x_K)$ and $z = (z_1, \ldots, z_K)$. The menu is designed such that for each $k \in \mathcal{K}$ it is for all types in $[\underline{p}_k, \bar{p}_k]$ most beneficial to choose contract (x_k, z_k). We have the following constraints for the menu:

$$\phi_B(x_k|p_k) - z_k \geq 0, \qquad \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k \in \mathcal{K}, \tag{4.1}$$

$$\phi_B(x_k|p_k) - z_k \geq \phi_B(x_l|p_k) - z_l, \qquad \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k, l \in \mathcal{K}, \tag{4.2}$$

$$x_k \geq 0, \qquad \forall\, k \in \mathcal{K}. \tag{4.3}$$

Constraints (4.3) simply enforce non-negative order quantities. The other constraints (4.1) and (4.2) affect which contract the buyer will choose. Constraints (4.1) ensure Individual Rationality (IR) for the buyer: for $k \in \mathcal{K}$ and $p_k \in [\underline{p}_k, \bar{p}_k]$ contract (x_k, z_k) must not give a lower net utility than the buyer's reservation level. If (4.1) does not hold, then type p_k will never accept contract (x_k, z_k). We make the conventional assumption that if the buyer has multiple options which all maximise his net utility, then the seller can convince the buyer to choose from these the most beneficial option to the seller. Consequently, (4.1) guarantees that all types will choose a contract from the menu. Constraints (4.2) ensure for $k \in \mathcal{K}$ that contract (x_k, z_k) has the highest net utility for all types in $[\underline{p}_k, \bar{p}_k]$. This is known as Incentive Compatibility (IC).

Thus, for a menu satisfying constraints (4.1)-(4.3) contract (x_k, z_k) is chosen by all types $[\underline{p}_k, \bar{p}_k]$. In other words, contract (x_k, z_k) is chosen by the buyer with probability

$$\omega_k \equiv \int_{\underline{p}_k}^{\bar{p}_k} \omega(p)\mathrm{d}p \qquad\qquad \forall\, k \in \mathcal{K}. \tag{4.4}$$

We consider two objective functions for the seller subject to constraints (4.1)-(4.3): he maximises either his expected net utility or his minimum net utility. With the above insight, the seller's expected net utility is given by

$$\sum_{k\in\mathcal{K}} \omega_k \left(\phi_S(x_k) + z_k\right). \tag{4.5}$$

This leads to the *Maximise Expected net utility* (ME) model: $\max\{(4.5) : (4.1)\text{-}(4.3)\}$. Similarly, the seller's minimum net utility is

$$\min_{k\in\mathcal{K}} \left(\phi_S(x_k) + z_k\right), \tag{4.6}$$

resulting in the *Maximise Minimum net utility* (MM) model: $\max\{(4.6) : (4.1)\text{-}(4.3)\}$.

It turns out that for a broad class of problems the MM model has multiple optimal solutions, as we will show. The seller can therefore choose from these optimal solutions based on a second criterion. In light of the seller's desire to maximise his utility, we consider the case that the seller selects the optimal MM solution with maximum *expected* net utility. This can be interpreted as a two-stage optimisation approach based on the MM and ME models.

In fact, we generalise this two-stage approach to a multi-objective approach by adding the constraint $\min_{k\in\mathcal{K}}(\phi_S(x_k) + z_k) \geq M$, or equivalently

$$\phi_S(x_k) + z_k \geq M, \qquad\qquad \forall\, k \in \mathcal{K}, \tag{4.7}$$

to the ME model for some parameter $M \in \mathbb{R}$. We note that (4.7) is also known as the *seller*'s individual rationality constraint, where M is the seller's reservation level. We call the resulting model the *Multi-Objective* (MO) model: $\max\{(4.5) : (4.7), (4.1)\text{-}(4.3)\}$. Notice that by choosing M sufficiently small/negative (4.7) is non-restrictive and the MO model becomes the ME model. Likewise, by setting M to the optimal MM objective value, the MO model has the above described two-stage interpretation and finds the optimal MM solution with maximum expected net utility. Hence, the parameter M allows the seller to analyse the trade-off between maximising expected or worst-case net utility in a multi-objective perspective.

We shall refer to the MM, ME, and MO models as pooling models in general. Our goal is to analyse these pooling models, determine the optimal solutions analytically, and apply the results to a concrete contracting problem. In particular, we want to analytically quantify the effect of pooling the buyer types, the chosen partition scheme, and the buyer's reservation level M.

4.1.1 Literature

In the MM model the seller maximises his minimum net utility, which is often called having a 'maximin' objective or being ambiguity averse in the literature. Here, the minimum is taken over all offered contracts or, equivalently, over all possible realisations of the buyer's type. Only the support $[\underline{p}, \bar{p}]$ of the distribution ω is needed. This is an extreme case of recent robust optimisation approaches to mechanism design, see for example Bergemann and Schlag (2011) and Pinar and Kizilkale (2017). In these models the minimum is taken over an uncertainty set for ω, e.g., the distribution ω cannot differ too much from a reference distribution. The resulting robust model is typically less conservative than the classical maximim model. For further references on using robust optimisation, see Aghassi and Bertsimas (2006), Ben-Tal et al. (2009), and Bergemann and Morris (2005).

Our pooling approach can be viewed as a different application of robust optimisation. Consider the classical discrete variant of the contracting problem (see for example Laffont and Martimort (2002)). Here, the buyer's type lies in the set $\{p_1, \ldots, p_K\}$ and follows a discrete distribution. If we associate an uncertainty set $[\underline{p}_k, \bar{p}_k]$ to type p_k, then our pooling model is the robust optimisation variant for the discrete model. By considering a continuum of types $[\underline{p}, \bar{p}]$ and using a partition scheme, we are restricting the uncertainty sets $[\underline{p}_k, \bar{p}_k]$ to form a partition of $[\underline{p}, \bar{p}]$.

A property of the pooling approach is to offer finitely many contracts to a continuum of buyer types. There are to our knowledge two papers in the literature that are strongly connected to this approach: Bergemann et al. (2011) and Wong (2014).

Bergemann et al. (2011) consider a linear-quadratic model with limited communication between the seller and the buyer based on Mussa and Rosen (1978). The seller wants to maximise his expected net utility. The limited communication restricts the seller to using a menu with finitely many contracts. In contrast to our pooling approach, they do not partition the types a priori. Instead, their menu maps each buyer type to one of the K contracts without any restrictions. By reformulating the problem into a mean square minimisation problem and applying quantisation theory, they are able to determine the optimal menu of contracts and the corresponding optimal mapping of buyer types to contracts. In particular, their results show that the restriction to K contracts leads to a loss in performance of the order $\Theta(1/K^2)$ compared to offering infinitely many contracts.

Wong (2014) uses the same modelling approach as Bergemann et al. (2011), but analyses a more general non-linear pricing problem (again maximising the seller's expected net utility). His analysis focuses on the loss in performance when restricting to K contracts instead of infinitely many contracts. In particular, he derives the same $\Theta(1/K^2)$ loss in performance as Bergemann et al. (2011), but under a more general setting.

For the ME model we have shown in Chapter 3 that the pooling approach and those of Bergemann et al. (2011) and Wong (2014) are equivalent provided that we use the K-partition of $[\underline{p}, \bar{p}]$ that maximises the seller's expected net utility. That is, both approaches lead to partitioning the buyer types. However, as argued in Wong (2014) and Chapter 3 determining the optimal K-partition is difficult in general. In case the optimal partition cannot be derived, the benefit of our pooling approach is that it allows for heuristic partition schemes in a simple and controlled way. As we will show, the complexity of the pooling models for a given partition is similar to classical discrete contracting models.

The previously discussed papers do not consider multi-objective optimisation. In terms of using a multi-objective approach, Zheng et al. (2015) has to our knowledge the strongest connection to our work. Zheng et al. (2015) consider a continuum of types $[\underline{p}, \bar{p}]$ and the seller offers a menu with infinitely many contracts, i.e., there is no a priori pooling. They suppose that the seller is not confident about the probability distribution ω and model this by a so-called ϵ-contamination: with probability $0 \leq \epsilon \leq 1$ the distribution ω is incorrect and the worst-case outcome of ω occurs. Consequently, the objective function is the weighted sum of the seller's expected net utility and the seller's minimum net utility. The assumed utility functions are $\phi_S(x) = -cx$ and $\phi_B(x|p) = p\chi(x)$, where χ is a strictly increasing, positive, and continuously differentiable concave function. Furthermore, the distribution ω has a non-decreasing hazard rate. They show that for $0 < \epsilon < 1$ the optimal menu effectively pools the types $[\underline{p}, p^*(\epsilon)]$ for some $\underline{p} < p^*(\epsilon) < \bar{p}$ and offers those types the same contract. For the types $(p^*(\epsilon), \bar{p}]$ infinitely many contracts are offered.

To compare their multi-objective approach to ours, we translate the model of Zheng et al. (2015) to our pooling setting. For given $0 \leq \epsilon \leq 1$ the resulting model is to maximise

$$(1-\epsilon)\sum_{k\in\mathcal{K}} \omega_k \left(\phi_S(x_k) + z_k\right) + \epsilon \min_{k\in\mathcal{K}} \left(\phi_S(x_k) + z_k\right)$$

subject to (4.1)-(4.3) with variables x and z. This is equivalent to maximising

$$(1-\epsilon)\sum_{k\in\mathcal{K}} \omega_k \left(\phi_S(x_k) + z_k\right) + \epsilon M$$

subject to (4.7) and (4.1)-(4.3) with variables x, z, and M. We refer to this model as the weighted objective model. Although this model is very similar to our MO model, there are differences. The most obvious difference is that for $\epsilon = 1$ the weighted objective model is our MM model, not our MO model (with correctly corresponding M). In this case, our MO model is a two-stage optimisation model which determines the optimal MM solution that maximises the seller's expected net utility. Typically, the MM model has multiple optimal solutions, whereas the MO model has just one (as we will show later). Furthermore, the parameter M in the MO model has a natural interpretation, namely the seller's reservation level. A similar interpretation of ϵ only follows indirectly after solving the weighted objective model and observing the corresponding optimal M. We will discuss further similarities and differences in more detail during our analysis.

Besides solving the pooling models for given number of contracts K, partition scheme, and seller's reservation level M, we want to quantify the effect of these choices on the corresponding optimal objective values. Due to the complexity we focus on a concrete contracting problem for such an analysis: the Linear-Quadratic-Uniform (LQU) problem adapted from Wong (2014). For the ME model variant of the LQU problem we also refer to Chapter 3, where we completed the analysis of Wong (2014). Since the MO model generalises the ME model, our results in this chapter supersede the mentioned analyses of the LQU problem. Furthermore, we are able to relate results for the LQU problem to Zheng et al. (2015).

4.1.2 Contribution

We analyse a contracting problem where the seller offers a menu of finitely many contracts to a buyer with a continuum $[\underline{p}, \bar{p}]$ of types. Here, the seller uses a partition scheme to pool the types a priori. Moreover, we consider a multi-objective approach for the seller's objective function which balances expected and worst-case (minimum) net utility. Compared to the literature, we extend related work by either considering a multi-objective approach (Bergemann et al. (2011) and Wong (2014)) or by pooling the buyer types with a partition scheme (Zheng et al. (2015)). Furthermore, there are differences in the modelling approaches as discussed in Section 4.1.1.

Under commonly used assumptions, we derive tractable reformulations for the pooling models and determine the optimal menu of contracts. The optimal menus all turn out to be the maxima of certain modified joint net utility functions. We apply and extend these results to a concrete contracting problem, namely the LQU problem. In particular, we derive the optimal partition scheme for the LQU problem and the corresponding optimal objective values. Consequently, we can analyse various performance measures that quantify the effect of the number of contracts K offered and of the seller's reservation level M. This leads to performance guarantees that give insight into the trade-off between maximising expected or worst-case net utility. All results are analytical and expressed in closed-form formulas.

The remainder of this chapter starts with the general analysis of the pooling models in Section 4.2, followed by the application to the LQU problem in Section 4.3. We conclude our findings in Section 4.4.

4.2 General analysis

In this section we analyse the three pooling models in detail. First, we present the essential details of the setting and the three models in Section 4.2.1. In Section 4.2.2 we derive tractable reformulations for the models under a common assumption on the buyer's utility function. Finally, we determine the optimal solution of the models for a broad class of problems in Section 4.2.3. All corresponding proofs are given in Appendix 4.A.

4.2.1 The models

As introduced in Section 4.1, we consider a seller with utility function $\phi_S : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and a buyer with utility function $\phi_B(\cdot|p) : \mathbb{R}_{\geq 0} \to \mathbb{R}$ for type $p \in [\underline{p}, \bar{p}] \subseteq \mathbb{R}$. The joint utility function is denoted by $\phi_J(\cdot|p) \equiv \phi_S(\cdot) + \phi_B(\cdot|p)$. The buyer's type follows a continuous distribution with strictly positive density function $\omega : [\underline{p}, \bar{p}] \to \mathbb{R}_{>0}$. The seller offers the buyer a menu with a limited number of contracts by pooling the buyer types a priori. In this menu, the k-th contract (x_k, z_k) specifies the order quantity $x_k \in \mathbb{R}_{\geq 0}$ and the side payment $z_k \in \mathbb{R}$ from the buyer to the seller. First, the seller chooses the number of contracts $K \in \mathbb{N}_{\geq 1}$ to offer. Second, the seller partitions $[\underline{p}, \bar{p}]$ into K subintervals $[\underline{p}_k, \bar{p}_k]$ with $\bar{p}_k > \underline{p}_k$ for $k \in \mathcal{K} = \{1, \dots, K\}$, leading to aggregate probabilities ω_k for $k \in \mathcal{K}$ given by (4.4). Given this partition/pooling scheme, the seller constructs a menu of K contracts by solving one of our pooling models: the MM model $\max\{(4.6) : (4.1)\text{-}(4.3)\}$, the ME model $\max\{(4.5) : (4.1)\text{-}(4.3)\}$, or the MO model $\max\{(4.5) : (4.7), (4.1)\text{-}(4.3)\}$.

We focus on the MO model:

$$\begin{aligned}
&\max_{x,z} && \sum_{k \in \mathcal{K}} \omega_k \left(\phi_S(x_k) + z_k\right) && \\
\text{s.t.} \quad & \phi_S(x_k) + z_k \geq M, && \forall\, k \in \mathcal{K}, && (4.7)\\
& \phi_B(x_k|p_k) - z_k \geq 0, && \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k \in \mathcal{K}, && (4.1)\\
& \phi_B(x_k|p_k) - z_k \geq \phi_B(x_l|p_k) - z_l, && \forall\, p_k \in [\underline{p}_k, \bar{p}_k], k, l \in \mathcal{K}, && (4.2)\\
& x_k \geq 0, && \forall\, k \in \mathcal{K}. && (4.3)
\end{aligned}$$

The MO model maximises the seller's expected net utility under individual rationality constraints for the seller (4.7) and the buyer (4.1), and under incentive compatibility constraints (4.2). Note that the buyer's reservation level in (4.1) is assumed to be zero, whereas the seller's reservation level $M \in \mathbb{R}$ in (4.7) is set by the seller. With the parameter M the seller can balance his expected net utility with his minimum net utility. In particular, by an appropriate choice of M we can solve the ME or MM model with the MO model. That is, if we increase M then the optimal solution transitions from an optimal ME solution to an optimal MM solution. As a final note, if M is too large, then the MO model is infeasible. These insights will be made more concrete in the following analysis.

4.2.2 Tractable reformulation

In order to obtain tractable reformulations of our models and the results to come, we need to assume additional structure on the buyer's utility function. Assumption 4.1 states that ϕ_B is non-decreasing in the buyer's type and satisfies the strictly increasing differences property.

Assumption 4.1. The buyer's utility function ϕ_B satisfies the following properties:

$$\begin{aligned}
\phi_B(x|\lambda) &\leq \phi_B(x|\mu) && \forall\, \lambda \leq \mu \in \mathbb{R}, x \geq 0, && (4.8)\\
\phi_B(x'|\lambda) - \phi_B(x|\lambda) &< \phi_B(x'|\mu) - \phi_B(x|\mu) && \forall\, \lambda < \mu \in \mathbb{R}, 0 \leq x < x'. && (4.9)
\end{aligned}$$

Note that we implicitly assume that $\phi_B(\cdot|\lambda)$ is defined for all $\lambda \in \mathbb{R}$, not just for $[\underline{p}, \bar{p}]$. However, it follows from the proofs that we only need to consider $2K$ instance-dependent values for λ. Assumption 4.1, or the stronger single-crossing condition, is common in the mechanism design literature and leads to non-decreasingness in the order quantities with respect to the buyer's type (see also Edlin and Shannon (1998), Laffont and Martimort (2002), and Schottmüller (2015)). This is also the case for the pooling models, as shown in Lemma 4.1.

Lemma 4.1. *Under Assumption 4.1, any x satisfies* (4.1)-(4.3) *if and only if* $0 \leq x_1 \leq \cdots \leq x_K$.

Furthermore, for fixed order quantities x, the IR and IC constraints (4.1) and (4.2) imply a dual shortest path problem structure on the side payments z. For non-pooling models this has been identified before, see for example Rochet and Stole (2003) and Vohra (2012). For our pooling models, the side payments are even more restricted, leading to the optimal formulas for z given in Lemma 4.2. In fact, we only need to assume (4.8) for this result, but we have chosen to merge certain assumptions for readability.

Lemma 4.2. *Consider the ME, MM, or MO model under Assumption 4.1. It is necessary and sufficient for optimality to set*

$$z_k = \phi_B(x_k|\underline{p}_k) - \sum_{i=1}^{k-1} \big(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\big) \qquad \forall\, k \in \mathcal{K}. \tag{4.10}$$

With Lemma 4.2 we can eliminate the side payments from our models. Using Lemma 4.1 we can then simplify the constraints from infinitely many to K linear constraints. This results in the tractable reformulations as shown in Theorem 4.3. Recall that $\phi_J(\cdot|p)$ is the joint utility function with respect to type p.

Theorem 4.3. *Under Assumption 4.1, the ME model is equivalent to*

$$\max_{0 \leq x_1 \leq \cdots \leq x_K} \sum_{k \in \mathcal{K}} \omega_k \left(\phi_J(x_k|\underline{p}_k) - \big(\phi_B(x_k|\bar{p}_k) - \phi_B(x_k|\underline{p}_k)\big) \sum_{i=k+1}^{K} \frac{\omega_i}{\omega_k} \right), \tag{4.11}$$

the MM model to

$$\max_{0 \leq x_1 \leq \cdots \leq x_K} \min_{k \in \mathcal{K}} \left(\phi_J(x_k|\underline{p}_k) - \sum_{i=1}^{k-1} \big(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\big) \right), \tag{4.12}$$

and the MO model to

$$\max_{x} \quad \sum_{k \in \mathcal{K}} \omega_k \left(\phi_J(x_k|\underline{p}_k) - \big(\phi_B(x_k|\bar{p}_k) - \phi_B(x_k|\underline{p}_k)\big) \sum_{i=k+1}^{K} \frac{\omega_i}{\omega_k} \right) \tag{4.13}$$

$$\text{s.t.} \quad \phi_J(x_k|\underline{p}_k) - \sum_{i=1}^{k-1} \big(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\big) \geq M, \qquad \forall\, k \in \mathcal{K}, \tag{4.14}$$

$$x_K \geq \cdots \geq x_1 \geq 0. \tag{4.15}$$

From the reformulations it is clear that the complexity of solving our pooling models depends on the shape of $\phi_J(\cdot|\underline{p}_k)$ and of $\phi_B(\cdot|\bar{p}_k) - \phi_B(\cdot|\underline{p}_k)$ for $k \in \mathcal{K}$. For example, if $\phi_J(\cdot|\underline{p}_k)$ is differentiable and concave, and if $\phi_B(\cdot|\bar{p}_k) - \phi_B(\cdot|\underline{p}_k)$ is linear, then all three models are concave optimisation problems with differential functions, which can be solved numerically in an efficient way. We focus on classifying problems for which the optimal solutions can be described in a unified way.

4.2.3 Optimal solutions

The next assumption excludes situations where the seller could potentially achieve infinite utility from the menu of contracts, see Assumption 4.2.

Assumption 4.2. For any $\lambda \in \mathbb{R}$ the joint utility function $\phi_J(\cdot|\lambda)$ has a maximum on $\mathbb{R}_{\geq 0}$ and on any closed subinterval of $\mathbb{R}_{\geq 0}$.

In Assumption 4.2, the existence of a maximum on any closed subinterval of $\mathbb{R}_{\geq 0}$ is needed because of a technicality (see the proof of Lemma 4.4 to come). In particular, Assumption 4.2 is satisfied if ϕ_S and ϕ_B are continuous functions in the order quantity.

The maximum of $\phi_J(\cdot|\lambda)$ for specific values of λ has a central role in the optimal solutions for our models. Therefore, we have an intermediate result on the maximisers of $\phi_J(\cdot|\lambda)$, see Lemma 4.4.

Lemma 4.4. *Under Assumptions 4.1-4.2, there exists a non-decreasing function $M^* : \mathbb{R} \to \mathbb{R}$ such that*

$$M^*(\lambda) \equiv \max_{x \geq 0}\{\phi_J(x|\lambda)\}.$$

For $M \leq M^*(\underline{p})$, there exists a non-decreasing function $x^*(\cdot|M) : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that

$$x^*(\lambda|M) \equiv \min \operatorname*{argmax}_{x \geq x^M}\{\phi_J(x|\lambda)\},$$

where $x^M \in \mathbb{R}_{\geq 0}$ is given by $x^M \equiv \min\left\{x \geq 0 : \phi_J(x|\underline{p}) \geq M\right\}$.

Recall that the seller's reservation level M only affects the MO model. For the MM and ME models, we can implicitly use an non-restrictive value for M, namely $M = -\infty$. For $M = -\infty$ we have $x^M = 0$, hence $x^*(\lambda|M)$ optimises over the entire domain $x \geq 0$. In this case we simplify our notation and use $x^*(\lambda)$ instead of $x^*(\lambda| - \infty)$. We can now express the optimal solution for the MM model, see Theorem 4.5.

Theorem 4.5. *Under Assumptions 4.1-4.2, the optimal objective value for the MM model is $M^*(\underline{p})$, which can be attained by offering a menu with a single contract with order quantity $x^*(\underline{p})$. Note that this does not depend on the partition of $[\underline{p}, \bar{p}]$. Another optimal solution is $x_k = x^*(\underline{p}_k)$ for $k \in \mathcal{K}$, which does depend on the partition.*

A consequence of Theorem 4.5 is that the complexity of solving the MM model is completely determined by the complexity of maximising $\phi_J(x|\underline{p})$ over $x \geq 0$. For a similar result for the ME and MO models, we need the assumption that each k-th term in the objective function can be written as $\phi_J(x_k|\lambda_k)$ for some λ_k non-decreasing in k. This assumption is formalised in Assumption 4.3.

Assumption 4.3. The density function ω and the buyer's utility function ϕ_B are such that there exist parameters $\pi_k \in \mathbb{R}$ for $k \in \mathcal{K}$ satisfying $\pi_1 \leq \cdots \leq \pi_K$ and

$$\phi_B(x|\pi_k) = \phi_B(x|\underline{p}_k) - \big(\phi_B(x|\bar{p}_k) - \phi_B(x|\underline{p}_k)\big) \sum_{i=k+1}^{K} \frac{\omega_i}{\omega_k} \quad \forall\, x \geq 0, k \in \mathcal{K}. \quad (4.16)$$

The parameters π_k are strongly related to the virtual valuation of the buyer types (see e.g. Laffont and Martimort (2002)). Under Assumptions 4.1 and 4.3 it is trivial to show that $\pi_k < \underline{p}_k$ for all $k \in \mathcal{K} \setminus \{K\}$ and $\pi_K \leq \underline{p}_K$ (see the proof of Theorem 4.7 to come). In Appendix 4.B we present an example problem class for which we prove that it satisfies Assumption 4.3 and provide a closed-form expression for π_k. In the example, the buyer's utility function is $\phi_B(x|p) = \psi(x) + p\chi(x)$ for some functions ψ and χ, where χ is strictly increasing and non-negative. Furthermore, ω is a continuous distribution with a non-decreasing hazard rate, e.g., the uniform distribution.

Under the additional assumption, we can derive the optimal solution for the ME model as shown in Theorem 4.6.

Theorem 4.6. *Under Assumptions 4.1-4.3, an optimal solution for the ME model is $x_k = x^*(\pi_k)$ for $k \in \mathcal{K}$.*

Compared to the MM model, where an optimal solution is given by the maximisers of $\phi_J(\cdot|\underline{p}_k)$ for $k \in \mathcal{K}$, an optimal ME solution is specified by the maximisers of $\phi_J(\cdot|\pi_k)$. In other words, we need to shift the buyer types downwards from $\underline{p}_k$ to π_k.

Last but not least, we have the optimal solution for the MO model. The MO model maximises the seller's expected net utility under the constraint that the seller's minimum net utility is at least his reservation level M. From Theorem 4.5 we know that the minimum net utility is at most $M^*(\underline{p})$, being the optimal objective value of the MM model. Therefore, any seller's reservation level $M \leq M^*(\underline{p})$ can be satisfied and any $M > M^*(\underline{p})$ is infeasible. Theorem 4.7 states the optimal MO solution.

Theorem 4.7. *Under Assumptions 4.1-4.3, the MO model is feasible if and only if $M \leq M^*(\underline{p})$, and an optimal solution for the MO model is $x_k = x^*(\pi_k|M)$ for $k \in \mathcal{K}$.*

In particular, if the seller sets $M = M^*(\underline{p})$, then the MO model is a two-stage optimisation which maximises first the seller's minimum net utility and second the seller's expected net utility. Hence, under Assumptions 4.1-4.3 we have identified a third optimal MM solution. This leads to the next straightforward corollary.

Corollary 4.8. *Under Assumptions 4.1-4.3, if for each $k \in \mathcal{K}$ the function*

$$\phi_J(x_k|\underline{p}_k) - \sum_{i=1}^{k-1} \big(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\big)$$

is concave on $x_K \geq \cdots \geq x_1 \geq 0$*, then any convex combination of*

$$\begin{aligned} x_k &= x^*(\underline{p}) && \forall\, k \in \mathcal{K}, \\ x_k &= x^*(\underline{p}_k) && \forall\, k \in \mathcal{K}, \\ x_k &= x^*(\pi_k | M^*(\underline{p})) && \forall\, k \in \mathcal{K}, \end{aligned}$$

is an optimal solution for the MM model.

Returning to Theorem 4.7, if $\phi_J(\cdot|\lambda)$ is concave, then the optimal MO solution can be found in two steps as follows. First, determine the optimal ME solution by using the shifted buyer types π_k, resulting in $x_k = x^*(\pi_k)$ for $k \in \mathcal{K}$. Second, set any $x_k < x^M$ to the threshold order quantity x^M in order to guarantee the seller's reservation level M. Zheng et al. (2015) derive a similar solution structure for their concave setting with infinitely many contracts, where the types $[\underline{p}, p^*]$ for some $\underline{p} < p^* < \bar{p}$ are offered the same contract (like x^M in our case). Returning to our result, the threshold x^M could lead to additional pooling of types as multiple contracts can specify the order quantity x^M. This implies that the original partition of $[\underline{p}, \bar{p}]$ can be improved to increase the seller's expected net utility.

This brings us to one of the decisions the seller has to make: the partition of $[\underline{p}, \bar{p}]$. Based on our results, we have the following strategy for the seller. First, the seller must determine the optimal MM objective value $M^*(\underline{p})$, which is independent of the partition. Second, he must decide on his reservation level $M \leq M^*(\underline{p})$. Third, he chooses the number of contracts K offered. Finally, the seller selects a partition and uses the above results to determine an optimal menu of contracts for the MO model.

Ideally, the seller optimises the partition such that his expected net utility is maximised. Unfortunately, such optimisation appears to be difficult in general. Given the complexity of the analysis, we focus on the so-called linear-quadratic-uniform problem adapted from Wong (2014). In Section 4.3 we derive the optimal partition and analyse performance guarantees for this specific problem.

4.3 Application to the LQU problem

In this section, we apply the results of Section 4.2 to a concrete contracting problem, called the Linear-Quadratic-Uniform (LQU) problem. We formalise the LQU problem in Section 4.3.1 and translate our general results from Section 4.2 to this setting in Section 4.3.2. In Sections 4.3.3-4.3.5 we continue the analysis, derive the optimal partition, and determine performance guarantees when using the optimal partition. All corresponding proofs are given in Appendix 4.C.

4.3.1 The linear-quadratic-uniform problem

In the *Linear-Quadratic-Uniform* (LQU) problem, the seller's utility function is linear in the order quantity: $\phi_S(x) = Px$, where $P \in \mathbb{R}_{>0}$ is the seller's utility per unit of sold product. The buyer's utility function is characterised by a saturation effect: the marginal utility of buying an additional product decreases linearly. That is, for order quantity $x \in \mathbb{R}_{\geq 0}$ the buyer's marginal utility of an additional product is $p - rx$. Here, $p \in [\underline{p}, \bar{p}] \subseteq \mathbb{R}_{>0}$ with $\bar{p} > \underline{p}$ is the buyer's (private) type and $r \in \mathbb{R}_{>0}$ is a saturation rate parameter. Note that p is strictly positive. The buyer's type is assumed to have a uniform distribution, i.e., $\omega(p) = 1/(\bar{p} - \underline{p})$. Consequently, the buyer's utility function is

$$\phi_B(x|p) = \int_0^x (p - ru)\mathrm{d}u = px - \tfrac{1}{2}rx^2.$$

Notice that for large order quantities the buyer's utility is negative, which models for example that excess products must be disposed of at a cost. Furthermore, ordering no products leads to zero utility for the buyer, which is his reservation level.

The pooling of contracts for the LQU problem has been analysed in Wong (2014) and under the name DMU-1 in Chapter 3, both with the goal to maximise the seller's expected net utility (the ME model). As mentioned in Section 4.1, we extend the analysis to the MO model, with which we can balance the maximisation of the seller's worst-case and expected net utility.

4.3.2 Optimal solutions

It is straightforward to verify that the LQU problem satisfies Assumptions 4.1-4.3 of Section 4.2. In particular, since ω is the uniform density function we have

$$\omega_k = \frac{\bar{p}_k - \underline{p}_k}{\bar{p} - \underline{p}},$$

and therefore (4.16) of Assumption 4.3 simplifies to

$$\pi_k x - \tfrac{1}{2}rx^2 = \underline{p}_k x - \tfrac{1}{2}rx^2 - (\bar{p}_k - \underline{p}_k)x \sum_{i=k+1}^{K} \frac{\bar{p}_i - \underline{p}_i}{\bar{p}_k - \underline{p}_k} = (\bar{p}_k + \underline{p}_k - \bar{p})x - \tfrac{1}{2}rx^2.$$

Hence, the parameters π_k are

$$\pi_k = \bar{p}_k + \underline{p}_k - \bar{p} \qquad\qquad \forall\, k \in \mathcal{K},$$

which satisfy $\pi_1 \leq \cdots \leq \pi_K$ and $\pi_k \leq \underline{p}_k$ for all $k \in \mathcal{K}$. In contrast to the buyer's type p the parameter π_k can be negative for some $k \in \mathcal{K}$, depending on the instance parameters and the partition.

Since $\phi_J(x|\lambda) = (P + \lambda)x - \frac{1}{2}rx^2$, the function M^* stated in Lemma 4.4 is

$$M^*(\lambda) = \begin{cases} \frac{1}{2r}(P+\lambda)^2 & \text{if } P + \lambda \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

For the MO model, we need to realise that for a non-negative seller's reservation level ($M \leq 0$) the seller's IR constraints (4.14) are non-restrictive. This follows from Theorem 4.7 and the definition of $x^*(\cdot|M)$ in Lemma 4.4. More precisely, since

$\phi_J(0|\underline{p}) = 0$ we have $x^M = 0$ for $M \leq 0$ and hence the optimal MO solution is the optimal ME solution. To conclude, the only proper values for M for the MO model are $0 \leq M \leq M^*(\underline{p})$.

Therefore, we only consider $M = \beta M^*(\underline{p})$ for some $\beta \in [0,1]$. For notational convenience, we often switch from M to β. We define $x^\beta \equiv x^{\beta M^*(\underline{p})}$ and $x^*(\cdot|\beta) \equiv x^*(\cdot|\beta M^*(\underline{p}))$, resulting in

$$\begin{aligned} x^\beta &= \min\left\{x \geq 0 : \phi_J(x|\underline{p}) \geq \beta M^*(\underline{p})\right\} \\ &= \min\left\{x \geq 0 : (P+\underline{p})x - \tfrac{1}{2}rx^2 \geq \beta\tfrac{1}{2r}(P+\underline{p})^2\right\} \\ &= \tfrac{1}{r}(1-\sqrt{1-\beta})(P+\underline{p}) \in [0, \tfrac{1}{r}(P+\underline{p})]. \end{aligned}$$

Thus, the function x^* stated in Lemma 4.4 is

$$x^*(\lambda|\beta) = \tfrac{1}{r}\max\left\{(1-\sqrt{1-\beta})(P+\underline{p}), P+\lambda\right\}. \tag{4.17}$$

In Corollary 4.9 we collect and translate the results of Theorems 4.5, 4.6, and 4.7 for the LQU problem.

Corollary 4.9. *For the MM-LQU model, any convex combination of*

$$\begin{aligned} x_k &= \tfrac{1}{r}(P+\underline{p}) && \forall\, k \in \mathcal{K}, \\ x_k &= \tfrac{1}{r}(P+\underline{p}_k) && \forall\, k \in \mathcal{K}, \\ x_k &= \tfrac{1}{r}\max\left\{P+\underline{p}, P-\bar{p}+\bar{p}_k+\underline{p}_k\right\} && \forall\, k \in \mathcal{K}, \end{aligned}$$

is an optimal solution. For the ME-LQU model, the only optimal solution is

$$x_k = \tfrac{1}{r}\max\left\{0, P-\bar{p}+\bar{p}_k+\underline{p}_k\right\} \qquad \forall\, k \in \mathcal{K}. \tag{4.18}$$

For the MO-LQU model and $M = \beta M^*(\underline{p})$ for some $\beta \in [0,1]$, the only optimal solution is

$$x_k = \tfrac{1}{r}\max\left\{(1-\sqrt{1-\beta})(P+\underline{p}), P-\bar{p}+\bar{p}_k+\underline{p}_k\right\} \qquad \forall\, k \in \mathcal{K}. \tag{4.19}$$

Note that the optimal solutions for the ME and MO models are unique due to the strict concavity of $\phi_J(\cdot|\lambda)$ (the details are given in the proof of Corollary 4.9). Furthermore, for certain instances the optimal ME solution (4.18) results in no trade with a range of buyer types (those for which $P - \bar{p} + \bar{p}_k + \underline{p}_k \leq 0$). Consequently, the seller's worst-case net utility is zero for such instances. It might be preferable to always trade with a potential buyer to at least make some revenue, even if this results in a potentially lower expected net utility. This is exactly what happens with the optimal MO solution (4.19) for $\beta > 0$: the optimal MO menu always instigates trade with the buyer if $\beta > 0$. A similar result is observed in Zheng et al. (2015) for their concave setting with infinitely many contracts.

We can use Corollary 4.9 to illustrate a difference between the approach of Zheng et al. (2015) and our MO model. If we translate the results of Zheng et al. (2015)

to our pooling setting, their multi-objective approach could lead to a discontinuity at $\beta = 1$ (the equivalent to their $\epsilon = 1$) in x_k as function of β. This is due to the fact that their approach leads to the MM model if $\beta = 1$, which has multiple optimal solutions, as seen in Corollary 4.9. In contrast, the MO solution is always unique and continuous in β.

Finally, for $\beta = 0$ the optimal MO solution is the optimal ME solution. Similarly, for $\beta = 1$ the optimal MO solution is the (unique) optimal MM solution that also maximises the seller's expected net utility as a two-stage optimisation process. For this reason, we focus completely on the MO model in the results to come.

4.3.3 Normalising the objective function

To construct an optimal menu of contracts, the seller has to decide on his reservation level M (or equivalently β), the number of contracts K offered, and the partition of the buyer types. To quantify the effect of these decisions, we need to express the optimal objective function value in terms of the stated decisions. We can simply substitute (4.19) in the objective function (4.13), but it turns out to be useful to normalise various parameters as follows. First, we redefine the partition as

$$\underline{p}_k = \underline{p} + \delta_{k-1}(\bar{p} - \underline{p}) \qquad \text{and} \qquad \bar{p}_k = \underline{p} + \delta_k(\bar{p} - \underline{p}),$$

where $\delta_0 = 0$, $\delta_k \in [0, 1]$ for $k = 1, \ldots, K-1$, and $\delta_K = 1$. For a proper K-partition, we have $0 = \delta_0 < \cdots < \delta_K = 1$. Consequently, $\omega_k = \delta_k - \delta_{k-1}$. Second, we introduce the normalisation factor ν and the aggregate instance parameter α:

$$\nu = \frac{2r}{(\bar{p} - \underline{p})^2} > 0 \qquad \text{and} \qquad \alpha = \frac{\bar{p} - \underline{p}}{P + \underline{p}} > 0.$$

As we will show, the (relative) performance measures of interest can be expressed completely in terms of α, β, and δ_k ($k \in \mathcal{K}$). The normalisation factor ν is used to simplify the expressions and cancels out in relative measures.

Let Γ_K be the optimal MO objective value when using a proper K-partition, i.e., using (4.19). We can express the *normalised* optimal MO objective value $\nu\Gamma_K$ in terms of the introduced normalised/aggregate parameters, see Lemma 4.10.

Lemma 4.10. *For any proper K-partition the normalised optimal MO-LQU objective value is given by*

$$\begin{aligned}\nu\Gamma_K = &\sum_{k=1}^{k^\beta}(\delta_k - \delta_{k-1})\left(\tfrac{\beta}{\alpha^2} + 2(\delta_k + \delta_{k-1} - 1)(1 - \sqrt{1-\beta})\tfrac{1}{\alpha}\right) \\ &+ \sum_{k=k^\beta+1}^{K}(\delta_k - \delta_{k-1})\left(\tfrac{1}{\alpha} + \delta_k + \delta_{k-1} - 1\right)^2,\end{aligned} \tag{4.20}$$

where k^β is the largest index affected by the seller's reservation level:

$$k^\beta = \max\left\{0, \max\left\{k \in \mathcal{K} : \delta_k + \delta_{k-1} < 1 - \tfrac{1}{\alpha}\sqrt{1-\beta}\right\}\right\}.$$

Notice that $k^\beta < K$, since $\delta_K = 1$. Furthermore, for instances with $\alpha \in (0, \sqrt{1-\beta}] \subseteq (0,1]$ we have $k^\beta = 0$, implying that the seller's reservation level is non-restrictive for all contracts.

Two extreme cases are $\nu\Gamma_1 = \alpha^{-2}$ and the limit of $\nu\Gamma_K$ for $K \to \infty$ using any sensible partition (such as $\delta_k = k/K$):

$$\nu\Gamma_\infty = \int_0^{\delta^\beta} \left(\tfrac{\beta}{\alpha^2} + (4\delta - 2)(1 - \sqrt{1-\beta})\tfrac{1}{\alpha}\right) \mathrm{d}\delta + \int_{\delta^\beta}^1 \left(\tfrac{1}{\alpha} + 2\delta - 1\right)^2 \mathrm{d}\delta,$$

where δ^β corresponds to the limit of k^β:

$$\delta^\beta = \max\left\{0, \tfrac{1}{2}(1 - \tfrac{1}{\alpha}\sqrt{1-\beta})\right\}.$$

Hence, we get

$$\nu\Gamma_\infty = \begin{cases} \frac{1}{\alpha^2} + \frac{1}{3} & \text{if } \alpha \leq \sqrt{1-\beta} \\ \frac{\beta}{\alpha^2} + \frac{1}{6}(1 + \frac{1}{\alpha}\sqrt{1-\beta})^3 & \text{if } \alpha > \sqrt{1-\beta} \end{cases}. \tag{4.21}$$

Notice that Γ_∞ is independent of the partition, as should be the case since we are effectively offering infinitely many contracts. Trivially, we have $\Gamma_K \leq \Gamma_\infty$ for any α, β, K, and partition. Hence, we can use Γ_∞ as a benchmark to evaluate the performance of the chosen partition scheme. Recalling the objective of the MO model, a natural choice for the partition is the one which maximises the seller's expected net utility Γ_K. For the LQU problem we are able to determine this optimal partition, as we will show in the next section.

4.3.4 Optimal partition

The goal of this section is to determine closed-form formulas for the partition that maximises Γ_K (or equivalently $\nu\Gamma_K$) for a given α, β, and K. As mentioned in Wong (2014) and Chapter 3, it seems to be difficult to determine such closed-form formulas in general, either due to complex system of equations needed to be solved or due to the existence of multiple local optima. However, the structure of the LQU problem allows us to determine closed-form formulas for the optimal partition.

First, we prove that offering the same contract multiple times is suboptimal and that we should use all available contracts. This is intuitively clear, but formalised in Lemma 4.11.

Lemma 4.11. *The optimal MO-LQU partition satisfies $k^\beta \in \{0,1\}$ and $0 = \delta_0 < \delta_1 < \cdots < \delta_{K-1} < \delta_K = 1$.*

Lemma 4.11 greatly restricts the value of k^β when determining the optimal partition, making the analysis manageable. We can now derive the optimal MO partition, see Theorem 4.12.

Theorem 4.12. *For* $0 < \alpha \leq \frac{K}{K-1}\sqrt{1-\beta}$ *the optimal MO-LQU partition satisfies* $k^\beta = 0$ *and is the equidistant partition:*

$$\delta_k^{\text{opt}} = \tfrac{k}{K} \qquad\qquad \forall\, k \in \mathcal{K}.$$

For $\alpha > \frac{K}{K-1}\sqrt{1-\beta}$ *the optimal MO-LQU partition satisfies* $k^\beta = 1$ *and is*

$$\delta_k^{\text{opt}} = 1 - \tfrac{K-k}{2K-1}(1 + \tfrac{1}{\alpha}\sqrt{1-\beta}) \qquad\qquad \forall\, k \in \mathcal{K}.$$

In particular, for $\beta = 1$ *the equidistant partition is suboptimal for all* $\alpha > 0$.

Sketch of the proof. Since $k^\beta \in \{0,1\}$ by Lemma 4.11 we only need to consider two variants for the formula of $\nu\Gamma_K$. For each variant we set the gradient to zero, leading to systems of linear equations after simplification, and determine the corresponding maximiser. The maximiser must be in line with the considered value of k^β, resulting in a specification of a valid range of instances: either $0 < \alpha \leq \frac{K}{K-1}\sqrt{1-\beta}$ or $\alpha > \frac{K}{K-1}\sqrt{1-\beta}$. These ranges are disjoint and capture all instances $\alpha > 0$, which completes the proof. □

The result in Theorem 4.12 (and its proof) is a generalisation of the derived optimal partition in Chapter 3. A remarkable property is that the equidistant partition is optimal for a range of instances, as specified by the relation between α, β, and K. This range of instances increases if β decreases (by lowering the seller's reservation level) or if K decreases (by offering less contracts). Moreover, if the equidistant partition is not optimal, then the optimal partition can be found by increasing δ_1 and partitioning the remaining subinterval $[\delta_1, 1]$ equidistantly for $\delta_2, \ldots, \delta_{K-1}$.

When using the optimal partition the expression for $\nu\Gamma_K$ (4.20) can be simplified to a similar expression as (4.21). This is shown in Corollary 4.13.

Corollary 4.13. *For the optimal partition the normalised optimal MO-LQU objective value* (4.20) *is*

$$\nu\Gamma_K^{\text{opt}} = \begin{cases} \frac{1}{\alpha^2} + \frac{1}{3}\left(1 - \frac{1}{K^2}\right) & \text{if } \alpha \leq \frac{K}{K-1}\sqrt{1-\beta} \\ \frac{\beta}{\alpha^2} + \frac{2}{3}\frac{K(K-1)}{(2K-1)^2}(1 + \frac{1}{\alpha}\sqrt{1-\beta})^3 & \text{if } \alpha > \frac{K}{K-1}\sqrt{1-\beta} \end{cases}. \tag{4.22}$$

Notice that (4.22) clearly converges to (4.21) if $K \to \infty$, as should be the case. Also, on certain intervals $\nu\Gamma_K^{\text{opt}}$ and $\nu\Gamma_\infty$ either differ by a constant $-1/(3K^2)$ or by a factor $4K(K-1)/(2K-1)^2$ in the cubic term. In the next section, we analyse the relative difference between Γ_K^{opt} and Γ_∞ in more detail to obtain performance guarantees.

4.3.5 Performance guarantees

In this section we analyse the performance of the optimal menu of contracts when using the optimal partition. We consider two performance measures: the pooling performance (Section 4.3.5.1) and the reservation level performance (Section 4.3.5.2).

Both measure the effectiveness of pooling the buyer types compared to offering infinitely many contracts in their own way. In particular, the first measure is useful when the seller's reservation level cannot be adjusted, whereas the second is insightful when the seller's reservation level is a decision variable.

4.3.5.1 Pooling performance

The *pooling performance* $\Gamma_K^{\text{opt}}/\Gamma_\infty$ is the fraction of the seller's expected net utility attained by offering K contracts instead of infinitely many contracts. Here, the seller uses the optimal menu of contracts and the optimal partition, as derived in Sections 4.3.2 and 4.3.4. In other words, it measures how much is lost due to the pooling of the buyer types by offering a limited number of contracts. Note that the seller's reservation level M (or β) must always be satisfied by both menus corresponding to Γ_K^{opt} and Γ_∞.

In Figure 4.1 we illustrate the attained pooling performance for two example instances in terms of β and K. Here, we use $\alpha = 2$ and $\alpha = \frac{9}{4} + \frac{3}{4}\sqrt{5} \approx 3.927$. We have chosen for $\alpha = 2$ because this is the threshold value in (4.22) for $K = 2$ and $\beta = 0$. The reason for the other chosen instance will be given in the next section. We observe in Figure 4.1 that for $\beta = 0$ the pooling performances are 88% ($K = 2$) and 96% ($K = 3$) for both instances. As β increases the pooling performance increases. However, the rate of increase differs significantly between $K = 2$ and $K = 3$, and between the two instances. Finally, notice that for fixed $K \in \{2, 3\}$ and for any $0 < \beta \leq 1$ the pooling performance is higher for $\alpha = 2$ than for the other instance.

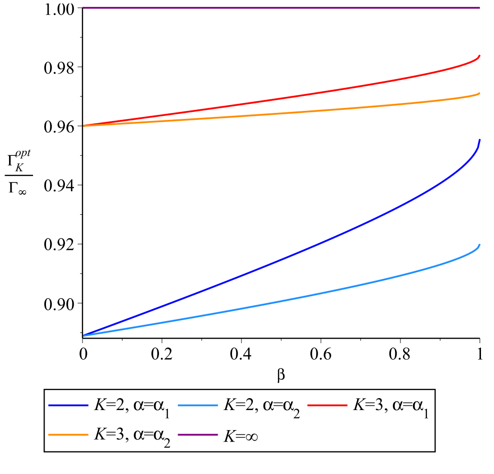


Figure 4.1: Pooling performance for the MO-LQU model with the optimal partition, where α is fixed to $\alpha_1 = 2$ or $\alpha_2 = \frac{9}{4} + \frac{3}{4}\sqrt{5}$.

By analysing expressions (4.21) and (4.22) for Γ_∞ and Γ_K^{opt}, respectively, we can generalise the above observations. Furthermore, we are able to derive guarantees for the pooling performance, see Theorem 4.14.

Theorem 4.14. *For the optimal partition the MO-LQU pooling performance* $\Gamma_K^{\text{opt}}/\Gamma_\infty$ *is continuous and non-increasing in* α*, and continuous and non-decreasing in* β*. In particular, we have the tight pooling performance guarantee*

$$\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty} \geq \frac{4K(K-1)}{(2K-1)^2} \qquad \forall\, \alpha > 0, 0 \leq \beta \leq 1. \tag{4.23}$$

Sketch of the proof. The idea is to consider all cases that occur based on (4.21) and (4.22). Analysing the closed-form and manageable formula for each case leads to the following insights.

- For $K = 1$ we have $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_1}{\Gamma_\infty} < 0$ for all $\alpha > 0$.
- For $K > 1$ we have
  - if $\beta = 0$: $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty} < 0$ for $0 < \alpha < \frac{K}{K-1}$ and $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty} = 0$ for $\alpha \geq \frac{K}{K-1}$,
  - if $0 < \beta \leq 1$: $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty} < 0$ for all $\alpha > 0$.
- We have $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty} = 0$ for $0 < \alpha \leq \sqrt{1-\beta}$ and $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty} > 0$ for $\alpha > \sqrt{1-\beta}$.

Hence, for any $0 \leq \beta \leq 1$ we obtain a tight pooling performance guarantee by taking the limit $\alpha \to \infty$, resulting in (4.23). The full proof is given in Appendix 4.C. □

In Figure 4.2 we show the attained pooling performance for various choices of α, β, and K. The aggregate instance parameter α increases when the seller's utility per unit of sold product P decreases or when the buyer's type interval $[\underline{p}, \bar{p}]$ widens (under certain conditions). We can interpret the first case (decreasing P) as a higher investment risk in products, since a product provides less utility. The second case (widening $[\underline{p}, \bar{p}]$) can be interpreted as an increase in uncertainty on the buyer's identity. Hence, Theorem 4.14 implies that an increase in investment risk or in the uncertainty on the buyer's identity has a negative effect on the pooling performance.

In contrast, increasing the seller's reservation level (encoded in β) has a positive effect on the pooling performance, provided that the seller's reservation level is restrictive ($\alpha > \sqrt{1-\beta}$). However, if we want to give a guarantee for the pooling performance that holds for any instance, then this positive effect has no influence. In fact, the seller's reservation level does not affect the pooling performance guarantee. This follows from the proof of Theorem 4.14, where we show that the guarantee (4.23) is tight for any $0 \leq \beta \leq 1$. Table 4.1 shows the values of this guarantee for $K = 1, \ldots, 5$.

From Figure 4.2 and Table 4.1 we can conclude that the seller should not offer a single contract due to poor pooling performance in general. In contrast, offering two contracts already leads to a pooling performance guarantee of 88% and offering three contracts results in 96%. Recall that for $\beta = 0$ the MO model is equivalent to the ME model. Therefore, the bounds for $\beta = 0$ correspond to the results in Wong (2014) and Chapter 3. Our analysis shows that the same bounds hold for the MO model for any $0 \leq \beta \leq 1$. Thus, the pooling of buyer types leads to a simpler menu of contracts and can be done with a controllable loss in the seller's expected net utility.

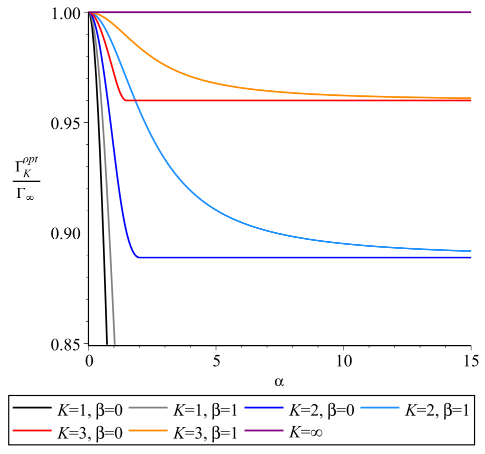


Figure 4.2: Attained pooling performance $\Gamma_K^{\text{opt}}/\Gamma_\infty$ for the MO-LQU model with the optimal partition.

K	Tight lower bound for $\Gamma_K^{\text{opt}}/\Gamma_\infty$
1	0
2	0.888
3	0.960
4	0.979
5	0.987
∞	1

Table 4.1: Pooling performance guarantees for the MO-LQU model with the optimal partition.

4.3.5.2 Reservation level performance

The *reservation level performance* $\Gamma_K^{\text{opt}}/\Gamma_\infty^{\beta=0}$ is similar to the pooling performance, except that the used benchmark $\Gamma_\infty^{\beta=0}$ disregards the seller's reservation level. That is, the numerator Γ_K^{opt} depends on β as before, but the denominator $\Gamma_\infty^{\beta=0}$ always uses $\beta = 0$. In particular, $\Gamma_\infty^{\beta=0}$ is the seller's maximum attainable expected net utility over all β and K. We can use the reservation level performance to quantify how much expected net utility is lost by the seller's reservation level, again in terms of the number of contracts offered.

We first consider the attained reservation level performance for two example instances, see Figure 4.3. As before, we use $\alpha = 2$ and $\alpha = \frac{9}{4} + \frac{3}{4}\sqrt{5} \approx 3.927$. Realise that for $\beta = 0$ the reservation level performance and the pooling performance are the same. In contrast to the pooling performance, the reservation level performance decreases when β increases, as seen in Figure 4.3. We observe that for $\alpha = 2$ the performance is less sensitive to changes in β for low values of β compared to the other instance. For both instances there is a steep decrease in performance when β approaches 1, i.e., when the seller fully considers his worst-case utility.

Similar to Theorem 4.14, we are able to generalise the above observations and determine guarantees for the reservation level performance. These results are shown in Theorem 4.15, where the term 'unimodal' refers to being non-increasing at first and then non-decreasing.

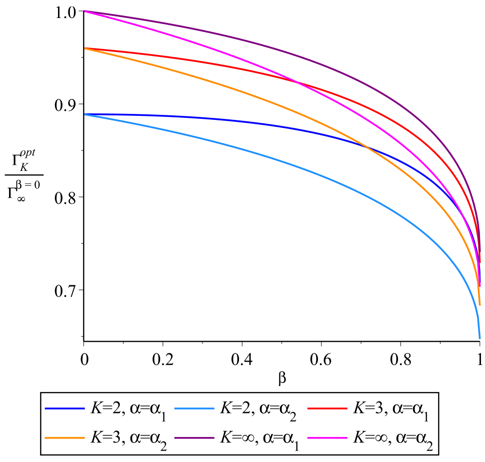


Figure 4.3: Reservation level performance for the MO-LQU model with the optimal partition, where α is fixed to $\alpha_1 = 2$ or $\alpha_2 = \frac{9}{4} + \frac{3}{4}\sqrt{5}$.

Theorem 4.15. *For the optimal partition the MO-LQU reservation level performance* $\Gamma_K^{\text{opt}}/\Gamma_\infty^{\beta=0}$ *is continuous and unimodal in* α*, and continuous and non-increasing in* β*. In particular, we have the tight reservation level performance guarantee*

$$\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} \geq \frac{8K(K-1)\left(4K(K-1)+(2K-1)\sqrt{2K^2-2K+1}+1\right)}{\left(6K(K-1)+(2K-1)\sqrt{2K^2-2K+1}+1\right)^2} \tag{4.24}$$

for all $\alpha > 0$ *and* $0 \leq \beta \leq 1$.

Sketch of the proof. We need to consider all cases that occur based on (4.21) and (4.22). By analysing each case, we obtain the following results.

- For $K = 1$ we have $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}} < 0$ and $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}} = 0$ for all $\alpha > 0$.
- For $K > 1$ we have
  - if $\beta = 0$: $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} < 0$ for $0 < \alpha < \frac{K}{K-1}$ and $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = 0$ for $\alpha \geq \frac{K}{K-1}$,
  - if $0 < \beta \leq 1$: $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} < 0$ for $0 < \alpha < \alpha^*$, $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = 0$ for $\alpha = \alpha^*$, and $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} > 0$ for $\alpha > \alpha^*$,
  - $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = 0$ for $0 < \alpha \leq \frac{K}{K-1}\sqrt{1-\beta}$ and $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} < 0$ for $\alpha > \frac{K}{K-1}\sqrt{1-\beta}$.

Here, the minimiser α^* is defined for $K > 1$ and $0 < \beta \leq 1$ by

$$\alpha^* = 1 + \frac{(2K(K-1)+1)\beta + (2K-1)\sqrt{\beta\left(2K(K-1)(1-\sqrt{1-\beta})+\beta\right)}}{2K(K-1)(1-\sqrt{1-\beta})}, \tag{4.25}$$

which satisfies $\alpha^* > 1$ and $\alpha^* > \frac{K}{K-1}\sqrt{1-\beta}$ if it exists.

Therefore, the reservation level performance guarantee is zero for $K = 1$ and for $K > 1$ it follows by taking $\beta = 1$ and evaluating the performance for $\alpha = \alpha^*$. The full proof is given in Appendix 4.C. □

Note that in Figures 4.1 and 4.3 the example instance with $\alpha = \frac{9}{4} + \frac{3}{4}\sqrt{5}$ corresponds to the minimiser (4.25) for $K = 2$ and $\beta = 1$. The attained reservation level performance for various choices of α, β, and K is depicted in Figure 4.4. For $\beta = 0$ the reservation level performance and the pooling performance are equivalent (see also Figure 4.2). For $0 < \beta \leq 1$ there is a unique minimiser for the reservation level performance, namely α^* stated in (4.25). Hence, for any given β we can easily determine the minimum reservation level performance. We omit the verbose exact expressions, simply use (4.21), (4.22), and (4.25). Instead, we depict the resulting minima in Figure 4.5, which are tight reservation level performance guarantees for each $0 \leq \beta \leq 1$.

In Figure 4.5 the values for $\beta = 0$ correspond to (4.23) and the values for $\beta = 1$ to (4.24). As seen in Figure 4.5, the reservation level performance guarantee decreases more rapidly for larger values of β. These guarantees are also given in Table 4.2 for certain choices of K and β. For example, increasing β from 0 to $\frac{3}{4}$ leads to approximately the same percentage point decrease in the guarantee as increasing β from $\frac{9}{10}$ to 1. Furthermore, notice that even with infinitely many contracts ($K = \infty$) it is not always possible to obtain full reservation level performance. If we let $K \to \infty$, then the bound in (4.24) is

$$\tfrac{8+4\sqrt{2}}{11+6\sqrt{2}} \approx 0.7009.$$

In other words, with infinitely many contracts the seller obtains at least 70% of the maximum expected net utility if he first maximises his worst-case net utility ($\beta = 1$) and this bound can be attained depending on the instance. Similarly, when using two (three) contracts, the seller achieves at least 64% (68%) when first maximising his worst-case net utility, and these bounds can be attained (see Table 4.2). Lowering the seller's reservation level raises these guarantees. For example, for $\beta = \frac{1}{2}$ these are 83%, 89%, and 92% for K equal to 2, 3, and ∞, respectively.

Overall, we conclude that the seller's reservation level has a significant impact on the seller's expected net utility, irrespective of the number of contracts offered. In terms of pooling performance, increasing the seller's reservation level has a positive effect, whereas it has a negative effect in terms of the reservation level performance. In any case, the seller should always offer at least two contracts in order to have a reasonable reservation level performance guarantee.

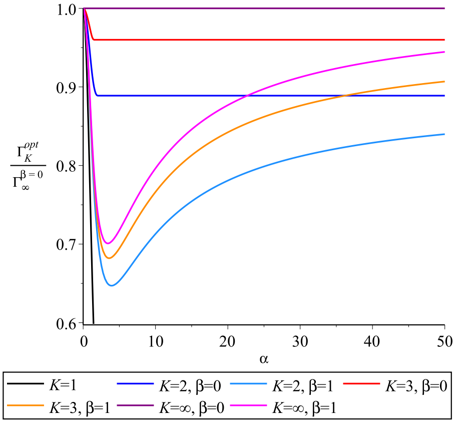


Figure 4.4: Attained reservation level performance $\Gamma_K^{\text{opt}}/\Gamma_\infty^{\beta=0}$ for the MO-LQU model with the optimal partition.

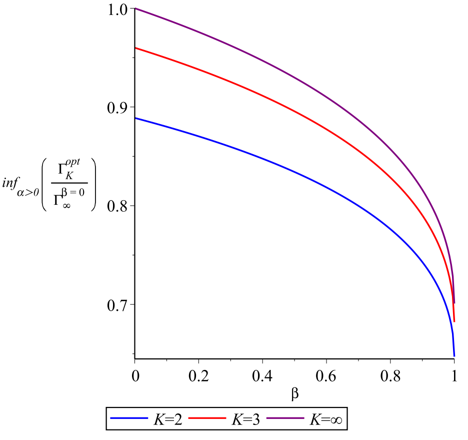


Figure 4.5: Tight reservation level performance guarantees in terms of β for the MO-LQU model with the optimal partition.

K	Tight lower bound for $\Gamma_K^{\text{opt}}/\Gamma_\infty^{\beta=0}$				
	$\beta=0$	$\beta=\frac{1}{2}$	$\beta=\frac{3}{4}$	$\beta=\frac{9}{10}$	$\beta=1$
1	0	0	0	0	0
2	0.888	0.834	0.788	0.742	0.647
3	0.960	0.895	0.842	0.790	0.682
4	0.979	0.912	0.857	0.802	0.691
5	0.987	0.919	0.863	0.808	0.695
∞	1	0.929	0.872	0.816	0.700

Table 4.2: Reservation level performance guarantees for the MO-LQU model with the optimal partition.

4.4 Concluding remarks

When faced with a continuum $[\underline{p}, \bar{p}]$ of buyer types, the seller can pool the buyer types to obtain a simpler menu of finitely many contracts. We analysed a pooling approach where the seller partitions the set of types $[\underline{p}, \bar{p}]$ a priori into $K \in \mathbb{N}_{\geq 1}$ subintervals $[\underline{p}_k, \bar{p}_k]$ for $k \in \{1, \ldots, K\}$. The resulting menu consists of K contracts and is designed such that the types in the k-th subinterval $[\underline{p}_k, \bar{p}_k]$ choose the k-th contract in the menu. In addition to pooling, we considered multiple objective functions for the seller: he maximises either his minimum net utility, his expected net utility, or a combination of both (resulting in a multi-objective approach). We modelled the multi-objective approach by maximising the seller's expected net utility under the additional constraint that his minimum net utility must be at least his reservation level. Here, the seller's reservation level is an additional model parameter decided by the seller.

Our analysis shows that under commonly used assumptions the three considered pooling models have tractable reformulations and that the optimal menus are maxima of certain modified joint net utility functions. In particular, the maximum obtainable minimum net utility is the maximum joint net utility with respect to the lowest buyer type $\underline{p}$. Using this property, the seller can fine-tune his reservation level in the multi-objective pooling model to balance his expected and worst-case net utility. Effectively, the multi-objective model encompasses the other models. With this model the seller can, for example, first maximise his minimum net utility, followed by his expected net utility (as a two-stage approach).

The considered pooling models depend on the chosen partition scheme and the seller's reservation level. We considered a contracting problem with quadratic utilities to quantify the effect of these decisions made by the seller. For this problem we first derived the optimal partition scheme, which then led to manageable formulas for the corresponding optimal objective value. In turn, these formulas can be used to determine various performance measures. We note that these results are analytical/exact and hold for any number of contracts.

We focused on two performance measures. The first is the pooling performance, which is the obtained expected net utility by offering K menus compared to infinitely many contracts. It quantifies purely the effect of pooling the buyer types. The second is the reservation level performance, which is the obtained expected net utility compared to ignoring the seller's reservation level and using infinitely many contracts. Here, the benchmark is the highest attainable expected net utility over all seller's reservation levels and all number of contracts. Both measures have been fully analysed, which resulted in performance guarantees (lower bounds) in terms of the number of contracts K offered. For example, offering a single contract has poor performance (in both measures) and is ill-advised. In contrast, offering two, three, or infinitely many contracts leads to a pooling performance guarantee of 88%, 96%, and 100%, respectively. The corresponding reservation level performance guarantees are 64%, 68%, and 70%, respectively. Note that the latter guarantees are overall bounds and can be made more specific for a fixed seller's reservation level. In particular, a reservation level near the maximum feasible value is costly in performance. All mentioned bounds can be attained for certain instances and are therefore tight.

From our analysis, we conclude that pooling of buyer types results in a simpler menu of contracts and any loss in performance can be controlled by the number of contracts offered. High performance can already be achieved with up to five contracts. Furthermore, a multi-objective optimisation approach can be performed by including the seller's reservation level as a decision parameter. The seller's reservation level has a significant impact on the seller's expected net utility, irrespective of the number of contracts offered. Increasing the reservation level has a positive effect on the pooling performance, but a negative effect on the reservation level performance. Therefore, the seller has to balance his expected and worst-case net utility and can use the stated performance measures to justify his choices.

Appendix

4.A Proofs of Section 4.2

This appendix contains the proofs of the results in Section 4.2. We note that certain proofs are similar to or generalisations of those found in Chapter 3.

Proof of Lemma 4.1. First, we show the necessity of $x_1 \leq \cdots \leq x_K$. Suppose $x_k > x_{k+1}$ for some $k, k+1 \in \mathcal{K}$ and consider (4.2) between contracts k with $\underline{p}_k$ and $k+1$ with $\underline{p}_{k+1}$. Adding both IC constraints leads to

$$\phi_B(x_k|\underline{p}_k) - \phi_B(x_{k+1}|\underline{p}_k) \geq \phi_B(x_k|\underline{p}_{k+1}) - \phi_B(x_{k+1}|\underline{p}_{k+1}).$$

Since $\underline{p}_k < \underline{p}_{k+1}$, this contradicts (4.9) of Assumption 4.1. Hence, $x_1 \leq \cdots \leq x_K$ must hold.

Second, we show sufficiency of $0 \leq x_1 \leq \cdots \leq x_K$. Let $x \in \mathbb{R}^K_{\geq 0}$ be non-decreasing and set $z \in \mathbb{R}^K$ to

$$z_k = \phi_B(x_k|\underline{p}_k) - \sum_{i=1}^{k-1} \big(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\big) \qquad \forall\, k \in \mathcal{K}. \tag{4.26}$$

It remains to check feasibility of (x, z). Fix $k \in \mathcal{K}$ and $p_k \in [\underline{p}_k, \bar{p}_k]$. For $l \in \mathcal{K}$ with $k < l$ we have

$$\begin{aligned}
z_k - z_l &\overset{(4.26)}{=} \phi_B(x_k|\underline{p}_k) - \phi_B(x_l|\underline{p}_l) + \sum_{i=k}^{l-1} \big(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\big) \\
&= \sum_{i=k+1}^{l} \big(\phi_B(x_{i-1}|\underline{p}_i) - \phi_B(x_i|\underline{p}_i)\big) \\
&\overset{(4.9)}{\leq} \sum_{i=k+1}^{l} (\phi_B(x_{i-1}|p_k) - \phi_B(x_i|p_k)) = \phi_B(x_k|p_k) - \phi_B(x_l|p_k).
\end{aligned}$$

Likewise, let $l \in \mathcal{K}$ with $l < k$, then

$$\begin{aligned}
z_k - z_l &\overset{(4.26)}{=} \phi_B(x_k|\underline{p}_k) - \phi_B(x_l|\underline{p}_l) - \sum_{i=l}^{k-1} \big(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\big) \\
&= \sum_{i=l+1}^{k} \big(\phi_B(x_i|\underline{p}_i) - \phi_B(x_{i-1}|\underline{p}_i)\big) \\
&\overset{(4.9)}{\leq} \sum_{i=l+1}^{k} (\phi_B(x_i|p_k) - \phi_B(x_{i-1}|p_k)) = \phi_B(x_k|p_k) - \phi_B(x_l|p_k).
\end{aligned}$$

Hence, all IC constraints (4.2) hold. Furthermore, we have

$$\phi_B(x_k|p_k) - z_k \overset{(4.2)}{\geq} \phi_B(x_1|p_k) - z_1 \overset{(4.8)}{\geq} \phi_B(x_1|\underline{p}_1) - z_1 \overset{(4.26)}{=} 0.$$

Thus, all IR constraints (4.1) are satisfied and the solution is feasible. $\square$

Proof of Lemma 4.2. Let $x \in \mathbb{R}^K_{\geq 0}$ be feasible, i.e., there exists a $z \in \mathbb{R}^K$ such that (x, z) satisfies (4.1)-(4.3) and for the MO model also (4.7). The proof consists of two parts: first we show that (4.10) holds for contract $k = 1$ and then for the other contracts in the menu ($k > 1$).

First, realise that for an optimal z at least one IR constraint (4.1) must hold with equality. If this is not the case, we can increase all z_k by adding some $\epsilon > 0$ until at least one IR constraint is tight. This new solution is still feasible, as (4.2) only considers the difference $z_k - z_l$, which is unaffected. For the MO model (4.7) would trivially still hold. Moreover, the objective value of the new solution is strictly larger for the ME, MM, and MO models. Hence, if no IR constraint is tight we have a contradiction.

Now, suppose that $z_1 < \phi_B(x_1|\underline{p}_1)$, then for $k \in \mathcal{K}$ we have for all $p_k \in [\underline{p}_k, \bar{p}_k]$ that

$$\phi_B(x_k|p_k) - z_k \overset{(4.2)}{\geq} \phi_B(x_1|p_k) - z_1 \overset{(4.8)}{\geq} \phi_B(x_1|\underline{p}_1) - z_1 > 0.$$

The result implies that no IR constraint is tight, which is suboptimal as argued above. Hence, for an optimal z it must hold that $z_1 = \phi_B(x_1|\underline{p}_1)$.

Second, fix $k \in \mathcal{K}$ with $k > 1$ and consider the following IC constraints (4.2) between contracts k and $k-1$:

$$\phi_B(x_k|\bar{p}_{k-1}) - \phi_B(x_{k-1}|\bar{p}_{k-1}) \leq z_k - z_{k-1} \leq \phi_B(x_k|\underline{p}_k) - \phi_B(x_{k-1}|\underline{p}_k).$$

Since $\bar{p}_{k-1} = \underline{p}_k$, this implies that

$$z_k - z_{k-1} = \phi_B(x_k|\underline{p}_k) - \phi_B(x_{k-1}|\underline{p}_k).$$

Using our earlier result that $z_1 = \phi_B(x_1|\underline{p}_1)$, we obtain the following formula:

$$z_k = \sum_{i=2}^{k} \big(\phi_B(x_i|\underline{p}_i) - \phi_B(x_{i-1}|\underline{p}_i)\big) + \phi_B(x_1|\underline{p}_1),$$

which can be rewritten into (4.10). □

Proof of Theorem 4.3. We use Lemmas 4.1 and 4.2 to eliminate the variable z. The equivalent MM model follows immediately. The ME model becomes

$$\max_{0 \leq x_1 \leq \cdots \leq x_K} \sum_{k \in \mathcal{K}} \omega_k \Big(\phi_J(x_k|\underline{p}_k) - \sum_{i=1}^{k-1} \big(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\big)\Big).$$

Collecting all x_k terms results in the stated formulation, where we use that $\omega_k > 0$ for all $k \in \mathcal{K}$. Finally, the MO model follows by combining these insights. □

Proof of Lemma 4.4. By Assumption 4.2 the maximum of $x \mapsto \phi_J(x|\lambda)$ is attainable for any $\lambda \in \mathbb{R}$. Thus, M^* is well-defined and non-decreasing since (4.8) holds by Assumption 4.1. Next, for $M \leq M^*(\underline{p})$ the threshold x^M is well-defined. By Assumptions 4.1 and 4.2, we can construct the stated function $x^*(\cdot|M)$ by selecting the smallest maximiser (if there are multiple). There is a technicality in this argument, which we discuss at the end of this proof. We continue with the proof of the non-decreasingness of $x^*(\cdot|M)$. Suppose the constructed $x^*(\cdot|M)$ is not non-decreasing, then there exist $\lambda < \mu$ with $x^*(\lambda|M) > x^*(\mu|M)$. By definition of the smallest maximiser, we have

$$\phi_J(x^*(\lambda|M)|\lambda) > \phi_J(x^*(\mu|M)|\lambda),$$
$$\phi_J(x^*(\mu|M)|\mu) \geq \phi_J(x^*(\lambda|M)|\mu).$$

Adding both inequalities and cancelling common terms leads to

$$\phi_B(x^*(\lambda|M)|\lambda) - \phi_B(x^*(\mu|M)|\lambda) > \phi_B(x^*(\lambda|M)|\mu) - \phi_B(x^*(\mu|M)|\mu),$$

which contradicts (4.9) of Assumption 4.1.

The technicality regarding the existence of $x^*(\cdot|M)$ is as follows. We have to show that $\phi_J(\cdot|\lambda)$ always has a maximum on $[x^M, \infty)$ for any $M \leq M^*(\underline{p})$ and any $\lambda \in \mathbb{R}$. First, for $M = -\infty$ we have $x^M = 0$ and Assumption 4.2 guarantees the existence of the maximum. Hence, the non-decreasing function $x^*(\cdot| -\infty)$ exists. For notational convenience, let $x^*(\lambda) = x^*(\lambda| -\infty)$. Second, $x^M \leq x^*(\underline{p})$ for any $M \leq M^*(\underline{p})$ by definition. This implies that $x^*(\lambda|M) = x^*(\lambda)$ for $\lambda \geq \underline{p}$, i.e., the restriction to $x \geq x^M$ has no effect for $\lambda \geq \underline{p}$. For $\lambda < \underline{p}$ we have that for $x \geq x^*(\underline{p})$

$$\phi_J(x|\lambda) - \phi_J(x^*(\underline{p})|\lambda) \overset{(4.9)}{\leq} \phi_J(x|\underline{p}) - \phi_J(x^*(\underline{p})|\underline{p}) \leq 0.$$

Here, the last inequality follows from the fact that $x^*(\underline{p})$ maximises $\phi_J(\cdot|\underline{p})$ by definition. Thus, the maximum of $\phi_J(\cdot|\lambda)$ on $[x^M, \infty)$ (if it exists) must be attained in the closed interval $[x^M, x^*(\underline{p})]$. By Assumption 4.2 this maximum exists. □

Proof of Theorem 4.5. First, we use induction to prove that the solution $x_k = x^*(\underline{p}_k)$ for $k \in \mathcal{K}$ is optimal. Then, we show that the resulting optimal objective value can also be attained using a menu with only a single contract.

In order to do so, we need the following insight. Suppose $x_{k+1} = x^*(\underline{p}_{k+1})$ and compare the k-th and $(k+1)$-th terms of (4.12). We claim that these two terms satisfy

$$\phi_J(x_k|\underline{p}_k) - \sum_{i=1}^{k-1} \left(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\right) \leq M^*(\underline{p}_{k+1}) - \sum_{i=1}^{k} \left(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\right),$$

where we have substituted $x_{k+1} = x^*(\underline{p}_{k+1})$, resulting in the term $M^*(\underline{p}_{k+1})$. The common terms cancel out in this expression, leading to

$$\phi_J(x_k|\bar{p}_k) \leq M^*(\underline{p}_{k+1}).$$

Since $\bar{p}_k = \underline{p}_{k+1}$, this inequality holds by definition of $M^*(\underline{p}_{k+1})$, which proves our claim. This implies that if $x_{k+1} = x^*(\underline{p}_{k+1})$ the $(k+1)$-th term does not affect the objective value and can be omitted in (4.12). Hence, if $x_l = x^*(\underline{p}_l)$ for all $l > k$ for some $k \in \mathcal{K}$, we only need to consider the first k terms of (4.12) for the remaining optimisation problem.

We continue with the induction proof that $x^*(\underline{p}_k)$ is optimal. First, we relax the feasibility constraint $x_1 \leq \cdots \leq x_K$. Second, notice that x_K only appears in the K-th term in (4.12). Therefore, we can optimise x_K independently for this term and optimally set $x_K = x^*(\underline{p}_K)$. Third, suppose that for some $k \in \mathcal{K}$ we have $x_l = x^*(\underline{p}_l)$ for all $l > k$. The remaining optimisation problem has decision variables $x_1, \ldots, x_k$. By the above mentioned insight, we only need to consider the first k terms of (4.12). As such, x_k only appears in the k-th term of (4.12) and we can optimise x_k independently as seen before. This results in $x_k = x^*(\underline{p}_k)$. By induction, we end up with $x_k = x^*(\underline{p}_k)$ for all $k \in \mathcal{K}$, which is optimal for the relaxed problem as the induction proof shows. Since x^* is non-decreasing by definition, $x_k = x^*(\underline{p}_k)$ is also feasible and optimal for the MM model.

Finally, by using the above mentioned insight it follows that for this optimum only the first term ($k = 1$) of (4.12) affects the objective value. Hence, the resulting optimal objective value is $M^*(\underline{p})$. The same objective value is attained by offering a single contract with order quantity $x = x^*(\underline{p})$, which does not depend on K or the partition of $[\underline{p}, \bar{p}]$. □

Proof of Theorem 4.6. Relax the feasibility constraint $x_1 \leq \cdots \leq x_K$ in (4.11) to obtain a separable optimisation problem for each $k \in \mathcal{K}$. By Assumption 4.3, the solution $x_k = x^*(\pi_k)$ for $k \in \mathcal{K}$ is optimal for this relaxed problem. Since $\pi_k \leq \pi_{k+1}$ and x^* is non-decreasing by definition, we have that $x_k \leq x_{k+1}$ for all $k \in \mathcal{K}$. Therefore, the relaxed optimum is feasible for the ME model and thus optimal. □

Proof of Theorem 4.7. Since $\underline{p}_1 = \underline{p}$ and x^M is the smallest value such that $\phi_J(x|\underline{p}) \geq M$, constraint (4.14) for $k = 1$ implies $x_1 \geq x^M$. Now relax all feasibility constraints (4.14)-(4.15), but add the implied constraints $x_k \geq x^M$ for $k \in \mathcal{K}$. By definition, $x_k = x^*(\pi_k|M)$ for $k \in \mathcal{K}$ is an optimal solution for the resulting separable optimisation problem.

It remains to verify that the proposed solution is also feasible for the MO model. Notice that $0 \leq x^M \leq x^*(\pi_1|M) \leq \cdots \leq x^*(\pi_K|M)$ by definition of $x^*(\cdot|M)$ and since $\pi_k \leq \pi_{k+1}$ for all $k \in \mathcal{K}$. Thus, we need to check if (4.14) holds for all $k \in \mathcal{K}$.

First, notice that $\pi_k \leq \underline{p}_k$ must hold for all $k \in \mathcal{K}$ by the assumptions. The proof is as follows. By (4.8) and (4.16) we have $\phi_B(\cdot|\pi_k) \leq \phi_B(\cdot|\underline{p}_k)$. If $\pi_k > \underline{p}_k$, then the previous result and (4.8) imply $\phi_B(\cdot|\pi_k) = \phi_B(\cdot|\underline{p}_k)$, which trivially violates (4.9). In fact, $\pi_k < \underline{p}_k$ must hold for all $k \in \mathcal{K} \setminus \{K\}$: if $\pi_k = \underline{p}_k$ for some $k < K$ then (4.16) implies $\phi_B(\cdot|\bar{p}_k) = \phi_B(\cdot|\underline{p}_k)$, which again trivially violates (4.9).

Second, we show a useful implication of the definition of $x^*(\cdot|M)$. For some $k \in \mathcal{K}$, consider any order quantity $\bar{x}$ with $x^M \leq \bar{x} \leq x_k = x^*(\pi_k|M)$. By definition of $x_k = x^*(\pi_k|M)$, we have

$$\phi_J(x_k|\pi_k) \geq \phi_J(\bar{x}|\pi_k). \tag{4.27}$$

Since $\pi_k \le \underline{p}_k$ and $\bar{x} \le x_k$, we have

$$
\begin{aligned}
\phi_J(x_k|\underline{p}_k) &= \phi_S(x_k) + \phi_B(x_k|\pi_k) + \big(\phi_B(x_k|\underline{p}_k) - \phi_B(x_k|\pi_k)\big) \\
&\overset{(4.27)}{\ge} \phi_S(\bar{x}) + \phi_B(\bar{x}|\pi_k) + \big(\phi_B(x_k|\underline{p}_k) - \phi_B(x_k|\pi_k)\big) \\
&\overset{(4.9)}{\ge} \phi_S(\bar{x}) + \phi_B(\bar{x}|\pi_k) + \big(\phi_B(\bar{x}|\underline{p}_k) - \phi_B(\bar{x}|\pi_k)\big) \\
&= \phi_J(\bar{x}|\underline{p}_k).
\end{aligned}
$$

Finally, the above result with $k = 1$ and $\bar{x} = x^M$ shows that (4.14) holds for $k = 1$:

$$\phi_J(x_1|\underline{p}_1) \ge \phi_J(x^M|\underline{p}_1) \ge M.$$

Likewise, using the above result for $k \in \mathcal{K}$ and $\bar{x} = x_{k-1}$ gives

$$\phi_J(x_k|\underline{p}_k) \ge \phi_J(x_{k-1}|\underline{p}_k).$$

Subtracting $(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i))$ for $i = 1, \ldots, k-1$ from both sides leads to

$$
\begin{aligned}
&\phi_J(x_k|\underline{p}_k) - \sum_{i=1}^{k-1} \big(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\big) \\
&\qquad \ge \phi_J(x_{k-1}|\underline{p}_{k-1}) - \sum_{i=1}^{k-2} \big(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\big).
\end{aligned}
$$

These are the left-hand sides of (4.14) for k and $k-1$. Repeatedly applying this result for $k, k-1, \ldots, 1$ gives

$$\phi_J(x_k|\underline{p}_k) - \sum_{i=1}^{k-1} \big(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\big) \ge \phi_J(x_1|\underline{p}_1) \ge M,$$

where we have derived the last inequality earlier. Hence, (4.14) holds for all $k \in \mathcal{K}$. We conclude that that $x_k = x^*(\pi_k|M)$ is feasible for the MO model and therefore optimal. □

Proof of Corollary 4.8. We can rewrite the MM model into

$$
\begin{aligned}
\max_{x,u} \quad & u \\
\text{s.t.} \quad & \phi_J(x_k|\underline{p}_k) - \sum_{i=1}^{k-1} \big(\phi_B(x_i|\bar{p}_i) - \phi_B(x_i|\underline{p}_i)\big) \ge u, && \forall\, k \in \mathcal{K}, \\
& x_K \ge \cdots \ge x_1 \ge 0.
\end{aligned}
$$

This is a concave optimisation problem by the additional assumption of this corollary, hence any convex combination of optimal solutions is also optimal. It remains to verify that all stated solutions are optimal for the MM model. The first two stated menus are optimal for the MM model as shown in Theorem 4.5. The third stated menu is also feasible and optimal for the MM model by construction, due to Assumption 4.3 and the choice of $M = M^*(\underline{p})$. See the proof of Theorem 4.7 for additional details regarding feasibility. $\square$

4.B Examples that satisfy Assumption 4.3

In this appendix, we give an example problem class that satisfies Assumptions 4.1 and 4.3. Consider a buyer with utility function given by $\phi_B(x|p) \equiv \psi(x) + p\chi(x)$, where the functions $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and $\chi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ do not depend on the type p. Furthermore, χ is strictly increasing and non-negative. Finally, ω is a (strictly positive) continuous distribution with a non-decreasing hazard rate. We show that the stated assumptions hold for this problem class.

We first verify that Assumption 4.1 holds, i.e., (4.8) and (4.9). For $\lambda \leq \mu \in \mathbb{R}$ and $x \geq 0$ we have

$$\phi_B(x|\lambda) - \phi_B(x|\mu) = (\lambda - \mu)\chi(x) \leq 0,$$

since χ is non-negative. For $\lambda < \mu \in \mathbb{R}$ and $0 \leq x < x'$ we get

$$\phi_B(x'|\lambda) - \phi_B(x|\lambda) - \phi_B(x'|\mu) + \phi_B(x|\mu) = (\lambda - \mu)(\chi(x') - \chi(x)) < 0,$$

as χ is strictly increasing. Thus, Assumptions 4.1 is satisfied.

Next, we show that Assumption 4.3 holds. In order to define π_k, we need to consider (4.16):

$$\begin{aligned}\phi_B(x|\pi_k) &= \phi_B(x|\underline{p}_k) - \big(\phi_B(x|\bar{p}_k) - \phi_B(x|\underline{p}_k)\big) \sum_{i=k+1}^{K} \frac{\omega_i}{\omega_k} \\ &= \psi(x) + \Big(\underline{p}_k - (\bar{p}_k - \underline{p}_k) \sum_{i=k+1}^{K} \frac{\omega_i}{\omega_k}\Big)\chi(x).\end{aligned}$$

Hence, π_k is the coefficient of $\chi(x)$ in the above expression:

$$\pi_k \equiv \underline{p}_k - (\bar{p}_k - \underline{p}_k) \sum_{i=k+1}^{K} \frac{\omega_i}{\omega_k} \qquad \forall\, k \in \mathcal{K}.$$

Notice that $\pi_k < \underline{p}_k$ for $k \in \mathcal{K} \setminus \{K\}$ and $\pi_K = \underline{p}_K$. In order to have $\pi_1 \leq \cdots \leq \pi_K$ and thus Assumption 4.3 to hold, we need conditions on the probability distribution ω. As stated, we assume that ω has a non-decreasing hazard rate, which implies that

$$\frac{\omega(v)}{1 - \int_{\underline{p}}^{v} \omega(p)\mathrm{d}p} \geq \frac{\omega(u)}{1 - \int_{\underline{p}}^{u} \omega(p)\mathrm{d}p} \qquad \forall\, u, v \in [\underline{p}, \bar{p}], u \leq v,$$

or equivalently

$$\frac{1}{\omega(u)}\int_u^{\bar{p}}\omega(p)\mathrm{d}p \geq \frac{1}{\omega(v)}\int_v^{\bar{p}}\omega(p)\mathrm{d}p \qquad\qquad \forall\, u,v\in[\underline{p},\bar{p}], u\leq v.$$

Since ω is assumed to be continuous, by the Mean Value Theorem there exist $\hat{p}_k \in (\underline{p}_k, \bar{p}_k)$ for $k\in\mathcal{K}$ such that

$$\omega(\hat{p}_k) = \frac{1}{\bar{p}_k-\underline{p}_k}\int_{\underline{p}_k}^{\bar{p}_k}\omega(p)\mathrm{d}p = \frac{\omega_k}{\bar{p}_k-\underline{p}_k}.$$

We now have for $k\in\mathcal{K}$ that

$$\begin{aligned}
\pi_k-\pi_{k+1} &= \underline{p}_k-(\bar{p}_k-\underline{p}_k)\sum_{i=k+1}^{K}\frac{\omega_i}{\omega_k}-\underline{p}_{k+1}+(\bar{p}_{k+1}-\underline{p}_{k+1})\sum_{i=k+2}^{K}\frac{\omega_i}{\omega_{k+1}}\\
&= \frac{\bar{p}_{k+1}-\underline{p}_{k+1}}{\omega_{k+1}}\sum_{i=k+2}^{K}\omega_i-\frac{\bar{p}_k-\underline{p}_k}{\omega_k}\sum_{i=k}^{K}\omega_i\\
&= \frac{1}{\omega(\hat{p}_{k+1})}\int_{\underline{p}_{k+1}}^{\bar{p}}\omega(p)\mathrm{d}p-\frac{1}{\omega(\hat{p}_k)}\int_{\underline{p}_k}^{\bar{p}}\omega(p)\mathrm{d}p\\
&< \frac{1}{\omega(\hat{p}_{k+1})}\int_{\hat{p}_{k+1}}^{\bar{p}}\omega(p)\mathrm{d}p-\frac{1}{\omega(\hat{p}_k)}\int_{\hat{p}_k}^{\bar{p}}\omega(p)\mathrm{d}p\leq 0.
\end{aligned}$$

Here, the first inequality follows from $\omega(p)>0$ for all $p\in[\underline{p},\bar{p}]$ and the last inequality from the non-decreasing hazard rate. Hence, $\pi_k<\pi_{k+1}$ for all $k\in\mathcal{K}$ and Assumption 4.3 is satisfied.

4.C Proofs of Section 4.3

In this appendix we give all proofs of the results in Section 4.3. We note that certain proofs are similar to or generalisations of those found in Chapter 3.

Proof of Corollary 4.9. First, the optimal ME and MO solutions follow from Theorems 4.6 and 4.7, and the optimal MM solutions from Corollary 4.8. Second, since the function $x\mapsto\phi_J(x|\lambda)$ for the LQU problem is strictly concave and differentiable for any $\lambda\in\mathbb{R}$, it has a unique maximiser. From the relaxations used in the proofs of Theorems 4.6 and 4.7 it follows that the stated optima are the only optima. □

Proof of Lemma 4.10. Consider the optimal MO solution (4.19). The first term in the maximisation corresponds to the case where the seller's reservation level is restrictive for contract $k\in\mathcal{K}$. This is the case if

$$\begin{aligned}
&& (1-\sqrt{1-\beta})(P+\underline{p}) &> P-\bar{p}+\bar{p}_k+\underline{p}_k\\
\iff && (1-\sqrt{1-\beta})(P+\underline{p}) &> P+\underline{p}+(\delta_k+\delta_{k-1}-1)(\bar{p}-\underline{p})\\
\iff && (1-\sqrt{1-\beta})\tfrac{1}{\alpha} &> \tfrac{1}{\alpha}+(\delta_k+\delta_{k-1}-1)\\
\iff && 1-\tfrac{1}{\alpha}\sqrt{1-\beta} &> \delta_k+\delta_{k-1}.
\end{aligned}$$

Since $\delta_k + \delta_{k-1} < \delta_{k+1} + \delta_k$ for all $k \in \mathcal{K}$, we can determine the largest index affected by the seller's reservation level:

$$k^\beta = \max\left\{0, \max\left\{k \in \mathcal{K} : \delta_k + \delta_{k-1} < 1 - \tfrac{1}{\alpha}\sqrt{1-\beta}\right\}\right\}.$$

Combining our results, the optimal MO objective value is (by recalling Assumption 4.3)

$$\begin{aligned}
\Gamma_K &= \sum_{k \in \mathcal{K}} \omega_k \phi_J(x_k|\pi_k) = \sum_{k \in \mathcal{K}} \omega_k \left((P+\pi_k)x_k - \tfrac{1}{2} r x_k^2\right) \\
&= \tfrac{1}{r} \sum_{k=1}^{k^\beta} \omega_k \left((P+\pi_k)(1-\sqrt{1-\beta})(P+\underline{p}) - \tfrac{1}{2}(1-\sqrt{1-\beta})^2 (P+\underline{p})^2\right) \\
&\quad + \tfrac{1}{2r} \sum_{k=k^\beta+1}^{K} \omega_k (P+\pi_k)^2.
\end{aligned}$$

Since $(1-\sqrt{1-\beta})^2 = -\beta + 2(1-\sqrt{1-\beta})$, $(\pi_k - \underline{p})/(\bar{p}-\underline{p}) = \delta_k + \delta_{k-1} - 1$, and

$$\frac{P+\pi_k}{\bar{p}-\underline{p}} = \frac{P+\bar{p}_k+\underline{p}_k-\bar{p}}{\bar{p}-\underline{p}} = \frac{P+2\underline{p}-\bar{p}}{\bar{p}-\underline{p}} + \delta_k + \delta_{k-1} = \frac{1}{\alpha} + \delta_k + \delta_{k-1} - 1,$$

the normalised optimal MO objective value is

$$\begin{aligned}
\nu\Gamma_K = &\sum_{k=1}^{k^\beta} (\delta_k - \delta_{k-1}) \left(\tfrac{\beta}{\alpha^2} + 2(\delta_k + \delta_{k-1} - 1)(1-\sqrt{1-\beta})\tfrac{1}{\alpha}\right) \\
&+ \sum_{k=k^\beta+1}^{K} (\delta_k - \delta_{k-1}) \left(\tfrac{1}{\alpha} + \delta_k + \delta_{k-1} - 1\right)^2.
\end{aligned}$$

This completes the proof. $\square$

Proof of Lemma 4.11. Let Δ be an optimal partition. First, suppose that $k^\beta(\Delta) \geq 2$. Construct a new partition $\hat{\Delta}$ with $\hat{\delta}_1 = \delta_{k^\beta}$, $\hat{\delta}_k = \delta_{k^\beta+1}$ for $1 < k \leq k^\beta$, and $\hat{\delta}_k = \delta_k$ otherwise. By construction, we have $k^\beta(\hat{\Delta}) = 1$, since

$$\begin{aligned}
\hat{\delta}_1 + \hat{\delta}_0 = \delta_{k^\beta} \leq \delta_{k^\beta} + \delta_{k^\beta-1} &< 1 - \tfrac{1}{\alpha}\sqrt{1-\beta}, \\
\hat{\delta}_k + \hat{\delta}_{k-1} \geq \delta_{k^\beta+1} + \delta_{k^\beta} &\geq 1 - \tfrac{1}{\alpha}\sqrt{1-\beta}, \qquad \text{for } k = 2, \ldots, K.
\end{aligned}$$

Here we use the definition of k^β. Since we have

$$\sum_{k=1}^{k^\beta} (\delta_k - \delta_{k-1})(\delta_k + \delta_{k-1}) - (\delta_{k^\beta} - \delta_0)(\delta_{k^\beta} + \delta_0) = \sum_{k=1}^{k^\beta} (\delta_k^2 - \delta_{k-1}^2) - \delta_{k^\beta}^2 = 0,$$

it is straightforward to verify that

$$\begin{aligned}\nu\Gamma_K(\Delta) - \nu\Gamma_K(\hat{\Delta}) &= \sum_{k=1}^{k^\beta}(\delta_k - \delta_{k-1})\left(\tfrac{\beta}{\alpha^2} + 2(\delta_k + \delta_{k-1} - 1)(1 - \sqrt{1-\beta})\tfrac{1}{\alpha}\right) \\ &\quad - (\delta_{k^\beta} - \delta_0)\left(\tfrac{\beta}{\alpha^2} + 2(\delta_{k^\beta} + \delta_0 - 1)(1 - \sqrt{1-\beta})\tfrac{1}{\alpha}\right) \\ &= 0.\end{aligned}$$

Hence, the new partition $\hat{\Delta}$ is also optimal and we can assume without loss of generality that $k^\beta(\Delta) \in \{0, 1\}$.

Second, suppose that $\delta_{i-1} = \delta_i < \delta_{i+1}$ for some $i \in \{1, \dots, K-1\}$. By the first part of this proof, we know that $i + 1 \geq 2 > k^\beta(\Delta) \in \{0, 1\}$. Case I: if $k^\beta < i$, then there exists an $0 < \epsilon < 1$ such that

$$(1-\epsilon)\delta_{i+1} + (1+\epsilon)\delta_i \geq 1 - \tfrac{1}{\alpha}\sqrt{1-\beta}.$$

We construct a new partition $\hat{\Delta}$ with $\hat{\delta}_i = (1-\epsilon)\delta_{i+1} + \epsilon\delta_i$ and $\hat{\delta}_k = \delta_k$ otherwise. By construction, we have $\hat{\delta}_{i-1} < \hat{\delta}_i < \hat{\delta}_{i+1}$ and $k^\beta(\hat{\Delta}) \leq k^\beta(\Delta) < i$. The difference in the resulting normalised optimal objective value is

$$\begin{aligned}\nu\Gamma_K(\Delta) - \nu\Gamma_K(\hat{\Delta}) &= 0 + (\delta_{i+1} - \delta_i)(\tfrac{1}{\alpha} + \delta_{i+1} + \delta_i - 1)^2 \\ &\quad - (\hat{\delta}_i - \hat{\delta}_{i-1})(\tfrac{1}{\alpha} + \hat{\delta}_i + \hat{\delta}_{i-1} - 1)^2 \\ &\quad - (\hat{\delta}_{i+1} - \hat{\delta}_i)(\tfrac{1}{\alpha} + \hat{\delta}_{i+1} + \hat{\delta}_i - 1)^2 \\ &= (\delta_{i+1} - \delta_i)(\tfrac{1}{\alpha} + \delta_{i+1} + \delta_i - 1)^2 \\ &\quad - (1-\epsilon)(\delta_{i+1} - \delta_i)(\tfrac{1}{\alpha} + (1-\epsilon)\delta_{i+1} + (1+\epsilon)\delta_i - 1)^2 \\ &\quad - \epsilon(\delta_{i+1} - \delta_i)(\tfrac{1}{\alpha} + (2-\epsilon)\delta_{i+1} + \epsilon\delta_i - 1)^2 \\ &< 0.\end{aligned}$$

Here, the inequality follows from the strict convexity of the quadratic function and contradicts the optimality of Δ. Case II: if $k^\beta = 1 = i$, then $\delta_1 = \delta_0 = 0$. Construct a new partition $\hat{\Delta}$ with $\hat{\delta}_1 = \delta_2$ and $\hat{\delta}_k = \delta_k$ otherwise. This leads to $k^\beta(\hat{\Delta}) = 0 < k^\beta(\Delta)$ and the same (optimal) objective value. We can now apply either the previous case or the following cases.

Finally, suppose $0 = \delta_0 < \cdots < \delta_{i-1} < \delta_i = \cdots = \delta_K = 1$ for some $i \in \{1, \dots, K-1\}$. Notice that $k^\beta(\Delta) < i$. Case I: if $i = 1$ and $\beta = 1$, then $k^\beta(\Delta) = 0$ and $\nu\Gamma_K(\Delta) = 1/\alpha^2$. Construct a new partition with $0 < \hat{\delta}_1 < 1$ and $\hat{\delta}_k = \delta_k = 1$ otherwise. This leads to $k^\beta(\hat{\Delta}) = 1$ and the following contradiction:

$$\begin{aligned}\nu\Gamma_K(\hat{\Delta}) &= \hat{\delta}_1(\tfrac{1}{\alpha^2} + 2(\hat{\delta}_1 - 1)\tfrac{1}{\alpha}) + (1 - \hat{\delta}_1)(\tfrac{1}{\alpha} + \hat{\delta}_1)^2 \\ &= \tfrac{1}{\alpha^2} + (1 - \hat{\delta}_1)\hat{\delta}_1^2 > \tfrac{1}{\alpha^2} = \nu\Gamma_K(\Delta).\end{aligned}$$

Case II: if $i > 1$ or $\beta < 1$, then there exists an $0 < \epsilon < 1$ such that

$$(1-\epsilon)\delta_i + (1+\epsilon)\delta_{i-1} \geq 1 - \tfrac{1}{\alpha}\sqrt{1-\beta}.$$

Construct a new partition with $\hat{\delta}_i = (1-\epsilon)\delta_i + \epsilon\delta_{i-1}$ and $\hat{\delta}_k = \delta_k$ otherwise. We have $\hat{\delta}_{i-1} < \hat{\delta}_i < \hat{\delta}_{i+1}$ and $k^\beta(\hat{\Delta}) = k^\beta(\Delta)$. Consequently, we get the contradiction

$$\begin{aligned}
\nu\Gamma_K(\Delta) - \nu\Gamma_K(\hat{\Delta}) &= (\delta_i - \delta_{i-1})(\tfrac{1}{\alpha} + \delta_i + \delta_{i-1} - 1)^2 + 0 \\
&\quad - (\hat{\delta}_i - \hat{\delta}_{i-1})(\tfrac{1}{\alpha} + \hat{\delta}_i + \hat{\delta}_{i-1} - 1)^2 \\
&\quad - (\hat{\delta}_{i+1} - \hat{\delta}_i)(\tfrac{1}{\alpha} + \hat{\delta}_{i+1} + \hat{\delta}_i - 1)^2 \\
&= (\delta_i - \delta_{i-1})(\tfrac{1}{\alpha} + \delta_i + \delta_{i-1} - 1)^2 \\
&\quad - (1-\epsilon)(\delta_i - \delta_{i-1})(\tfrac{1}{\alpha} + (1-\epsilon)\delta_i + (1+\epsilon)\delta_{i-1} - 1)^2 \\
&\quad - \epsilon(\delta_i - \delta_{i-1})(\tfrac{1}{\alpha} + (2-\epsilon)\delta_i + \epsilon\delta_{i-1} - 1)^2 \\
&< 0.
\end{aligned}$$

To conclude, an optimal partition Δ must satisfy $k^\beta(\Delta) \in \{0, 1\}$ and $0 < \delta_1 < \cdots < \delta_{K-1} < 1$. □

Proof of Theorem 4.12. By Lemma 4.11 we only need to consider the cases $k^\beta = 0$ and $k^\beta = 1$. Case I: suppose $k^\beta = 0$, then

$$\nu\Gamma_K = \sum_{k=1}^{K} (\delta_k - \delta_{k-1})(\tfrac{1}{\alpha} + \delta_k + \delta_{k-1} - 1)^2.$$

This expression is quadratic in δ_k for $k \in \{1, \ldots, K-1\}$, since the cubic terms cancel out. Setting the gradient to zero, leads to

$$(\delta_{k+1} - \delta_{k-1})(\delta_{k+1} + \delta_{k-1} - 2\delta_k) = 0, \qquad \forall\, k \in \{1, \ldots, K-1\}.$$

Since $\delta_{k+1} > \delta_{k-1}$ by Lemma 4.11, $\delta_k = \frac{1}{2}(\delta_{k+1} + \delta_{k-1})$ must hold. Solving this system of linear equalities, results in the equidistant partition:

$$\delta_k = \tfrac{k}{K}.$$

Since $k^\beta = 0$, it must hold that

$$\tfrac{1}{K} = \delta_1 = \delta_1 + \delta_0 \geq 1 - \tfrac{1}{\alpha}\sqrt{1-\beta} \qquad \Longleftrightarrow \qquad \alpha \leq \tfrac{K}{K-1}\sqrt{1-\beta}.$$

Case II: suppose $k^\beta = 1$, then

$$\nu\Gamma_K = \delta_1 \left(\beta\tfrac{1}{\alpha^2} + 2(\delta_1 - 1)(1 - \sqrt{1-\beta})\tfrac{1}{\alpha}\right) + \sum_{k=2}^{K} (\delta_k - \delta_{k-1})(\tfrac{1}{\alpha} + \delta_k + \delta_{k-1} - 1)^2. \tag{4.28}$$

This expression is cubic in δ_1 and quadratic in δ_k for $k \in \{2, \ldots, K-1\}$. Setting the gradient to zero leads to

$$-3\delta_1^2 - 2(\delta_2 + 2\tfrac{1}{\alpha}\sqrt{1-\beta} - 2)\delta_1 + \delta_2^2 - 1 + 2\tfrac{1}{\alpha}\sqrt{1-\beta} - \tfrac{1-\beta}{\alpha^2} = 0$$

and

$$(\delta_{k+1}-\delta_{k-1})(\delta_{k+1}+\delta_{k-1}-2\delta_k)=0, \qquad \forall\, k\in\{2,\ldots,K-1\}.$$

The roots for the first equation are $\delta_1 = \frac{1}{3}(\delta_2 + 1 - \frac{1}{\alpha}\sqrt{1-\beta})$ and $\delta_1 = 1-\delta_2-\frac{1}{\alpha}\sqrt{1-\beta}$, where the first root is the largest. The second set of equations are as before, implying $\delta_k = \frac{1}{2}(\delta_{k+1}+\delta_{k-1})$ for $k>1$. By substituting $\delta_K = 1$ we can express δ_k as an affine function of δ_{k-1} for $k>1$. Consequently, δ_k is an affine function of δ_1 for $k>1$. Hence, after substitution of δ_k, (4.28) remains a cubic function in δ_1, whose leading term is $-\delta_1^3$. We conclude that the optimal value for δ_1 is the larger root of the corresponding derivative.

The resulting system of linear equations, $\delta_1 = \frac{1}{3}(\delta_2+1-\frac{1}{\alpha}\sqrt{1-\beta})$ and $\delta_k = \frac{1}{2}(\delta_{k+1}+\delta_{k-1})$ for $k\in\{2,\ldots,K-1\}$, has the following solution:

$$\delta_k = 1-\tfrac{K-k}{2K-1}(1+\tfrac{1}{\alpha}\sqrt{1-\beta}), \qquad \forall\, k\in\{1,\ldots,K-1\}.$$

For this partition to be valid with $k^\beta=1$, we must have

$$1-\tfrac{K-1}{2K-1}(1+\tfrac{1}{\alpha}\sqrt{1-\beta}) = \delta_1 = \delta_1+\delta_0 < 1-\tfrac{1}{\alpha}\sqrt{1-\beta} \quad\Longleftrightarrow\quad \alpha > \tfrac{K}{K-1}\sqrt{1-\beta}.$$

Likewise, we have for $k\in\{2,\ldots,K-1\}$ that

$$\alpha>0 \quad\Longrightarrow\quad 2-\tfrac{2K-2k+1}{2K-1}(1+\tfrac{1}{\alpha}\sqrt{1-\beta}) = \delta_k+\delta_{k-1} \ge 1-\tfrac{1}{\alpha}\sqrt{1-\beta}.$$

This implies that for the partition indeed $k^\beta=1$. Finally, clearly $\delta_{K-1}<1$, so we only need to verify that $\delta_1>0$:

$$1-\tfrac{K-1}{2K-1}(1+\tfrac{1}{\alpha}\sqrt{1-\beta})>0 \quad\Longleftrightarrow\quad \alpha>\tfrac{K-1}{K}\sqrt{1-\beta}.$$

Thus, the partition is valid for $\alpha > \frac{K}{K-1}\sqrt{1-\beta}$.

Notice that both cases for k^β are disjoint and cover all possible values of $\alpha>0$. This completes the proof. $\square$

Proof of Corollary 4.13. By Theorem 4.12, for $\alpha \le \frac{K}{K-1}\sqrt{1-\beta}$ we have $k^\beta=0$ and $\delta_k = k/K$. The expressions (4.20) simplifies to

$$\nu\Gamma_K^{\text{opt}} = \sum_{k=1}^{K}\tfrac{1}{K}\left(\tfrac{1}{\alpha}+\tfrac{2k-1}{K}-1\right)^2 = \tfrac{1}{\alpha^2}+\tfrac{1}{3}\left(1-\tfrac{1}{K^2}\right).$$

For $\alpha>\frac{K}{K-1}\sqrt{1-\beta}$ we have $k^\beta=1$ and (4.20) becomes

$$\begin{aligned}\nu\Gamma_K^{\text{opt}} &= \left(1-\tfrac{K-1}{2K-1}(1+\tfrac{1}{\alpha}\sqrt{1-\beta})\right)\left(\tfrac{\beta}{\alpha^2}-\tfrac{2K-2}{2K-1}(1+\tfrac{1}{\alpha}\sqrt{1-\beta})(1-\sqrt{1-\beta})\tfrac{1}{\alpha}\right)\\ &\quad+\sum_{k=2}^{K}\tfrac{1}{2K-1}(1+\tfrac{1}{\alpha}\sqrt{1-\beta})\left(\tfrac{1}{\alpha}+1-\tfrac{2K-2k+1}{2K-1}(1+\tfrac{1}{\alpha}\sqrt{1-\beta})\right)^2\\ &= \tfrac{\beta}{\alpha^2}+\tfrac{2}{3}\tfrac{K(K-1)}{(2K-1)^2}(1+\tfrac{1}{\alpha}\sqrt{1-\beta})^3.\end{aligned}$$

In particular, these expressions converge to (4.21) as $K\to\infty$, as should be the case. $\square$

Proof of Theorem 4.14. Continuity of $\Gamma_K^{\mathrm{opt}}/\Gamma_\infty$ is trivially verified by (4.21) and (4.22). For readability, we state the properties that will be proved in the end:

- For $K = 1$ we have $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_1}{\Gamma_\infty} < 0$ for all $\alpha > 0$.
- For $K > 1$ we have
  - if $\beta = 0$: $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty} < 0$ for $0 < \alpha < \frac{K}{K-1}$ and $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty} = 0$ for $\alpha \geq \frac{K}{K-1}$,
  - if $0 < \beta \leq 1$: $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty} < 0$ for all $\alpha > 0$.
- We have $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty} = 0$ for $0 < \alpha \leq \sqrt{1-\beta}$ and $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty} > 0$ for $\alpha > \sqrt{1-\beta}$.

We start with the proof for $K = 1$, which is considered separately to prevent issues with division by zero. Note that $\nu\Gamma_1 = \alpha^{-2}$ and that there is no partition to optimise in this case. Therefore, for $0 < \alpha \leq \sqrt{1-\beta}$ it is trivial to show that

$$\frac{\Gamma_1}{\Gamma_\infty} = \frac{3}{\alpha^2+3}, \qquad \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_1}{\Gamma_\infty} = -\frac{6\alpha}{(\alpha^2+3)^2} < 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_1}{\Gamma_\infty} = 0.$$

For $\alpha > \sqrt{1-\beta}$, we have

$$\frac{\Gamma_1}{\Gamma_\infty} = \frac{6\alpha}{6\alpha\beta + (\alpha+\sqrt{1-\beta})^3}.$$

The corresponding derivatives are

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_1}{\Gamma_\infty} &= -\frac{6(2\alpha-\sqrt{1-\beta})(\alpha+\sqrt{1-\beta})^2}{\left(6\alpha\beta+(\alpha+\sqrt{1-\beta})^3\right)^2} < 0,\\ \frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_1}{\Gamma_\infty} &= \frac{9\alpha(\alpha-\sqrt{1-\beta})^2}{\sqrt{1-\beta}\left(6\alpha\beta+(\alpha+\sqrt{1-\beta})^3\right)^2} > 0,\end{aligned}$$

since $\alpha > \sqrt{1-\beta}$. Hence, for any $0 \leq \beta \leq 1$ the infimum for the pooling performance is

$$\inf_{\alpha>0}\frac{\Gamma_1}{\Gamma_\infty} = \lim_{\alpha\to\infty}\frac{\Gamma_1}{\Gamma_\infty} = 0.$$

We continue with the proof for $K > 1$. Based on (4.21) and (4.22) we need to differentiate three cases.

Case I: for $0 < \alpha \leq \sqrt{1-\beta}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty} = -\frac{6\alpha}{K^2(\alpha^2+3)^2} < 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty} = 0.$$

Case II: for $\sqrt{1-\beta} < \alpha \leq \frac{K}{K-1}\sqrt{1-\beta}$, which can only occur for $0 \leq \beta < 1$, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty} &= 6\frac{(K^2-1)\sqrt{1-\beta}\alpha^4 + 2\left((K^2-1)\beta-1\right)\alpha^3}{K^2\left(6\alpha\beta+(\alpha+\sqrt{1-\beta})^3\right)^2}\\ &\quad + 6\frac{-\left(K^2(\beta+2)+1-\beta\right)\sqrt{1-\beta}\alpha^2 + K^2(1-\beta)^{3/2}}{K^2\left(6\alpha\beta+(\alpha+\sqrt{1-\beta})^3\right)^2}.\end{aligned} \tag{4.29}$$

We claim that (4.29) is strictly negative on $\sqrt{1-\beta} < \alpha < \frac{K}{K-1}\sqrt{1-\beta}$ and that (4.29) at $\alpha = \frac{K}{K-1}\sqrt{1-\beta}$ is either zero (if $\beta = 0$) or strictly negative (if $0 < \beta < 1$). Clearly, the denominator is always strictly positive. Hence, it is sufficient to focus on the numerator. Let $f(\alpha) = c_4\alpha^4 + c_3\alpha^3 + c_2\alpha^2 + c_0$ denote the numerator. Recall that $K > 1$. Since this case cannot occur for $\beta = 1$, we have $c_4 > 0$, $c_3 \in \mathbb{R}$, $c_2 < 0$, and $c_0 > 0$.

First, by Descartes' Sign Rule the number of positive real roots of f is bounded by 2, namely by the number of sign changes in the sequence c_4, c_3, c_2, and c_0.

Second, we evaluate the numerator f for certain values for α:

$$\lim_{\alpha \downarrow 0} f(\alpha) = 6K^2(1-\beta)^{3/2} > 0, \qquad \lim_{\alpha\to\infty} f(\alpha) = \infty > 0,$$
$$f(\sqrt{1-\beta}) = -24(1-\beta)^{3/2} < 0,$$
$$f\left(\tfrac{K}{K-1}\sqrt{1-\beta}\right) = -6\frac{K^2(K+1)\beta(1-\beta)^{3/2}}{(K-1)^3} \leq 0,$$

where we use that $K > 1$ and that this case cannot occur for $\beta = 1$. By continuity of f we conclude that there is a positive real root on $(0, \sqrt{1-\beta})$ and on $[\frac{K}{K-1}\sqrt{1-\beta}, \infty)$.

Thus, f has exactly two positive real roots. If $0 < \beta < 1$ both fall outside of $(\sqrt{1-\beta}, \frac{K}{K-1}\sqrt{1-\beta}]$. Furthermore, since f is strictly negative on the borders of this interval, it is strictly negative on the entire interval. If $\beta = 0$, one of the roots is the border point $\frac{K}{K-1}$. The same conclusions hold for (4.29).

Furthermore, using $\alpha > \sqrt{1-\beta}$ we have that the derivative to β is

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty} = \frac{3\alpha(\alpha-\sqrt{1-\beta})^2((K^2-1)\alpha^2+3K^2)}{K^2\sqrt{1-\beta}\left(6\alpha\beta+(\alpha+\sqrt{1-\beta})^3\right)^2} > 0.$$

Case III: for $\alpha > \frac{K}{K-1}\sqrt{1-\beta}$ it holds that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty} = -\frac{6\beta\left(2\alpha^3+3\sqrt{1-\beta}\alpha^2-(1-\beta)^{3/2}\right)}{(2K-1)^2\left(6\alpha\beta+(\alpha+\sqrt{1-\beta})^3\right)^2} \leq 0, \tag{4.30}$$
$$\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty} = \frac{3\alpha(2\sqrt{1-\beta}\alpha^3+3(2-\beta)\alpha^2+6\sqrt{1-\beta}\alpha+2-\beta^2-\beta)}{(2K-1)^2\sqrt{1-\beta}\left(6\alpha\beta+(\alpha+\sqrt{1-\beta})^3\right)^2} > 0,$$

where we use that $\alpha > \frac{K}{K-1}\sqrt{1-\beta} > \sqrt{1-\beta}$. Note that (4.30) is zero if $\beta = 0$ and strictly negative if $0 < \beta \leq 1$.

We conclude that the derivative of the pooling performance is non-negative (if $\beta = 0$) or strictly negative (if $0 < \beta \leq 1$) with respect to α in all cases. Hence, the infimum is reached for $\alpha \to \infty$:

$$\inf_{\alpha>0}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty} = \lim_{\alpha\to\infty}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty} = \frac{4K(K-1)}{(2K-1)^2}.$$

Here, the limit trivially follows from (4.21) and (4.22). Furthermore, notice that this infimum holds for any $0 \leq \beta \leq 1$, implying that this bound is tight for any $0 \leq \beta \leq 1$. □

Proof of Theorem 4.15. Continuity of $\Gamma_K^{\mathrm{opt}}/\Gamma_\infty^{\beta=0}$ is trivially verified by (4.21) and (4.22). For readability, we make the following claims, which are all proved in the end:

- For $K=1$ we have $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}}<0$ and $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}}=0$ for all $\alpha>0$.
- For $K>1$ we have
  - if $\beta=0$: $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty^{\beta=0}}<0$ for $0<\alpha<\frac{K}{K-1}$ and $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty^{\beta=0}}=0$ for $\alpha\geq\frac{K}{K-1}$,
  - if $0<\beta\leq 1$: $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty^{\beta=0}}<0$ for $0<\alpha<\alpha^*$, $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty^{\beta=0}}=0$ for $\alpha=\alpha^*$, and $\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty^{\beta=0}}>0$ for $\alpha>\alpha^*$,
  - $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty^{\beta=0}}=0$ for $0<\alpha\leq\frac{K}{K-1}\sqrt{1-\beta}$ and $\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty^{\beta=0}}<0$ for $\alpha>\frac{K}{K-1}\sqrt{1-\beta}$.

Here, the minimiser α^* is defined for $K>1$ and $0<\beta\leq 1$ by

$$\alpha^*=1+\frac{(2K(K-1)+1)\beta+(2K-1)\sqrt{\beta\left(2K(K-1)(1-\sqrt{1-\beta})+\beta\right)}}{2K(K-1)(1-\sqrt{1-\beta})}$$

and we claim that $\alpha^*>1$ and $\alpha^*>\frac{K}{K-1}\sqrt{1-\beta}$ if it exists.

We first focus on the tight reservation level performance guarantee. Using the above claims, in particular on the derivative to β, we conclude that it is sufficient to consider $\beta=1$ to derive the tight guarantee for the reservation level performance. Therefore, we have to consider two cases: $K=1$ and $K>1$ with $\beta=1$.

First, consider the case $K=1$. We have for $0<\alpha\leq 1$ that

$$\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}}=\frac{3}{\alpha^2+3},\qquad \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}}=-\frac{6\alpha}{(\alpha^2+3)^2}<0,\qquad \frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}}=0.$$

Likewise, for $\alpha>1$

$$\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}}=\frac{6\alpha}{(\alpha+1)^3},\qquad \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}}=-\frac{6(2\alpha-1)}{(\alpha+1)^4}<0,\qquad \frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}}=0.$$

Thus, for any $0\leq\beta\leq 1$ the reservation level performance guarantee is

$$\inf_{\alpha>0}\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}}=\lim_{\alpha\to\infty}\frac{\Gamma_1}{\Gamma_\infty^{\beta=0}}=0. \tag{4.31}$$

Second, consider $K>1$ and $\beta=1$. We need to discern two cases based on α.

Case I: for $0<\alpha\leq 1$ we have

$$\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty^{\beta=0}}=\frac{2K(K-1)\alpha^2+3(2K-1)^2}{(2K-1)^2(\alpha^2+3)},\qquad \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty^{\beta=0}}=-\frac{6(2K^2-2K+1)\alpha}{(2K-1)^2\left(\alpha^2+3\right)^2}<0.$$

Case II: for $\alpha > 1$ we get

$$\begin{aligned}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} &= \frac{4K(K-1)\alpha^3 + 6(2K-1)^2\alpha}{(2K-1)^2(\alpha+1)^3},\\ \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} &= 6\frac{2K(K-1)\alpha^2 - (8K^2-8K+2)\alpha + 4K^2 - 4K + 1}{(2K-1)^2(\alpha+1)^4}.\end{aligned} \tag{4.32}$$

The derivative (4.32) has roots

$$\alpha^{\pm} = 2 + \frac{1 \pm (2K-1)\sqrt{2K^2-2K+1}}{2K(K-1)}.$$

Note that α^+ corresponds to α^* for this case. Evaluating the formula in (4.32) for $\alpha = 1$ gives

$$-\frac{3}{8}\frac{2K(K-1)+1}{(2K-1)^2} < 0.$$

Since (4.32) is a parabola that opens upward, we have $\alpha^- < 1 < \alpha^+$. Hence, the reservation level performance has a minimum at α^+. By combining Case I and Case II, we conclude that α^+ is the global minimum for $K > 1$ and $\beta = 1$.

As argued above, for $K > 1$ the tight reservation level performance guarantee follows from evaluating $\Gamma_K^{\text{opt}}/\Gamma_\infty^{\beta=0}$ at $\alpha = \alpha^+$ and $\beta = 1$, resulting in

$$\inf_{0\le\beta\le1}\inf_{\alpha>0}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = \frac{8K(K-1)\left(4K(K-1) + (2K-1)\sqrt{2K^2-2K+1}+1\right)}{\left(6K(K-1) + (2K-1)\sqrt{2K^2-2K+1}+1\right)^2}.$$

This formula also works for $K = 1$ (resulting in a value of 0, see also (4.31)).

It remains to prove all our claims. The proofs for $K = 1$ have already been given. Therefore, consider the case $K > 1$. Unfortunately, the proofs are somewhat tedious work. We have to distinguish four cases.

Case I: for $0 < \alpha \le 1$ and $\alpha \le \frac{K}{K-1}\sqrt{1-\beta}$ we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = -\frac{6\alpha}{K^2\left(\alpha^2+3\right)^2} < 0, \qquad \frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = 0.$$

Case II: for $1 < \alpha \le \frac{K}{K-1}\sqrt{1-\beta}$ the derivatives are

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = 6\frac{(K^2-1)\alpha^2 - K^2(2\alpha-1)}{K^2(\alpha+1)^4}, \qquad \frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\text{opt}}}{\Gamma_\infty^{\beta=0}} = 0.$$

The roots of the derivative to α are $\frac{K}{K+1} < 1$ and $\frac{K}{K-1} \ge \frac{K}{K-1}\sqrt{1-\beta}$. Notice that the numerator is a parabola that opens upward. For $\beta = 0$ the derivative to α is strictly negative on $1 < \alpha < \frac{K}{K-1}$ and zero at $\alpha = \frac{K}{K-1}$. For $0 < \beta \le 1$ it is strictly negative on the entire interval $1 < \alpha \le \frac{K}{K-1}\sqrt{1-\beta}$.

Case III: for $\frac{K}{K-1}\sqrt{1-\beta} < \alpha \leq 1$ we get

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty^{\beta=0}} = -6\frac{K(K-1)\sqrt{1-\beta}\left(\alpha^4-(\beta+2)\alpha^2+(1-\beta)\right)+(2K^2-2K+1)\beta\alpha^3}{(2K-1)^2\alpha^2\left(\alpha^2+3\right)^2}. \tag{4.33}$$

Let f be the numerator of (4.33). For $\beta = 0$ this case cannot occur. For $\beta = 1$ the function f simplifies to a cubic function with roots $\alpha = 0$. Hence, it follows trivially that (4.33) is strictly negative for $\alpha > 0$. For $0 < \beta < 1$ it holds that $f(\alpha) < 0$ on $\alpha > 0$ if and only if

$$g(\alpha) = -6\alpha^4 - 6\frac{2K^2-2K+1}{K(K-1)}\frac{\beta}{\sqrt{1-\beta}}\alpha^3 + 6(\beta+2)\alpha^2 - 6(1-\beta) < 0 \quad \forall\, \alpha > 0,$$

where the quartic function g differs from f by a positive factor. Let the quartic function h be defined by

$$h(\alpha) = -6\alpha^4 - 12\tfrac{\beta}{\sqrt{1-\beta}}\alpha^3 + 6(\beta+2)\alpha^2 - 6(1-\beta).$$

Since $(2K^2-2K+1)/(K(K-1)) > 2$ for $K > 1$, we have $g(\alpha) < h(\alpha)$ for all $\alpha > 0$. The discriminant of h is zero. By using well-known properties of quartic formulas we conclude that h has two distinct real roots and one double real root. The shape of h now follows from evaluating it for certain points:

$$\lim_{\alpha\to-\infty} h(\alpha) = -\infty < 0, \quad h(-1) = 12\beta\left(1+\tfrac{1}{\sqrt{1-\beta}}\right) > 0, \quad h(0) = -6(1-\beta) < 0.$$

This trivially implies that $h(\alpha) \leq 0$ for all $\alpha > 0$ by evaluating all possible shapes of h. Thus, $g(\alpha) < 0$, $f(\alpha) < 0$, and (4.33) is strictly negative for all $\alpha > 0$.

The derivative to β is given by

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty^{\beta=0}} = -3\frac{K(K-1)\alpha^2-(2K^2-2K+1)\sqrt{1-\beta}\alpha+K(K-1)(1-\beta)}{(2K-1)^2\sqrt{1-\beta}\alpha(\alpha^2+3)}, \tag{4.34}$$

which has roots $\frac{K-1}{K}\sqrt{1-\beta}$ and $\frac{K}{K-1}\sqrt{1-\beta}$ (both smaller than the considered α). Note that the numerator of (4.34) is a parabola that opens downward. Hence, (4.34) is strictly negative.

Case IV: for $\alpha > \frac{K}{K-1}\sqrt{1-\beta}$ and $\alpha > 1$ it holds that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty^{\beta=0}} = 6&\frac{2K(K-1)(1-\sqrt{1-\beta})\alpha^2}{(2K-1)^2(\alpha+1)^4}\\ &- 6\frac{\left((4K^2-4K+2)\beta+4K(K-1)(1-\sqrt{1-\beta})\right)\alpha}{(2K-1)^2(\alpha+1)^4}\\ &+ 6\frac{-2K(K-1)(1-\beta)^{3/2}+2K(K-1)(1+\beta)+\beta}{(2K-1)^2(\alpha+1)^4}.\end{aligned} \tag{4.35}$$

If $\beta = 0$, then (4.35) is always equal to zero. For $\beta > 0$ the numerator of (4.35) is a parabola that opens upward with roots

$$\alpha^{\pm} = 1 + \frac{\big(2K(K-1)+1\big)\beta \pm (2K-1)\sqrt{\beta\big(2K(K-1)(1-\sqrt{1-\beta})+\beta\big)}}{2K(K-1)(1-\sqrt{1-\beta})}.$$

We claim that α^+ is a local minimiser for the reservation level performance, which turns out to be the global minimiser by checking all other cases. This claim is proved by showing that $\alpha^- < 1 < \alpha^+$ and, if needed, that $\alpha^- < \frac{K}{K-1}\sqrt{1-\beta} < \alpha^+$. This implies that (4.35), with a parabola that opens upward as numerator, is strictly negative for $\alpha < \alpha^+$, zero at $\alpha = \alpha^+$, and strictly positive for $\alpha > \alpha^+$. Hence, α^+ is a local minimiser.

We continue to prove our claim. Evaluating (4.35) for $\alpha = 1$ results in the value

$$-\frac{3}{8}\beta\frac{2K(K-1)(1-\sqrt{1-\beta})+1}{(2K-1)^2} < 0.$$

This implies that $\alpha^- < 1 < \alpha^+$. If $\frac{K}{K-1}\sqrt{1-\beta} \leq 1$ the proof for this case if complete. Otherwise, $\frac{K}{K-1}\sqrt{1-\beta} > 1$ or equivalently

$$\beta < \frac{2K-1}{K^2}.$$

Evaluating (4.35) for $\alpha = \frac{K}{K-1}\sqrt{1-\beta}$ gives

$$-6\frac{(K-1)^3((K+1)\beta - 2K(1-\sqrt{1-\beta}))}{(K(1+\sqrt{1-\beta})-1)^4}, \tag{4.36}$$

which is zero only if $\beta = 0$ or if $\beta = \frac{4K}{(K+1)^2}$ and strictly negative in between these values. Since $\frac{2K-1}{K^2} = \frac{4K}{(K+1)^2}$ only if $K = 1$ (excluding negative values), we conclude that we are considering β satisfying

$$0 < \beta < \frac{2K-1}{K^2} < \frac{4K}{(K+1)^2}.$$

For such β the value (4.36) is strictly negative. This implies that $\alpha^- < \frac{K}{K-1}\sqrt{1-\beta} < \alpha^+$, completing the proof for this case.

The derivative to β is

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\frac{\Gamma_K^{\mathrm{opt}}}{\Gamma_\infty^{\beta=0}} = -6\frac{K(K-1)\alpha^2 - (2K^2-2K+1)\sqrt{1-\beta}\alpha + K(K-1)(1-\beta)}{(2K-1)^2\sqrt{1-\beta}(\alpha+1)^3}, \tag{4.37}$$

with roots $\frac{K-1}{K}\sqrt{1-\beta}$ and $\frac{K}{K-1}\sqrt{1-\beta}$ (both smaller than the considered α). As seen before, the numerator is a parabola that opens downward, which implies that (4.37) is strictly negative. $\square$

Chapter 5

Two-echelon lot-sizing with asymmetric information and continuous type space

Abstract

In this chapter, we analyse a two-echelon discrete lot-sizing problem with a supplier and a retailer under information asymmetry. We assume that all cost parameters are time independent and that the retailer has single-dimensional continuous private information, namely either his setup cost or his holding cost. The supplier uses mechanism design to determine a menu of contracts that minimises his expected costs, where each contract specifies the retailer's procurement plan and a side payment to the retailer. There is no restriction on the number of contracts in the menu.

To optimally solve this contracting problem we present a two-stage approach, based on a theoretical analysis. The first stage generates a list of procurement plans that is sufficient to solve the contracting problem to optimality. The second stage optimally assigns these plans to the retailer types and determines all side payments. The result is an optimal menu with finitely many contracts that pools retailer types. We identify cases for which the contracting problem can be solved in polynomial time and provide the corresponding algorithms. Furthermore, our analysis reveals that information asymmetry leads to atypical structures in the plans of the optimal menu, e.g., plans violating the zero-inventory property. Our solution approach and several results are directly applicable to more general problems as well.

This chapter is based on Kerkkamp et al. (2018b).

Contents

5.1 Introduction

We consider a two-echelon supply chain consisting of a supplier and a retailer under a discrete lot-sizing setting with asymmetric information. The supply chain must come to an agreement for a joint procurement plan to satisfy market demand for a single indivisible product for a given time horizon. We assume that the market demand can be modelled as demand in discrete time periods and is known up front, leading to a discrete lot-sizing problem for the supply chain. That is, the joint procurement plan specifies the following for each time period up to the planning horizon. For the supplier, the options are to produce new products, to keep products in inventory for later time periods, and to transfer products to the retailer. For the retailer, these are to receive products from the supplier, to keep products in inventory, and to satisfy market demand. We assume all market demand must be satisfied and back-ordering is not allowed.

If all information is shared among the two parties in the supply chain, we have the traditional joint lot-sizing problem, which is well known and analysed thoroughly in the literature, e.g., in Zangwill (1969). We consider the case where there is only partial cooperation between the supplier and the retailer. Namely, the retailer has private information on his cost structure that he does not share with the supplier. Furthermore, we assume that the supplier and the retailer both act individually rationally and want to minimise their own costs. This partial cooperation typically leads to inefficiencies for the supply chain, see for example Inderfurth et al. (2013) and Perakis and Roels (2007). However, we consider the problem from the supplier's point of view, who wants to minimise his own costs, and thus perfect supply chain coordination is not a goal.

We assume that the supply chain uses a pull ordering strategy, i.e., the retailer has the initiative and the market power to place orders at the supplier. The supplier must satisfy these orders. Hence, by default the retailer will order according to his own individually optimal procurement plan, which is typically suboptimal for the supplier. The supplier has a single opportunity to offer the retailer a menu of contracts to persuade him to change his procurement plan. A single contract specifies the retailer's orders at the supplier and a side payment from the supplier to the retailer. By using a large enough side payment the supplier can convince the retailer to accept a different procurement plan. The menu can contain any number of contracts. However, since the retailer can reject any offered contract and has private information on his cost structure, it is not trivial to design a menu of contracts that minimises the supplier's costs.

We consider the case where all cost parameters are time independent. Consequently, as we will show in Section 5.2, the only relevant costs are the supplier's setup cost of production, the retailer's setup cost for an order, and the holding costs for the inventory of both parties. We assume that the retailer's private information is either his setup cost or his holding cost, and lies in a certain interval. Thus, the private information is single dimensional, bounded, and continuous. We also assume that the supplier has a probability distribution for the retailer's private information.

The supplier uses mechanism design (see Laffont and Martimort (2002)) to construct a menu of contracts that minimises his expected costs, which requires solving a specific optimisation problem. We call this optimisation problem the *contracting problem*, which can be formulated as a mixed integer linear program with infinitely many variables and constraints. All details of the setting and the model will be given in Section 5.2.

Our goals are to analyse the contracting problem to obtain a tractable formulation and to determine efficiently solvable cases. Of particular interest is whether the information asymmetry changes the complexity class of the underlying optimisation problem. That is, if all information is shared the corresponding contracting problem turns out to be a traditional joint lot-sizing problem, which is solvable in polynomial time (see Zangwill (1969)). With information asymmetry the contracting problem is non-trivial, but can it still be solved in polynomial time? Before we state our main results for these questions, we first discuss the related literature to position our contribution.

5.1.1 Related literature

At its core, the described lot-sizing contracting problem is strongly related to the two-echelon lot-sizing problem. Also notice that the retailer's default plan, being individually optimal, follows from solving a single-level lot-sizing problem. Both these traditional lot-sizing problems have been analysed in detail in the literature. We refer to Wagner and Whitin (1958) and Zangwill (1969) for solution methods. As we will show, we need to solve several joint lot-sizing subproblems where either the number of retailer setups or the amount of retailer inventory is fixed. In a way, this relates to a parametric analysis (see Van Hoesel and Wagelmans (2000)) and stability regions of solutions (see Richter and Vörös (1989)). However, certain properties used in the previous references do not hold in general for our subproblems. For example, in specific cases the optimal menu of contracts contains procurement plans that do not satisfy the so-called zero-inventory property, implying that the retailer has unnecessary inventory when considered in isolation (see also Section 5.3).

The lot-sizing contracting problem fits in the broader research field of the application of mechanism design to traditional optimisation problems. We focus on literature that considers related supply chain procurement problems with asymmetric information. See also Laffont and Martimort (2002) and Leng and Parlar (2005) for more general references on this topic.

Perhaps one of the most fundamental researched problems is the economic order quantity (EOQ) problem under information asymmetry. Compared to our lot-sizing setting, the EOQ problem considers a constant demand rate over time, an infinite time horizon, and divisible products. Several variations have been researched, such as the private information being continuous or discrete, and single- or two-dimensional (see for example Corbett and de Groote (2000), Inderfurth et al. (2013), and Pishchulov and Richter (2016) and Chapter 2). Another setting is the newsvendor problem under information asymmetry, which considers a single period but with uncertain demand. This problem has been analysed in Burnetas et al. (2007), Cachon (2003), and Cakanyildirim et al. (2012) among others.

In these models a (single) order quantity describes the entire procurement plan and the total costs of each party have closed-form expressions in terms of this order quantity. In contrast, for the lot-sizing problem the total costs of each party cannot be expressed as (manageable) closed-form formulas. Instead, the costs follow from solving a combinatorial optimisation problem. This requires a different solution approach.

In Albrecht (2017) a coordination problem based on lot-sizing between a supplier and retailer is considered. Both parties only communicate the desired or supplied order quantities, no other information is shared. The focus lies on a heuristic coordination scheme which might lead to an optimal procurement plan for the entire supply chain. Certain conditions are identified for which this is indeed the case. In the proposed scheme, the retailer determines a list of individually optimal retailer plans, where each plan has a fixed number of retailer setups. The list is then offered to the supplier, who determines his optimal response (a supplier plan) for each retailer plan. Finally, both parties jointly decide which of the resulting joint procurement plans is executed. These final negotiations should also include a way to divide the resulting profit gained from the coordination, for which several strategies are suggested but not analysed. Similar coordination and negotiation settings are analysed in for example Buer et al. (2013), Dudek and Stadtler (2005), and Dudek and Stadtler (2007).

The setting in Albrecht (2017) differs significantly from ours: our goal is to minimise the supplier's costs, not to achieve perfect supply chain coordination, and more information is available to the supplier. However, the coordination scheme has the following similarity to our case. As we will show in Section 5.3, with private setup cost it is sufficient for optimality to design a list of T plans, namely one plan for each possible number of retailer setups. In contrast, these plans follow from joint lot-sizing problems and are not individually optimal plans.

To our knowledge, only the works of Mobini et al. (2014) and Phouratsamay (2017) consider similar discrete lot-sizing problems under information asymmetry. In the setting of Mobini et al. (2014) the costs are time dependent and the retailer's private information is discrete and multi-dimensional. Several conditions are identified under which the retailer's behaviour, regarding the selection of contracts, is more structured. Furthermore, the case with private demand information is analysed. Phouratsamay (2017) also considers the lot-sizing contracting problem with time-dependent costs and discrete private information. Three contract variations are analysed: contracts without side payments, contracts where the side payments can only compensate the retailer's holding costs, and contracts with unrestricted side payments. If all information is shared among the supplier and retailer, the variant with restricted side payments is NP-hard and the other two are solvable in polynomial time. For the private information case, the variant without side payments is polynomially solvable, but the complexity for the others remain open. For all these cases a numerical study is performed, showing that using restricted side payments performs only slightly worse than using unrestricted side payments. We complement the work of Mobini et al. (2014) and Phouratsamay (2017) by considering continuous private information, which requires a different solution approach.

5.1.2 Contribution

We present and analyse a two-echelon discrete lot-sizing problem where the retailer has single-dimensional continuous private information. In this principal-agent contracting problem either the retailer's setup cost or his holding cost is private. To our knowledge, this type of problem has not been researched in the literature, and we are the first to analyse a principal-agent contracting problem with an underlying combinatorial structure and continuous private information. Based on a theoretical analysis, we propose a two-stage solution approach consisting of a plan-generation stage and a plan-assignment stage. We identify cases where these stages can be solved in polynomial time and give the corresponding algorithms. This provides further insights into the complexity of lot-sizing models with asymmetric information.

Moreover, we observe structural differences compared to traditional lot-sizing problems due to the information asymmetry, such as optimal menus with plans that violate the zero-inventory property. Furthermore, the contracting problem and several of our results have an intuitive graphical interpretation, which is also applicable to other problem settings. Therefore, we also describe a more general setting for which the (conceptual) model, the solution approach, and certain results are applicable as well.

The remainder is organised as follows. In Section 5.2 we formally introduce the setting of our problem and the associated optimisation model. In Section 5.3 we analyse this model, derive a solution approach, and prove complexity results. The generalisability of our results is the central topic of the discussion in Section 5.4, in which we also conclude our results.

5.2 The contracting problem

In this section we formalise the contracting problem described in the introduction. In Section 5.2.1 we specify the lot-sizing setting, the two players involved in the problem, and their possible actions. The corresponding optimisation model is given in Section 5.2.2.

5.2.1 The setting

Our setting considers a discrete lot-sizing problem between a supplier and a retailer for a finite planning horizon $T \in \mathbb{N}_{\geq 1}$. The retailer needs to satisfy market demand $d_t \in \mathbb{N}_{>0}$ in each time period $t \in \mathcal{T} = \{1, \dots, T\}$ in the planning horizon. We assume that the products are indivisible, leading to discrete demand, and that this demand is strictly positive and deterministic in each period. The strict positivity of the demand streamlines certain results and proofs, and will be discussed in Section 5.4.1. The market demand can be satisfied either from the retailer's inventory, i.e., surplus available from the previous time period, or directly from a retailer's order at the supplier. In turn, the supplier satisfies the retailer's orders either from available inventory or by setting up a new production.

In the entire supply chain lead-times are zero, all demand or orders must be met, and back-ordering is not allowed. Furthermore, in the first time period the starting inventory of the supplier and retailer are assumed to be zero and the retailer must end with zero inventory after the final time period T. Thus, to satisfy the market demand a procurement plan for the supply chain must be made. This plan specifies for each period $t \in \mathcal{T}$ the supplier's production quantity $x_t^S \in \mathbb{N}$ and the retailer's order quantity $x_t^R \in \mathbb{N}$ at the supplier. Since the demand is deterministic and there are no back-orders, these order quantities completely determine the flow of products in the supply chain. Hence, a procurement plan prescribes the setups, the order quantities, and the resulting inventory for the entire planning horizon.

In our setting all costs and revenues are time independent. Consequently, we can assume without loss of generality that the variable procurement costs and the revenue from sold products are zero in the supply chain. We will elaborate on this after giving the optimisation model. Therefore, there are two relevant types of costs involved for the supplier and retailer, namely setup cost and holding cost. If the retailer places an order he incurs a setup cost of $f \in \mathbb{R}_{>0}$ and keeping a unit of products in inventory costs $h \in \mathbb{R}_{>0}$ per time period. Similarly, for the supplier we have setup cost $F \in \mathbb{R}_{>0}$ and holding cost $H \in \mathbb{R}_{>0}$.

As mentioned in the introduction, the retailer has single-dimensional private information, i.e, either his setup cost f or his holding cost h is private. To handle both cases, we use θ for his private cost and call θ the retailer's *type*. To be precise, if the setup cost f is private we define $f(\theta) = \theta$ and $h(\theta) = h$. In the other case, with private holding cost h, we define $f(\theta) = f$ and $h(\theta) = \theta$. We assume that the supplier has estimated the retailer's private information θ to follow a strictly positive continuous distribution $\omega : \Theta \to \mathbb{R}_{>0}$ on a closed interval $\Theta = [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}_{>0}$.

By assumption, the retailer has the market power to enforce any retailer's procurement plan onto the supplier. Consequently, by default the retailer orders according to his individually optimal plan, which depends on his type. The corresponding retailer's default costs are denoted by $\phi^*(\theta)$ for type $\theta \in \Theta$ and follow from solving a traditional single-level lot-sizing problem. We refer to ϕ^* as the retailer's *default option*, also known as his reservation level. The supplier uses mechanism-design techniques by offering the retailer a menu of contracts to incentivise the retailer to alter his procurement plan. The menu effectively assigns a contract to each type $\theta \in \Theta$, where a contract prescribes the retailer's order quantities $x_t^R(\theta)$, $t \in \mathcal{T}$, and a side payment $z(\theta) \in \mathbb{R}$ from the supplier to the retailer. However, the retailer has the power to choose any of the offered contracts or his default option, whichever minimises his own costs. Therefore, this menu of contracts has to be specifically designed by the supplier, as will be made clear in the next section when discussing the optimisation model.

The overall goal of the supplier is to design a menu of contracts that minimises the supplier's expected net costs whilst ensuring that the retailer can satisfy the market demand. There is no restriction on the number of contracts in the menu, but an optimal menu with fewer contracts is preferred. Finally, the menu can be offered only once and there are no renegotiations.

5.2.2 The contracting model

We formulate an optimisation model to determine an optimal menu of contracts that minimises the supplier's expected net costs, as described in the previous section. To this end, let $y_t^R \in \mathbb{B}$ denote whether the retailer has a setup (places an order) at time $t \in \mathcal{T}$ and let $I_t^R \in \mathbb{N}$ be the retailer's ending inventory at time $t \in \mathcal{T}$. Similarly, we have the setup indicator $y_t^S \in \mathbb{B}$ and the ending inventory $I_t^S \in \mathbb{N}$ for the supplier. Recall that $x_t^S, x_t^R \in \mathbb{N}$ are the order quantities and $z \in \mathbb{R}$ is the side payment. The *contracting model* is defined as follows:

$$\min \int_{\underline{\theta}}^{\bar{\theta}} \omega(\theta)\Big(F \sum_{t\in\mathcal{T}} y_t^S(\theta) + H \sum_{t\in\mathcal{T}} I_t^S(\theta) + z(\theta)\Big)\mathrm{d}\theta \tag{5.1}$$

subject to

$$\begin{aligned}
I_0^S(\theta) &= 0, && \forall\,\theta \in \Theta, && (5.2)\\
I_{t-1}^S(\theta) + x_t^S(\theta) &= I_t^S(\theta) + x_t^R(\theta), && \forall\,\theta \in \Theta, t \in \mathcal{T}, && (5.3)\\
x_t^S(\theta) &\le M y_t^S(\theta), && \forall\,\theta \in \Theta, t \in \mathcal{T}, && (5.4)\\
I_0^R(\theta) = I_T^R(\theta) &= 0, && \forall\,\theta \in \Theta, && (5.5)\\
I_{t-1}^R(\theta) + x_t^R(\theta) &= I_t^R(\theta) + d_t, && \forall\,\theta \in \Theta, t \in \mathcal{T}, && (5.6)\\
y_t^R(\theta) \le x_t^R(\theta) &\le M y_t^R(\theta), && \forall\,\theta \in \Theta, t \in \mathcal{T}, && (5.7)\\
y_t^S(\theta), y_t^R(\theta) &\in \mathbb{B}, && \forall\,\theta \in \Theta, t \in \mathcal{T}, && (5.8)\\
x_t^S(\theta), x_t^R(\theta), I_t^S(\theta), I_t^R(\theta) &\in \mathbb{N}, && \forall\,\theta \in \Theta, t \in \mathcal{T}, && (5.9)\\
f(\theta)\sum_{t\in\mathcal{T}} y_t^R(\hat{\theta}) + h(\theta)\sum_{t\in\mathcal{T}} I_t^R(\hat{\theta}) &\equiv \phi(\boldsymbol{x}(\hat{\theta})|\theta), && \forall\,\theta, \hat{\theta} \in \Theta, && (5.10)\\
\phi(\boldsymbol{x}(\theta)|\theta) - z(\theta) &\le \phi^*(\theta), && \forall\,\theta \in \Theta, && (5.11)\\
\phi(\boldsymbol{x}(\theta)|\theta) - z(\theta) &\le \phi(\boldsymbol{x}(\hat{\theta})|\theta) - z(\hat{\theta}), && \forall\,\theta, \hat{\theta} \in \Theta. && (5.12)
\end{aligned}$$

Here, the objective (5.1) is to minimise the supplier's expected net costs, which consists of setup and holding costs and the side payment paid to the retailer. Constraints (5.2)-(5.9) are the lot-sizing constraints for the procurement plan of each contract, constraints (5.10) are for notational convenience, and constraints (5.11)-(5.12) are the mechanism-design constraints.

In particular, constraints (5.2) make sure that the supplier's inventory at the start of the planning horizon is zero. Constraints (5.3) model the supplier's inventory balance, i.e., the flow of products on the supplier's level. Next, constraints (5.4) enforce that a setup takes place if at least one unit of products is produced. Here, M is a suitably large number, e.g., $M = \sum_{t\in\mathcal{T}} d_t$.

Constraints (5.5)-(5.7) are similar and correspond to the retailer. Note that by assumption the supplier can only prescribe the retailer's order quantities $x_t^R(\theta)$, so he cannot force the retailer to incur the setup cost f by using a dummy order of zero products. This is reflected in the model by $y_t^R(\theta) \le x_t^R(\theta)$ in (5.7), which is explicitly needed for correctness. We also enforce our assumption that $I_T^R(\theta) = 0$ in (5.5). This is in contrast to traditional lot-sizing models without information asymmetry.

Moreover, for given order quantities $(x_t^R,\, t \in \mathcal{T})$ the rest of the retailer's procurement plan $(y_t^R,\, I_t^R,\, t \in \mathcal{T})$ is fixed. In other words, there is a bijection between the retailer's order quantities and his procurement plan. Therefore, we denote a contract by $(\boldsymbol{x}(\theta), z(\theta))$, where $\boldsymbol{x}(\theta)$ encodes the retailer's procurement plan. We omit the superscript for the retailer in $\boldsymbol{x}(\theta)$ to simplify our notation.

In constraints (5.10) we define $\phi(\boldsymbol{x}(\hat{\theta})|\theta)$ as the retailer's lot-sizing costs when using plan $\boldsymbol{x}(\hat{\theta})$ and being type $\theta \in \Theta$. Next, constraints (5.11) are the Individual Rationality (IR) constraints, which imply that for retailer type θ the contract $(\boldsymbol{x}(\theta), z(\theta))$ leads to net costs that do not exceed his default costs $\phi^*(\theta)$. Constraints (5.12) are the Incentive Compatibility (IC) constraints and require for retailer type θ that contract $(\boldsymbol{x}(\theta), z(\theta))$ has the lowest net costs of all contracts. Thus, (5.11) and (5.12) ensure that the retailer of type θ will accept his intended contract $(\boldsymbol{x}(\theta), z(\theta))$.

We conclude this section with several remarks on the model related to information asymmetry. First, notice that the supplier in fact faces a bi-level optimisation problem. In our case, the retailer's response to a contract can be easily incorporated by (5.5)-(5.7), leading to a single-level optimisation model. However, care has to be taken to enforce the proper behaviour, i.e., no dummy setups and no excess supply of products, as explained above. Second, as $\phi^*(\theta) \leq \phi(\boldsymbol{x}|\theta)$ by definition for any feasible plan $\boldsymbol{x}$, any feasible contract has a non-negative side payment by (5.11). Finally, in the previous section we claimed that any time-independent variable cost or revenue can be assumed to be zero. It is trivial to verify that $\sum_{t\in\mathcal{T}} x_t^S(\theta) = \sum_{t\in\mathcal{T}} x_t^R(\theta) = \sum_{t\in\mathcal{T}} d_t$ for all $\theta \in \Theta$ in any optimal solution. Therefore, a non-zero time-independent variable cost/revenue either leads to a constant term in the objective or cancels out in (5.11) and (5.12). This is also the case if that cost/revenue is private information. Hence, we only need to include setup and holding costs.

Clearly, complicating factors in solving the contracting model are the infinitely many variables and constraints. In the next section, we describe a solution approach which leads to polynomial-time algorithms in certain cases.

5.3 Solution approach

To solve the contracting model introduced in Section 5.2.2 we propose a two-stage approach. In the first stage, a list of procurement plans for the supply chain is constructed such that the list is sufficient for solving the contracting problem in the second stage. Next, in the second stage, the plans are assigned to retailer types and appropriate side payments are determined. To justify this approach, we start by analysing the contracting model in Section 5.3.1. The plan assignment is discussed in Section 5.3.2 and the plan generation in Section 5.3.3. All corresponding proofs are given in Appendix 5.A.

5.3.1 Analysis

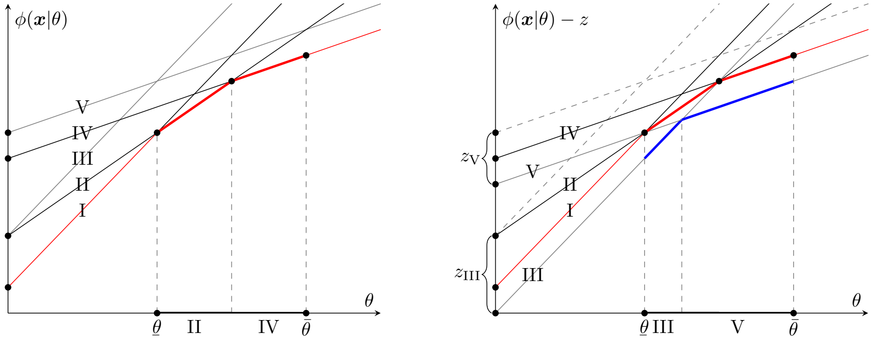
First of all, let us state some well-known properties of the retailer's default option ϕ^*, i.e., it being the lower envelope of at most T linear functions in $\theta \in \Theta$, see Lemma 5.1. The stated *zero-inventory property* means that a setup only occurs if there is no inventory from the previous time period. In this case, it only refers to the retailer's level, i.e., $y_t^R I_{t-1}^R = 0$ for all $t \in \mathcal{T}$.

Lemma 5.1 (Van Hoesel and Wagelmans (2000))**.** *The retailer's default option $\phi^*(\theta)$ is piecewise linear, non-decreasing, concave, and continuous in the retailer type $\theta \in \Theta$. It consists of at most T linear segments and the corresponding retailer's default plans satisfy the zero-inventory property. A complete specification of ϕ^* can be determined in $\mathcal{O}(T^2)$ time.*

We have a graphical interpretation of Lemma 5.1 which is also useful for the contracting model. For any retailer plan we can plot its costs as a function of θ, see Figure 5.1a for a conceptual example. In this example, we assume that only the 5 shown retailer plans exist to keep the figure legible. In a real example, there could be excessively, but still finitely, many feasible plans. The horizontal axis is the type space which contains Θ. The vertical axis is the retailer's total costs $\phi(\boldsymbol{x}|\theta)$. Each line is a retailer plan $\boldsymbol{x}$ with the following properties. In case of private setup cost, the slope is equal to the number of retailer setups $\sum_{t\in\mathcal{T}} y_t^R$ and the intersection with the vertical axis is the retailer's total holding costs $h\sum_{t\in\mathcal{T}} I_t^R$. In case of private holding cost, the slope is the total amount of retailer inventory $\sum_{t\in\mathcal{T}} I_t^R$ and the intersection with the vertical axis is the retailer's total setup costs $f\sum_{t\in\mathcal{T}} y_t^R$. In either case, the slope implies the retailer's total *private* costs and the intersection with the vertical axis is equal to the retailer's total *public* costs.

In Figure 5.1a plans I, II, and IV form the retailer's default option ϕ^* (shown in red). Plans III and V are never optimal for the retailer. For the contracting model the optimal supplier plans are determined for these retailer plans. If, for example, plans III and V result in very low costs for the supplier, he can use side payments to incentivise the retailer to accept these plans instead of I, II, and IV, as shown in blue in Figure 5.1b. Side payments shift the lines vertically, leading to a new lower envelope, which must lie under ϕ^* for $\theta \in \Theta$ by constraints (5.11).

From the graphical interpretation it follows intuitively that the optimal menu leads to a piecewise linear, non-decreasing, concave, and continuous function (a lower envelope) in terms of $\theta \in \Theta$, which lies below ϕ^* in Θ. Hence, the slopes of the segments must be non-increasing. Furthermore, if multiple segments have the same slope, only one with the lowest supplier's net costs is required. This implies a strong ordering in the slopes, i.e., either the number of retailer setups (private setup cost) or the retailer inventory (private holding cost) is strictly decreasing. This result is formalised in Lemma 5.2.



(a) Without side payments the retailer accepts contracts II or IV.

(b) Using side payments to incentivise the retailer to accept contracts III or V.

Figure 5.1: Conceptual graphical interpretation of the contracting model.

Lemma 5.2. *Without loss of optimality, any two distinct contracts* $(\boldsymbol{x}(\theta), z(\theta))$ *and* $(\boldsymbol{x}(\hat{\theta}), z(\hat{\theta}))$ *for some* $\theta < \hat{\theta} \in \Theta$ *in an optimal menu satisfy*

$$\begin{cases} \sum_{t\in\mathcal{T}} y_t^R(\theta) > \sum_{t\in\mathcal{T}} y_t^R(\hat{\theta}) & \text{if setup cost } f \text{ is private} \\ \sum_{t\in\mathcal{T}} I_t^R(\theta) > \sum_{t\in\mathcal{T}} I_t^R(\hat{\theta}) & \text{if holding cost } h \text{ is private} \end{cases}. \tag{5.13}$$

A direct consequence of the strict ordering in Lemma 5.2 and the discrete nature of the involved quantities is that offering only a limited number of contracts is sufficient for optimality for the contracting problem. By doing so, multiple retailer types will be assigned the same contract, which is called *pooling*. Moreover, it follows that this pooling occurs in a structured way: the interval $[\underline{\theta}, \bar{\theta}]$ is partitioned into subintervals and each subinterval is assigned a unique contract. This effect is again intuitively clear from the graphical interpretation in Figure 5.1b. More details are provided in Corollary 5.3.

Corollary 5.3. *Without loss of optimality, an optimal menu partitions (pools) the retailer types into subintervals and consists of at most*

$$\begin{cases} T & \text{contracts if setup cost } f \text{ is private} \\ 1 + \sum_{t\in\mathcal{T}} (t-1)d_t & \text{contracts if holding cost } h \text{ is private} \end{cases}.$$

In particular, for such an optimal menu consisting of $K \in \mathbb{N}_{\geq 1}$ *distinct contracts the types* $[\underline{\theta}, \bar{\theta}]$ *are partitioned into* K *closed subintervals* $[\underline{\theta}_k, \bar{\theta}_k]$, $k \in \{1, \dots, K\}$, *where the* k*-th contract is the most preferred contract for all types in the* k*-th subinterval* $[\underline{\theta}_k, \bar{\theta}_k]$.

The maximum number of contracts stated in Corollary 5.3 is the number of feasible slopes that can be achieved. By Lemma 5.2, it is sufficient for optimality to design a single plan for each feasible slope, i.e., for each feasible number of retailer setups $\sum_{t\in\mathcal{T}} y_t^R$ (private setup cost) or retailer inventory $\sum_{t\in\mathcal{T}} I_t^R$ (private holding cost). It turns out that the procurement plans in an optimal menu can be determined independently from each other, each following from a modified joint lot-sizing problem, see Theorem 5.4.

Theorem 5.4. *Without loss of optimality, the lot-sizing variables of a contract* $(\boldsymbol{x}(\theta), z(\theta))$ *in an optimal menu satisfying*

$$\begin{cases} \sum_{t\in\mathcal{T}} y_t^R(\theta) = n & \text{if setup cost } f \text{ is private} \\ \sum_{t\in\mathcal{T}} I_t^R(\theta) = n & \text{if holding cost } h \text{ is private} \end{cases} \tag{5.14}$$

are determined by solving a corresponding joint lot-sizing problem, namely minimising

$$\begin{cases} \sum_{t\in\mathcal{T}} \Big(Fy_t^S(\theta) + HI_t^S(\theta) + hI_t^R(\theta)\Big) & \text{if setup cost } f \text{ is private} \\ \sum_{t\in\mathcal{T}} \Big(Fy_t^S(\theta) + HI_t^S(\theta) + fy_t^R(\theta)\Big) & \text{if holding cost } h \text{ is private} \end{cases} \tag{5.15}$$

under the constraints (5.2)-(5.9) *and* (5.14).

We call the joint lot-sizing problem of Theorem 5.4 the *n-plan generation problem.* Notice that the n-plan generation problem only includes the supplier's setup and holding costs and the retailer's *public* costs. The retailer's *private* costs are fixed by (5.14). We can obtain this result in the graphical interpretation as well. Consider the situation in Figure 5.1a. We can 'normalise' all plans by shifting them downwards so they intersect with the origin, by setting the side payment equal to the retailer's public costs of the plan. All plans with the same slope (see (5.14)) are now essentially equivalent and it is optimal to only use the plan with the lowest supplier's 'normalised' costs. That is, the plan for which (5.15) is minimal, as the normalisation incorporates the retailer's public costs into the supplier's costs.

From the above theoretical results, we conclude that it is sufficient for optimality to solve the n-plan generation problem for each feasible slope n and use these plans to design a menu of contracts. The described plan generation is the first stage of our solution approach. We postpone the analysis of the n-plan generation problem to Section 5.3.3. First, we continue in Section 5.3.2 with the second stage of the solution approach: the *plan assignment problem*, where we need to assign the plans to the retailer types by using side payments, leading to a menu of contracts.

5.3.2 Plan assignment

From Section 5.3.1 we can assume without loss of optimality that we have a finite list of procurement plans, obtained from the plan-generation stage. The next step is the *plan assignment* stage where we need to decide which plans of the list will be incorporated into contracts and how these plans/contracts are assigned to the retailer types. Before we state the plan assignment model in Section 5.3.2.2, we derive two properties in Section 5.3.2.1 that will simplify the model.

5.3.2.1 Properties

We first introduce additional notation. Let $K \in \mathbb{N}_{\geq 1}$ be the number of plans in the considered list. For now, assume that each plan is included into a contract. We can index and sort the contracts by decreasing slope of the retailer plan, resulting in $(\boldsymbol{x}_k, z_k)$ for $k \in \mathcal{K} = \{1, \ldots, K\}$. By Lemma 5.2 and Corollary 5.3, an optimal menu will partition $[\underline{\theta}, \bar{\theta}]$ into K subintervals $[\underline{\theta}_k, \bar{\theta}_k]$, where the k-th contract will be assigned to types in $[\underline{\theta}_k, \bar{\theta}_k]$.

For the plan assignment, we need to take the IR and IC constraints (5.11)-(5.12) into account. These infinitely many constraints can be made tractable by using the partition structure described above. See Lemma 5.5 for the result.

Lemma 5.5. *When determining an optimal menu with K distinct contracts $(\boldsymbol{x}_k, z_k)$, $k \in \mathcal{K} = \{1, \ldots, K\}$, and corresponding partition subintervals $[\underline{\theta}_k, \bar{\theta}_k]$, the IR and IC constraints* (5.11)-(5.12) *are equivalent to:*

$$
\begin{aligned}
\phi(\boldsymbol{x}_1|\underline{\theta}) - z_1 &\leq \phi^*(\underline{\theta}), && \\
\phi(\boldsymbol{x}_k|\underline{\theta}_k) - z_k &\leq \phi^*(\underline{\theta}_k), && \forall\, k \in \mathcal{K} \setminus \{1\}, \\
\phi(\boldsymbol{x}_K|\bar{\theta}) - z_K &\leq \phi^*(\bar{\theta}), && \\
\phi(\boldsymbol{x}_k|\underline{\theta}_k) - z_k &= \phi(\boldsymbol{x}_{k-1}|\underline{\theta}_k) - z_{k-1}, && \forall\, k \in \mathcal{K} \setminus \{1\}. \qquad (5.16)
\end{aligned}
$$

Consider Lemma 5.5 in the graphical interpretation. The plan assignment problem essentially consists of shifting the lines in Figure 5.1a vertically to construct an optimal lower envelope for domain Θ (seen in blue in Figure 5.1b). From the piecewise linearity, concavity, and continuity of ϕ^* and the new lower envelope, it follows immediately that we only need to consider the IR constraints at the breakpoints. Furthermore, (5.16) relates to the continuity of the constructed lower envelope.

The second property concerns redundant plans included in the list. In an optimal menu it might be the case that not all provided plans are assigned to retailer types. Ideally, having these redundant plans included in the list should not interfere with the optimisation process. Lemma 5.6 shows that this is indeed the case: redundant plans can safely be added without affecting the optimum.

Lemma 5.6. *Having redundant plans/contracts does not affect the plan assignment problem.*

Graphically, the lines of redundant plans can/are placed tangent to the constructed lower envelope. This does not affect the lower envelope (the optimum), but ensures feasibility according to the equivalent IR and IC constraints stated in Lemma 5.5. From this point onwards, given a menu of contracts, a plan k in the menu is called *assigned* if $\bar{\theta}_k > \underline{\theta}_k$ and *redundant* if $\bar{\theta}_k = \underline{\theta}_k$.

5.3.2.2 The plan assignment model

We can now formulate the plan assignment model. To do so in a unified way for both cases of private information, we introduce new notation for the supplier's lot-sizing costs, the retailer's public lot-sizing costs, and the slope corresponding to the private information. For a procurement plan $\boldsymbol{x}$ the corresponding supplier's costs are

$$C = \sum_{t\in\mathcal{T}} \Big(Fy_t^S + HI_t^S\Big).$$

If setup cost f is private, we define the retailer's public costs c^{pub} and the slope n by

$$c^{\text{pub}} = h\sum_{t\in\mathcal{T}} I_t^R, \qquad\qquad n = \sum_{t\in\mathcal{T}} y_t^R.$$

Otherwise, if holding cost h is private, we have

$$c^{\text{pub}} = f\sum_{t\in\mathcal{T}} y_t^R, \qquad\qquad n = \sum_{t\in\mathcal{T}} I_t^R.$$

Hence, by definition, we have $\phi(\boldsymbol{x}_k|\theta) = c_k^{\text{pub}} + n_k\theta$ for $k \in \mathcal{K}$ and $\theta \in \Theta$. Note that in the plan-assignment stage C_k, c_k^{pub}, and n_k are all known parameters and follow from the provided list of plans. It is essential that the plans are sorted such that $n_k > n_{k+1}$ for all $k \in \mathcal{K}$ to ensure that the model is in line with Lemma 5.2.

The *plan assignment model* is given by:

$$\min \sum_{k\in\mathcal{K}} \Big(\int_{\underline{\theta}_k}^{\bar{\theta}_k} \omega(\theta)\mathrm{d}\theta\Big)(C_k + z_k) \tag{5.17}$$

subject to

$$\begin{aligned}
c_1^{\text{pub}} + n_1\underline{\theta} - z_1 &\le \phi^*(\underline{\theta}), && && (5.18)\\
c_k^{\text{pub}} + n_k\underline{\theta}_k - z_k &\le \phi^*(\underline{\theta}_k), && \forall\, k \in \mathcal{K}\setminus\{1\}, && (5.19)\\
c_K^{\text{pub}} + n_K\bar{\theta} - z_K &\le \phi^*(\bar{\theta}), && && (5.20)\\
c_{k-1}^{\text{pub}} - c_k^{\text{pub}} + (n_{k-1} - n_k)\underline{\theta}_k &= z_{k-1} - z_k, && \forall\, k \in \mathcal{K}\setminus\{1\} && (5.21)\\
\underline{\theta}_1 &= \underline{\theta}, && && (5.22)\\
\underline{\theta}_k &\le \bar{\theta}_k, && \forall\, k \in \mathcal{K}, && (5.23)\\
\bar{\theta}_k &= \underline{\theta}_{k+1}, && \forall\, k \in \mathcal{K}\setminus\{K\}, && (5.24)\\
\bar{\theta}_K &= \bar{\theta}. && && (5.25)
\end{aligned}$$

Here, (5.18)-(5.20) are the IR constraints and (5.21) the IC constraints as described in Lemma 5.5. The constraints (5.22)-(5.25) model the partition of $[\underline{\theta}, \bar{\theta}]$ and the corresponding assignment of contracts to subintervals as stated in Corollary 5.3. Consequently, the integral in the objective (5.17) is the probability that the retailer accepts contract $(\boldsymbol{x}_k, z_k)$.

We emphasise again that the model is only correct if $n_k > n_{k+1}$ for all $k \in \mathcal{K}$ (by Lemma 5.2). Also, by Lemma 5.6 redundant plans/contracts can be added without affecting the optimum, provided that the ordering in n_k is maintained. Moreover, note that $\phi^*(\underline{\theta}_k)$ in (5.19) can be modelled with at most T linear constraints for each $k \in \mathcal{K}$ (see Lemma 5.1). Namely, replace (5.19) by

$$c_k^{\text{pub}} + n_k\underline{\theta}_k - z_k \leq c_l^* + n_l^*\underline{\theta}_k, \qquad \forall\, l \in \mathcal{L}, k \in \mathcal{K} \setminus \{1\},$$

where the functions $\theta \mapsto c_l^* + n_l^*\theta$ for $l \in \mathcal{L}$ ($|\mathcal{L}| \leq T$) correspond to the retailer's default plans and whose lower envelope is ϕ^*. Finally, by combining (5.21) the IC constraints imply for $k \in \mathcal{K}$ that

$$z_k = z_1 + c_k^{\text{pub}} - c_1^{\text{pub}} - \sum_{i=2}^{k}(n_{i-1} - n_i)\underline{\theta}_i.$$

Substituting this expression in the objective function, results in a separable non-linear objective:

$$\begin{aligned} &z_1 - c_1^{\text{pub}} + \sum_{k\in\mathcal{K}} \Big(\int_{\underline{\theta}_k}^{\bar{\theta}_k} \omega(\theta)\mathrm{d}\theta\Big)\Big(C_k + c_k^{\text{pub}} - \sum_{i=2}^{k}(n_{i-1} - n_i)\underline{\theta}_i\Big) \\ &\quad = z_1 - c_1^{\text{pub}} + \sum_{k\in\mathcal{K}} \Big(\int_{\underline{\theta}_k}^{\bar{\theta}_k} \omega(\theta)\mathrm{d}\theta\Big)\Big(C_k + c_k^{\text{pub}}\Big) - \sum_{k=2}^{K}\Big(\int_{\underline{\theta}_k}^{\bar{\theta}} \omega(\theta)\mathrm{d}\theta\Big)(n_{k-1} - n_k)\underline{\theta}_k. \end{aligned} \tag{5.26}$$

Thus, the plan assignment model has a formulation with linear constraints and a non-linear separable objective function. In general, such optimisation models are difficult to solve to optimality, but several (heuristic) solution approaches have been designed (see for example Bradley et al. (1977), Byrd et al. (2003), and Kolda et al. (2007)).

If the retailer type distribution ω is uniform, the plan assignment model has a hidden convexity. The standard formulation is still non-convex, but by using the reformulated objective function (5.26) we obtain a linearly-constrained convex-quadratic model. It is well known that these models can be solved efficiently (see Ye and Tse (1989)). This result is captured in Theorem 5.7 and its proof contains the details of the convex formulation.

Theorem 5.7. *If ω is a uniform distribution, then the plan assignment model can be formulated as a linearly-constrained convex-quadratic model. It can be solved in polynomial time in the number of contracts K by interior-point methods.*

5.3.3 Plan generation

In the plan-generation stage we need to solve several joint lot-sizing problems as described in Theorem 5.4. In Section 5.3.3.1 we first give properties of this problem that are common for the two private information cases. Then we focus on each case separately: private setup cost in Section 5.3.3.2 and private holding cost in Section 5.3.3.3.

5.3.3.1 Common properties

In a standard joint lot-sizing problem, i.e., without constraint (5.14), it is well known that there exists an optimal solution that satisfies the zero-inventory property. Such an optimal solution can be found in polynomial time using dynamic programming by its decomposition into independent subplans. In contrast, for the n-plan generation problem the optimal solution might not satisfy the zero-inventory property, as we will show later. However, the zero-inventory property always holds for the supplier's lot-sizing plan, see Lemma 5.8.

Lemma 5.8. *For an optimal solution for the n-plan generation problem, the supplier's lot-sizing plan must satisfy the zero-inventory property.*

Another property in certain joint lot-sizing problems is that the joint plan is *nested*. This means that a supplier setup implies a retailer setup in the same time period: $y_t^S = 1$ implies $y_t^R = 1$ for $t \in \mathcal{T}$. This property holds for the n-plan generation problem, as shown in Lemma 5.9.

Lemma 5.9. *For an optimal solution for the n-plan generation problem, the joint lot-sizing plan must be nested.*

The properties in Lemmas 5.8 and 5.9 imply that the main focus of the remaining analysis is the retailer's plan. In particular, how does the constraint on either the retailer setups or the retailer inventory affect the solution structure? We continue with analysing the n-plan generation problem separately for the two private information cases.

5.3.3.2 Private setup cost

In this section we prove that for private setup cost the plan generation problem can be solved in polynomial time by a dynamic-programming algorithm. An essential part of this algorithm is that we can decompose an optimal solution of the n-plan generation into independent subplans. An *independent subplan*, denoted by $(\underline{t}, \bar{t}, n)$, only considers the subproblem with time periods $\{\underline{t}, \dots, \bar{t}\} \subseteq \mathcal{T}$. It has a single supplier setup, in the initial time period $\underline{t}$, from which exactly all demand $\sum_{t=\underline{t}}^{\bar{t}} d_t$ is satisfied. Also, there is no inventory transferred to/from time periods not belonging to the subproblem. Finally, the subplan must have exactly n retailer setups. The decomposable structure into independent subplans is proven in Lemma 5.10.

Lemma 5.10. *Any optimal solution of the n-plan generation problem can be decomposed into independent subplans.*

The result of Lemma 5.10 implies that the optimal solution of the n-plan generation problem can be found by solving several appropriately chosen subproblems independently. In order to solve such a subproblem we need to determine the structure of its optimal solutions. The next result, Lemma 5.11, shows that the structure depends on whether $H \leq h$ or $H > h$.

Lemma 5.11. *Consider an optimal independent subplan prescribing exactly n retailer setups. If $H \leq h$ then this subplan satisfies the zero-inventory property (without loss of optimality if $H = h$). If $H > h$ then this subplan is unique: the retailer has setups only in the first n periods, where the post-initial orders are 1 unit of supply. In this case, the retailer's plan might not satisfy the zero-inventory property.*

Lemma 5.11 states that if $H \leq h$ the optimal solution satisfies the zero-inventory property. Hence, this case is similar to traditional joint lot-sizing problems and can be solved by dynamic programming. However, if $H > h$ there is a unique and straightforward optimal solution, which might violate the zero-inventory property. See Figure 5.2 for an example with $T = 5$ and $n = 3$. This figure is a *network flow graph*, where the arrows indicate strictly positive flow of products through the supply chain. That is, a vertical arrow is a setup and a horizontal arrow implies having inventory at that time period. The upper layer is the supplier's lot-sizing plan and the lower layer the retailer's plan. At the bottom the time periods are displayed. In Appendix 5.B we give an example where the (unique) optimal menu contains such a contract that violates the zero-inventory property.

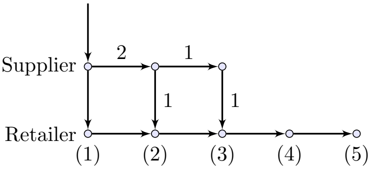


Figure 5.2: The unique optimal subplan in the case of private setup cost, $H > h$, $T = 5$, and $n = 3$.

From Lemmas 5.10 and 5.11 the dynamic-programming approach should be clear. First, solve all related independent subproblems and then use these optimal subplans to construct an optimal solution for the n-plan generation problem by dynamic programming. Since we need to solve all n-plan generation problems ($n \in \{1, \dots, T\}$) we can reuse many computations. The approach is similar to the dynamic-programming algorithm in Zangwill (1969), but we need to fix the number of retailer setups and take the two cases $H \leq h$ and $H > h$ into account. Theorem 5.12 concludes these insights and its proof contains the specification of the dynamic-programming algorithm.

Theorem 5.12. *Solving all n-plan generation problems can be done in $\mathcal{O}(T^4)$ time by dynamic programming.*

To conclude the private setup cost case, we can use the dynamic-programming algorithm stated in the proof of Theorem 5.12 to construct a list of procurement plans that is sufficient for optimality for the contracting problem. This list can then be used in the plan-assignment stage to determine the optimal allocation of contracts to the retailer types and solve the contracting problem. In particular, if ω is a uniform distribution, the entire contracting problem can be solved to optimality in polynomial time by Theorems 5.7 and 5.12. We state this result in the next theorem.

Theorem 5.13. *If ω is a uniform distribution, then the contracting problem can be solved in polynomial time.*

5.3.3.3 Private holding cost

The case that the holding cost is private information appears to be more complicated than having private setup cost. In particular, a similar result as Lemma 5.10 does not hold for the n-plan generation problem in general. For example, certain amounts of retailer inventory (n values) cannot be achieved with plans that satisfy the zero-inventory property. Furthermore, if the supplier's setup cost F is appropriately chosen, then it would be optimal to have several supplier setups in such a (sub)plan, disproving that the decomposition structure holds in general. The smallest example is $T = 2$, $d_1 = 1$, $d_2 = 2$, $F < H$, and the optimal plan for $n = 1$, see Figure 5.3c.

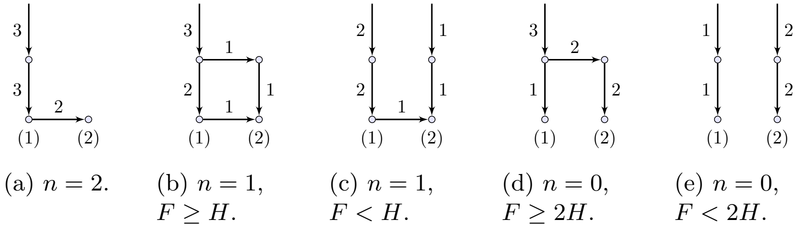


Figure 5.3: The optimal solutions for the n-plan generation problems in the case of private holding cost with $T = 2$, $d_1 = 1$, and $d_2 = 2$.

Moreover, we potentially need to solve pseudo-polynomially many n-plan generation problems by Corollary 5.3. We have not been able to determine an efficient combinatorial algorithm to solve all n-plan generation problems. However, there seems to be a redundancy in the complete list of plans. For example, the plans in Figures 5.3c and 5.3e lead to the same supplier's lot-sizing costs, but the retailer's lot-sizing costs are lower for the plan for $n = 0$ (Figure 5.3e). In other words, the plan of Figure 5.3c has a form of inefficiency. Unfortunately, it is not directly clear whether we can omit this plan, since the side payments need to be taken into account.

If we assume a uniform distribution for the retailer's type, then the following lemma provides further indications that certain plans are redundant. Lemma 5.14 states a necessary condition for assigning a plan with a slope that does not occur in ϕ^*.

Lemma 5.14. *Assume that ω is a uniform distribution. Consider $k \in \{2, \dots, K-1\}$ such that ϕ^* has no slopes n^* with $n_{k-1} > n^* > n_{k+1}$. If plan k is assigned, i.e., $\bar{\theta}_k > \underline{\theta}_k$, then the following must hold:*

$$\frac{(C_k + c_k^{\text{pub}}) - (C_{k-1} + c_{k-1}^{\text{pub}})}{n_{k-1} - n_k} + \frac{(C_k + c_k^{\text{pub}}) - (C_{k+1} + c_{k+1}^{\text{pub}})}{n_k - n_{k+1}} < 0. \tag{5.27}$$

In particular, if we consider *all* possible plans ($n_{k+1} = n_k - 1$), Lemma 5.14 implies that an assigned plan $k \in \mathcal{K}$ such that n_k is not a slope of ϕ^* must satisfy

$$2(C_k + c_k^{\text{pub}}) - (C_{k-1} + c_{k-1}^{\text{pub}}) - (C_{k+1} + c_{k+1}^{\text{pub}}) < 0.$$

We can apply this to the example in Figure 5.3. Realise that $n = 1$ is not a slope of ϕ^*. The condition of Lemma 5.14 for $n = 1$ is

$$\begin{aligned}0 &> 2(F + \min\{F, H\} + 2f) - (F + f) - (F + \min\{F, 2H\} + 2f) \\ &= f + 2\min\{F, H\} - \min\{F, 2H\} \geq f > 0.\end{aligned}$$

This is a contradiction. Hence, the plan for $n = 1$ is never assigned.

Under the additional assumption that the supplier's setup cost F is high enough to prevent any additional supplier setups, we can solve the plan generation problem in polynomial time. In this case, all optimal plans have exactly a single supplier setup. The idea is to use Lemma 5.14 to exclude many plans and show that the remaining plans can be determined efficiently. Of particular interest are so-called extreme plans. We call a plan *m-extreme* if it has minimal retailer inventory with m retailer setups, where $m \in \{1, \ldots, T\}$. It is trivial that these extreme plans must satisfy the zero-inventory property. Note that all default plans of ϕ^* are extreme plans. Lemma 5.15 shows that under the mentioned assumptions T extreme plans are sufficient for optimality for the contracting problem.

Lemma 5.15. *Assume that ω is a uniform distribution and $F > H \max_{\tau \in \mathcal{T}}\{(\tau - 1)\sum_{t=\tau}^{T} d_t\}$. A list consisting of an m-extreme plan for each $m \in \{1, \ldots, T\}$ is sufficient for optimality for the contracting problem.*

Under the assumptions of Lemma 5.15, all m-extreme plans for fixed m have the same supplier's costs and retailer's public costs. Consequently, it is sufficient to determine any m-extreme plan. In this case, we can solve the (entire) plan generation problem by determining these T extreme plans, which can be done by dynamic programming. The result is a polynomial-time algorithm for the plan generation under the specified assumptions, see Lemma 5.16.

Lemma 5.16. *Assume that ω is a uniform distribution and $F > H \max_{\tau \in \mathcal{T}}\{(\tau - 1)\sum_{t=\tau}^{T} d_t\}$. Generating plans sufficient for optimality for the contracting problem can be done in $\mathcal{O}(T^3)$ time by dynamic programming.*

By combining Theorem 5.7 and Lemmas 5.15 and 5.16, we conclude that under the stated conditions the contracting problem can be solved in polynomial time, see Corollary 5.17.

Corollary 5.17. *If ω is a uniform distribution and $F > H \max_{\tau \in \mathcal{T}}\{(\tau - 1)\sum_{t=\tau}^{T} d_t\}$, then the contracting problem can be solved in polynomial time.*

From numerical experiments we have indications that similar results hold without the condition on the supplier's setup cost F. Furthermore, we have the following property. Consider a list containing the plans for *all* slopes of ϕ^*. Now keep all side payments fixed and focus on the plans with slopes different from ϕ^*. These other plans share a special property: they can always be removed from a feasible menu to obtain a new feasible menu (without changing the side payments). This property is obvious from the graphical interpretation and can potentially be used to exclude plans from consideration. To conclude, we conjecture that plans with the same slopes as ϕ^* are essential for optimality for the contracting problem. However, more research needs to be done for a formal proof and for other distributions for ω.

5.4 Discussion and conclusion

The modelling concept and solution approach is applicable to a broader range of problems. In this section, we discuss the generalisability of our results, propose research directions, and conclude our findings.

5.4.1 Demand assumption

One of our assumptions is that the demand in each period is strictly positive. This is not without loss of generality, especially due to the time-independent holding costs. We will discuss the consequences if demand can be zero.

First, we often use that the number of retailer setups lies between 1 and T. Instead, there is a maximum feasible number of retailer setups $1 \leq K \leq T$. This has no significant impact on the results. Second, the dynamic-programming algorithms for both plan generation problems need to be adjusted slightly to prevent dummy retailer setups. Consequently, fewer options need to be considered during the algorithm, so the complexity results still hold.

Only our results for the plan generation for private setup cost are significantly affected. In the proofs of Lemmas 5.10 and 5.11 we have explicitly mentioned where we use that demand is strictly positive. There is a very specific case for which the two stated proofs do not hold if demand can be zero: there needs to be a substructure that violates the zero-inventory property where this retailer inventory cannot be decreased without invalidating a retailer setup. All details are provided in Appendix 5.C, which we summarise here.

A common assumption for lot-sizing problems is that value is added to the product as it moves downstream in the supply chain, increasing the holding cost. In other words, $H \leq h$ holds. Another interpretation is that the supplier benefits from economies of scale to have less holding costs. In Appendix 5.C.1 we prove for the case $H \leq h$ that without loss of optimality a plan is assigned in an optimal menu only if it satisfies the zero-inventory property. We conclude that, when demand is non-negative and $H \leq h$, the plan generation problem is solvable in $\mathcal{O}(T^4)$ time by dynamic programming.

The other case, $H > h$, can be analysed using techniques similar to those in our proofs. The (unique) optimal n-plan can be non-decomposable, as shown in Appendix 5.C.2. However, we show that an optimal plan consists of substructures similar to Figure 5.2. That is, the solution is fixed when we know the supplier setups and how many retailer setups occur in between supplier setups. This allows for a dedicated dynamic-programming algorithm with $\mathcal{O}(T^5)$ running time, which can potentially be improved.

We conclude that our results are still valid when demand is non-negative, albeit that some modifications are needed.

5.4.2 Generalisability

For several of our results we did not use any property of the lot-sizing problem, implying that these results are also valid for other problems. Here, we discuss the generalisability of our approach. We still refer to the two involved parties by the supplier and the retailer.

The more general setting is as follows. Given the decision variables (the plan) of the supplier, the retailer needs to solve a Mixed Integer Linear Programming (MILP) problem to determine his optimal plan, and vice versa. Both the supplier and the retailer want to minimise costs as their objective. We assume that for any retailer plan there exists a feasible supplier plan, in order to have a well-defined default option.

The retailer has single-dimensional private information $\theta \in [\underline{\theta}, \bar{\theta}] = \Theta \subseteq \mathbb{R}$, which must only affect his objective value (his costs). Let $\boldsymbol{x}$ denote all decision variables from the supplier and the retailer. The retailer's costs for type $\theta \in \Theta$ are

$$\phi(\boldsymbol{x}|\theta) = (a^\top \boldsymbol{x} + a_0)\theta + (b^\top \boldsymbol{x} + b_0),$$

for given vectors a and b, and scalars a_0 and b_0. Hence, we have public costs $c^{\text{pub}} = b^\top \boldsymbol{x} + b_0$ and slope $n = a^\top \boldsymbol{x} + a_0$. We assume that this slope $a^\top \boldsymbol{x} + a_0$ only takes on finitely many values over all feasible plans $\boldsymbol{x}$, which is essential. Consequently, the retailer's default option ϕ^* is the lower envelope of finitely many linear functions.

Under the described setting and assumptions, we can apply the same two-stage solution approach consisting of a plan-generation stage and a plan-assignment stage. In particular, all results related to the graphical interpretation and the plan assignment are valid, since these are independent of the lot-sizing setting. The main difference in the difficulty in solving the contracting problem lies in the plan generation. If the bi-level optimisation problem with the additional constraint that $(a^\top \boldsymbol{x} + a_0) = n$ for some slope n can be solved by a single-level MILP problem, then general MILP solvers can be used. Whether polynomial-time algorithms exist highly depends on the underlying optimisation problem. Also, if the number of plans to generate is high, it might be useful to use heuristics instead, which we will discuss in the next section.

To conclude the generalisability, we provide the details on which results are still valid. First, since the default option ϕ^* is the lower envelope of finitely many linear functions, equivalent properties as in Lemma 5.1 hold (except for the complexity result). By trivially modifying the cost functions in our proofs, we obtain the following similar results. As in Lemma 5.2, the slopes of the assigned plans must be strictly decreasing. By the finitely many possible slopes, we get a bound on the number of contracts and properties similar to Corollary 5.3. In the equivalent of Theorem 5.4, the slope $a^\top \boldsymbol{x} + a_0$ must be fixed to n and the joint optimisation problem minimises the sum of the supplier's costs and the retailer's public costs $b^\top \boldsymbol{x} + b_0$. Regarding the plan assignment, Lemmas 5.5, 5.6, and 5.14, and Theorem 5.7 are valid, since they do not use any lot-sizing properties. The other results concern the plan generation and are specific to the lot-sizing problem.

5.4.3 Heuristics

Our analysis provides several research directions for heuristics. First of all, the plan generation can be restricted to (potentially suboptimal) plans that can be constructed efficiently, e.g., lot-sizing plans satisfying the zero-inventory property. Also, we can generate plans only for slopes with intuitive interpretations. For example, only plans with slopes that appear in ϕ^* or in the optima of the unrestricted traditional joint lot-sizing problem, when a sensitivity analysis is performed on the private cost parameter.

Second, by rescaling the side payment z_k as $\tilde{z}_k = z_k - c_k^{\text{pub}}$, the constraints of the plan assignment problem only depend on the slopes of the included plans. This leads the following idea. First, add a place-holder plan for each possible slope. Second, determine lower bounds on the joint costs $C_k + c_k^{\text{pub}}$ for each $k \in \mathcal{K}$. These costs appear in the objective after rescaling the side payment. This results in a relaxation for the contracting problem. Then, we determine the exact joint costs for (a subset of) the place-holder plans assigned in the optimum of the relaxation. If we calculate the joint costs of all assigned plans, then we obtain an upper bound. By repeating this process, we get better lower and upper bounds, and a solution approach.

Third, we can design an iterative heuristic as follows. If we fix the number of contracts and their assignment to retailer types, then the resulting optimisation model is a mixed integer linear program. This model is given in Appendix 5.D. For a given partition, we solve this model to obtain procurement plans. Then, solve the plan assignment model for these plans, resulting in a new partition. Switching between these models leads to an iterative heuristic.

5.4.4 Concluding remarks

If all information is shared in the supply chain, then it is trivial to show that the supplier's contracting problem can be solved by a single joint lot-sizing problem. Namely, compared to the model with information asymmetry in Section 5.2.2, there is only a single type, a single IR constraint, and no IC constraints. In any optimal solution the IR constraint binds, i.e., $z = \phi(\boldsymbol{x}) - \phi^*$, where we omit the single type. Substituting this in the objective function results in a traditional joint lot-sizing problem that minimises the sum of the supplier's costs and *all* retailer's costs. As discussed before, this problem can be solved in polynomial time.

Our analysis and obtained results show that information asymmetry does not necessarily change the complexity class of the underlying optimisation problem. That is, we have identified cases for which we can solve the contracting problem under information asymmetry in polynomial time. However, clearly it is more complicated to determine the optimal solution. Not only do we need to solve multiple joint lot-sizing problems to generate a sufficient list of plans, we also need to solve the assignment model. Furthermore, the interdependence between the contracts through the side payments results in offering atypical procurement plans. For example, plans that do no satisfy the zero-inventory property or that are not decomposable into independent subproblems.

Although the plan generation for private setup cost can be solved in polynomial time for any instance, further research might narrow down which n-plans are sufficient for optimality for the contracting problem. For the case of private holding cost more research is needed to prove polynomial-time complexity under less restrictive assumptions (if valid at all). Finally, the above described generalisability provides research directions to other, more general, problem settings.

Appendix

5.A Proofs of Section 5.3

This appendix contains all proofs of Section 5.3. Note that many results also have an intuitive graphical interpretation for which we refer to the main text.

5.A.1 Proofs of Section 5.3.1

Proof of Lemma 5.1. The default option ϕ^* follows from the single-level lot-sizing problem on the retailer's level. It is trivial that there are finitely many feasible procurement plans and each feasible plan is linear and non-decreasing in θ. By definition, ϕ^* is the point-wise minimum (the lower envelope) of these finitely many linear functions. Only the plans that for a given number of setups (ranging from 1 to T) minimise the retailer's holding costs can be minimisers and form the lower envelope. Consequently, these plans must satisfy the zero-inventory property. This proves the stated properties of ϕ^*. Finally, ϕ^* can be determined efficiently in $\mathcal{O}(T^2)$ time by using the method described in Van Hoesel and Wagelmans (2000). □

Proof of Lemma 5.2. First, realise that by definition we have

$$\begin{aligned}&\phi(\boldsymbol{x}(\theta)|\theta) - \phi(\boldsymbol{x}(\hat{\theta})|\theta) + \phi(\boldsymbol{x}(\hat{\theta})|\hat{\theta}) - \phi(\boldsymbol{x}(\theta)|\hat{\theta}) \\ &\quad = (f(\theta) - f(\hat{\theta}))\Big(\sum_{t\in\mathcal{T}} y_t^R(\theta) - \sum_{t\in\mathcal{T}} y_t^R(\hat{\theta})\Big) + (h(\theta) - h(\hat{\theta}))\Big(\sum_{t\in\mathcal{T}} I_t^R(\theta) - \sum_{t\in\mathcal{T}} I_t^R(\hat{\theta})\Big).\end{aligned} \tag{5.28}$$

Hence, if

$$\begin{cases} \sum_{t\in\mathcal{T}} y_t^R(\theta) = \sum_{t\in\mathcal{T}} y_t^R(\hat{\theta}) & \text{if setup cost } f \text{ is private} \\ \sum_{t\in\mathcal{T}} I_t^R(\theta) = \sum_{t\in\mathcal{T}} I_t^R(\hat{\theta}) & \text{if holding cost } h \text{ is private} \end{cases} \tag{5.29}$$

then the right-hand side of (5.28) is equal to zero and

$$\phi(\boldsymbol{x}(\theta)|\theta) - \phi(\boldsymbol{x}(\hat{\theta})|\theta) = \phi(\boldsymbol{x}(\theta)|\hat{\theta}) - \phi(\boldsymbol{x}(\hat{\theta})|\hat{\theta}).$$

Second, the IC conditions state

$$\phi(\boldsymbol{x}(\theta)|\theta) - z(\theta) \leq \phi(\boldsymbol{x}(\hat{\theta})|\theta) - z(\hat{\theta}), \qquad \phi(\boldsymbol{x}(\hat{\theta})|\hat{\theta}) - z(\hat{\theta}) \leq \phi(\boldsymbol{x}(\theta)|\hat{\theta}) - z(\theta),$$

implying that

$$\phi(\boldsymbol{x}(\theta)|\theta) - \phi(\boldsymbol{x}(\hat{\theta})|\theta) \leq z(\theta) - z(\hat{\theta}) \leq \phi(\boldsymbol{x}(\theta)|\hat{\theta}) - \phi(\boldsymbol{x}(\hat{\theta})|\hat{\theta}). \tag{5.30}$$

So if (5.29) is true, then (5.30) holds with equalities, resulting in

$$\phi(\boldsymbol{x}(\theta)|\theta) - z(\theta) = \phi(\boldsymbol{x}(\hat{\theta})|\theta) - z(\hat{\theta}), \qquad \phi(\boldsymbol{x}(\hat{\theta})|\hat{\theta}) - z(\hat{\theta}) = \phi(\boldsymbol{x}(\theta)|\hat{\theta}) - z(\theta).$$

In other words, both types θ and $\hat{\theta}$ are indifferent to each other's contracts and these contracts can be interchanged without affecting feasibility.

Now consider an optimal menu of contracts. If (5.29) holds for types θ and $\hat{\theta}$, then assigning both types either contract $(\boldsymbol{x}(\theta), z(\theta))$ or $(\boldsymbol{x}(\hat{\theta}), z(\hat{\theta}))$ is feasible as shown above. Assigning the contract that leads to the lowest supplier's net costs cannot result in a worse objective value, i.e., the new menu must be optimal as well. By repeating this argument, we conclude that without loss of optimality distinct contracts do not satisfy (5.29).

Finally, from (5.30) it follows that (5.28) must be non-positive. For types $\theta < \hat{\theta}$ with distinct contracts we have either $f(\theta) < f(\hat{\theta})$ or $h(\theta) < h(\hat{\theta})$, depending on which cost parameter is private. Furthermore, with the above insight (5.28) must be strictly negative (without loss of optimality). Thus, (5.13) holds without loss of optimality. □

Proof of Corollary 5.3. By Lemma 5.2 we can bound the number of distinct contracts. First, the total number of retailer setups lies between 1 and T. Since $d_t > 0$ for all $t \in \mathcal{T}$, all numbers $1, \ldots, T$ of retailer setups are feasible. Second, the total retailer inventory lies between 0 (use the maximum number of setups) and $\sum_{t \in \mathcal{T}} (t-1) d_t$ (use one setup). All discrete intermediate values are also feasible by appropriately delaying parts of the orders (starting with a single setup). Here, we use our assumption that the products are indivisible, i.e., the retailer order quantities must be discrete. Consequently, by the finite bounds given above and the discrete nature, Lemma 5.2 implies that there are only finitely many contracts in an optimal menu (without loss of optimality). Hence, retailer types must be pooled, i.e., some are offered the same contract.

The partitioning of the retailer types follows trivially from the ordering implied by (5.13). Only the technicality that we can use closed subintervals remains to be shown. Consider the case that the k-th contract $(\boldsymbol{x}_k, z_k)$ is the most preferred contract for all types $(\underline{\theta}_k, \bar{\theta}_k]$, but not for type $\underline{\theta}_k$. Instead, type $\underline{\theta}_k$ strictly prefers the l-th contract $(\boldsymbol{x}_l, z_l)$:

$$\phi(\boldsymbol{x}_l|\underline{\theta}_k) - z_l < \phi(\boldsymbol{x}_k|\underline{\theta}_k) - z_k.$$

However, we also have

$$\begin{aligned} & \phi(\boldsymbol{x}_k|\theta_k) - z_k \leq \phi(\boldsymbol{x}_l|\theta_k) - z_l && \forall\, \theta_k \in (\underline{\theta}_k, \bar{\theta}_k] \\ \Longrightarrow \quad & \phi(\boldsymbol{x}_k|\underline{\theta}_k) - \phi(\boldsymbol{x}_l|\underline{\theta}_k) = \lim_{\theta_k \to \underline{\theta}_k} \{\phi(\boldsymbol{x}_k|\theta_k) - \phi(\boldsymbol{x}_l|\theta_k)\} \leq z_k - z_l. && \end{aligned}$$

Here, the continuity of the limit follows from the fact that $\phi(\boldsymbol{x}_k|\theta_k) - \phi(\boldsymbol{x}_l|\theta_k)$ is continuous in θ_k. The result contradicts that $\underline{\theta}_k$ strictly prefers the l-th contract. Similar arguments hold for the other cases. Thus, each subinterval is closed. $\square$

Proof of Theorem 5.4. Consider a feasible menu and any of its contracts $(\boldsymbol{x}(\theta), z(\theta))$. Modify the lot-sizing variables of contract $(\boldsymbol{x}(\theta), z(\theta))$, resulting in $\bar{\boldsymbol{x}}(\theta)$, in any way such that it is a feasible lot-sizing plan satisfying (5.2)-(5.9) and (5.14). Adjust the side payment to $\bar{z}(\theta)$ to compensate for the change in costs for type θ:

$$\bar{z}(\theta) = z(\theta) + \phi(\bar{\boldsymbol{x}}(\theta)|\theta) - \phi(\boldsymbol{x}(\theta)|\theta).$$

By construction, we have

$$\begin{aligned}
&\phi(\bar{\boldsymbol{x}}(\theta)|\theta) - \bar{z}(\theta) = \phi(\boldsymbol{x}(\theta)|\theta) - z(\theta) \leq \phi^*(\theta), \\
&\phi(\bar{\boldsymbol{x}}(\theta)|\theta) - \bar{z}(\theta) = \phi(\boldsymbol{x}(\theta)|\theta) - z(\theta) \leq \phi(\boldsymbol{x}(\hat{\theta})|\theta) - z(\hat{\theta}), \qquad \forall\, \hat{\theta} \in \Theta.
\end{aligned}$$

Furthermore, for $\hat{\theta} \in \Theta$ we get

$$\begin{aligned}
\phi(\bar{\boldsymbol{x}}(\theta)|\hat{\theta}) - \bar{z}(\theta) &= \phi(\bar{\boldsymbol{x}}(\theta)|\hat{\theta}) - \phi(\bar{\boldsymbol{x}}(\theta)|\theta) + \phi(\boldsymbol{x}(\theta)|\theta) - \phi(\boldsymbol{x}(\theta)|\hat{\theta}) + \phi(\boldsymbol{x}(\theta)|\hat{\theta}) - z(\theta) \\
&= (f(\hat{\theta}) - f(\theta))\Big(\sum_{t\in\mathcal{T}} \bar{y}_t^R(\theta) - \sum_{t\in\mathcal{T}} y_t^R(\theta)\Big) \\
&\quad + (h(\hat{\theta}) - h(\theta))\Big(\sum_{t\in\mathcal{T}} \bar{I}_t^R(\theta) - \sum_{t\in\mathcal{T}} I_t^R(\theta)\Big) \\
&\quad + \phi(\boldsymbol{x}(\theta)|\hat{\theta}) - z(\theta) \\
&= \phi(\boldsymbol{x}(\theta)|\hat{\theta}) - z(\theta) \geq \phi(\boldsymbol{x}(\hat{\theta})|\hat{\theta}) - z(\hat{\theta}).
\end{aligned}$$

Here, the last equality holds since both plans satisfy (5.14) and the inequality follows from the IC constraints. To conclude, the menu with the modified contract is feasible.

Finally, consider an optimal menu. We can modify each contract sequentially as described above, where the corresponding term in the objective is

$$\begin{aligned}
&\sum_{t\in\mathcal{T}} \Big(F\bar{y}_t^S(\theta) + H\bar{I}_t^S(\theta)\Big) + \bar{z}(\theta) \\
&\quad = \sum_{t\in\mathcal{T}} \Big(F\bar{y}_t^S(\theta) + H\bar{I}_t^S(\theta)\Big) + z(\theta) + \phi(\bar{\boldsymbol{x}}(\theta)|\theta) - \phi(\boldsymbol{x}(\theta)|\theta),
\end{aligned}$$

where $z(\theta)$ and $\phi(\boldsymbol{x}(\theta)|\theta)$ are now constants. Furthermore, in

$$\phi(\bar{\boldsymbol{x}}(\theta)|\theta) = f(\theta)\sum_{t\in\mathcal{T}} \bar{y}_t^R(\theta) + h(\theta)\sum_{t\in\mathcal{T}} \bar{I}_t^R(\theta)$$

one of these terms is constant (equal to n times the retailer's type) by (5.14) and the other term does not depend on the retailer's type. The result now follows. $\square$

5.A.2 Proofs of Section 5.3.2

Proof of Lemma 5.5. For $k \in \mathcal{K}$ the IR and IC constraints must hold for all types $\theta_k \in [\underline{\theta}_k, \bar{\theta}_k]$ with respect to contract $(\boldsymbol{x}_k, z_k)$. First, we consider the IC constraints. For $k \in \mathcal{K} \setminus \{1\}$ two (adjacent) IC constraints state

$$\phi(\boldsymbol{x}_k|\underline{\theta}_k) - \phi(\boldsymbol{x}_{k-1}|\underline{\theta}_k) \le z_k - z_{k-1} \le \phi(\boldsymbol{x}_k|\bar{\theta}_{k-1}) - \phi(\boldsymbol{x}_{k-1}|\bar{\theta}_{k-1}).$$

Since $\bar{\theta}_{k-1} = \underline{\theta}_k$, equality holds throughout and we obtain

$$z_k - z_{k-1} = \phi(\boldsymbol{x}_k|\underline{\theta}_k) - \phi(\boldsymbol{x}_{k-1}|\underline{\theta}_k). \tag{5.31}$$

This shows necessity of (5.31) and we continue with proving its sufficiency. We denote the lot-sizing variables of retailer plan $\boldsymbol{x}_i$ by $y_t^{R(i)}$ and $I_t^{R(i)}$. For $l < k$ we have

$$\begin{aligned}
z_k - z_l = \sum_{i=l+1}^{k} (z_i - z_{i-1}) &\overset{(5.31)}{=} \sum_{i=l+1}^{k} (\phi(\boldsymbol{x}_i|\underline{\theta}_i) - \phi(\boldsymbol{x}_{i-1}|\underline{\theta}_i)) \\
&= \sum_{i=l+1}^{k} \Big(f(\underline{\theta}_i)\Big(\sum_{t\in\mathcal{T}} y_t^{R(i)} - \sum_{t\in\mathcal{T}} y_t^{R(i-1)}\Big) + h(\underline{\theta}_i)\Big(\sum_{t\in\mathcal{T}} I_t^{R(i)} - \sum_{t\in\mathcal{T}} I_t^{R(i-1)}\Big)\Big) \\
&\overset{(5.13)}{\le} \sum_{i=l+1}^{k} \Big(f(\theta_l)\Big(\sum_{t\in\mathcal{T}} y_t^{R(i)} - \sum_{t\in\mathcal{T}} y_t^{R(i-1)}\Big) + h(\theta_l)\Big(\sum_{t\in\mathcal{T}} I_t^{R(i)} - \sum_{t\in\mathcal{T}} I_t^{R(i-1)}\Big)\Big) \\
&= \phi(\boldsymbol{x}_k|\theta_l) - \phi(\boldsymbol{x}_l|\theta_l),
\end{aligned}$$

for any $\theta_l \in [\underline{\theta}_l, \bar{\theta}_l]$. Likewise, for $l > k$ we get

$$\begin{aligned}
z_k - z_l = -\sum_{i=k+1}^{l} (z_i - z_{i-1}) &\overset{(5.31)}{=} -\sum_{i=k+1}^{l} (\phi(\boldsymbol{x}_i|\underline{\theta}_i) - \phi(\boldsymbol{x}_{i-1}|\underline{\theta}_i)) \\
&= -\sum_{i=k+1}^{l} \Big(f(\underline{\theta}_i)\Big(\sum_{t\in\mathcal{T}} y_t^{R(i)} - \sum_{t\in\mathcal{T}} y_t^{R(i-1)}\Big) + h(\underline{\theta}_i)\Big(\sum_{t\in\mathcal{T}} I_t^{R(i)} - \sum_{t\in\mathcal{T}} I_t^{R(i-1)}\Big)\Big) \\
&\overset{(5.13)}{\le} -\sum_{i=k+1}^{l} \Big(f(\theta_l)\Big(\sum_{t\in\mathcal{T}} y_t^{R(i)} - \sum_{t\in\mathcal{T}} y_t^{R(i-1)}\Big) + h(\theta_l)\Big(\sum_{t\in\mathcal{T}} I_t^{R(i)} - \sum_{t\in\mathcal{T}} I_t^{R(i-1)}\Big)\Big) \\
&= \phi(\boldsymbol{x}_k|\theta_l) - \phi(\boldsymbol{x}_l|\theta_l),
\end{aligned}$$

for any $\theta_l \in [\underline{\theta}_l, \bar{\theta}_l]$. Thus, all IC constraints are implied by (5.31).

Second, the IR constraints for $k \in \mathcal{K}$ are

$$\begin{aligned}
& \phi(\boldsymbol{x}_k|\theta_k) - z_k \le \phi^*(\theta_k) \qquad \forall\, \theta_k \in [\underline{\theta}_k, \bar{\theta}_k] \\
\iff \quad & \sup\{\phi(\boldsymbol{x}_k|\theta_k) - \phi^*(\theta_k) : \theta_k \in [\underline{\theta}_k, \bar{\theta}_k]\} \le z_k.
\end{aligned}$$

Since $\phi^*(\theta_k)$ is concave by Lemma 5.1 and $\phi(\boldsymbol{x}_k|\theta_k)$ is linear in θ_k, the difference $\phi(\boldsymbol{x}_k|\theta_k) - \phi^*(\theta_k)$ is a convex function in θ_k. The stated supremum is therefore attained at one of the border points $\underline{\theta}_k$ or $\bar{\theta}_k$ and the other IR constraints are redundant. In fact, more IR constraints are redundant, provided that the IC constraints hold. For $k \in \mathcal{K} \setminus \{K\}$ we have

$$\phi(\boldsymbol{x}_{k+1}|\underline{\theta}_{k+1}) - z_{k+1} \overset{(5.31)}{=} \phi(\boldsymbol{x}_k|\underline{\theta}_{k+1}) - z_k = \phi(\boldsymbol{x}_k|\bar{\theta}_k) - z_k.$$

This implies that we only need one of the IR constraints corresponding to types $\bar{\theta}_k$ and $\underline{\theta}_{k+1}$. □

Proof of Lemma 5.6. Suppose $\underline{\theta}_k = \bar{\theta}_k$, i.e., the k-th contract is effectively not assigned and is redundant. First, we consider the IC constraints. From (5.16) we have

$$\begin{aligned} z_{k+1} - z_k &= \phi(\boldsymbol{x}_{k+1}|\underline{\theta}_{k+1}) - \phi(\boldsymbol{x}_k|\underline{\theta}_{k+1}), \\ z_k - z_{k-1} &= \phi(\boldsymbol{x}_k|\underline{\theta}_k) - \phi(\boldsymbol{x}_{k-1}|\underline{\theta}_k). \end{aligned}$$

Adding both equalities and using $\underline{\theta}_k = \bar{\theta}_k = \underline{\theta}_{k+1}$ results in

$$\begin{aligned} z_{k+1} - z_{k-1} &= \phi(\boldsymbol{x}_{k+1}|\underline{\theta}_{k+1}) - \phi(\boldsymbol{x}_k|\underline{\theta}_{k+1}) + \phi(\boldsymbol{x}_k|\underline{\theta}_k) - \phi(\boldsymbol{x}_{k-1}|\underline{\theta}_k) \\ &= \phi(\boldsymbol{x}_{k+1}|\underline{\theta}_{k+1}) - \phi(\boldsymbol{x}_k|\underline{\theta}_{k+1}) + \phi(\boldsymbol{x}_k|\underline{\theta}_{k+1}) - \phi(\boldsymbol{x}_{k-1}|\underline{\theta}_{k+1}) \\ &= \phi(\boldsymbol{x}_{k+1}|\underline{\theta}_{k+1}) - \phi(\boldsymbol{x}_{k-1}|\underline{\theta}_{k+1}). \end{aligned}$$

This is the IC constraint if we would omit the redundant k-th contract from the model.

Second, consider the IR constraints. If $k \in \mathcal{K} \setminus \{K\}$ we have

$$\phi(\boldsymbol{x}_k|\underline{\theta}_k) - z_k = \phi(\boldsymbol{x}_k|\bar{\theta}_k) - z_k = \phi(\boldsymbol{x}_k|\underline{\theta}_{k+1}) - z_k \overset{(5.16)}{=} \phi(\boldsymbol{x}_{k+1}|\underline{\theta}_{k+1}) - z_{k+1}.$$

Thus, the corresponding IR constraints are implied by others and have no effect. Similarly, if $k = K$ it holds that

$$\phi(\boldsymbol{x}_K|\bar{\theta}_K) - z_K = \phi(\boldsymbol{x}_K|\underline{\theta}_K) - z_K = \phi(\boldsymbol{x}_K|\bar{\theta}_{K-1}) - z_K \overset{(5.16)}{=} \phi(\boldsymbol{x}_{K-1}|\bar{\theta}_{K-1}) - z_{K-1}.$$

Notice that this leads exactly to the second IR constraint required for type $K-1$ if we would omit the K-th contract from the model. We conclude that the redundant contract has no effect on the feasible region.

Finally, the probability that the k-th contract is selected by the retailer is equal to $\int_{\underline{\theta}_k}^{\bar{\theta}_k} \omega(\theta)\mathrm{d}\theta$. If $\underline{\theta}_k = \bar{\theta}_k$ then this probability is zero. Furthermore, other contracts are unaffected as argued above. Therefore, such redundant contracts do not affect the optimisation problem. □

Proof of Theorem 5.7. First, define $\bar{\theta}_0 = \underline{\theta}$ for notational convenience. Since ω is a uniform distribution, the objective function in (5.17) becomes

$$\begin{aligned} &\sum_{k \in \mathcal{K}} \frac{\bar{\theta}_k - \bar{\theta}_{k-1}}{\bar{\theta} - \underline{\theta}}(C_k + z_k) \\ &\quad = \sum_{k=1}^{K-1} \frac{(C_k - C_{k+1} + z_k - z_{k+1})\bar{\theta}_k}{\bar{\theta} - \underline{\theta}} + \frac{(C_K + z_K)\bar{\theta} - (C_1 + z_1)\underline{\theta}}{\bar{\theta} - \underline{\theta}}. \end{aligned}$$

We can use (5.21) to get

$$\begin{aligned}(C_k - C_{k+1} + z_k - z_{k+1})\bar{\theta}_k &= \left(C_k - C_{k+1} + c_k^{\text{pub}} - c_{k+1}^{\text{pub}} + (n_k - n_{k+1})\underline{\theta}_{k+1}\right)\bar{\theta}_k \\ &= (n_k - n_{k+1})\bar{\theta}_k^2 + \left(C_k + c_k^{\text{pub}} - C_{k+1} - c_{k+1}^{\text{pub}}\right)\bar{\theta}_k.\end{aligned}$$

Furthermore, we can repeatedly use (5.21) to obtain

$$\begin{aligned}z_k &= z_1 + \sum_{i=2}^{k}\left(c_i^{\text{pub}} - c_{i-1}^{\text{pub}} - (n_{i-1} - n_i)\underline{\theta}_i\right) = z_1 + c_k^{\text{pub}} - c_1^{\text{pub}} - \sum_{i=2}^{k}(n_{i-1} - n_i)\underline{\theta}_i \\ &= z_1 + c_k^{\text{pub}} - c_1^{\text{pub}} - \sum_{i=1}^{k-1}(n_i - n_{i+1})\bar{\theta}_i. \end{aligned} \tag{5.32}$$

Combining all results and substituting z_K, leads to the objective function

$$\begin{aligned}z_1 - c_1^{\text{pub}} &+ \frac{(C_K + c_K^{\text{pub}})\bar{\theta} - (C_1 + c_1^{\text{pub}})\underline{\theta}}{\bar{\theta} - \underline{\theta}} \\ &+ \sum_{k=1}^{K-1} \frac{(n_k - n_{k+1})\bar{\theta}_k^2 + \left(C_k + c_k^{\text{pub}} - C_{k+1} - c_{k+1}^{\text{pub}} - (n_k - n_{k+1})\bar{\theta}\right)\bar{\theta}_k}{\bar{\theta} - \underline{\theta}}.\end{aligned} \tag{5.33}$$

This objective is convex quadratic, as $n_k > n_{k+1}$ holds.

After eliminating unnecessary variables and using (5.32), the quadratic formulation is to minimise (5.33) subject to

$$\begin{aligned}
n_1\underline{\theta} + c_1^{\text{pub}} - z_1 &\leq \phi^*(\underline{\theta}), && && [\lambda_0]\\
n_k\bar{\theta}_k + \sum_{i=1}^{k-1}(n_i - n_{i+1})\bar{\theta}_i + c_1^{\text{pub}} - z_1 &\leq \phi^*(\bar{\theta}_k), && \forall\, k \in \{1,\ldots,K-1\}, && [\lambda_k]\\
n_K\bar{\theta} + \sum_{i=1}^{K-1}(n_i - n_{i+1})\bar{\theta}_i + c_1^{\text{pub}} - z_1 &\leq \phi^*(\bar{\theta}), && && [\lambda_K]\\
\underline{\theta} - \bar{\theta}_1 &\leq 0, && && [\mu_{0,1}]\\
\bar{\theta}_k - \bar{\theta}_{k+1} &\leq 0, && \forall\, k \in \{1,\ldots,K-2\}, && [\mu_{k,k+1}]\\
\bar{\theta}_{K-1} - \bar{\theta} &\leq 0. && && [\mu_{K-1,K}]
\end{aligned}$$

Note that we do not need the IC constraints in the model any more. Also, the IR constraints have been rewritten by using

$$n_k\underline{\theta}_k + \sum_{i=1}^{k-1}(n_i - n_{i+1})\bar{\theta}_i = n_{k-1}\bar{\theta}_{k-1} + \sum_{i=1}^{k-2}(n_i - n_{i+1})\bar{\theta}_i.$$

Finally, the default values $\phi^*(\bar{\theta}_k)$ can each be modelled using at most T linear constraints, as shown in Section 5.3.2.2.

Thus, if ω is a uniform distribution, then the plan assignment model can be formulated as a linearly-constrained convex-quadratic model with $(K-1)+1$ continuous variables and $2+(K-1)T+K$ linear constraints. Furthermore, the coefficients of the model are polynomial in the input size of the contracting problem as they relate to total costs in lot-sizing plans. This convex-quadratic model can be solved in polynomial time in the number of contracts K by interior-point methods, see for example Ye and Tse (1989). $\square$

5.A.3 Proofs of Section 5.3.3

Proof of Lemma 5.8. Suppose there exists an optimal solution where the supplier's lot-sizing plan does not satisfy the zero-inventory property. Consequently, there exists a time $\tau \in \mathcal{T}$ such that $I^S_{\tau-1} > 0$ and $y^S_\tau = 1$. We can strictly improve the contract by shifting $I^S_{\tau-1}$ units from the origin order(s) to time τ, leading to a reduction in the total costs by at least $H > 0$. This contradicts the optimality of the original lot-sizing plan. Hence, the supplier's plan must satisfy the zero-inventory property in the optimal solution. $\square$

Proof of Lemma 5.9. Suppose there is an optimal solution where $y^S_\tau = 1$ and $y^R_\tau = 0$ for some $\tau \in \mathcal{T}$. The supplier order must transfer to the retailer at some time, so it must hold that $\tau < T$. Furthermore, as $y^R_\tau = 0$ it is feasible to delay the considered supplier order. By shifting the supplier order from time τ to time $\tau+1$ the total costs reduce by $Hx^S_\tau > 0$, contradicting optimality. $\square$

Proof of Lemma 5.10. Consider an optimal solution and suppose there exist two supplier setups part of the same subplan. Let $t_1 < t_2 \in \mathcal{T}$ be the time periods of the first of such setups: $y^S_{t_1} = y^S_{t_2} = 1$ and $y^S_t = 0$ for all $t_1 < t < t_2$. By Lemmas 5.8 and 5.9 we conclude that $I^S_{t_1-1} = I^S_{t_2-1} = 0$ and $y^R_{t_1} = y^R_{t_2} = 1$. By assumption, both setups are part of the same subplan, so it must hold that $I^R_{t_2-1} > 0$. Furthermore, $I^R_{t_1-1} = 0$ and the origin of these $I^R_{t_2-1}$ products is the supplier order at time t_1, since times t_1 and t_2 correspond the first occurrence of the described supplier setups.

Now, realise that we can feasibly shift one unit of supply from the retailer's inventory $I^R_{t_2-1}$ to the supplier and retailer setups at time t_2 without invalidating any retailer setups in time periods $t_1, \dots, t_2-1$ (for details, see below). This results in a total cost reduction of at least $h > 0$, contradicting optimality. Hence, no two supplier setups can be part of the same subplan, implying that an optimal solution results in the stated decomposition into independent subplans. Note that each subplan must prescribe the number of retailer setups to ensure that the retailer uses exactly n setups in total.

The feasibility of the described shift is guaranteed by the assumption that $d_t > 0$ for all $t \in \mathcal{T}$. The details are as follows. Arbitrarily follow one unit of supply from the supplier order at time t_1 to the retailer's inventory at time t_2-1, which prevents time t_2 from being the start of a new independent subplan. Let τ ($t_1 \le \tau < t_2$) be the time this supply is transferred to the retailer's level, so $x^R_\tau > 0$. If $x^R_\tau = 1$ we cannot use this path to remove the considered unit of supply as that would remove the retailer setup at time τ, causing issues with the fact that the total number of retailer setups is fixed. However, by assumption we have $I^R_{t-1} + x^R_t = I^R_t + d_t > I^R_t$ for all $t \in \mathcal{T}$. In particular, $I^R_{\tau-1} + 1 = I^R_{\tau-1} + x^R_\tau = I^R_\tau + d_\tau > I^R_\tau \geq 1$, implying $I^R_{\tau-1} > 0$. Thus, there exists an alternative supply path. By repeating this argument and using $I^R_{t_1-1} = 0$, we conclude that there must exist a supply path for which $x^R_\tau > 1$. This path can be used to feasibly shift supply. □

Proof of Lemma 5.11. The idea of the proof is as follows. Consider such an optimal independent subplan, which by definition must have exactly one supplier setup (in the initial period). Hence, all post-initial demand is supplied from inventory (on the supplier's or retailer's level).

If $H < h$ it is strictly optimal to shift as much inventory as possible to the supplier's level, keeping the prescribed number of retailer setups in mind. Therefore, an optimal subplan must satisfy the zero-inventory property. If $H = h$ the described shifts are weakly optimal, so the zero-inventory property holds without loss of optimality.

If $H > h$ then it is strictly optimal to shift as much inventory as possible to the retailer's level. This implies placing all prescribed retailer setups as early as possible, i.e., in the first n periods, and reducing the post-initial retailer orders to the minimum of 1 unit of products. This unique optimal subplan satisfies the zero-inventory property only under very specific conditions.

We continue with a formal proof of the above argument. To simplify notation, we assume without loss of generality (by independence) that the considered optimal independent subplan spans all time periods $1, \ldots, T$. Thus, by definition of the subplan: $y^S_1 = 1$, $y^S_t = 0$ for $1 < t \leq T$, $I^S_0 = I^S_T = 0$, $y^R_1 = 1$, $\sum_{t \in \mathcal{T}} y^R_t = n$, $I^R_0 = I^R_T = 0$.

Case I: suppose $H \leq h$.

Suppose the optimal subplan does not satisfy the zero-inventory property. Let $t_2 \in \mathcal{T}$ be the first time that the zero-inventory property is violated, which must be in the retailer's lot-sizing plan by Lemma 5.8. Let $1 \leq t_1 < t_2$ be the time of the preceding retailer setup, which must exist. Hence, we have $y^R_{t_1} = y^R_{t_2} = 1$, $y^R_t = 0$ for all $t_1 < t < t_2$, $I^R_{t_1-1} = 0$, and $I^R_{t_2-1} > 0$. We emphasise that by assumption $d_{t_1} > 0$ so $x^R_{t_1} > I^R_{t_2-1}$ must hold, which is essential.

Now, shift $I^R_{t_2-1}$ units of inventory to the supplier instead of the retailer as follows. We reduce the retailer order at time t_1 by $I^R_{t_2-1}$ and keep these units in the supplier's inventory, resulting in a change in total costs of $(H - h)(t_2 - t_1)I^R_{t_2-1} \leq 0$. If $H < h$ then this is a strict inequality, which contradicts optimality. If $H = h$ then this shift does not affect the total costs, but removes this violation of the zero-inventory property. By repeating this argument, an optimal subplan can be constructed which satisfies the zero-inventory property.

Case II: suppose $H > h$.

First, we show that all post-initial retailer orders must be the minimum of 1 unit of products. Suppose $1 < \tau \leq T$ exists such that $x^R_\tau > 1$. Shift a unit of supply from the retailer order at time τ to time 1, resulting in change in costs of $(\tau-1)(h-H) < 0$. This contradicts optimality, so it must hold that $x^R_t \in \mathbb{B}$ for all $1 < t \leq T$.

Second, we prove that all n retailer setups must be in the first n periods. Suppose there exist $t_1, t_2 \in \mathcal{T}$ with $t_2 > t_1 + 1$, $x^R_{t_1} = x^R_{t_2} = 1$, and $x^R_t = 0$ for all $t_1 < t < t_2$.

Shifting the retailer order from time t_2 to $t_1 + 1$ results in a change in costs of $(h - H)(t_2 - t_1 - 1) < 0$, which contradicts optimality. We conclude that all n retailer setups are in the first n periods.

To conclude this case, if $H > h$ there is a unique optimal solution given as follows (see Figure 5.2). The retailer setups are all in the first n periods: $y_t^R = 1$ if and only if $1 \leq t \leq n$. All post-initial retailer orders have size 1: $x_t^R = 1$ for all $1 < t \leq n$. All remaining demand is supplied from the retailer's inventory, by the order in the initial period: $x_1^R = \sum_{t \in \mathcal{T}} d_t - (n-1)$. The total costs for the subplan are

$$F + nf + (H-h)\sum_{t=1}^{n}(t-1) + h\sum_{t=1}^{T}(t-1)d_t.$$

This subplan, the unique optimum, satisfies the zero-inventory property if and only if $n = 1$, or $n = T$ and $d_t = 1$ for all $1 < t \leq T$. Note that if the subplan spans a subset of the time periods, these expressions need to be trivially adjusted accordingly. $\square$

Proof of Theorem 5.12. The Dynamic-Programming (DP) algorithm that provides optimal solutions for all n-plan generation problems is based on Zangwill (1969). From Lemma 5.3 and Theorem 5.4 we know that we need to solve several joint lot-sizing problems where the number of retailer setups is fixed to $1, \ldots, T$. The DP relies on the fact that any optimal solution is decomposable into independent subplans, see Lemma 5.10. By considering all possible decompositions, we can determine an optimal solution. As such, we need to solve the corresponding subproblems, for which we use the insights of Lemma 5.11.

We now present the DP algorithm. The DP states are $(\underline{t}, n)$ which corresponds to the joint lot-sizing problem with time periods $\{\underline{t}, \ldots, T\} \subseteq \mathcal{T}$ and prescribes having exactly $1 \leq n \leq T$ retailer setups. Let $v(\underline{t}, n)$ be the corresponding optimal objective value. Thus, our list of optimal plans follows from the states $(1, n)$ with $n = 1, \ldots, T$.

We also need states related to the subproblems. For the subproblems, we have DP states $(\underline{t}, \bar{t}, n)$ which corresponds to the joint lot-sizing problem with time periods $\{\underline{t}, \ldots, \bar{t}\} \subseteq \mathcal{T}$ and where the number of retailer setups is fixed to $1 \leq n \leq T$. Let $w(\underline{t}, \bar{t}, n)$ be the optimal objective value *minus* supplier setup cost F of the corresponding optimal independent subplan.

The DP initialisation is given by

$$v(\underline{t}, 1) = F + w(\underline{t}, T, 1),$$

where we consider all feasible states: $1 \leq \underline{t} \leq T$. That is, if only a single retailer setup is allowed, it must be an independent subplan. The DP recursion is:

$$\begin{aligned} v(\underline{t}, n) = F + \min \Big\{ & w(\underline{t}, T, n), \\ & \min_{\underline{t} \leq \tau < T} \Big\{ \min_{\substack{1 \leq m \leq 1 + \tau - \underline{t} \\ n + \tau - T \leq m \leq n-1}} \{ w(\underline{t}, \tau, m) + v(\tau + 1, n - m) \} \Big\} \Big\}, \end{aligned}$$

where we consider all feasible states ($n = 1$ is the initialisation): $1 \leq \underline{t} \leq T$ and $1 < n \leq (1 + T - \underline{t})$. Essentially, the DP recursion compares the non-decomposable subplan $(\underline{t}, T, n)$ to all other feasibly decomposable subplans.

The solutions to the subproblems can be determined as follows. First, if only a single retailer setup is allowed ($n = 1$), there is a single feasible solution. We have for all feasible states $1 \leq \underline{t} \leq \bar{t} \leq T$:

$$w(\underline{t}, \bar{t}, 1) = f + h \sum_{t=\underline{t}}^{\bar{t}} (t - \underline{t}) d_t.$$

Next, we use Lemma 5.11 and need to consider two cases. If $H > h$ then we directly know the unique optimal solution and obtain

$$w(\underline{t}, \bar{t}, n) = nf + (H - h) \sum_{t=1}^{n} (t - 1) + h \sum_{t=\underline{t}}^{\bar{t}} (t - \underline{t}) d_t,$$

where we consider all feasible states (except $n = 1$): $1 \leq \underline{t} < \bar{t} \leq T$ and $1 < n \leq (1 + \bar{t} - \underline{t})$.

Otherwise, if $H \leq h$ there exists an optimal solution that satisfies the zero-inventory property and we can use a straightforward modification of the standard joint lot-sizing DP:

$$\begin{aligned} w(\underline{t}, \bar{t}, n) = \min_{\underline{t} \leq \tau \leq \bar{t}+1-n} \Bigg\{ \Bigg(f + h \sum_{t=\underline{t}}^{\tau} (t - \underline{t}) d_t + H(\tau + 1 - \underline{t}) \sum_{t=\tau+1}^{\bar{t}} d_t \Bigg) \\ + w(\tau + 1, \bar{t}, n - 1) \Bigg\}, \end{aligned}$$

where we consider all feasible states (except $n = 1$): $1 \leq \underline{t} < \bar{t} \leq T$ and $1 < n \leq (1 + \bar{t} - \underline{t})$.

The optimal plans are constructed by keeping track of the optimal choices made during the DP. It remains to determine the complexity of this DP. Precomputing the summations of demand takes $\mathcal{O}(T^2)$ time. The calculation of one $w(\underline{t}, \bar{t}, n)$ value then takes at most $\mathcal{O}(T)$ time, leading to $\mathcal{O}(T^4)$ time to determine w. Next, one $v(\underline{t}, n)$ value needs $\mathcal{O}(T^2)$ time, resulting in $\mathcal{O}(T^4)$ time for a complete specification of v. Thus, the total complexity is polynomial, namely $\mathcal{O}(T^4)$ time. □

Proof of Theorem 5.13. First, realise that solving all n-plan generation problems results in a list of T plans, so the number of contracts K is equal to T and is polynomial in the input size. The theorem now follows immediately by combining Theorems 5.7 and 5.12. □

Proof of Lemma 5.14. If ω is a uniform distribution the plan assignment model can be reformulated into a convex model. We refer to the proof of Theorem 5.7 for the model and its dual variables. We will use the Karush-Kuhn-Tucker (KKT) conditions (see Karush (1939) and Kuhn and Tucker (1951)) to prove the lemma.

For dual feasibility, all dual variables λ_k ($k \in \{0, \ldots, K\}$) and $\mu_{k,k+1}$ ($k \in \{0, \ldots, K-1\}$) must be non-negative. The KKT stationarity condition for z_1 is

$$1 - \sum_{k=0}^{K} \lambda_k = 0. \tag{5.34}$$

The stationarity conditions for $\bar{\theta}_k$, $k \in \{1, \ldots, K-1\}$, are

$$0 \in \frac{1}{\bar{\theta} - \underline{\theta}} \Big(2(n_k - n_{k+1})\bar{\theta}_k + \big((C_k + c_k^{\text{pub}}) - (C_{k+1} + c_{k+1}^{\text{pub}}) - (n_k - n_{k+1})\bar{\theta}\big)\Big) \\ + \Big(n_k - \partial\phi^*(\bar{\theta}_k)\Big)\lambda_k + (n_k - n_{k+1}) \sum_{i=k+1}^{K} \lambda_i - \mu_{k-1,k} + \mu_{k,k+1}. \qquad (5.35)$$

Here, we need the subdifferential $\partial\phi^*(\bar{\theta}_k)$, since ϕ^* is not differentiable in all points. In our case, $\partial\phi^*(\bar{\theta}_k)$ is the closed interval

$$\partial\phi^*(\bar{\theta}_k) = \Big[\lim_{\theta_k \downarrow \bar{\theta}_k} \frac{\mathrm{d}}{\mathrm{d}\theta_k}\phi^*(\theta_k), \lim_{\theta_k \uparrow \bar{\theta}_k} \frac{\mathrm{d}}{\mathrm{d}\theta_k}\phi^*(\theta_k)\Big].$$

Hence, in the optimal solution there exist subgradients, for simplicity denoted by $\frac{\mathrm{d}}{\mathrm{d}\bar{\theta}_k}\phi^*(\bar{\theta}_k) \in \partial\phi^*(\bar{\theta}_k)$ for $k \in \mathcal{K}$, such that (5.35) is satisfied. In other words, for these subgradients it must hold that

$$\bar{\theta}_k = \frac{(C_{k+1} + c_{k+1}^{\text{pub}}) - (C_k + c_k^{\text{pub}})}{2(n_k - n_{k+1})} + \frac{1}{2}\Big(\bar{\theta} - (\bar{\theta} - \underline{\theta}) \sum_{i=k+1}^{K} \lambda_i\Big) \\ - \frac{(\bar{\theta} - \underline{\theta})}{2(n_k - n_{k+1})} \left(\Big(n_k - \frac{\mathrm{d}}{\mathrm{d}\bar{\theta}_k}\phi^*(\bar{\theta}_k)\Big)\lambda_k - \mu_{k-1,k} + \mu_{k,k+1}\right). \qquad (5.36)$$

Now consider $k \in \{2, \ldots, K-1\}$ such that ϕ^* has no slopes n^* with $n_{k-1} > n^* > n_{k+1}$. If plan k is assigned, we have $\bar{\theta}_k > \bar{\theta}_{k-1}$ and $\mu_{k-1,k} = 0$. Substituting (5.36) in $\bar{\theta}_k > \bar{\theta}_{k-1}$ results in the condition

$$\frac{(C_k + c_k^{\text{pub}}) - (C_{k-1} + c_{k-1}^{\text{pub}})}{2(n_{k-1} - n_k)} + \frac{(C_k + c_k^{\text{pub}}) - (C_{k+1} + c_{k+1}^{\text{pub}})}{2(n_k - n_{k+1})} \\ + \frac{(\bar{\theta} - \underline{\theta})}{2(n_k - n_{k+1})} \left(\Big(n_{k+1} - \frac{\mathrm{d}}{\mathrm{d}\bar{\theta}_k}\phi^*(\bar{\theta}_k)\Big)\lambda_k + \mu_{k,k+1}\right) \\ + \frac{(\bar{\theta} - \underline{\theta})}{2(n_{k-1} - n_k)} \left(-\Big(n_{k-1} - \frac{\mathrm{d}}{\mathrm{d}\bar{\theta}_{k-1}}\phi^*(\bar{\theta}_{k-1})\Big)\lambda_{k-1} + \mu_{k-2,k-1}\right) < 0. \qquad (5.37)$$

Note that the summations over λ_i almost cancel out: only λ_k remains which explains why the term $n_{k+1}\lambda_k$ is present. We claim that the terms in (5.37) containing λ_{k-1} and λ_k are non-negative.

First, if $\lambda_{k-1} > 0$ then the IR constraint binds at $\bar{\theta}_{k-1}$. This is only possible if $n_{k-1} \geq \frac{\mathrm{d}}{\mathrm{d}\bar{\theta}_{k-1}}\phi^*(\bar{\theta}_{k-1})$. If $n_{k-1} = \frac{\mathrm{d}}{\mathrm{d}\bar{\theta}_{k-1}}\phi^*(\bar{\theta}_{k-1})$, then the term with λ_{k-1} is zero. Otherwise, if $n_{k-1} > \frac{\mathrm{d}}{\mathrm{d}\bar{\theta}_{k-1}}\phi^*(\bar{\theta}_{k-1})$, we get a contradiction with our assumptions. Namely, in this case it must hold that $\lim_{\theta_{k-1} \downarrow \bar{\theta}_{k-1}} \frac{\mathrm{d}}{\mathrm{d}\theta_{k-1}}\phi^*(\theta_{k-1}) \leq n_{k+1} < n_k$ as there are no larger eligible slopes of ϕ^* by assumption. Assigning plan k will violate the IR constraints, which leads to a contradiction. We conclude that $\big(n_{k-1} - \frac{\mathrm{d}}{\mathrm{d}\bar{\theta}_{k-1}}\phi^*(\bar{\theta}_{k-1})\big)\lambda_{k-1} = 0$.

Second, if $\lambda_k > 0$ then $n_k > \frac{\mathrm{d}}{\mathrm{d}\bar{\theta}_k}\phi^*(\bar{\theta}_k)$, where the inequality is strict since n_k is not a slope of ϕ^*. In particular, it must hold that $\frac{\mathrm{d}}{\mathrm{d}\bar{\theta}_k}\phi^*(\bar{\theta}_k) \leq n_{k+1}$. This proves our claim.

Thus, all terms with dual variables in (5.37) are non-negative. Therefore, the other terms must be strictly negative in total, i.e., (5.27) must hold. □

Proof of Lemma 5.15. First of all, realise that $\max_{\tau \in \mathcal{T}}\{(\tau - 1)\sum_{t=\tau}^{T} d_t\}$ is the maximum inventory that can be rerouted to a new supplier setup. So the assumption on the supplier's setup cost implies that it is never optimal to have more than a single supplier setup in a lot-sizing plan. Consequently, any optimal solution for the n-plan generation problem always has total costs

$$F + \Big(\sum_{t \in \mathcal{T}}(t-1)d_t - n\Big)H \tag{5.38}$$

for the supplier. Only the number of retailer setups can be minimised. Let $m_k \in \mathbb{N}_{\geq 1}$ be the number of retailer setups of the optimal n_k-plan.

Suppose we have determined a minimal list of plans sufficient for optimality for the contracting problem. Hence, each of these plans is assigned to retailer types. Add to this list the following T plans if not yet present: for $m \in \{1, \ldots, T\}$ add any m-extreme plan, i.e., a plan that has minimal retailer inventory with m retailer setups. Denote the indices of the combined list by $\mathcal{K}$ and the resulting indices of the extreme plans by $\mathcal{L} \subseteq \mathcal{K}$. It can be verified from the properties of ϕ^* that the slopes of ϕ^* are contained in $\{n_l : l \in \mathcal{L}\}$.

By definition of the extreme plans, we have $n_1 = \sum_{t \in \mathcal{T}}(t-1)d_t$, $n_K = 0$, and $1, K \in \mathcal{L}$. By our assumptions, and consequently by (5.38), all m-extreme plans have the same supplier's costs and retailer's public costs for fixed m. Therefore, it does not matter which m-extreme plan is added. Also, any two extreme plans in $\mathcal{L}$ have different amounts of retailer inventory: the plan with the most retailer setups has the lowest retailer inventory. This trivially follows from adding retailer setups to the plan with less retailer setups, which must decrease the retailer inventory.

Consider a non-extreme plan $k \in \mathcal{K} \setminus \mathcal{L}$, so $1 < k < K$ holds. Realise that we can apply Lemma 5.14 to this plan. If this plan would be assigned in the optimal menu, then Lemma 5.14 states that the following condition must be met:

$$\begin{aligned}
& \frac{(C_k + c_k^{\mathrm{pub}}) - (C_{k-1} + c_{k-1}^{\mathrm{pub}})}{n_{k-1} - n_k} + \frac{(C_k + c_k^{\mathrm{pub}}) - (C_{k+1} + c_{k+1}^{\mathrm{pub}})}{n_k - n_{k+1}} < 0 \\
\iff \quad & \frac{(n_{k-1} - n_k)H + (m_k - m_{k-1})f}{n_{k-1} - n_k} + \frac{(n_{k+1} - n_k)H + (m_k - m_{k+1})f}{n_k - n_{k+1}} < 0 \\
\iff \quad & \frac{m_k - m_{k-1}}{n_{k-1} - n_k} + \frac{m_k - m_{k+1}}{n_k - n_{k+1}} < 0.
\end{aligned} \tag{5.39}$$

Define $\underline{k}, \bar{k} \in \mathcal{L}$ such that $n_{\underline{k}} > n_k > n_{\bar{k}}$ and $m_{\bar{k}} = m_{\underline{k}} + 1$, so $\underline{k} < k < \bar{k}$. Note that these must exist. By definition, (5.39) must hold for each $\underline{k} < k < \bar{k}$. Adding these inequalities results in

$$\frac{m_{\underline{k}+1} - m_{\underline{k}}}{n_{\underline{k}} - n_{\underline{k}+1}} + \frac{m_{\bar{k}-1} - m_{\bar{k}}}{n_{\bar{k}-1} - n_{\bar{k}}} < 0,$$

as all interior terms cancel out. We claim that this is a contradiction. By definition, $n_{\underline{k}}$ is the minimum amount of retailer inventory when using $m_{\underline{k}}$ retailer setups, and likewise for $n_{\bar{k}}$ and $m_{\bar{k}}$. Since $n_k < n_{\underline{k}}$ for all $k > \underline{k}$ it follows that $m_k > m_{\underline{k}}$ for $k > \underline{k}$. Also, by definition we have $m_{\bar{k}} = m_{\underline{k}} + 1$ and, as a consequence, $m_k \geq m_{\bar{k}}$ for $k > \underline{k}$. By combining these insights, we obtain the following contradiction:

$$0 > \frac{m_{\underline{k}+1} - m_{\underline{k}}}{n_{\underline{k}} - n_{\underline{k}+1}} + \frac{m_{\bar{k}-1} - m_{\bar{k}}}{n_{\bar{k}-1} - n_{\bar{k}}} \geq \frac{1}{n_{\underline{k}} - n_{\underline{k}+1}} > 0.$$

We conclude that we only need to include plans which have minimal retailer inventory for each number of retailer setups. □

Proof of Lemma 5.16. Recall that under the assumptions all m-extreme plan have the same supplier's costs and retailer's public costs for fixed m. Therefore, we only need to determine a plan that minimises the retailer's inventory and uses exactly m retailer setups. The Dynamic-Programming (DP) algorithm is as follows. The DP states are $(\underline{t}, m)$, which corresponds to the joint lot-sizing problem with time periods $\{\underline{t}, \dots, T\} \subseteq \mathcal{T}$ and prescribes having exactly $1 \leq m \leq T$ retailer setups. The corresponding minimal retailer inventory is denoted by $v(\underline{t}, m)$. Hence, our list of extreme plans follows from the states $(1, m)$ with $m = 1, \dots, T$.

The DP initialisation for $1 \leq \underline{t} \leq T$ is

$$v(\underline{t}, 1) = \sum_{t=\underline{t}}^{T} (t - \underline{t}) d_t.$$

The DP recursion is given by

$$v(\underline{t}, m) = \min_{\underline{t} \leq \tau \leq T+1-m} \Big\{ \sum_{t=\underline{t}}^{\tau} (t - \underline{t}) d_t + v(\tau + 1, m - 1) \Big\},$$

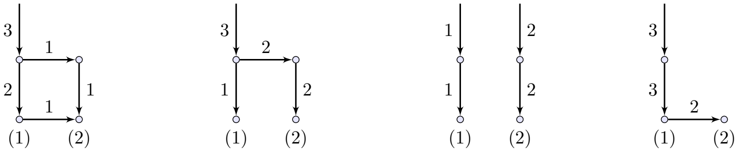
where we consider all feasible states (except $m = 1$): $1 \leq \underline{t} < T$ and $1 < m \leq (1 + T - \underline{t})$.

The plans are constructed by keeping track of the optimal choices made during the DP algorithm. Precomputing the summations of demand takes $\mathcal{O}(T^2)$ time. The calculation of each DP state then requires $\mathcal{O}(T)$ time, leading to $\mathcal{O}(T^3)$ time in total for the algorithm. □

Proof of Corollary 5.17. By Lemma 5.15 it is sufficient for optimality to use a list consisting of an m-extreme plan for each $m \in \{1, \dots, T\}$. This list consists of T plans and it can be constructed in polynomial time by Lemma 5.16. Thus, the number of contracts K is equal to T and polynomial in the input size. The result now follows from Theorem 5.7. □

5.B Example for private setup cost

In this appendix we give an example for the private setup cost case where the optimal menu assigns a contract that violates the zero-inventory property. We consider the smallest example by setting $T = 2$, $d_1 = 1$, and $d_2 = 2$, see Figure 5.4 for the optimal solution for each n-plan generation problem. Note that the plan in Figure 5.4a does not satisfy the zero-inventory property.



(a) $n = 2$, $H > h$, $F > H + h$. (b) $n = 2$, $H \le h$, $F > 2H$. (c) $n = 2$, $F \le 2H$, $F \le H + h$. (d) $n = 1$.

Figure 5.4: The optimal solutions for the n-plan generation problems in the case of private setup cost with $T = 2$, $d_1 = 1$, and $d_2 = 2$.

The cost parameters are as follows: $\underline{\theta} = 1$, $\bar{\theta} = 5$, $F = 4$, $H = 2$, and $h = 1$. It is trivial to verify that the default option ϕ^* is given by

$$\phi^*(\theta) = \begin{cases} 2\theta & \text{if } \theta \in [1, 2] \\ \theta + 2 & \text{if } \theta \in (2, 5] \end{cases}.$$

Since $H > h$ and $F > H + h$ the plan in Figure 5.4a is the only optimum for $n = 2$. Furthermore, we let ω be a uniform distribution. For the first contract we have $n_1 = 2$ retailer setups, $C_1 = F + H = 6$, and $c_1^{\text{pub}} = h = 1$. For the second contract these parameters are $n_2 = 1$, $C_2 = F = 4$, and $c_2^{\text{pub}} = 2h = 2$.

If the optimal $\bar{\theta}_1$ lies in $[1, 2)$, then the first segment of ϕ^* binds. By (5.19) and (5.21) the side payments must be

$$z_2 = c_2^{\text{pub}} + n_2\underline{\theta}_2 - \phi^*(\underline{\theta}_2) = 2 + \bar{\theta}_1 - 2\bar{\theta}_1 = 2 - \bar{\theta}_1,$$
$$z_1 = z_2 + c_1^{\text{pub}} - c_2^{\text{pub}} + (n_1 - n_2)\underline{\theta}_2 = (2 - \bar{\theta}_1) + 1 - 2 + \bar{\theta}_1 = 1.$$

Consequently, the objective function is

$$\tfrac{1}{\bar{\theta} - \underline{\theta}}\Big((\bar{\theta}_1 - \underline{\theta})(C_1 + z_1) + (\bar{\theta} - \bar{\theta}_1)(C_2 + z_2)\Big) = \tfrac{1}{4}\Big(\bar{\theta}_1^2 - 4\bar{\theta}_1 + 23\Big).$$

Since the derivative is $\frac{1}{4}(2\bar{\theta}_1 - 4) < 0$ on $[1, 2)$, it is optimal set $\bar{\theta}_1$ as large as possible in $[1, 2)$.

If the optimal $\bar{\theta}_1$ lies in $(2, 5]$, then the second segment of ϕ^* binds. Similar to the previous case, this leads to side payments $z_2 = 2 + \bar{\theta}_1 - (\bar{\theta}_1 + 2) = 0$ and $z_1 = \bar{\theta}_1 - 1$. The resulting objective function is $\frac{1}{4}(\bar{\theta}_1^2 + 15)$, which has strictly positive derivative on $(2, 5]$. Hence, it is optimal to set $\bar{\theta}_1$ as small as possible in $(2, 5]$.

Finally, it can be similarly verified that if $\bar{\theta}_1 = 2$, then $z_1 = 1$, $z_2 = 0$, and the objective value is $\frac{19}{4}$. This corresponds to the limits of the above two cases. We conclude that this is the unique optimal solution, which assigns the plan in Figure 5.4a to types $[1, 2]$ and the plan in Figure 5.4d to types $[2, 5]$. Thus, the optimal menu assigns a plan that violates the zero-inventory property.

5.C Relaxing the demand assumption

If demand can be zero certain results are no longer valid or need to be adjusted. This appendix describes the required changes. In Assumption 5.1 we state the properties of the considered demand.

Assumption 5.1. The (integer) demand is strictly positive in the first and last period, and non-negative otherwise, i.e., $d_1, d_T \in \mathbb{N}_{>0}$ and $d_t \in \mathbb{N}$ for $1 < t < T$.

Since we have time-independent costs, the assumption $d_1, d_T > 0$ is without loss of generality. If either is zero, that period can be removed from the problem. Under Assumption 5.1 only the results for the plan generation for private setup cost are significantly affected, i.e., Lemmas 5.10 and 5.11, and Theorem 5.12. To address the issues we condition on whether $H \leq h$ (Section 5.C.1) or $H > h$ (Section 5.C.2). Before we do so, we mention the minor adjustments needed for the other results.

Throughout our results and proofs we use that the number of retailer setups lies between 1 and T. If some demand is zero, there is a maximum feasible number of retailer setups $1 \leq K \leq T$, which should be used instead. This has no significant impact on the results. Furthermore, dummy setups (with zero order quantity) need to be prevented in any dynamic-programming algorithm. This slight adjustment does not affect the complexity results.

For the remainder of this appendix we consider the plan generation problem for private setup cost f under Assumption 5.1. For the proofs we often refer to or reuse parts of the proofs for Lemmas 5.10 and 5.11, and Theorem 5.12.

5.C.1 Case $H \leq h$

This case is more regular and essentially all results still hold. In Lemma 5.18 we will show that any plan assigned in the optimal menu must satisfy the zero-inventory property (and thus be decomposable). This lemma replaces Lemmas 5.10 and 5.11 for this case, and leads to the same overall conclusion. Consequently, Theorem 5.12 is unaffected and this case is solvable in $\mathcal{O}(T^4)$ time by dynamic programming. We continue with the lemma and its proof.

Lemma 5.18. *Under Assumption 5.1, private setup cost f, and $H \leq h$, it suffices for optimality for the contracting problem to restrict the plan generation to plans satisfying the zero-inventory property.*

Proof. The idea is as follows. First, we show that without loss of optimality an optimal n-plan violates the zero-inventory property only if n is strictly larger than the maximum slope of ϕ^*. This implies that this plan has unnecessary many retailer setups. Next, we show that such plans are never assigned in an optimal menu. The details are given below.

Part 1: violating the zero-inventory property implies a larger slope than ϕ^*.

We first introduce some notation. For $t \in \mathcal{T}$ with $I^R_{t-1} > 0$ we can backtrack the flow of a unit of products from the inventory I^R_{t-1} to its origin at the supplier. At some time period $\tau \in \{1, \ldots, t-1\}$ this unit of products is transferred from the supplier to the retailer by a retailer order, which we call the transfer time. Since there can be multiple options when backtracking, we define the indices $\mathcal{L}(t)$ such that $\{\tau_l : l \in \mathcal{L}(t)\} \subseteq \{1, \ldots, t-1\}$ are *all possible* transfer times for a unit of products of the inventory I^R_{t-1}. That is, any path in the network flow graph from the supplier to the retailer's inventory I^R_{t-1} transfers to the retailer's layer at some time τ_l, $l \in \mathcal{L}(t)$. By definition, we have $x^R_{\tau_l} \geq 1$ for all $l \in \mathcal{L}(t)$. We denote the first and last transfer times by $\tau_1(t)$ and $\tau_L(t)$, respectively.

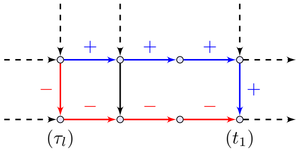
Suppose an optimal solution to the n-plan generation problem exists that violates the zero-inventory property, say at time $t_1 \in \mathcal{T}$ (and potentially elsewhere). By Lemma 5.8 this implies that $1 < t_1 \leq T$, $x^R_{t_1} > 0$, and $I^R_{t_1-1} > 0$.

If $l \in \mathcal{L}(t_1)$ with $x^R_{\tau_l} > 1$ exists, then we can reduce the inventory $I^R_{t_1-1}$ by shifting a unit of products to the supplier's inventory at time τ_l instead of the retailer's inventory (see Figure 5.5a). The costs change by $(t_1 - \tau_l)(H - h) \leq 0$. Since $\tau_l < t_1$ this contradicts optimality if $H < h$, otherwise (if $H = h$) we can exclude this solution without loss of optimality. Hence, for any time $t \in \mathcal{T}$ where the plan violates the zero-inventory property ($x^R_t > 0$ and $I^R_{t-1} > 0$) it must hold that $x^R_{\tau_l} = 1$ for all $l \in \mathcal{L}(t)$.

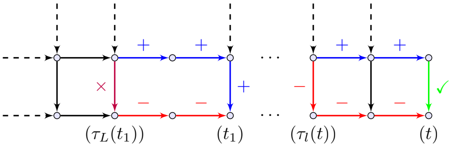
Suppose there exists a time $t \in \mathcal{T}$ with $y^R_t = 0$, $I^R_{t-1} > 0$, and $x^R_{\tau_l(t)} > 1$ for some $l \in \mathcal{L}(t)$. See also Figure 5.5b. Decrease $I^R_{t_1-1}$ by shifting one unit of the flow through the latest time $\tau_L(t_1)$ to the supplier's inventory instead. This removes the retailer setup at time $\tau_L(t_1)$, but still leads to a feasible plan with a change in costs of $(H - h)(t_1 - \tau_L(t_1)) \leq 0$. Reinsert the removed retailer setup at time t and shift a single unit of products from time $\tau_l(t) < t$ (which has $x^R_{\tau_l(t)} > 1$ by assumption) to this new setup. The result is a feasible n-plan with a change in costs of $(H - h)(t - \tau_l(t)) + (H - h)(t_1 - \tau_L(t_1)) \leq 0$, compared to the original n-plan. This contradicts the optimality of the original plan if $H < h$ or we can exclude it without loss of optimality ($H = h$).

Thus, by combining the above, we have for all $t \in \mathcal{T}$ with $I^R_{t-1} > 0$ that $x^R_{\tau_l} = 1$ for all $l \in \mathcal{L}(t)$. Suppose there exists $t \in \mathcal{T}$ with $d_t > 0$ and $y^R_t = 0$, which implies that $I^R_{t-1} > 0$. Hence, $x^R_{\tau_l} = 1$ for all $l \in \mathcal{L}(t)$ must hold. Realise that this is only possible if $d_{\tau_1(t)} = 0$. We can reposition this retailer setup at time $\tau_1(t)$ to time t resulting in a change in costs of $(H - h)(t - \tau_1(t)) \leq 0$. Again, this results in a contradiction to optimality ($H < h$) or can be excluded without loss of optimality ($H = h$).

We conclude that $y^R_t = 1$ for all $t \in \mathcal{T}$ with $d_t > 0$. Furthermore, it must hold that $d_{\tau_1(t_1)} = 0$ as $x^R_{\tau_1(t_1)} = 1$. Therefore, the plan must have strictly more retailer setups than the number of time periods with strictly positive demand. In other words, the plan has enough retailer setups to directly satisfy all demand without any retailer inventory. However, there are redundant retailer setups (such as at $\tau_1(t_1)$) that only increase the number of retailer setups and lead to higher costs for both the supplier and the retailer. Consequently, the number of retailer setups n of this optimal plan exceeds the maximum slope of ϕ^*, which is essential.



(a) Without changing retailer setups.



(b) A retailer setup is removed at time $\tau_L(t_1)$ and reinserted at time t.

Figure 5.5: Shifting flow in procurement plans. Reduced flows are red, increased flows are blue, purple setups are removed, and green setups are new.

Part 2: plans violating the zero-inventory property are not assigned in an optimal menu.

Suppose the considered plan, which violates the zero-inventory property, is assigned to retailer types in an optimal solution to the contracting problem. Let it be the k-th contract in the optimal menu and let n_k denote its number of retailer setups. By Lemma 5.6 we assume without loss of generality that the $(k+1)$-th contract has a plan with one less retailer setup, $n_{k+1} = n_k - 1$, but is potentially not assigned to retailer types. Consider the k-th plan and remove a unit of inventory from $I^R_{t_1-1}$, remove the retailer setup at time $\tau_L(t_1)$, and shift that supply to the supplier's inventory. We obtain a feasible $(n_k - 1)$-plan with a change in costs of $(H - h)(t_1 - \tau_L(t_1)) \leq 0$. Hence, it must hold that

$$C_{k+1} + c^{\text{pub}}_{k+1} \leq C_k + c^{\text{pub}}_k + (H - h)(t_1 - \tau_L(t_1)) \leq C_k + c^{\text{pub}}_k.$$

By also using (5.21), we have

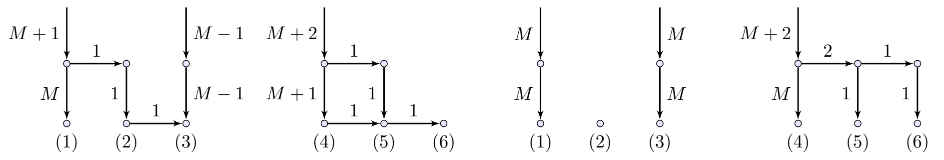
$$\begin{aligned} C_{k+1} + z_{k+1} &= C_{k+1} + c^{\text{pub}}_{k+1} - c^{\text{pub}}_{k+1} + z_{k+1} \leq C_k + c^{\text{pub}}_k - c^{\text{pub}}_{k+1} + z_{k+1} \\ &= C_k + z_k - (n_k - n_{k+1})\underline{\theta}_{k+1} < C_k + z_k. \end{aligned} \tag{5.40}$$

Thus, keeping the side payments $z_{k+1}, \ldots, z_K$ constant and replacing the k-th contract by the $(k+1)$-th leads to a strictly better objective value when considered in isolation. In terms of the partition, we only change $\underline{\theta}_{k+1}$ to $\underline{\theta}_k$, so the k-th contract is no longer assigned. However, the side payments of contracts $1, \ldots, k$ also need to be adjusted for feasibility. This can be done by decreasing $z_1, \ldots, z_k$ by $(n_k - n_{k+1})(\underline{\theta}_{k+1} - \underline{\theta}_k) > 0$ according to (5.21). Namely, consider (5.21) for $k+1$. Since $\underline{\theta}_{k+1}$ is changed to $\underline{\theta}_k$, z_k needs to decrease by $(n_k - n_{k+1})(\underline{\theta}_{k+1} - \underline{\theta}_k)$. Now consider (5.21) for k. As $\underline{\theta}_k$ remains unchanged, we need to decrease z_{k-1} by $(n_k - n_{k+1})(\underline{\theta}_{k+1} - \underline{\theta}_k)$. The previous argument also holds for all contracts $1, \ldots, k-1$. Graphically, we are shifting all lines $1, \ldots, k$ vertically upwards such that the k-th line is no longer essential for the lower envelope. By (5.40) and the decrease in side payments, these modifications improve the objective value. All these modifications to the menu are feasible with respect to the IR constraints, since the plans of *all* contracts $1, \ldots, k+1$ have a slope of at least the maximum slope of ϕ^*. The latter property is essential for this argument.

We conclude that we have constructed a strictly better feasible menu where the considered k-th contract is not assigned to retailer types. This contradiction implies that it suffices for optimality for the contracting problem to restrict the plan generation to plans that satisfy the zero-inventory property. □

5.C.2 Case $H > h$

In this case, there are instances for which any optimal n-plan is non-decomposable into independent subplans for certain values of n. For example, by appropriately choosing $H > h$, M, and F, the plan in Figure 5.6a is the unique optimum. In particular, for any $H > h$ this example can be extended such that the optimum is non-decomposable. Thus, Lemma 5.10 does not hold in general. The details are as follows.



(a) Non-decomposable solution. (b) Decomposable solution.

Figure 5.6: Two possible solutions for the 5-plan generation problem in the case of private setup cost with $T = 6$, $d = (M, 0, M, M, 1, 1)$.

For given H and h with $H > h$, let $T \geq 6$ be such that

$$H > \frac{T-3}{T-5} h,$$

and set the demand to $d = (M, 0, M, M, 1, \ldots, 1)$ for some $M \in \mathbb{N}_{>0}$ large enough. The parameters M and F are chosen such that it is optimal to have a supplier setup only at the time periods with demand M, i.e., time periods 1, 3, and 4.

Now consider the $(T-1)$-plan generation problem. Any plan that does not have a supplier and retailer setup at times 1, 3, and 4 is suboptimal due to the choice of M and the resulting large holding costs. Suppose an optimal $(T-1)$-plan does not have a retailer setup at time 2, then it must have retailer setups in all other periods (similar to Figure 5.6b). Perform the following shift to obtain a feasible plan similar to Figure 5.6a. Shift the supply provided by x_T^R to x_4^R, reposition the retailer setup at time T to time 2, and set $x_2^R = 1$ by shifting a unit of supply from the orders at time 3 to time 1. This leads to a change in costs of $(h - H)(T - 4) + (H + h) = h(T - 3) - H(T - 5) < 0$ by choice of T. From this contradiction to optimality, we conclude that any optimal $(T-1)$-plan is non-decomposable.

There are also instances where this non-decomposable plan is necessary for optimality for the contracting problem, i.e., any solution can be improved by adding it to the menu. For example, let ω be a uniform distribution and take $T = 6$, $d = (20, 0, 20, 20, 1, 1)$, $F = 20$, $H = 7$, $h = 2$, and $f \in [1, 20] = \Theta$. These parameters satisfy the conditions given above, so contract 2 with $n_2 = T - 1 = 5$ is non-decomposable. Furthermore, an optimal solution is to assign contracts 2, 3, and 4 (with $n_2 = 5$, $n_3 = 4$, and $n_4 = 3$) to retailer types $[1, 2]$, $[2, 4]$, and $[4, 20]$, respectively. The side payments are $z = (3, 6, 2, 0, 24, 172)$ and the optimal objective value is 62. Removing the non-decomposable plan from the menu, leads to an optimal objective value of approximately 62.1, showing the necessity to include this plan.

With these insights, we provide a dynamic-programming algorithm to solve all n-plan generation problems. The running time is polynomial, namely $\mathcal{O}(T^5)$. The presented algorithm suffices for our goal to show that this case is also efficiently solvable. The result is stated in Lemma 5.19.

Lemma 5.19. *Under Assumption 5.1, private setup cost f, and $H > h$, solving all n-plan generation problems can be done in $\mathcal{O}(T^5)$ time by dynamic programming.*

Proof. For $H > h$ and some demand being zero, the optimal n-plan solution might be non-decomposable into independent subproblems (recall the example in Figure 5.6a). Therefore, a new approach is needed. The idea is as follows.

First, we show that an optimal plan consists of substructures as illustrated in Figure 5.7, which is similar to Figure 5.2. In particular, the optimal solution is fixed when we are given the supplier setups and how many retailer setups occur in between the supplier setups. Here, it is essential that $H > h$, so we know that as much inventory as possible is placed at the retailer (without invalidating retailer setups).

Next, we describe a Dynamic-Programming (DP) algorithm similar to that in the proof of Theorem 5.12. Since we might have a non-decomposable solution, we need to add the available inventory to the DP states. In an optimal solution these inventory states are non-zero only if removing that inventory would lead to dummy retailer setups. Hence, the inventory states are bounded by T, which is essential to obtain a polynomial-time algorithm. Below we give all the details.

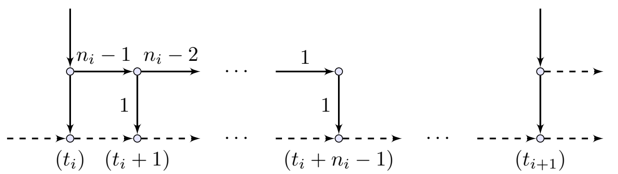


Figure 5.7: The unique optimal substructure in the case of private setup cost and $H > h$. There are sequential supplier setups at time periods t_i and t_{i+1}, and n_i retailer setups in time periods $t_i, \ldots, t_{i+1} - 1$.

Part 1: properties of the optimal n-plan.

Consider an optimal solution of the n-plan generation problem. Assume that $t_1 < t_2 \in \mathcal{T}$ exist such that $y^S_{t_1} = y^S_{t_2} = 1$ and $y^S_t = 0$ for $t_1 < t < t_2$. By Lemmas 5.8 and 5.9 we know that $I^S_{t_1-1} = I^S_{t_2-1} = 0$ and $y^R_{t_1} = y^R_{t_2} = 1$. Let $n_1 \in \{1, \ldots, t_2 - t_1\}$ be the number of retailer setups in $t_1, \ldots, t_2 - 1$. Note that if such t_1 and t_2 do not exist, then the optimum is an independent subplan and we can apply Lemma 5.11 to obtain similar properties as derived below.

First, we show that all retailer orders in periods $t_1 + 1, \ldots, t_2 - 1$ must be the minimum of 1 unit of products. Suppose $t_1 < \tau < t_2$ exists such that $x^R_\tau > 1$. These products are satisfied from the supplier's inventory. Shift a unit of supply from the retailer order at time τ to time t_1, resulting in change in costs of $(\tau - t_1)(h - H) < 0$. This contradicts optimality, so it must hold that $x^R_t \in \mathbb{B}$ for all $t_1 < t < t_2$.

Second, we prove that all n_1 retailer setups must be in the initial periods. Suppose there exist $t_1 \leq \tau_1 < \tau_2 < t_2$ with $\tau_2 > \tau_1 + 1$, $x^R_{\tau_1} = x^R_{\tau_2} = 1$, and $x^R_t = 0$ for all $\tau_1 < t < \tau_2$. Again, these products are satisfied from the supplier's inventory. Shifting the retailer order from time τ_2 to $\tau_1 + 1$ results in a change in costs of $(h - H)(\tau_2 - \tau_1 - 1) < 0$, which contradicts optimality. We conclude that all n_1 retailer setups are in the first n_1 periods of the considered time periods.

For the considered optimal solution we have for some $N \in \{1, \ldots, T\}$ the supplier setups $t_1, \ldots, t_N \in \mathcal{T}$ and the number of retailer setups $n_1, \ldots, n_N \in \{1, \ldots, T\}$ such that n_i retailer setups take place in time periods $t_i, \ldots, t_{i+1} - 1$ for $i \in \{1, \ldots, N - 1\}$ and n_N in $t_N, \ldots, T$. From the above, we must have $x^S_{t_i} > 0$, $x^R_{t_i} > 0$, $x^R_{t_i+1} = \cdots = x^R_{t_i+n_i-1} = 1$ for $i \in \{1, \ldots, N\}$ and zero otherwise.

Now realise that the retailer order $x^R_{t_i}$ needs to be minimal in the optimal solution. Otherwise there is unnecessary retailer inventory at time $t_{i+1} - 1$ which can be transferred to the supplier and retailer setup at time t_{i+1}. This strictly reduces the total costs, contradicting optimality. The minimal order $x^R_{t_i}$ is at least 1 (to keep it a valid setup) and is such that all demand in $t_i, \ldots, t_{i+1} - 1$ is satisfied, taking into account the supply from any available inventory $I^R_{t_i-1}$ and any additional retailer setups $x^R_{t_i+1} = \cdots = x^R_{t_i+n_i-1} = 1$.

In particular, if $I^R_{t_i-1} > \sum_{j=1}^{i-1} n_j$ for some $i \in \{1, \ldots, N\}$, then the retailer inventory $I^R_{t_i-1}$ can be reduced, which contradicts optimality. Namely, consider the artificial situation that $d_t = 0$ for all $1 \leq t < t_i$. The $\sum_{j=1}^{i-1} n_j$ retailer setups need to provide at least 1 supply each, which is transferred by retailer inventory $I^R_{t_i-1}$ to time t_i. If more inventory is supplied, then a unit of products can be shifted from a previous order to the order at time t_i, without invalidating a retailer setup. This strictly reduces the costs, proving the claim. Note that a better bound exists, but this suffices to obtain polynomial running time.

Part 2: the dynamic-programming algorithm.

We can now formulate a DP algorithm with a polynomial running time that solves all n-plan generation problems. The DP states are $(\underline{t}, n, m_{\text{in}})$ which corresponds to the joint lot-sizing subproblem with time periods $\{\underline{t}, \ldots, T\} \subseteq \mathcal{T}$ and prescribes having exactly $1 \leq n \leq T$ retailer setups. Furthermore, the supplier inventory satisfies $I^S_{\underline{t}-1} = I^S_T = 0$ and for the retailer inventory we have $I^R_{\underline{t}-1} = m_{\text{in}} \in \{0, \ldots, \underline{t}-$

$1\}$ and $I_T^R = 0$. Note that the bound on m_{in} follows from the above arguments on minimal remaining inventory. Let $v(\underline{t}, n, m_{\text{in}})$ be the corresponding optimal objective value. Our list of optimal plans follows from the states $(1, n, 0)$ with $n = 1, \dots, T$ (or up to the maximum feasible number of retailer setups).

We also need states $(\underline{t}, \bar{t}, n, m_{\text{in}})$ for $\underline{t} \leq \bar{t} \in \mathcal{T}$, $n \in \{1, \dots, 1 + \bar{t} - \underline{t}\}$, and $m_{\text{in}} \in \{0, \dots, \underline{t} - 1\}$ as follows. They correspond to subproblems spanning time periods $\underline{t}, \dots, \bar{t}$ with a supplier setup only at time $\underline{t}$: $y_{\underline{t}}^S = 1$ and $y_t^S = 0$ for $\underline{t} < t \leq \bar{t}$. Furthermore, there are exactly n retailer setups: $\sum_{t=\underline{t}}^{\bar{t}} y_t^R = n$. Finally, the supplier inventory satisfies $I_{\underline{t}-1}^S = I_{\bar{t}}^S = 0$ and for the retailer inventory we have $I_{\underline{t}-1}^R = m_{\text{in}}$.

By the above analysis there exists a unique optimal subplan for each feasible state $(\underline{t}, \bar{t}, n, m_{\text{in}})$. Let $w(\underline{t}, \bar{t}, n, m_{\text{in}})$ denote the corresponding costs. Furthermore, we have the corresponding (minimal) remaining inventory $I_{\bar{t}}^R$ denoted by $m_{\text{out}}(\underline{t}, \bar{t}, n, m_{\text{in}})$. Any infeasible states are assigned the value infinity, i.e., they are omitted when determining the optimal plans.

The DP initialisation for feasible states is given by

$$v(\underline{t}, 1, m_{\text{in}}) = w(\underline{t}, T, 1, m_{\text{in}}).$$

The DP recursion for feasible states is:

$$\begin{aligned} v(\underline{t}, n, m_{\text{in}}) = \min \Big\{ & w(\underline{t}, T, n, m_{\text{in}}), \\ & \min_{\underline{t} \leq \tau < T} \Big\{ \min_{1 \leq \kappa \leq n-1} \{ w(\underline{t}, \tau, \kappa, m_{\text{in}}) + v(\tau + 1, n - \kappa, m_{\text{out}}(\underline{t}, \tau, \kappa, m_{\text{in}})) \} \Big\} \Big\}. \end{aligned}$$

Certain options in the shown ranges might lead to infeasible states and should be omitted. There are $\mathcal{O}(T^3)$ many DP states and each takes $\mathcal{O}(T^2)$ time to compute. Thus, to determine v we need $\mathcal{O}(T^5)$ time.

It remains to solve the subproblems related to w and m_{out}. For a feasible state $(\underline{t}, \bar{t}, n, 0)$ the supplier order must be $x_{\underline{t}}^S = \max\{n, \sum_{t=\underline{t}}^{\bar{t}} d_t\}$ in order to supply all demand in $\underline{t}, \dots, \bar{t}$ and to have no dummy retailer setups. We can construct the corresponding subplan in $\mathcal{O}(T)$ time, from which we obtain I_t^R, $t \in \{\underline{t}, \dots, \bar{t}\}$, and in particular the remaining inventory $m_{\text{out}}(\underline{t}, \bar{t}, n, 0)$. The corresponding costs are

$$w(\underline{t}, \bar{t}, n, 0) = F + nf + H \sum_{t=1}^{n} (t - 1) + h \sum_{t=\underline{t}}^{\bar{t}} I_t^R.$$

Thus, calculating $w(\underline{t}, \bar{t}, n, 0)$ and $m_{\text{out}}(\underline{t}, \bar{t}, n, 0)$ for all $\underline{t}$, $\bar{t}$, and n takes $\mathcal{O}(T^4)$ time. The other feasible states follow from

$$w(\underline{t}, \bar{t}, n, m + 1) = \begin{cases} w(\underline{t}, \bar{t}, n, m) & \text{if } x_{\underline{t}}^R > 1 \text{ in state } (\underline{t}, \bar{t}, n, m) \\ w(\underline{t}, \bar{t}, n, m) + h(1 + \bar{t} - \underline{t}) & \text{otherwise} \end{cases},$$

$$m_{\text{out}}(\underline{t}, \bar{t}, n, m + 1) = \begin{cases} m_{\text{out}}(\underline{t}, \bar{t}, n, m) & \text{if } x_{\underline{t}}^R > 1 \text{ in state } (\underline{t}, \bar{t}, n, m) \\ m_{\text{out}}(\underline{t}, \bar{t}, n, m) + 1 & \text{otherwise} \end{cases}.$$

That is, if $x_{\underline{t}}^R > 1$ and if m_{in} increases by 1, then this additional initial inventory is used to satisfy demand in $\underline{t}, \dots, \bar{t}$. Furthermore, we decrease $x_{\underline{t}}^S$ and $x_{\underline{t}}^R$ by 1. The inventory I_t^R remains unchanged for $t \in \{\underline{t}, \dots, \bar{t}\}$. If $x_{\underline{t}}^R = 1$, the previously described changes would lead to a dummy retailer setup at time $\underline{t}$. Hence, the additional initial inventory is kept in inventory throughout the subplan.

In total, determining w and m_{out} takes $\mathcal{O}(T^4)$ time. Overall, the DP takes $\mathcal{O}(T^5)$ time, which can potentially be reduced since many m_{in} values are infeasible or not used. □

5.D Fixed partition model

In this appendix we formulate the *fixed partition model*. In this model the number of contracts and their assignment to the retailer types is fixed, but the lot-sizing plans and side payments of these contracts need to be determined.

For $K \in \mathbb{N}_{\geq 1}$ we are given a K-partition of Θ, denoted by $[\underline{\theta}_k, \bar{\theta}_k]$ with $k \in \mathcal{K} = \{1, \dots, K\}$. It is allowed to have $\underline{\theta}_k = \bar{\theta}_k$, since (the proof of) Lemma 5.6 also holds for the fixed partition model. The fixed partition model is given by

$$\min \sum_{k \in \mathcal{K}} \Big(\int_{\underline{\theta}_k}^{\bar{\theta}_k} \omega(\theta) \mathrm{d}\theta \Big) \Big(F \sum_{t \in \mathcal{T}} y_t^{S(k)} + H \sum_{t \in \mathcal{T}} I_t^{S(k)} + z_k \Big)$$

subject to

$$\begin{aligned}
I_0^{S(k)} &= 0, && \forall\, k \in \mathcal{K},\\
I_{t-1}^{S(k)} + x_t^{S(k)} &= I_t^{S(k)} + x_t^{R(k)}, && \forall\, k \in \mathcal{K}, t \in \mathcal{T},\\
x_t^{S(k)} &\leq M y_t^{S(k)}, && \forall\, k \in \mathcal{K}, t \in \mathcal{T},\\
I_0^{R(k)} = I_T^{R(k)} &= 0, && \forall\, k \in \mathcal{K},\\
I_{t-1}^{R(k)} + x_t^{R(k)} &= I_t^{R(k)} + d_t, && \forall\, k \in \mathcal{K}, t \in \mathcal{T},\\
y_t^{R(k)} \leq x_t^{R(k)} &\leq M y_t^{R(k)}, && \forall\, k \in \mathcal{K}, t \in \mathcal{T},\\
y_t^{S(k)}, y_t^{R(k)} &\in \mathbb{B}, && \forall\, k \in \mathcal{K}, t \in \mathcal{T},\\
x_t^{S(k)}, x_t^{R(k)}, I_t^{S(k)}, I_t^{R(k)} &\in \mathbb{N}, && \forall\, k \in \mathcal{K}, t \in \mathcal{T},\\
f(\theta_k) \sum_{t \in \mathcal{T}} y_t^{R(l)} + h(\theta_k) \sum_{t \in \mathcal{T}} I_t^{R(l)} &= \phi(\boldsymbol{x}_l | \theta_k), && \forall\, \theta_k \in [\underline{\theta}_k, \bar{\theta}_k], k, l \in \mathcal{K},\\
\phi(\boldsymbol{x}_k | \theta_k) - z_k &\leq \phi^*(\theta_k), && \forall\, \theta_k \in [\underline{\theta}_k, \bar{\theta}_k], k \in \mathcal{K},\\
\phi(\boldsymbol{x}_k | \theta_k) - z_k &\leq \phi(\boldsymbol{x}_l | \theta_k) - z_l, && \forall\, \theta_k \in [\underline{\theta}_k, \bar{\theta}_k], k, l \in \mathcal{K}.
\end{aligned}$$

By realising that Lemma 5.5 is also valid for this model, the IR and IC constraints can be replaced by the following finitely many constraints:

$$\begin{aligned}
\phi(\boldsymbol{x}_1|\underline{\theta}) - z_1 &\leq \phi^*(\underline{\theta}), && \\
\phi(\boldsymbol{x}_k|\underline{\theta}_k) - z_k &\leq \phi^*(\underline{\theta}_k), && \forall\, k \in \mathcal{K} \setminus \{1\}, \\
\phi(\boldsymbol{x}_K|\bar{\theta}) - z_K &\leq \phi^*(\bar{\theta}), && \\
\phi(\boldsymbol{x}_k|\underline{\theta}_k) - z_k &= \phi(\boldsymbol{x}_{k-1}|\underline{\theta}_k) - z_{k-1}, && \forall\, k \in \mathcal{K} \setminus \{1\}.
\end{aligned}$$

The resulting model is a standard mixed integer linear program, provided that we use the linear modelling of ϕ^* as shown in Section 5.3.2.2.

Chapter 6

Conclusion

Contents

6.1 Main findings

We have analysed a variety of contracting problems between two parties who do not share all their information. These two parties form a supply chain, but act individually rationally, i.e., they only care about their own interests. Therefore, any coordination takes place for their own benefit. The considered supply chain coordination problems are viewed from the upstream party's perspective, implying that the downstream party has private information. The upstream party deals with this information asymmetry by applying mechanism-design techniques to construct a menu of contracts, which is offered to the downstream party. Each contract specifies the procurement plan for the supply chain and a side payment. These side payments are the incentive mechanism to persuade the downstream party to accept a contract from the menu.

In this setting, there are several modelling choices to be made, in particular on the private information and on the requirements of the menu of contracts. In Chapter 1 we introduced three modelling approaches, in which the private information lies either in a finite discrete set or in a bounded interval, and in which the menu contains finitely or infinitely many contracts. Chapters 2-5 each consider one of these contracting models under various problem settings, requiring different techniques for the analysis. Below, we will discuss the main findings of these chapters.

In Chapter 2 we considered a contracting model with a supplier who must satisfy a retailer's demand in the economic order quantity setting. In this setting, both the supplier and the retailer want to minimise their own costs consisting of ordering and holding costs. The retailer has single-dimensional private information, which affects either his ordering costs or his holding costs, resulting in two considered cases. The retailer's private information is assumed to lie in a finite discrete set, i.e., there are finitely many retailer types. The number of contracts in the menu is equal to the number of retailer types.

First, we showed that both cases of private information are equivalent by rescaling the parameters and variables. Next, a substitution in the side payment variables revealed a hidden convexity of the contracting model. Consequently, the contracting problem can be solved efficiently for any number of retailer types by using numerical optimisation methods. The remainder of Chapter 2 focuses on determining structural properties of the optimal menus, leading to a sufficient condition to guarantee unique contracts in the menu. We used the analysis to derive closed-form formulas for the case with two retailer types and a minimal list of candidate menus for the case with three types.

In Chapter 3 we analysed several contracting models with a seller and a buyer, in which the seller either maximises his utility or minimises his costs. Here, the buyer's private information is single dimensional and lies in a bounded interval. In order to offer finitely many contracts, the seller pools the buyer types a priori by partitioning the type space into subintervals. The number of contracts is equal to the number of subintervals and thus controllable by the seller.

Under a condition on the buyer's utility/cost function, related to monotonicity in the buyer type, the model has a tractable reformulation with finitely many linear constraints. In addition, we derived the optimal menu for certain concave utility (or convex cost) functions. These results are all for a given partition of the type space. The next step is to also optimise this partition, either analytically or numerically. Unfortunately, our analysis suggests that it is difficult to guarantee finding the optimal partition in general, due to the possible existence of multiple local optima. Therefore, we focused on two specific problem settings: certain concave utility functions with a decreasing marginal gain in the order quantity, and convex cost functions based on the economic order quantity setting similar to Chapter 2. For these problems, we determined the optimal partition for any number of contracts. This allowed us to evaluate the performance of a simple heuristic, namely using the equidistant partition. The heuristic performs well when offering at least 5 contracts, compared to the optimal partition. Finally, we quantified the effect of pooling, i.e., of offering a limited number of contracts.

Chapter 4 expands on Chapter 3 by considering a multi-objective variant of the previous contracting model. Namely, the seller wants to balance his expected and worst-case net utility. We used a constraint-wise formulation, i.e., the seller maximises his expected net utility under the additional constraint that his worst-case net utility meets a certain threshold. By varying this threshold, the seller can quantify his trade-off between expected and worst-case net utility.

Under weaker assumptions than those in Chapter 3, we determined a tractable reformulation of the model, and derived the optimal menu of contracts for a given threshold and a given partition of the buyer types. To gain further insights into the multi-objective trade-off, we revisited the problem setting with decreasing marginal gains of Chapter 3 and generalised the associated results. In particular, we derived the optimal partition and its performance guarantees related to the effects of pooling and the threshold constraint. The results show that there is a significant trade-off to be made between the seller's expected and worst-case net utility, regardless of the number of contracts in the menu.

Finally, in Chapter 5 we switched to a contracting model where the supply chain coordination has an underlying combinatorial optimisation problem, namely the discrete lot-sizing problem. As in Chapter 2, this setting has a supplier and a retailer, both minimising their own costs consisting of ordering and holding costs. However, the problem considers discrete time periods, a finite planning horizon, and indivisible products, i.e., integer order quantities. Furthermore, the retailer's private information is single dimensional and lies in an interval. In contrast to Chapters 3 and 4, there are no restrictions on the number of contracts in the menu.

By exploiting the discrete nature of the underlying lot-sizing problem, we showed that there exists an optimal menu consisting of finitely many contracts, which will necessarily pool the retailer types. We focused on such optimal menus, derived properties of the corresponding optimal contracts, and devised a two-stage solution approach. The first stage of this approach generates a list of lot-sizing plans that is sufficient for optimality for the contracting problem. The second stage assigns these plans to the retailer types and constructs the corresponding menu of contracts by determining the side payments. We identified cases where these stages can be solved in polynomial time and provided the associated algorithms.

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Abstract

In a supply chain consisting of individualistic parties, the sharing of information is not always beneficial to each party. If a party discloses his private information it could undermine his bargaining position in the supply chain. Consequently, there is no incentive to share information, which results in information asymmetry between the involved parties. We consider a two-echelon supply chain setting viewed from the upstream party's perspective, who faces an individualistic downstream party with private information. The upstream party uses mechanism-design techniques to maximise his own benefit by designing a menu of contracts, which is offered to the downstream party. Each contract specifies the procurement plan for the supply chain and a side payment. These side payments are the incentive mechanism to persuade the downstream party to accept a contract from the menu.

We consider this principal-agent contracting problem for several utility maximisation or cost minimisation problem settings. The goal is to determine a menu of contracts that is the most beneficial to the upstream party, whilst still being acceptable for the downstream party. To achieve this goal, we analyse a variety of optimisation models, which differ in the requirements of the menu of contracts. Our analysis provides insights into modelling approaches, structural properties of optimal menus, and solution methods.

Abstract

In een productieketen met individualistische partijen is het uitwisselen van informatie niet altijd gunstig voor elke partij. Als een partij zijn private informatie vrijgeeft, dan kan dit zijn onderhandelingspositie in de productieketen ondermijnen. Dit heeft als gevolg dat er geen stimulans is om informatie te delen, resulterend in asymmetrische informatie tussen de betrokken partijen. Wij beschouwen een leverancier die producten levert aan een afnemer met private informatie. De leverancier gebruikt mechanismen met stimulansen om zijn eigen voordeel te maximaliseren door een menu van contracten te ontwerpen en aan te bieden aan de afnemer. Elk contract specificeert een leveringsplan en een bijbetaling. De bijbetalingen zorgen voor financiële stimulansen om de afnemer ervan te overtuigen akkoord te gaan met een contract uit het menu.

We beschouwen dit principaal-agent-contracteringsprobleem voor verschillende situaties. Het doel is om een menu van contracten te bepalen dat het voordeligst is voor de leverancier, maar dat nog wel door de afnemer geaccepteerd zal worden. Wij analyseren hiervoor een aantal optimalisatiemodellen die zich onderscheiden in de voorwaarden waaraan het menu moet voldoen. Onze analyse geeft inzicht in modelleringsaanpakken, oplossingsmethoden en structurele eigenschappen van de optimale menu's.

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Rutger Kerkkamp holds an MSc degree in Econometrics and Management Science from Erasmus University Rotterdam and an MSc degree in Applied Mathematics from Delft University of Technology, both with a specialisation in optimisation, operations research, and logistics. His research interests range from theoretical proofs in optimisation to applied logistical problems, and include the design and implementation of algorithms.

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In his free time he visits the university to assist his colleagues in improving their skills in time management, multitasking, and linguistics, among others.

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In a supply chain consisting of individualistic parties, the sharing of information is not always beneficial to each party. If a party discloses his private information it could undermine his bargaining position in the supply chain. Consequently, there is no incentive to share information, which results in information asymmetry between the involved parties. We consider a two-echelon supply chain setting viewed from the upstream party's perspective, who faces an individualistic downstream party with private information. The upstream party uses mechanism-design techniques to maximise his own benefit by designing a menu of contracts, which is offered to the downstream party. Each contract specifies the procurement plan for the supply chain and a side payment. These side payments are the incentive mechanism to persuade the downstream party to accept a contract from the menu.

We consider this principal-agent contracting problem for several utility maximisation or cost minimisation problem settings. The goal is to determine a menu of contracts that is the most beneficial to the upstream party, whilst still being acceptable for the downstream party. To achieve this goal, we analyse a variety of optimisation models, which differ in the requirements of the menu of contracts. Our analysis provides insights into modelling approaches, structural properties of optimal menus, and solution methods.

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