
Optimistic Concurrency Control for Distributed Unsupervised Learning

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Abstract

Research on distributed machine learning algorithms has focused primarily on one of two extremes—algorithms that obey strict concurrency constraints or algorithms that obey few or no such constraints. We consider an intermediate alternative in which algorithms optimistically assume that conflicts are unlikely and if conflicts do arise a conflict-resolution protocol is invoked. We view this “optimistic concurrency control” paradigm as particularly appropriate for large-scale machine learning algorithms, particularly in the unsupervised setting. We demonstrate our approach in three problem areas: clustering, feature learning and online facility location. We evaluate our methods via large-scale experiments in a cluster computing environment.

1 Introduction

The desire to apply machine learning to increasingly larger datasets has pushed the machine learning community to address the challenges of distributed algorithm design: partitioning and coordinating computation across the processing resources. In many cases, when computing statistics of iid data or transforming features, the computation factors according to the data and coordination is only required during aggregation. For these *embarrassingly parallel* tasks, the machine learning community has embraced the map-reduce paradigm, which provides a template for constructing distributed algorithms that are fault tolerant, scalable, and *easy to study*.

However, in pursuit of richer models, we often introduce statistical dependencies that require more sophisticated algorithms (e.g., collapsed Gibbs sampling or coordinate ascent) which were developed and studied in the *serial* setting. Because these algorithms iteratively transform a global state, parallelization can be challenging and often requires frequent and complex coordination.

Recent efforts to distribute these algorithms can be divided into two primary approaches. The **mutual exclusion** approach, adopted by [1] and [2], guarantees a *serializable* execution preserving the theoretical properties of the serial algorithm but at the expense of parallelism and costly locking overhead. Alternatively, in the **coordination-free** approach, proposed by [3] and [4], processors communicate frequently without coordination minimizing the cost of contention but leading to stochasticity, data-corruption, and requiring potentially complex analysis to prove algorithm correctness.

In this paper we explore a third approach, **optimistic concurrency control** (OCC) [5] which offers the performance gains of the coordination-free approach while at the same time ensuring a serializable execution and preserving the theoretical properties of the serial algorithm. Like the coordination-free approach, OCC exploits the infrequency of data-corrupting operations. However, instead of allowing occasional data-corruption, OCC detects data-corrupting operations and applies correcting computation. As a consequence, OCC automatically ensures correctness, and the analysis is only necessary to guarantee optimal scaling performance.

We apply OCC to distributed nonparametric unsupervised learning—including but not limited to clustering—and implement distributed versions of the DP-Means [6], BP-Means [7], and online facility location (OFL) algorithms. We demonstrate how to analyze OCC in the context of the DP-Means algorithm and evaluate the empirical scalability of the OCC approach on all three of the proposed algorithms. The primary contributions of this paper are:

1. Concurrency control approach to distributing unsupervised learning algorithms.
2. Reinterpretation of online nonparametric clustering in the form of facility location with approximation guarantees.
3. Analysis of optimistic concurrency control for unsupervised learning.
4. Application to feature modeling and clustering.

2 Optimistic Concurrency Control

Many machine learning algorithms iteratively transform some global state (e.g., model parameters or variable assignment) giving the illusion of serial dependencies between each operation. However, due to sparsity, exchangeability, and other symmetries, it is often the case that many, *but not all*, of the state-transforming operations can be computed concurrently while still preserving **serializability**: the equivalence to some serial execution where individual operations have been reordered.

This opportunity for serializable concurrency forms the foundation of distributed database systems. For example, two customers may concurrently make purchases exhausting the inventory of unrelated products, but if they try to purchase the same product then we may need to serialize their purchases to ensure sufficient inventory. One solution (*mutual exclusion*) associates locks with each product type and forces each purchase of the same product to be processed serially. This might work for an unpopular, rare product but if we are interested in selling a popular product for which we have a large inventory the serialization overhead could lead to unnecessarily slow response times. To address this problem, the database community has adopted **optimistic concurrency control** (OCC) [5] in which the system tries to satisfy the customers requests without locking and corrects transactions that could lead to negative inventory (e.g., by forcing the customer to checkout again).

Optimistic concurrency control exploits situations where most operations can execute concurrently without conflicting or violating serialization invariants. For example, given sufficient inventory the order in which customers are satisfied is immaterial and concurrent operations can be executed serially to yield the same final result. However, in the rare event that inventory is nearly depleted two concurrent purchases may not be serializable since the inventory can never be negative. By shifting the cost of concurrency control to rare events we can admit more costly concurrency control mechanisms (e.g., re-computation) in exchange for an efficient, simple, coordination-free execution for the majority of the events.

Formally, to apply OCC we must define a set of **transactions** (i.e., operations or collections of operations), a mechanism to detect when a transaction violates serialization invariants (i.e., cannot be executed concurrently), and a method to correct (e.g., rollback) transactions that violate the serialization invariants. Optimistic concurrency control is most effective when the cost of validating concurrent transactions is small and conflicts occur infrequently.

Machine learning algorithms are ideal for optimistic concurrency control. The conditional independence structure and sparsity in our models and data often leads to sparse parameter updates substantially reducing the chance of conflicts. Similarly, symmetry in our models often provides the flexibility to reorder serial operations while preserving algorithm invariants. Because the models encode the dependency structure, we can easily detect when an operation violates serial invariants and correct by rejecting the change and rerunning the computation. Alternatively, we can exploit the semantics of the operations to resolve the conflict by accepting a modified update. As a consequence OCC allows us to easily construct provably correct and efficient distributed algorithms without the need to develop new theoretical tools to analyze complex non-deterministic distributed behavior.

2.1 The OCC Pattern for Machine Learning

Optimistic concurrency control can be distilled to a simple pattern (meta-algorithm) for the design and implementation of distributed machine learning systems. We begin by evenly partitioning N data points (and the corresponding computation) across the P available processors. Each processor maintains a replicated view of the global state and *serially* applies the learning algorithm as a sequence of operations on its assigned data and *the global state*. If an operation mutates the global state in a way that preserves the serialization invariants then the operation is accepted locally and its effect on the global state, if any, is eventually replicated to other processors.

However, if an operation could potentially conflict with operations on other processors then it is sent to a unique serializing processor where it is rejected or corrected and the resulting global state change is eventually replicated to the rest of the processors. Meanwhile the originating processor either tentatively accepts the state change (if a rollback operator is defined) or proceeds as though the operation has been deferred to some point in the future.

While it is possible to execute this pattern asynchronously with minimal coordination, for simplicity we adopt the bulk-synchronous model of [8] and divide the computation into *epochs*. Within an epoch t , b data points $\mathcal{B}(p, t)$ are evenly assigned to each of the P processors. Any state changes or serialization operations are transmitted at the end of the epoch and processed before the next epoch. While potentially slower than an asynchronous execution, the bulk-synchronous execution is deterministic and can be easily expressed using existing systems like Hadoop or Spark [9].

3 OCC for Unsupervised Learning

Much of the existing literature on distributed machine learning algorithms has focused on classification and regression problems, where the underlying model is continuous. In this paper we apply the OCC pattern to machine learning problems that have a more discrete, combinatorial flavor—in particular unsupervised clustering and latent feature learning problems. These problems exhibit symmetry via their invariance to both data permutation and cluster or feature permutation. Together with the sparsity of interacting operations in their existing serial algorithms, these problems offer a unique opportunity to develop OCC algorithms.

The K-means algorithm provides a paradigm example; here the inferential goal is to partition the data. Rather than focusing solely on K-means, however, we have been inspired by recent work in which a general family of K-means-like algorithms have been obtained by taking Bayesian nonparametric (BNP) models based on combinatorial stochastic processes such as the Dirichlet process, the beta process, and hierarchical versions of these processes, and subjecting them to *small-variance asymptotics* where the posterior probability under the BNP model is transformed into a cost function that can be optimized [7]. The algorithms considered to date in this literature have been developed and analyzed in the serial setting; our goal is to explore distributed algorithms for optimizing these cost functions that preserve the structure and analysis of their serial counterparts.

3.1 OCC DP-Means

We first consider the *DP-means* algorithm (Alg. 1) introduced by [6]. Like the K-means algorithm, DP-Means alternates between updating the cluster assignment z_i for each point x_i and recomputing the centroids $\mathcal{C} = \{\mu_k\}_{k=1}^K$ associated with each clusters. However, DP-Means differs in that the number of clusters is not fixed a priori. Instead, if the distance from a given data point to all existing cluster centroids is greater than a parameter λ , then a new cluster is created. While the second phase is trivially parallel, the process of introducing clusters in the first phase is inherently serial. However, clusters tend to be introduced infrequently, and thus DP-Means provides an opportunity for OCC.

In Alg. 3 we present an OCC parallelization of the DP-Means algorithm in which each iteration of the serial DP-Means algorithm is divided into $N/(Pb)$ bulk-synchronous epochs. The data is evenly partitioned $\{x_i\}_{i \in \mathcal{B}(p,t)}$ across processor-epochs into blocks of size $b = |\mathcal{B}(p,t)|$. During each epoch t , each processor p evaluates the cluster membership of its assigned data $\{x_i\}_{i \in \mathcal{B}(p,t)}$ using the cluster centers \mathcal{C} from the previous epoch and optimistically proposes a new set of cluster centers $\hat{\mathcal{C}}$. At the end of each epoch the proposed cluster centers, $\hat{\mathcal{C}}$, are *serially* validated using Alg. 2.

Algorithm 1: Serial DP-means

Input: data $\{x_i\}_{i=1}^N$, threshold λ
 $\mathcal{C} \leftarrow \emptyset$
while *not converged* **do**
 for $i = 1$ to N **do**
 $\mu^* \leftarrow \operatorname{argmin}_{\mu \in \mathcal{C}} \|x_i - \mu\|$
 if $\|x_i - \mu^*\| > \lambda$ **then**
 $z_i \leftarrow x_i$
 $\mathcal{C} \leftarrow \mathcal{C} \cup x_i$ // New cluster
 else $z_i \leftarrow \mu^*$ // Use nearest
 for $\mu \in \mathcal{C}$ **do** // Recompute Centers
 $\mu \leftarrow \operatorname{Mean}(\{x_i \mid z_i = \mu\})$

Output: Accepted cluster centers \mathcal{C}

Algorithm 2: DPValidate

Input: Set of proposed cluster centers $\hat{\mathcal{C}}$
 $\mathcal{C} \leftarrow \emptyset$
for $x \in \hat{\mathcal{C}}$ **do**
 $\mu^* \leftarrow \operatorname{argmin}_{\mu \in \mathcal{C}} \|x - \mu\|$
 if $\|x_i - \mu^*\| < \lambda$ **then** // Reject
 $\operatorname{Ref}(x) \leftarrow \mu^*$ // Rollback Assgs
 else $\mathcal{C} \leftarrow \mathcal{C} \cup x$ // Accept

Output: Accepted cluster centers \mathcal{C}

Algorithm 3: Parallel DP-means

Input: data $\{x_i\}_{i=1}^N$, threshold λ
Input: Epoch size b and P processors
Input: Partitioning $\mathcal{B}(p, t)$ of data $\{x_i\}_{i \in \mathcal{B}(p, t)}$ to processor-epochs where $b = |\mathcal{B}(p, t)|$
 $\mathcal{C} \leftarrow \emptyset$
while *not converged* **do**
 for epoch $t = 1$ to $N/(Pb)$ **do**
 $\hat{\mathcal{C}} \leftarrow \emptyset$ // New candidate centers
 for $p \in \{1, \dots, P\}$ **do in parallel**
 // Process local data
 for $i \in \mathcal{B}(p, t)$ **do**
 $\mu^* \leftarrow \operatorname{argmin}_{\mu \in \mathcal{C}} \|x_i - \mu\|$
 // Optimistic Transaction
 if $\|x_i - \mu^*\| > \lambda$ **then**
 $z_i \leftarrow \operatorname{Ref}(x_i)$
 $\hat{\mathcal{C}} \leftarrow \hat{\mathcal{C}} \cup x_i$
 else $z_i \leftarrow \mu^*$ // Always Safe
 // Serially validate clusters
 $\mathcal{C} \leftarrow \mathcal{C} \cup \operatorname{DPValidate}(\hat{\mathcal{C}})$
 for $\mu \in \mathcal{C}$ **do** // Recompute Centers
 $\mu \leftarrow \operatorname{Mean}(\{x_i \mid z_i = \mu\})$

Output: Accepted cluster centers \mathcal{C}

Figure 1: The Serial DP-Means algorithm and distributed implementation using the OCC pattern.

The validation process accepts cluster centers that are not covered by (i.e., not within λ of) already accepted cluster centers. When a cluster center is rejected we update its reference to point to the already accepted center, thereby correcting the original point assignment.

3.2 OCC Facility Location

The DP-Means objective turns out to be equivalent to the classic Facility Location (FL) objective: $J(\mathcal{C}) = \sum_{x \in X} \min_{\mu \in \mathcal{C}} \|x - \mu\|^2 + \lambda^2 |\mathcal{C}|$, which selects the set of cluster centers (facilities) $\mu \in \mathcal{C}$ that minimizes the shortest distance $\|x - \mu\|^2$ to each point (customer) x as well as the penalized cost of the clusters $\lambda^2 |\mathcal{C}|$. However, while DP-Means allows the clusters to be arbitrary points (e.g., $\mathcal{C} \in \mathbb{R}^D$), FL constrains the clusters to be points $\mathcal{C} \subseteq \mathcal{F}$ in a set of candidate locations \mathcal{F} . Hence, we obtain a link between combinatorial Bayesian models and FL allowing us to apply algorithms with known approximation bounds to Bayesian inspired nonparametric models. As we will see in Section 4, our OCC algorithm provides constant-factor approximations for both FL and DP-means.

Facility location has been studied intensely. We build on the *online* facility location (OFL) algorithm described by Meyerson [10]. The OFL algorithm processes each data point x serially in a single pass by either adding x to the set of clusters with probability $\min(1, \min_{\mu \in \mathcal{C}} \|x - \mu\|^2 / \lambda^2)$ or assigning x to the nearest existing cluster. Using OCC we are able to construct a distributed OFL algorithm (Alg. 4) which is nearly identical to the OCC DP-Means algorithm (Alg. 3) but which provides strong approximation bounds. The OCC OFL algorithm differs only in that clusters are introduced and validated stochastically—the validation process ensures that the new clusters are accepted with probability equal to the serial algorithm.

3.3 OCC BP-Means

BP-means is an algorithm for learning collections of latent binary *features*, providing a way to define groupings of data points that need not be mutually exclusive or exhaustive like clusters.

Algorithm 4: Parallel OFL

Input: Same as DP-Means

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for epoch  $t = 1$  to  $N/(Pb)$  do  $\hat{C} \leftarrow \emptyset$ 
  for  $p \in \{1, \dots, P\}$  do in parallel
    for  $i \in \mathcal{B}(p, t)$  do
       $d \leftarrow \min_{\mu \in \mathcal{C}} \|x_i - \mu\|$ 
      with probability  $\min\{d^2, \lambda^2\} / \lambda^2$ 
       $\hat{C} \leftarrow \hat{C} \cup (x_i, d)$ 
   $\mathcal{C} \leftarrow \mathcal{C} \cup \text{OFLValidate}(\hat{C})$ 

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Output: Accepted cluster centers \mathcal{C}

Algorithm 5: OFLValidate

Input: Set of proposed cluster centers \hat{C} $\mathcal{C} \leftarrow \emptyset$

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for  $(x, d) \in \hat{C}$  do
   $d^* \leftarrow \min_{\mu \in \mathcal{C}} \|x - \mu\|$ 
  with probability  $\min\{d^{*2}, d^2\} / d^2$ 
   $\mathcal{C} \leftarrow \mathcal{C} \cup x$  // Accept

```

Output: Accepted cluster centers \mathcal{C}

Figure 2: The OCC algorithm for Online Facility Location (OFL).

As with serial DP-means, there are two phases in serial BP-means (Alg. 6). In the first phase, each data point x_i is labeled with binary assignments from a collection of features ($z_{ik} = 0$ if x_i doesn't belong to feature k ; otherwise $z_{ik} = 1$) to construct a representation $x_i \approx \sum_k z_{ik} f_k$. In the second phase, parameter values (the feature means $f_k \in \hat{C}$) are updated based on the assignments. The first step also includes the possibility of introducing an additional feature. While the second phase is trivially parallel, the inherently serial nature of the first phase combined with the infrequent introduction of new features points to the usefulness of OCC in this domain.

The OCC parallelization for BP-means follows the same basic structure as OCC DP-means. Each transaction operates on a data point x_i in two phases. In the first, analysis phase, the optimal representation $\sum_k z_{ik} f_k$ is found. If x_i is not well represented (i.e., $\|x_i - \sum_k z_{ik} f_k\| > \lambda$), the difference is proposed as a new feature in the second validation phase. At the end of epoch t , the proposed features $\{f_i^{new}\}$ are serially validated to obtain a set of accepted features \tilde{C} . For each proposed feature f_i^{new} , the validation process first finds the optimal representation $f_i^{new} \approx \sum_{f_k \in \tilde{C}} z_{ik} f_k$ using newly accepted features. If f_i^{new} is not well represented, the difference $f_i^{new} - \sum_{f_k \in \tilde{C}} z_{ik} f_k$ is added to \tilde{C} and accepted as a new feature.

Finally, to update the feature means, let F be the K -row matrix of feature means. The feature means update $F \leftarrow (Z^T Z)^{-1} Z^T X$ can be evaluated as a single transaction by computing the sums $Z^T Z = \sum_i z_i z_i^T$ (where z_i is a $K \times 1$ column vector so $z_i z_i^T$ is a $K \times K$ matrix) and $Z^T X = \sum_i z_i x_i^T$ in parallel.

We present the pseudocode for the OCC parallelization of BP-means in Appendix A.

4 Analysis of Correctness and Scalability

In contrast to the coordination-free pattern in which scalability is trivial and correctness often requires strong assumptions or holds only in expectation, the OCC pattern leads to simple proofs of correctness and challenging scalability analysis. However, in many cases it is preferable to have algorithms that are correct and probably fast rather than fast and possibly correct. We first establish serializability:

Theorem 4.1 (Serializability). *The distributed DP-means, OFL, and BP-means algorithms are serially equivalent to DP-means, OFL and BP-means, respectively.*

The proof (Appendix B) of Theorem 4.1 is relatively straightforward and is obtained by constructing a permutation function that describes an equivalent serial execution for each distributed execution. The proof can easily be extended to many other machine learning algorithms.

Serializability allows us to easily extend important theoretical properties of the serial algorithm to the distributed setting. For example, by invoking serializability, we can establish the following result for the OCC version of the online facility location (OFL) algorithm:

Theorem 4.2. *If the data is randomly ordered, then the OCC OFL algorithm provides a constant-factor approximation for the DP-means objective. If the data is adversarially ordered, then OCC OFL provides a log-factor approximation to the DP-means objective.*

The proof (Appendix B) of Theorem 4.2 is first derived in the serial setting then extended to the distributed setting through serializability. In contrast to divide-and-conquer schemes, whose approximation bounds commonly depend *multiplicatively* on the number of levels [11], Theorem 4.2 is unaffected by distributed processing and has no communication or coarsening tradeoffs. Furthermore, to retain the same factors as a batch algorithm on the full data, divide-and-conquer schemes need a large number of preliminary centers at lower levels [11, 12]. In that case, the communication cost can be high, since all proposed clusters are sent at the same time, as opposed to the OCC approach. We address the communication overhead (the number of rejections) for our scheme next.

Scalability The scalability of the OCC algorithms depends on the number of transactions that are rejected during validation (i.e., the rejection rate). While a general scalability analysis can be challenging, it is often possible to gain some insight into the asymptotic dependencies by making simplifying assumptions. In contrast to the coordination-free approach, we can still *safely* apply OCC algorithms in the absence of a scalability analysis or when simplifying assumptions do not hold.

To illustrate the techniques employed in OCC scalability analysis we study the DP-Means algorithm, whose scalability limiting factor is determined by the number of points that must be serially validated. We show that the communication cost only depends on the number of clusters and processing resources and does not directly depend on the number of data points. The proof is in Appendix C.

Theorem 4.3 (DP-Means Scalability). *Assume N data points are generated iid to form a random number (K_N) of well-spaced clusters of diameter λ : λ is an upper bound on the distances within clusters and a lower bound on the distance between clusters. Then the expected number of serially validated points is bounded above by $Pb + \mathbf{E}[K_N]$ for P processors and b points per epoch.*

Under the separation assumptions of the theorem, the number of clusters present in N data points, K_N , is exactly equal to the number of clusters found by DP-Means in N data points; call this latter quantity k_N . The experimental results in Figure 3 suggest that the bound of $Pb + k_N$ may hold more generally beyond the assumptions above. Since the master must process at least k_N points, the overhead caused by rejections is Pb and independent of N .

5 Evaluation

For our experiments, we generated synthetic data for clustering (DP-means and OFL) and feature modeling (BP-means). The cluster and feature proportions were generated nonparametrically as described below. All data points were generated in \mathbb{R}^{16} space. We fixed threshold parameter $\lambda = 1$.

Clustering: The cluster proportions and indicators were generated simultaneously using the stick-breaking procedure for Dirichlet processes—‘sticks’ are ‘broken’ on-the-fly to generate new clusters as necessary. For our experiments, we used a fixed concentration parameter $\theta = 1$. Cluster means were sampled $\mu_k \sim N(0, I_{16})$, and data points were generated at $x_i \sim N(\mu_{z_i}, \frac{1}{4}I_{16})$.

Feature modeling: We use the stick-breaking procedure of [13] to generate feature weights. Unlike with Dirichlet processes, we are unable to perform stick-breaking on-the-fly with Beta processes. Instead, we generate enough features so that with high probability (> 0.9999) the remaining non-generated features will have negligible weights (< 0.0001). The concentration parameter was also fixed at $\theta = 1$. We generated feature means $f_k \sim N(0, I_{16})$ and data points $x_i \sim N(\sum_k z_{ik} f_k, \frac{1}{4}I_{16})$.

5.1 Simulated experiments

To test the efficiency of our algorithms, we simulated the first iteration (one complete pass over all the data, where most clusters / features are created and thus greatest coordination is needed) of each algorithm in MATLAB. The number of data points, N , was varied from 256 to 2560 in intervals of 256. We also varied Pb , the number of data points processed in one epoch, from 16 to 256 in powers of 2. For each value of N and Pb , we empirically measured k_N , the number of accepted clusters /

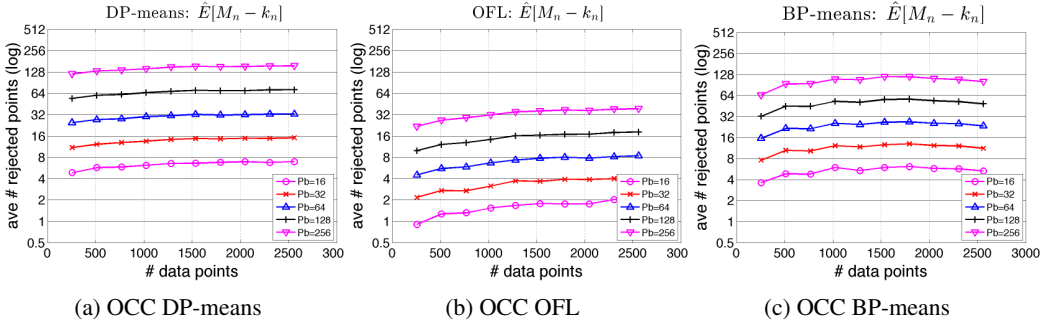


Figure 3: Simulated distributed DP-means, OFL and BP-means: expected number of data points proposed but not accepted as new clusters / features is independent of size of data set.

features, and M_N , the number of proposed clusters / features. This was repeated 400 times to obtain the empirical average $\hat{\mathbb{E}}[M_N - k_N]$ of the number of rejections.

For OCC DP-means, we observe $\hat{\mathbb{E}}[M_N - k_N]$ is bounded above by Pb (Fig. 3a), and that this bound is independent of the data set size, even when the assumptions of Thm 4.3 are violated. (We also verified that similar empirical results are obtained when the assumptions are not violated; see Appendix C.) The same behavior is observed for the other two OCC algorithms (Fig. 3b and Fig. 3c).

5.2 Distributed implementation and experiments

We also implemented¹ the distributed algorithms in Spark [9], an open-source cluster computing system. The DP-means and BP-means algorithms were initialized by pre-processing a small number of data points (1/16 of the first Pb points)—this reduces the number of data points sent to the master on the first epoch, while still preserving serializability of the algorithms. Our Spark implementations were tested on Amazon EC2 by processing a fixed data set on 1, 2, 4, 8 m2.4xlarge instances. Ideally, to process the same amount of data, an algorithm and implementation with perfect scaling would take half the runtime on 8 machines as it would on 4, and so on. The plots in Figure 4 shows this comparison by dividing all runtimes by the runtime on one machine.

DP-means: We ran the distributed DP-means algorithm on $2^{27} \approx 134\text{M}$ data points, using $\lambda = 2$. The block size b was chosen to keep $Pb = 2^{23} \approx 8\text{M}$ constant. The algorithm was run for 5 iterations (complete pass over all data in 16 epochs). We were able to get perfect scaling (Figure 4a) in all but the first iteration, when the master has to perform the most synchronization of proposed centers.

OFL: The distributed OFL algorithm was run on $2^{20} \approx 1\text{M}$ data points, using $\lambda = 2$. Unlike DP-means and BP-means, OFL is a single-pass algorithm and we did not perform any initialization clustering. The block size b was chosen such that $Pb = 2^{16} \approx 66\text{K}$ data points are processed each epoch, which gives us 16 epochs. Figure 4b shows that we get no scaling in the first epoch, where all Pb data points are sent to the master. Scaling improves in the later epochs, as the master’s workload decreases with fewer proposals but the workers’ workload increases with more centers.

BP-means: Distributed BP-means was run on $2^{23} \approx 8\text{M}$ data points, with $\lambda = 1$; block size was chosen such that $Pb = 2^{19} \approx 0.5\text{M}$ is constant. Five iterations were run, with 16 epochs per iteration. As with DP-means, we were able to achieve nearly perfect scaling; see Figure 4c.

6 Related work

Others have proposed alternatives to mutual exclusion and coordination-free parallelism for machine learning algorithm design. [14] proposed transforming the underlying model to expose additional parallelism while preserving the marginal posterior. However, such constructions can be challenging or infeasible and many hinder mixing or convergence. Likewise, [15] proposed a reparameterization of the underlying model to expose additional parallelism through conditional independence. Additional

¹Code will be made available at our project page <https://amplab.cs.berkeley.edu/projects/ccml/>.

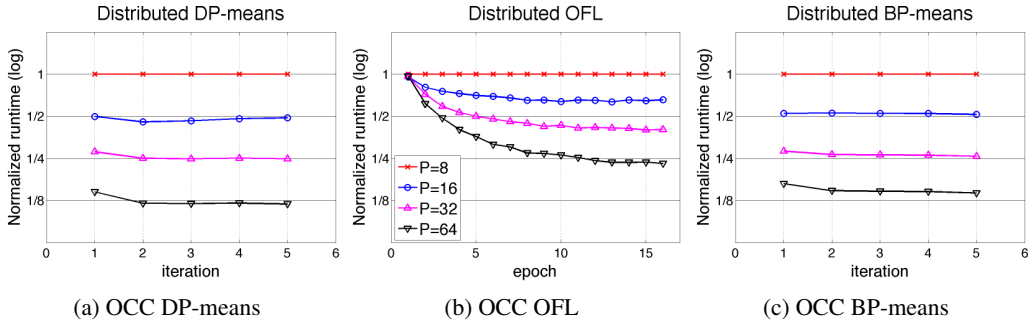


Figure 4: Normalized runtime for distributed algorithms. Runtime of each iteration / epoch is divided by that using 1 machine ($P = 8$). Ideally, the runtime with 2, 4, 8 machines ($P = 16, 32, 64$) should be respectively 1/2, 1/4, 1/8 of the runtime using 1 machine. OCC DP-means and BP-means obtain nearly perfect scaling for all iterations. OCC OFL rejects a lot initially, but quickly gets better in later epochs.

work similar in spirit to ours using OCC-like techniques includes [16] who proposed an approximate parallel sampling algorithm for the IBP which is made exact by introducing an additional Metropolis-Hastings step, and [17] who proposed a look-ahead strategy in which future samples are computed optimistically based on the likely outcomes of current samples.

There has been substantial work on scalable clustering algorithms [18, 19, 20]. Several authors [11, 21, 22, 12] have proposed streaming approximation algorithms that rely on hierarchical divide-and-conquer schemes. The approximation factors in these algorithms are multiplicative in the hierarchy and demand a careful tradeoff between communication and approximation quality which is obviated in the OCC framework. Several methods [12, 23, 21] first collect and then re-cluster a set of centers, and therefore need to communicate all intermediate centers. Our approach avoids these stages, since a center causes no rejections in the epochs after it is established: the rejection rate does not grow with K . Finally, the OCC framework can easily integrate and exploit many of the ideas in the cited works.

7 Discussion

In this paper we have shown how optimistic concurrency control can be usefully employed in the design of distributed machine learning algorithms. As opposed to previous approaches, this preserves correctness, in most cases at a small cost. We established the equivalence of our distributed OCC DP-means, OFL and BP-means algorithms to their serial counterparts, thus preserving their theoretical properties. In particular, the strong approximation guarantees of serial OFL translate immediately to the distributed algorithm. Our theoretical analysis ensures OCC DP-means achieves high parallelism without sacrificing correctness. We implemented and evaluated all three OCC algorithms on a distributed computing platform and demonstrate strong scalability in practice.

We believe that there is much more to do in this vein. Indeed, machine learning algorithms have many properties that distinguish them from classical database operations and may allow going beyond the classic formulation of OCC. In particular we may be able to partially or *probabilistically* accept non-serializable operations in a way that preserves underlying algorithm invariants. Laws of large numbers and concentration theorems may provide tools for designing such operations. Moreover, the conflict detection mechanism can be treated as a control knob, allowing us to softly switch between stable, theoretically sound algorithms and potentially faster coordination-free algorithms.

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References

- [1] J. Gonzalez, Y. Low, A. Gretton, and C. Guestrin. Parallel Gibbs sampling: From colored fields to thin junction trees. In *Proceedings of the 14th International Conference on Artificial Intelligence and Statistics (AISTATS)*, pages 324–332, 2011.
- [2] Yucheng Low, Joseph Gonzalez, Aapo Kyrola, Danny Bickson, Carlos Guestrin, and J. M. Hellerstein. Distributed GraphLab: A framework for machine learning and data mining in the cloud. In *Proceedings of the 38th International Conference on Very Large Data Bases (VLDB)*, Istanbul, 2012.
- [3] Benjamin Recht, Christopher Re, Stephen J. Wright, and Feng Niu. Hogwild: A lock-free approach to parallelizing stochastic gradient descent. In *Advances in Neural Information Processing Systems (NIPS) 24*, pages 693–701, Granada, 2011.
- [4] Amr Ahmed, Mohamed Aly, Joseph Gonzalez, Shравan Narayanamurthy, and Alexander J. Smola. Scalable inference in latent variable models. In *Proceedings of the 5th ACM International Conference on Web Search and Data Mining (WSDM)*, 2012.
- [5] Hsiang-Tsung Kung and John T Robinson. On optimistic methods for concurrency control. *ACM Transactions on Database Systems (TODS)*, 6(2):213–226, 1981.
- [6] Brian Kulis and Michael I. Jordan. Revisiting k-means: New algorithms via Bayesian nonparametrics. In *Proceedings of 29th International Conference on Machine Learning (ICML)*, Edinburgh, 2012.
- [7] Tamara Broderick, Brian Kulis, and Michael I. Jordan. MAD-bayes: MAP-based asymptotic derivations from Bayes. In *Proceedings of the 30th International Conference on Machine Learning (ICML)*, 2013.
- [8] Leslie G. Valiant. A bridging model for parallel computation. *Communications of the ACM*, 33(8):103–111, 1990.
- [9] Matei Zaharia, Mosharaf Chowdhury, Michael J Franklin, Scott Shenker, and Ion Stoica. Spark: Cluster computing with working sets. In *Proceedings of the 2nd USENIX Conference on Hot Topics in Cloud Computing*, 2010.
- [10] A. Meyerson. Online facility location. In *Proceedings of the 42nd Annual Symposium on Foundations of Computer Science (FOCS)*, Las Vegas, 2001.
- [11] A. Meyerson, N. Mishra, R. Motwani, and L. O’Callaghan. Clustering data streams: Theory and practice. *IEEE Transactions on Knowledge and Data Engineering*, 15(3):515–528, 2003.
- [12] N. Ailon, R. Jaiswal, and C. Monteleoni. Streaming k-means approximation. In *Advances in Neural Information Processing Systems (NIPS) 22*, Vancouver, 2009.
- [13] John Paisley, David Blei, and Michael I Jordan. Stick-breaking Beta processes and the Poisson process. In *Proceedings of the 15th International Conference on Artificial Intelligence and Statistics (AISTATS)*, 2012.
- [14] D. Newman, A. Asuncion, P. Smyth, and M. Welling. Distributed inference for Latent Dirichlet Allocation. In *Advances in Neural Information Processing Systems (NIPS) 20*, Vancouver, 2007.
- [15] D. Lovell, J. Malmaud, R. P. Adams, and V. K. Mansinghka. ClusterCluster: Parallel Markov chain Monte Carlo for Dirichlet process mixtures. *ArXiv e-prints*, April 2013.
- [16] F. Doshi-Velez, D. Knowles, S. Mohamed, and Z. Ghahramani. Large scale nonparametric Bayesian inference: Data parallelisation in the Indian Buffet process. In *Advances in Neural Information Processing Systems (NIPS) 22*, Vancouver, 2009.
- [17] Tianbing Xu and Alexander Ihler. Multicore Gibbs sampling in dense, unstructured graphs. In *Proceedings of the 14th International Conference on Artificial Intelligence and Statistics (AISTATS)*. 2011.
- [18] I. Dhillon and D.S. Modha. A data-clustering algorithm on distributed memory multiprocessors. In *Workshop on Large-Scale Parallel KDD Systems*, 2000.
- [19] A. Das, M. Datar, A. Garg, and S. Ragarajam. Google news personalization: Scalable online collaborative filtering. In *Proceedings of the 16th World Wide Web Conference*, Banff, 2007.
- [20] A. Ene, S. Im, and B. Moseley. Fast clustering using MapReduce. In *Proceedings of the 17th ACM SIGKDD Conference on Knowledge Discovery and Data Mining*, San Diego, 2011.
- [21] M. Shindler, A. Wong, and A. Meyerson. Fast and accurate k -means for large datasets. In *Advances in Neural Information Processing Systems (NIPS) 24*, Granada, 2011.
- [22] Moses Charikar, Liadan O’Callaghan, and Rina Panigrahy. Better streaming algorithms for clustering problems. In *Proceedings of the 35th Annual ACM Symposium on Theory of Computing (STOC)*, 2003.
- [23] B. Bahmani, B. Moseley, A. Vattani, R. Kumar, and S. Vassilvitskii. Scalable kmeans++. In *Proceedings of the 38th International Conference on Very Large Data Bases (VLDB)*, Istanbul, 2012.

A Pseudocode for OCC BP-means

Algorithm 6: Serial BP-means

Input: data $\{x_i\}_{i=1}^N$, threshold λ
Initialize $z_{i1} = 1, f_1 = N^{-1} \sum_i x_i, K = 1$
while not converged do
 for $i = 1$ **to** N **do**
 for $k = 1$ **to** K **do**
 Set z_{ik} to minimize $\|x_i - \sum_{j=1}^K z_{ij} f_j\|_2^2$
 if $\|x_i - \sum_{j=1}^K z_{ij} f_{i,j}\|_2^2 > \lambda^2$ **then**
 Set $K \leftarrow K + 1$
 Create feature $f_K \leftarrow x_i - \sum_{k=1}^K z_{ik} f_j$
 Assign $z_{iK} \leftarrow 1$ (and $z_{iK} \leftarrow 0$ for $j \neq i$)
 $F \leftarrow (Z^T Z)^{-1} Z^T X$

Algorithm 7: BPValidate

Input: Set of proposed feature centers $\hat{\mathcal{C}}$
 $\mathcal{C} \leftarrow \emptyset$
for $f^{new} \in \hat{\mathcal{C}}$ **do**
 for $f_{k'} \in \mathcal{C}$ **do**
 Set $z_{ik'}$ to minimize $\|f^{new} - \sum_{f_j \in \mathcal{C}} z_{ij} f_j\|_2^2$
 if $\|f^{new} - \sum_{f_j \in \mathcal{C}} z_{ij} f_j\|_2^2 > \lambda^2$ **then**
 $\mathcal{C} \leftarrow \mathcal{C} \cup \{f^{new} - \sum_{f_j \in \mathcal{C}} z_{ij} f_j\}$
 $\text{Ref}(f^{new}) \leftarrow \{z_{ij}\}_{f_j \in \mathcal{C}}$

Output: Accepted feature centers \mathcal{C}

Algorithm 8: Parallel BP-means

Input: data $\{x_i\}_{i=1}^N$, threshold λ
Input: Epoch size b and P processors
Input: Partitioning $\mathcal{B}(p, t)$ of data $\{x_i\}_{i \in \mathcal{B}(p, t)}$ to processor-epochs where $b = |\mathcal{B}(p, t)|$
 $\mathcal{C} \leftarrow \emptyset$
while not converged do
 for epoch $t = 1$ **to** $N/(Pb)$ **do**
 $\hat{\mathcal{C}} \leftarrow \emptyset$ // New candidate features
 for $p \in \{1, \dots, P\}$ **do in parallel**
 // Process local data
 for $i \in \mathcal{B}(p, t)$ **do**
 // Optimistic Transaction
 for $f_k \in \mathcal{C}$ **do**
 Set z_{ik} to minimize $\|x_i - \sum_j z_{ij} f_j\|_2^2$
 if $\|x_i - \sum_j z_{ij} f_j\|_2^2 > \lambda^2$ **then**
 $f_i^{new} \leftarrow x_i - \sum_j z_{ij} f_j$
 $z_i \leftarrow z_i \oplus \text{Ref}(f_i^{new})$
 $\hat{\mathcal{C}} \leftarrow \hat{\mathcal{C}} \cup f_i^{new}$
 // Serially validate features
 $\mathcal{C} \leftarrow \mathcal{C} \cup \text{DPValidate}(\hat{\mathcal{C}})$
 Compute $Z^T Z = \sum_i z_i z_i^T$ and $Z^T X = \sum_i z_i x_i^T$ in parallel
 Re-estimate features $F \leftarrow (Z^T Z)^{-1} Z^T X$

Output: Accepted feature centers \mathcal{C}

Figure 5: The Serial BP-Means algorithm and parallel implementation using the OCC pattern, similar to OCC DP-means. Instead of proposing new clusters centered at the data point x_i , in OCC BP-means we propose features f_i^{new} that allow us to obtain perfect representations of the data point. The validation process continues to improve on the representation $x_i \approx \sum_k z_{ik} f_k$ by using the most recently accepted features $f_{k'} \in \hat{\mathcal{C}}$, and only accepts a proposed feature if the data point is still not well-represented.

B Proof of serializability of distributed algorithms

B.1 Proof of Theorem 4.1 for DP-means

We note that both distributed DP-means and BP-means iterate over z -updates and cluster / feature means re-estimation until convergence. In each iteration, distributed DP-means and BP-means perform the same set of updates as their serial counterparts. Thus, it suffices to show that each iteration of the distributed algorithm is serially equivalent to an iteration of the serial algorithm.

Consider the following ordering on transactions:

- Transactions on individual data points are ordered before transactions that re-estimate cluster / feature means are ordered.
- A transaction on data point x_i is ordered before a transaction on data point x_j if
 1. x_i is processed in epoch t , x_j is processed in epoch t' , and $t < t'$
 2. x_i and x_j are processed in the same epoch, x_i and x_j are not sent to the master for validation, and $i < j$
 3. x_i and x_j are processed in the same epoch, x_i is not sent to the master for validation but x_j is

4. x_i and x_j are processed in the same epoch, x_i and x_j are sent to the master for validation, and the master serially validates x_i before x_j

We show below that the distributed algorithms are equivalent to the serial algorithms under the above ordering, by inductively demonstrating that the outputs of each transaction is the same in both the distributed and serial algorithms.

Denote the set of clusters after the t epoch as \mathcal{C}^t .

The first transaction on x_j in the serial ordering has \mathcal{C}^0 as its input. By definition of our ordering, this transaction belongs the first epoch, and is either (1) not sent to the master for validation, or (2) the first data point validated at the master. Thus in both the serial and distributed algorithms, the first transaction either (1) assigns x_j to the closest cluster in \mathcal{C}^0 if $\min_{\mu_k \in \mathcal{C}^0} \|x_j - \mu_k\| < \lambda$, or (2) creates a new cluster with center at x_j otherwise.

Now consider any other transaction on x_j in epoch t .

Case 1: x_j is not sent to the master for validation.

In the distributed algorithm, the input to the transaction is \mathcal{C}^{t-1} . Since the transaction is not sent to the master for validation, we can infer that there exists $\mu_k \in \mathcal{C}^{t-1}$ such that $\|x_j - \mu_k\| < \lambda$.

In the serial algorithm, x_j is ordered after any x_i if (1) x_i was processed in an earlier epoch, or (2) x_i was processed in the same epoch but not sent to the master (i.e. does not create any new cluster) and $i < j$. Thus, the input to this transaction is the set of clusters obtained at the end of the previous epoch, \mathcal{C}^{t-1} , and the serial algorithm assigns x_j to the closest cluster in \mathcal{C}^{t-1} (which is less than λ away).

Case 2: x_j is sent to the master for validation.

In the distributed algorithm, x_j is not within λ of any cluster center in \mathcal{C}^{t-1} . Let $\hat{\mathcal{C}}^t$ be the new clusters created at the master in epoch t before validating x_j . The distributed algorithm either (1) assigns x_j to $\mu_{k^*} = \operatorname{argmin}_{\mu_k \in \hat{\mathcal{C}}^t} \|x_j - \mu_k\|$ if $\|x_j - \mu_{k^*}\| \leq \lambda$, or (2) creates a new cluster with center at x_j otherwise.

In the serial algorithm, x_j is ordered after any x_i if (1) x_i was processed in an earlier epoch, or (2) x_i was processed in the same epoch t , but x_i was not sent to the master (i.e. does not create any new cluster), or (3) x_i was processed in the same epoch t , x_i was sent to the master, and serially validated at the master before x_j . Thus, the input to the transaction is $\mathcal{C}^{t-1} \cup \hat{\mathcal{C}}^t$. We know that x_j is not within λ of any cluster center in \mathcal{C}^{t-1} , so the outcome of the transaction is either (1) assign x_j to $\mu_{k^*} = \operatorname{argmin}_{\mu_k \in \hat{\mathcal{C}}^t} \|x_j - \mu_k\|$ if $\|x_j - \mu_{k^*}\| \leq \lambda$, or (2) create a new cluster with center at x_j otherwise. This is exactly the same as the distributed algorithm.

B.2 Proof of Theorem 4.1 for BP-means

The serial ordering for BP-means is exactly the same as that in DP-means. The proof for the serializability of BP-means follows the same argument as in the DP-means case, except that we perform feature assignments instead of cluster assignments.

B.3 Proof of Theorem 4.1 for OFL

Here we prove Theorem 4.1 that the distributed OFL algorithm is equivalent to a serial algorithm.

(Theorem 4.1, OFL). We show that with respect to the returned centers (facilities), the distributed OFL algorithm is equivalent to running the serial OFL algorithm on a particular permutation of the input data. We assume that the input data is randomly permuted and the indices i of the points x_i refer to this permutation. We assign the data points to processors by assigning the first b points to processor p_1 , the next b points to processor p_2 , and so on, cycling through the processors and assigning them batches of b points, as illustrated in Figure 6. In this respect, our ordering is generic, and can be adapted to any assignments of points to processors. We assume that each processor visits its points in

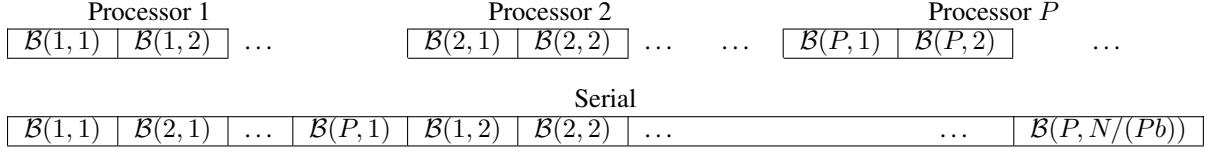


Figure 6: Illustration of distributed and serial order of blocks $\mathcal{B}(i, t)$ of length b for OFL. The order within each block is maintained. Block $\mathcal{B}(i, t)$ is processed in epoch t by processor p_i .

the order induced by the indices, and likewise the master processes the points of an epoch in that order.

For the serial algorithm, we will use the following ordering of the data: Point x_i precedes point x_j if

1. x_i is processed in epoch t and x_j is processed in epoch t' , and $t < t'$, or
2. x_i and x_j are processed in the same epoch and $i < j$.

If the data is assigned to processors as outlined above, then the serial algorithm will process the points exactly in the order induced by the indices. That means the set of points processed in any given epoch t is the same for the serial and distributed algorithm. We denote by \mathcal{C}^t the global set of validated centers collected by OCC OFL up to (including) epoch t , and by $\tilde{\mathcal{C}}^i$ the set of centers collected by the serial algorithm up to (including) point x_i .

We will prove the equivalence inductively.

Epoch $t = 1$. In the first epoch, all points are sent to the master. These are the first Pb points. Since the master processes them in the same order as the serial algorithm, the distributed and serial algorithms are equivalent.

Epoch $t > 1$. Assume that the algorithms are equivalent up to point x_{i-1} in the serial order, and point x_i is processed in epoch t . By assumption, the set \mathcal{C}^{t-1} of global facilities for the distributed algorithm is the same as the set $\tilde{\mathcal{C}}^{(t-1)Pb}$ collected by the serial algorithm up to point $x_{(t-1)Pb}$. For notational convenience, let $D(x_i, \mathcal{C}^t) = \min_{\mu \in \mathcal{C}^t} \|x_i - \mu\|^2$ be the distance of x_i to the closest global facility.

The essential issue to prove is the following claim:

Claim 1. *If the algorithms are equivalent up to point x_{i-1} , then the probability of x_i becoming a new facility is the same for the distributed and serial algorithm.*

The serial algorithm accepts x_i as a new facility with probability $\min\{1, D(x_i, \tilde{\mathcal{C}}^{i-1})/\lambda^2\}$. The distributed algorithm sends x_i to the master with probability $\min\{1, D(x_i, \mathcal{C}^{t-1})/\lambda^2\}$. The probability of ultimate acceptance (validation) of x_i as a global facility is the probability of being sent to the master *and* being accepted by the master. In epoch t , the master receives a set of candidate facilities with indices between $(t-1)Pb + 1$ and tPb . It processes them in the order of their indices, i.e., all candidates x_j with $j < i$ are processed before i . Hence, the assumed equivalence of the algorithms up to point x_{i-1} implies that, when the master processes x_i , the set $\mathcal{C}^{t-1} \cup \hat{\mathcal{C}}$ equals the set of facilities $\tilde{\mathcal{C}}^{i-1}$ of the serial algorithm. The master consolidates x_i as a global facility with probability 1 if $D(x_i, \tilde{\mathcal{C}}^{i-1} \cup \hat{\mathcal{C}}) > \lambda^2$ and with probability $D(x_i, \tilde{\mathcal{C}}^{i-1} \cup \hat{\mathcal{C}})/D(x_i, \mathcal{C}^{t-1})$ otherwise.

We now distinguish two cases. If the serial algorithm accepts x_i because $D(x_i, \tilde{\mathcal{C}}^{i-1}) \geq \lambda^2$, then for the distributed algorithm, it holds that

$$D(x_i, \mathcal{C}^{t-1}) \geq D(x_i, \mathcal{C}^{t-1} \cup \hat{\mathcal{C}}) = D(x_i, \tilde{\mathcal{C}}^{i-1}) \geq \lambda^2 \quad (1)$$

and therefore the distributed algorithm also always accepts x_i .

Otherwise, if $D(x_i, \tilde{\mathcal{C}}^{i-1}) < \lambda^2$, then the serial algorithm accepts with probability $D(x_i, \tilde{\mathcal{C}}^{i-1})/\lambda^2$. The distributed algorithm accepts with probability

$$\mathbb{P}(x_i \text{ accepted}) = \mathbb{P}(x_i \text{ sent to master}) \cdot \mathbb{P}(x_i \text{ accepted at master} \mid x_i \text{ sent}) \quad (2)$$

$$= \frac{D(x_i, \mathcal{C}^{t-1})}{\lambda^2} \cdot \frac{D(x_i, \tilde{\mathcal{C}}^{i-1} \cup \hat{\mathcal{C}})}{D(x_i, \mathcal{C}^{t-1})} \quad (3)$$

$$= \frac{D(x_i, \tilde{\mathcal{C}}^{i-1})}{\lambda^2}. \quad (4)$$

This proves the claim.

The claim implies that if the algorithms are equivalent up to point x_{i-1} , then they are also equivalent up to point x_i . This proves the theorem. \square

B.4 Proof of Theorem 4.2 (Approximation bound)

We begin by relating the results of facility location algorithms and DP-means. Recall that the objective of DP-means and FL is

$$J(\mathcal{C}) = \sum_{x \in X} \min_{\mu \in \mathcal{C}} \|x - \mu\|^2 + \lambda^2 |\mathcal{C}|. \quad (5)$$

In FL, the facilities may only be chosen from a pre-fixed set of centers (e.g., the set of all data points), whereas DP-means allows the centers to be arbitrary, and therefore be the empirical mean of the points in a given cluster. However, choosing centers from among the data points still gives a factor-2 approximation. Once we have established the corresponding clusters, shifting the means to the empirical cluster centers never hurts the objective. The following proposition has been a useful tool in analyzing clustering algorithms:

Proposition B.1. *Let \mathcal{C}^* be an optimal solution to the DP-means problem (5), and let \mathcal{C}^{FL} be an optimal solution to the corresponding FL problem, where the centers are chosen from the data points. Then*

$$J(\mathcal{C}^{FL}) \leq 2J(\mathcal{C}^*).$$

Proof. (Proposition B.1) It is folklore that Proposition B.1) holds for the K-means objective, i.e.,

$$\min_{\mathcal{C} \subseteq X, |\mathcal{C}|=k} \sum_{i=1}^n \min_{\mu \in \mathcal{C}} \|x_i - \mu\|^2 \leq 2 \min_{\mathcal{C} \subseteq X} \sum_{i=1}^n \min_{\mu \in \mathcal{C}} \|x_i - \mu\|^2. \quad (6)$$

In particular, this holds for the optimal number $K^* = \lfloor \mathcal{C}^* \rfloor$. Hence, it holds that

$$J(\mathcal{C}^{FL}) \leq \min_{\mathcal{C} \subseteq X, |\mathcal{C}|=K^*} \sum_{i=1}^n \min_{\mu \in \mathcal{C}} \|x_i - \mu\|^2 + \lambda^2 K^* \leq 2J(\mathcal{C}^*). \quad (7)$$

\square

With this proposition at hand, all that remains is to prove an approximation factor for the FL problem.

Proof. (Theorem 4.2) First, we observe that the proof of Theorem 4.1 implies that, for any random order of the data, the OCC and serial algorithm process the data in exactly the same way, performing ultimately exactly the same operations. Therefore, any approximation factor that holds for the serial algorithm straightforwardly holds for the OCC algorithm too.

Hence, it remains to prove the approximation factor of the serial algorithm. Let $\mathcal{C}_1^{FL}, \dots, \mathcal{C}_k^{FL}$ be the clusters in an optimal solution to the FL problem, with centers $\mu_1^{FL}, \dots, \mu_k^{FL}$. We analyze each optimal cluster individually. The proof follows along the lines of the proofs of Theorems 2.1 and 4.2 in [10], adapting it to non-metric squared distances. We show the proof for the constant factor, the logarithmic factor follows analogously by using the ring-splitting as in [10].

First, we see that the expected total cost of any point x is bounded by the distance to the closest open facility y that is present when x arrives. If we always count in the distance of $\|x - y\|^2$ into the cost of x , then the expected cost is $\gamma(x) = \lambda^2 \|x - y\|^2 / \lambda^2 + \|x - y\|^2 = 2\|x - y\|^2$.

We consider an arbitrary cluster C_i^* and divide it into $|C_i^*|/2$ good points and $|C_i^*|/2$ bad points. Let $D_i = \frac{1}{|C_i^*|} \sum_{x \in C_i^*} \|x - \mu_i\|$ be the average service cost of the cluster, and let d_g and d_b be the service cost of the good and bad points, respectively (i.e., $D_i = (d_g + d_b)/|C_i^*|$). The good points satisfy $\|x - \mu_i^{\text{FL}}\| \leq 2D_i$. Suppose the algorithm has chosen a center, say y , from the points C_i^{FL} . Then any other point $x \in C_i^{\text{FL}}$ can be served at cost at most

$$\|x - y\|^2 \leq \left(\|x - \mu_i^{\text{FL}}\| + \|y - \mu_i^{\text{FL}}\| \right)^2 \leq 2\|x - \mu_i^{\text{FL}}\|^2 + 4D_i. \quad (8)$$

That means once the algorithm has established a good center within C_i^{FL} , all other good points together may be serviced within a constant factor of the total optimal service cost of C_i^{FL} , i.e., at $2d_g + 4(d_g + d_b)$. The assignment cost of all the good points in C_i^{FL} that are passed before opening a good facility is, by construction of the algorithm and expected waiting times, in expectation λ^2 . Hence, in expectation, the cost of the good points in C_i^{FL} will be bounded by $\sum_{x \text{ good}} \gamma(x) \leq 2(2d_g + 4d_g + 4d_b + \lambda^2)$.

Next, we bound the expected cost of the bad points. We may assume that the bad points are injected randomly in between the good points, and bound the servicing cost of a bad point $x_b \in C_i^{\text{FL}}$ in terms of the closest good point $x_g \in C_i^{\text{FL}}$ preceding it in our data sequence. Let y be the closest open facility to μ_i^{FL} when y arrives. Then

$$\|x_b - y\|^2 \leq 2\|y - \mu_i^{\text{FL}}\|^2 + 2\|x_b - \mu_i^{\text{FL}}\|^2. \quad (9)$$

Now assume that x_g was assigned to y' . Then

$$\|y - \mu_i^{\text{FL}}\|^2 \leq \|y' - \mu_i^{\text{FL}}\|^2 \leq 2\|y' - x_g\|^2 + 2\|x_g - \mu_i^{\text{FL}}\|^2. \quad (10)$$

From (9) and (8), it then follows that

$$\|x_b - y\|^2 \leq 4\|y' - x_g\|^2 + 4\|x_g - \mu_i^{\text{FL}}\|^2 + 2\|x_b - \mu_i^{\text{FL}}\|^2 \quad (11)$$

$$= 2\gamma(x_g) + 4\|x_g - \mu_i^{\text{FL}}\|^2 + 2\|x_b - \mu_i^{\text{FL}}\|^2. \quad (12)$$

Since the data is randomly permuted, x_g could be, with equal probability, any good point, and in expectation we will average over all good points.

Finally, with probability $2/|C_i^{\text{FL}}|$ there is no good point before x_g . In that case, we will count in x_b as the most costly case of opening a new facility, incurring cost λ^2 . In summary, we can bound the expected total cost of C_i^{FL} by

$$\sum_{x \text{ good}} \gamma(x) + \sum_{x \text{ bad}} \gamma(x) \leq 12d_g + 8d_b + \lambda^2 + \frac{2C_i^{\text{FL}}}{2C_i^{\text{FL}}} \lambda^2 + 2\left(2\frac{|C_i^{\text{FL}}|}{2|C_i^{\text{FL}}|}\right) (12d_g + 8d_b + \lambda^2) + 4d_g + 2d_b \quad (13)$$

$$\leq 68d_g + 42d_b + 4\lambda^2 \leq 68J(C_i^{\text{FL}}). \quad (14)$$

This result together with Proposition B.1 proves the theorem. \square

C Master processing bound for DP-means (Theorem 4.3)

We restate Theorem 4.3 here for convenience.

DP-Means Scalability. Assume N data points are generated iid to form a random number (K_N) of well-spaced clusters of diameter λ : λ is an upper bound on the distances within clusters and a lower bound on the distance between clusters. Then the expected number of *serially* validated points is bounded above by $Pb + \mathbf{E}[K_N]$ for P processors and b points per epoch.

Proof. As in the theorem statement, we assume P processors, b points assigned to each processor per epoch, and N total data points. We further assume a generative model for the cluster memberships: namely, that they are generated iid from an arbitrary distribution $(\pi_j)_{j=1}^{\infty}$. That is, we have $\sum_{j=1}^{\infty} \pi_j = 1$ and, for each j , $\pi_j \in [0, 1]$. We see that there are perhaps infinitely many latent clusters. Nonetheless, in any data set of finite size N , there will of course be only finitely many clusters to which any data point in the set belongs. Call the number of such clusters K_N .

Consider any particular cluster indexed by j . At the end of the first epoch in which a worker sees j , that worker (and perhaps other workers) will send some data point from j to the master. By construction, some data point from j will belong to the collection of cluster centers at the master by the end of the processing done at the master and therefore by the beginning of the next epoch. It follows from our assumption (all data points within a single cluster are within a λ diameter) that no other data point from cluster j will be sent to the master in future epochs. It follows from our assumption about the separation of clusters that no points in other clusters will be covered by any data point from cluster j .

Let S_j represent the (random) number of points from cluster j sent to the master. Since there are Pb points processed by workers in a single epoch, N_j is constrained to take values between 0 and Pb . Further, note that there are a total of $N/(Pb)$ epochs.

Let $A_{j,s,t}$ be the event that the master is sent s data points from cluster j in epoch t . All of the events $\{A_{j,s,t}\}$ with $s = 1, \dots, Pb$ and $t = 1, \dots, N/(Pb)$ are disjoint. Define $A'_{j,0}$ to be the event that, for all epochs $t = 1, \dots, N/(Pb)$, zero data points are sent to the master; i.e., $A'_{j,0} := \bigcup_t A_{j,0,t}$. Then $A'_{j,0}$ is also disjoint from the events $\{A_{j,s,t}\}$ with $s = 1, \dots, Pb$ and $t = 1, \dots, N/(Pb)$. Finally,

$$A'_{j,0} \cup \bigcup_{s=1}^{Pb} \bigcup_{t=1}^{N/(Pb)} A_{j,s,t}$$

covers all possible data configurations. It follows that

$$\mathbb{E}[S_j] = 0 * \mathbb{P}[A'_{j,0}] + \sum_{s=1}^{Pb} \sum_{t=1}^{N/(Pb)} s \mathbb{P}[A_{j,s,t}] = \sum_{s=1}^{Pb} \sum_{t=1}^{N/(Pb)} s \mathbb{P}[A_{j,s,t}]$$

Note that, for s points from cluster j to be sent to the master at epoch t , it must be the case that no points from cluster j were seen by workers during epochs $1, \dots, t-1$, and then s points were seen in epoch t . That is, $\mathbb{P}[A_{j,s,t}] = (1 - \pi_j)^{Pb(t-1)} \cdot \binom{Pb}{s} \pi_j^s (1 - \pi_j)^{Pb-s}$.

Then

$$\begin{aligned} \mathbb{E}[S_j] &= \left(\sum_{s=1}^{Pb} s \binom{Pb}{s} \pi_j^s (1 - \pi_j)^{Pb-s} \right) \cdot \left(\sum_{t=1}^{N/(Pb)} (1 - \pi_j)^{Pb(t-1)} \right) \\ &= \pi_j Pb \cdot \frac{1 - (1 - \pi_j)^{Pb \cdot N/(Pb)}}{1 - (1 - \pi_j)^{Pb}}, \end{aligned}$$

where the last line uses the known, respective forms of the expectation of a binomial random variable and of the sum of a geometric series.

To proceed, we make use of a lemma.

Lemma C.1. *Let m be a positive integer and $\pi \in (0, 1]$. Then*

$$\frac{1}{1 - (1 - \pi)^m} \leq \frac{1}{m\pi} + 1.$$

Proof. A particular subcase of Bernoulli's inequality tells us that, for integer $l \leq 0$ and real $x \geq -1$, we have $(1 + x)^l \geq 1 + lx$. Choose $l = -m$ and $x = -\pi$. Then

$$\begin{aligned} (1 - \pi)^m &\leq \frac{1}{1 + m\pi} \\ \Leftrightarrow 1 - (1 - \pi)^m &\geq 1 - \frac{1}{1 + m\pi} = \frac{m\pi}{1 + m\pi} \\ \Leftrightarrow \frac{1}{1 - (1 - \pi)^m} &\leq \frac{m\pi + 1}{m\pi} = \frac{1}{m\pi} + 1. \end{aligned}$$

□

We can use the lemma to find the expected total number of data points sent to the master:

$$\begin{aligned}
\mathbb{E} \sum_{j=1}^{\infty} S_j &= \sum_{j=1}^{\infty} \mathbb{E} S_j = \sum_{j=1}^{\infty} \pi_j P b \cdot \frac{1 - (1 - \pi_j)^N}{1 - (1 - \pi_j)^{P b}} \\
&\leq \sum_{j=1}^{\infty} \pi_j P b \cdot \left(1 + \frac{1}{\pi_j P b}\right) \cdot (1 - (1 - \pi_j)^N) \\
&= P b \sum_{j=1}^{\infty} \pi_j (1 - (1 - \pi_j)^N) + \sum_{j=1}^{\infty} (1 - (1 - \pi_j)^N) \\
&\leq P b + \sum_{j=1}^{\infty} \mathbb{P}(\text{cluster } j \text{ occurs in the first } N \text{ points}) \\
&= P b + \mathbb{E}[K_N].
\end{aligned}$$

Conversely,

$$\mathbb{E} \sum_{j=1}^{\infty} S_j \geq \sum_{j=1}^{\infty} \pi_j P b = P b.$$

□

To analyze the total running time, we note that after each of the $N/(Pb)$ epochs the master and workers must communicate. Each worker must process N/P data points, and the master sees at most $k_N + Pb$ points. Thus, the total expected running time is $O(N/(Pb) + N/P + Pb)$.

C.1 Experiment – under assumptions of theorem

To demonstrate the bound on the expected number of data points proposed but not accepted as new centers, we generated synthetic data with separable clusters. Cluster proportions are generated using the stick-breaking procedure for the Dirichlet process, with concentration parameter $\theta = 1$. Cluster means are set at $\mu_k = (2k, 0, 0, \dots, 0)$, and generated data uniformly in a ball of radius $1/2$ around each center. Thus, all data points from the same cluster are at most distance 1 from one another, and more than distance of 1 from any data point from a different cluster.

We follow the same experimental framework in Section 5.1.

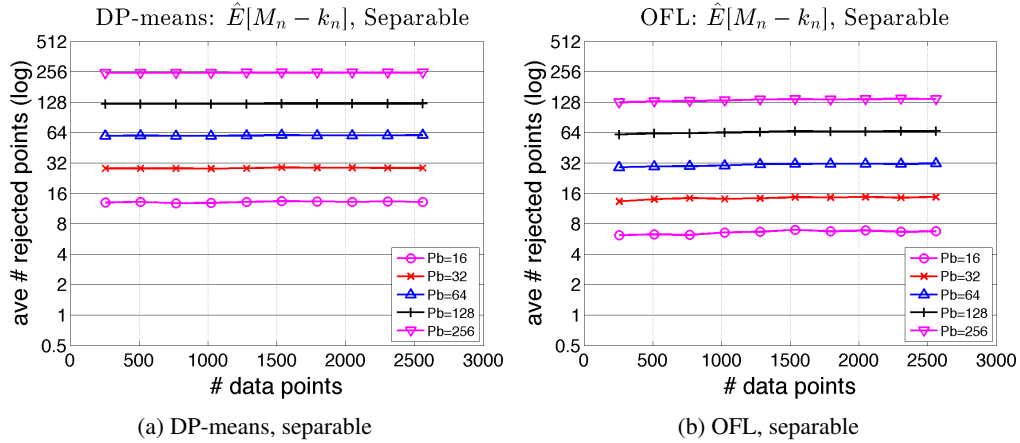


Figure 7: Simulated distributed DP-means and OFL: expected number of data points proposed but not accepted as new clusters is independent of size of data set.

In the case where we have separable clusters (Figure 7), $\hat{\mathbb{E}}[M_N - k_N]$ is bounded from above by Pb , which is in line with the above Theorem 4.3.

C.2 Experiment – assumptions of theorem violated

We further examine the rejection rate of distributed DP-means when the assumptions of the theorem are violated. Specifically, we vary the separation between cluster centers, and the algorithm choice of λ relative to the cluster diameter.

Data is synthetically generated using the stick-breaking process with $\theta = 1$, truncated to a maximum of 16 clusters in 16 dimensions. Cluster centers are chosen to keep all pairwise distances at a fixed separation. Data points are generated in balls of radius 1 around the cluster centers.

Separation between cluster centers is varied between 0 (completely overlapping), 1 (partially overlapping), and 2 (almost separated). The choice of λ is varied between 0.5 (smaller than actual radius), 1, 2, and 4. Note that data points from the same cluster are within a distance of 2 from one another, and all data points are within a distance of 4 from one another.

Figure 8 shows the number of cluster centers that are rejected by distributed DP-means. We observe that the bounds of the theorem hold even when the assumptions are violated.

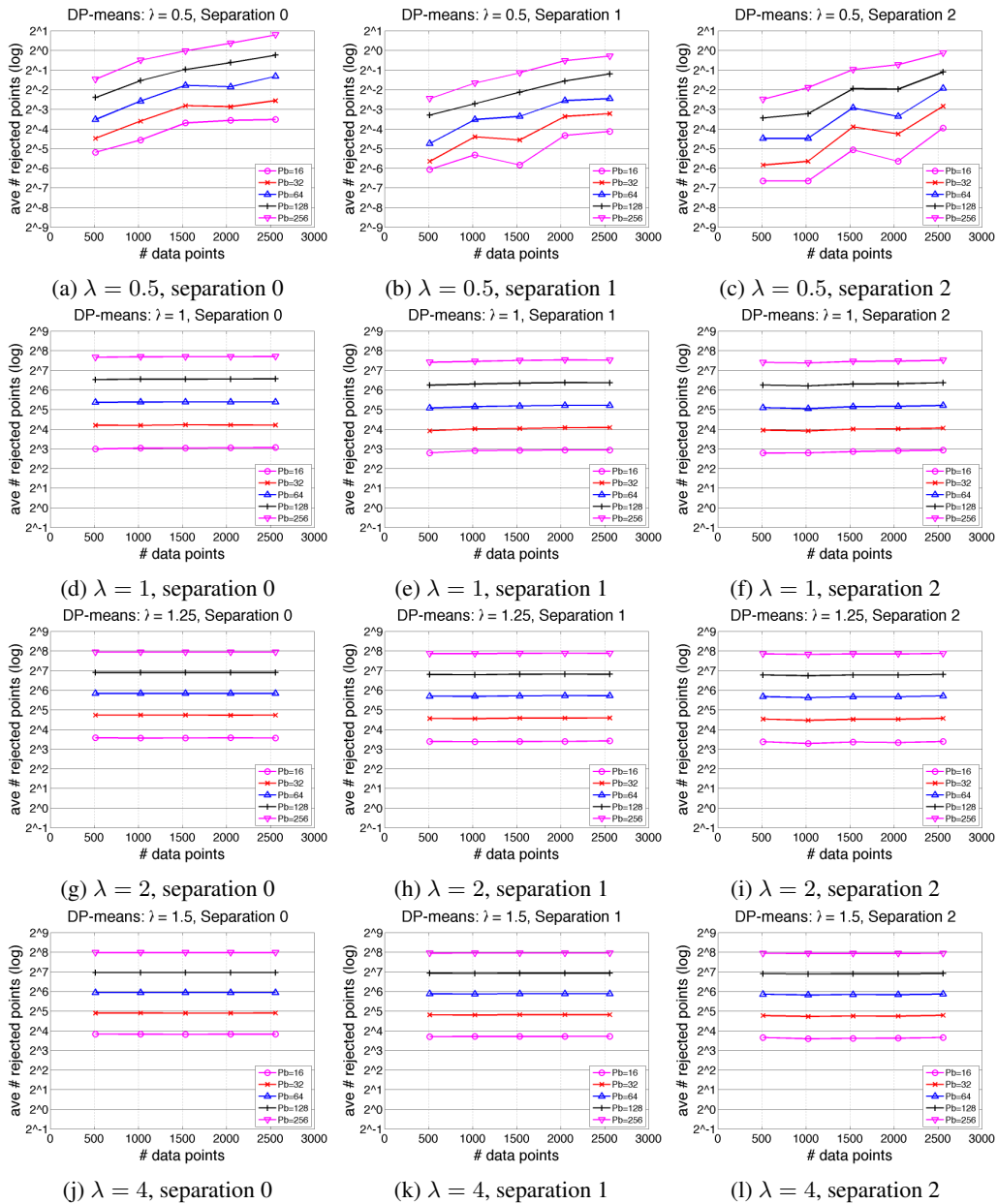


Figure 8: Number of proposed but rejected cluster centers, for various choices of λ and underlying separation between cluster centers. Note that the y-axis for $\lambda = 0.5$ is different than for other choices of λ .