OPTIMISTIC LIMITS OF THE COLORED JONES POLYNOMIALS

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ABSTRACT. We show that the optimistic limits of the colored Jones polynomials of the hyperbolic knots coincide with the optimistic limits of the Kashaev invariants modulo $4\pi^2$.

1. Introduction

1.1. Preliminaries

Kashaev conjectured the following relation in [5]:

$$\text{vol}(L) = 2\pi \lim_{N \to \infty} \frac{\log |\langle L \rangle_N|}{N},$$

where L is a hyperbolic link, $\text{vol}(L)$ is the hyperbolic volume of $S^3 - L$, and $\langle L \rangle_N$ is the N-th Kashaev invariant. After, a generalized conjecture was proposed in [12] that

$$i(\text{vol}(L) + i \, \text{cs}(L)) \equiv 2\pi i \lim_{N \to \infty} \frac{\log \langle L \rangle_N}{N} \pmod{\pi^2},$$

where $\text{cs}(L)$ is the Chern-Simons invariant of $S^3 - L$ defined in [7].

The calculation of the actual limit of the Kashaev invariant is very hard, and only several cases are known. On the other hand, while proposing the conjecture, Kashaev used a formal approximation to predict the actual limit. His formal approximation was formulated as *optimistic limit* by H. Murakami

in [9]. This method can be summarized in the following way. First, we fix an expression of $\langle L \rangle_N$, then apply the following *formal substitution*

$$(1) \qquad (q)_k \sim \exp\left\{\frac{N}{2\pi i}\left(-\mathrm{Li}_2(q^k) + \frac{\pi^2}{6}\right)\right\},$$

$$(q^{-1})_k \sim \exp\left\{\frac{N}{2\pi i}\left(\mathrm{Li}_2(q^{-k}) - \frac{\pi^2}{6}\right)\right\},$$

$$q^{kl} \sim \exp\left\{\frac{N}{2\pi i}\left(\log q^k \cdot \log q^l\right)\right\},$$

to the expression, where $q = \exp(2\pi i/N)$, $\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-t)}{t}dt$ for $z \in \mathbb{C}$, $[k]$ is the residue of an integer k modulo N, $(q)_k = \prod_{n=1}^{[k]}(1-q^n)$ and $(q)_0 = 1$. Then by substituting each q^k with a complex variable z, we obtain a potential function $\exp\left\{\frac{N}{2\pi i}F(\ldots, z, \ldots)\right\}$. Finally, let

$$F_0(\ldots, z, \ldots) := F - \sum_z \left(z\frac{\partial F}{\partial z}\right)\log z$$

and evaluate F_0 for an *appropriate solution* of the equations $\left\{\exp\left(z\frac{\partial F}{\partial z}\right) = 1\right\}$. Then the resulting complex number is called *the optimistic limit*.

For example, the optimistic limit of the Kashaev invariant of the 5_2 knot was calculated in [5] and [13] as follows. By the formal substitution,

$$\langle 5_2 \rangle_N = \sum_{k \leq l} \frac{(q)_l^2}{(q^{-1})_k} q^{-k(l+1)}$$

$$\sim \exp\left\{\frac{N}{2\pi i}\left(-2\mathrm{Li}_2(q^l) - \mathrm{Li}_2(\frac{1}{q^k}) - \log q^l \log q^k + \frac{\pi^2}{2}\right)\right\}.$$

By substituting $z = q^l$ and $u = q^k$, we obtain

$$F(z, u) = -2\mathrm{Li}_2(z) - \mathrm{Li}_2(\frac{1}{u}) - \log z \log u + \frac{\pi^2}{2},$$

and

$$F_0(z, u) = F(z, u) - \left(z\frac{\partial F}{\partial z}\right)\log z - \left(u\frac{\partial F}{\partial u}\right)\log u.$$

For the choice of a solution $(z_0, u_0) = (0.3376\cdots - i\,0.5623\cdots, 0.1226\cdots + i\,0.7449\cdots)$ of the equations $\left\{\exp\left(z\frac{\partial F}{\partial z}\right) = 1,\ \exp\left(u\frac{\partial F}{\partial u}\right) = 1\right\}$, the optimistic limit becomes

$$F_0(z_0, u_0) = i\,(2.8281\cdots - i\,3.0241\cdots) \equiv i(\mathrm{vol}(5_2) + i\,\mathrm{cs}(5_2)) \pmod{\pi^2}.$$

As seen above, the optimistic limit depends on the expression and the choice of the solution, so it is not well-defined. However, Yokota made a very useful way to determine the optimistic limit of a hyperbolic knot K in [17] and [18] by defining a potential function $V(z_1, \ldots, z_g)$ of the knot diagram, which also

comes from the formal substitution of certain expression of the Kashaev invariant $\langle K\rangle_N$ (the definition of $V(z_1,\ldots,z_g)$ will be given in Section 3.1). As above, he also defined

$$V_0(z_1,\ldots,z_g) := V - \sum_{k=1}^{g}\left(z_k\frac{\partial V}{\partial z_k}\right)\log z_k$$

and

$$\mathcal{H}_1 := \left\{\exp(z_k\frac{\partial V}{\partial z_k}) = 1 \mid k = 1,\ldots,g\right\}.$$

After proving that $\mathcal{H}_1$ is the hyperbolicity equation of Yokota triangulation, he chose the geometric solution $\mathbf{z}^{(0)} = (z_1^{(0)},\ldots,z_g^{(0)})$ of $\mathcal{H}$ (Yokota triangulation will be discussed in Section 2.1. The hyperbolicity equation consists of edge relations and the cusp conditions of a triangulation, and the geometric solution is the one which gives the hyperbolic structure of the triangulation. Details are in Section 4). Then he proved

(2) $$V_0(\mathbf{z}^{(0)}) \equiv i(\mathrm{vol}(K) + i\,\mathrm{cs}(K))\ (\mathrm{mod}\ \pi^2)$$

in [18]. Therefore, we denote

$$2\pi i \underset{N\to\infty}{\mathrm{o\text{-}lim}}\frac{\log\langle K\rangle_N}{N} := V_0(\mathbf{z}^{(0)})$$

and call it *the optimistic limit of the Kashaev invariant* $\langle K\rangle_N$.

To obtain (2), Yokota assumed several assumptions on the knot diagram and the existence of an essential solution of $\mathcal{H}_1$. The assumptions on the diagram essentially reduce redundant crossings of the diagram before finding the potential function V. Exact statements are **Assumption 1.1–1.4.** and **Assumption 2.2.** in [18]. We remark that these assumptions are needed so that, after the collapsing process, Yokota triangulation becomes a topological triangulation of the knot complement $S^3 - K$ (see Section 3.1 for details).

As mentioned before, the set of equations $\mathcal{H}_1$ becomes the hyperbolicity equation of Yokota triangulation. Therefore, each solution $\mathbf{z} = (z_1,\ldots,z_g)$ of $\mathcal{H}_1$ determines the shape parameters of the ideal tetrahedra of the triangulation and the parameters are expressed by the ratios of $z_1,\ldots,z_g$ (details are in Section 4). We call a solution $\mathbf{z}$ of $\mathcal{H}_1$ *essential* if no shape parameters are in $\{0,1,\infty\}$, which implies no edges of the triangulation are homotopically nontrivial. A well-known fact is that if the hyperbolicity equation has an essential solution, then there is a unique geometric solution $\mathbf{z}^{(0)}$ of $\mathcal{H}_1$ (for details, see [16, Section 2.8]). Therefore, to guarantee the existence of the geometric solution, Yokota assumed the existence of an essential solution.

On the other hand, it is proved in [11] that

$$J_L(N;\exp\frac{2\pi i}{N}) = \langle L\rangle_N,$$

where $J_L(N;x)$ is the N-th colored Jones polynomial of the link L with a complex variable x. Therefore, it is natural to define the optimistic limit of

the colored Jones polynomial so that it gives the volume and the Chern-Simons invariant. Although it looks trivial, due to the ambiguity of the optimistic limit, only few results are known. It was numerically confirmed for few examples in [12], actually proved only for the volume part of two bridge links in [13] and for the Chern-Simons part of twist knots in [2]. In a nutshell, the purpose of this paper is to propose a general method to define the optimistic limit of the colored Jones polynomial of a hyperbolic knot K and to prove the following relation:

$$(3) \qquad 2\pi i \operatorname*{o-lim}_{N\to\infty} \frac{\log\langle K\rangle_N}{N} \equiv 2\pi i \operatorname*{o-lim}_{N\to\infty} \frac{\log J_K(N; \exp\frac{2\pi i}{N})}{N} \pmod{4\pi^2}.$$

1.2. Main result

For a hyperbolic knot K, we define a potential function $W(w_1, \dots, w_m)$ of a knot diagram in Section 3.2, which also comes from the formal substitution of certain expression of the colored Jones polynomial $J_L(N; \exp\frac{2\pi i}{N})$. We define

$$W_0(w_1, \dots, w_m) := W - \sum_{l=1}^{m} \left(w_l \frac{\partial W}{\partial w_l} \right) \log w_l$$

and

$$\mathcal{H}_2 := \left\{ \exp\left(w_l \frac{\partial W}{\partial w_l} \right) = 1 \mid l = 1, \dots, m \right\}.$$

Also, we discuss Thurston triangulation of the knot complement $S^3 - K$ in Section 2.2, which was introduced in [14].

Proposition 1.1. *For a hyperbolic knot K with a fixed diagram, we assume the diagram satisfies **Assumption 1.1.–1.4.** and **Assumption 2.2.** in [18]. For the potential function $W(w_1, \dots, w_m)$ of the diagram, $\mathcal{H}_2$ becomes the hyperbolicity equation of Thurston triangulation.*

Proof of Proposition 1.1 will be given in Section 4.

Each solution $\mathbf{w} = (w_1, \dots, w_m)$ of $\mathcal{H}_2$ determines the shape parameters of the ideal tetrahedra of Thurston triangulation and the parameters are expressed by the ratios of $w_1, \dots, w_m$ (details are in Section 4). We call a solution $\mathbf{w}$ of $\mathcal{H}_2$ *essential* if no shape parameters are in $\{0, 1, \infty\}$. Comparing Yokota triangulation and Thurston triangulation, we obtain the following lemma.

Lemma 1.2. *For a hyperbolic knot K with a fixed diagram and the assumptions of Proposition 1.1, an essential solution $\mathbf{z} = (z_1, \dots, z_g)$ of $\mathcal{H}_1$ determines the unique solution $\mathbf{w} = (w_1, \dots, w_m)$ of $\mathcal{H}_2$, and vice versa. Furthermore, if the determined solution $\mathbf{w}$ is essential, then $\mathbf{w}$ also induces $\mathbf{z}$, and vice versa.*

Proof of Lemma 1.2 will be given in Section 5. Although there is a possibility that an essential solution $\mathbf{z}$ of $\mathcal{H}_1$ determines a non-essential solution $\mathbf{w}$ of $\mathcal{H}_2$, we expect this not to happen in almost all cases (this is discussed in Appendix A.2). In this paper, we only consider the case when the determined solution $\mathbf{w}$ is essential.

Theorem 1.3. *For a hyperbolic knot K with a fixed diagram, assume the assumptions of Proposition 1.1. Let $V(z_1, \ldots, z_g)$ and $W(w_1, \ldots, w_m)$ be the potential functions of the knot diagram. Also assume the hyperbolicity equation $\mathcal{H}_1$ has an essential solution $\mathbf{z} = (z_1, \ldots, z_g)$ and let $\mathbf{z}^{(0)} = (z_1^{(0)}, \ldots, z_g^{(0)})$ be the geometric solution of $\mathcal{H}_1$. From Lemma 1.2, let $\mathbf{w} = (w_1, \ldots, w_g)$ and $\mathbf{w}^{(0)} = (w_1^{(0)}, \ldots, w_m^{(0)})$ be the corresponding solutions of $\mathcal{H}_2$ determined by $\mathbf{z}$ and by $\mathbf{z}^{(0)}$, respectively. We also assume $\mathbf{w}$ and $\mathbf{w}^{(0)}$ are essential. Then*

(1) *$V_0(\mathbf{z}) \equiv W_0(\mathbf{w}) \pmod{4\pi^2}$,*
(2) *$\mathbf{w}^{(0)}$ is the geometric solution of $\mathcal{H}_2$ and*

$$W_0(\mathbf{w}^{(0)}) \equiv i(\mathrm{vol}(K) + i\,\mathrm{cs}(K)) \pmod{\pi^2}.$$

The proof is in Section 5. We denote

$$2\pi i \underset{N \to \infty}{\text{o-lim}} \frac{\log J_K(N; \exp \frac{2\pi i}{N})}{N} := W_0(\mathbf{w}^{(0)})$$

and call it *the optimistic limit of the colored Jones polynomial $J_K(N; \exp \frac{2\pi i}{N})$*. With this definition, Theorem 1.3 implies (3). Also, we obtain the colored Jones polynomial version of [1, Corollary 1.4] as follows.

Corollary 1.4. *For a hyperbolic knot K with a fixed diagram, assume the assumptions of Proposition 1.1. Let $\mathbf{w}$ be an essential solution of $\mathcal{H}_2$, $\mathbf{w}^{(0)}$ be the geometric solution of $\mathcal{H}_2$, and $\rho_{\mathbf{w}} : \pi_1(S^3 - K) \to \mathrm{PSL}(2, \mathbb{C})$ be the parabolic representation induced by $\mathbf{w}$. Also, assume the corresponding solutions $\mathbf{z}$ and $\mathbf{z}^{(0)}$ of $\mathcal{H}_1$ determined by $\mathbf{w}$ and by $\mathbf{w}^{(0)}$, respectively, from Lemma 1.2 are essential. Then*

$$W_0(\mathbf{w}) \equiv i\left(\mathrm{vol}(\rho_{\mathbf{w}}) + i\,\mathrm{cs}(\rho_{\mathbf{w}})\right) \pmod{\pi^2},$$

where $\mathrm{vol}(\rho_{\mathbf{w}}) + i\,\mathrm{cs}(\rho_{\mathbf{w}})$ is the complex volume of $\rho_{\mathbf{w}}$ defined in [19]. Furthermore, the following inequality holds:

(4) $$\mathrm{Im}\, W_0(\mathbf{w}) \leq \mathrm{Im}\, W_0(\mathbf{w}^{(0)}) = \mathrm{vol}(K).$$

The equality in (4) holds if and only if $\mathbf{w} = \mathbf{w}^{(0)}$.

Proof. It is a well-known fact that the hyperbolic volume is the maximal volume of all possible $\mathrm{PSL}(2, \mathbb{C})$ representations and that the maximum happens if and only if the representation is discrete and faithful (for the proof and details, see [4]).

From the proof of Lemma 1.2, if $\mathbf{w}$ and $\mathbf{z}$ are essential, then the shapes of each (collapsed) octahedra in Figure 2 and Figure 10 of Yokota and Thurston triangulations coincide. Therefore, these triangulations form the same geometric shape, and the parabolic representation $\rho_{\mathbf{w}}$ coincides with $\rho_{\mathbf{z}}$ up to conjugate, where $\rho_{\mathbf{w}}$ and $\rho_{\mathbf{z}}$ are the parabolic representations induced by $\mathbf{w}$ and by $\mathbf{z}$, respectively. This also implies that $\mathbf{z}^{(0)}$ is the geometric solution of $\mathcal{H}_1$.

Yokota proved

$$V_0(\mathbf{z}^{(\mathbf{0})}) \equiv i\,(\mathrm{vol}(K) + i\,\mathrm{cs}(K))\ (\mathrm{mod}\ \pi^2)$$

in [18] using Zickert's formula of [19], but the formula also holds for any parabolic representation $\rho_{\mathbf{z}}$ induced by $\mathbf{z}$. Therefore, Yokota's proof also implies

$$V_0(\mathbf{z}) \equiv i\,(\mathrm{vol}(\rho_{\mathbf{z}}) + i\,\mathrm{cs}(\rho_{\mathbf{z}}))\ (\mathrm{mod}\ \pi^2).$$

Among the essential solutions $\mathbf{z}$ of $\mathcal{H}_1$, only the geometric solution $\mathbf{z}^{(\mathbf{0})}$ induces the discrete faithful representation. Therefore, applying Theorem 1.3, we complete the proof.   □

This paper consists of the following contents. In Section 2, we describe Yokota triangulation and Thurston triangulation, which correspond to the Kashaev invariant and the colored Jones polynomial, respectively. We show that these two triangulations are related by finite steps of 3-2 moves and 4-5 moves on some crossings. In Section 3, the potential functions V and W are defined. In Section 4, we explain the geometries of V and W, and prove Proposition 1.1. In Section 5, we introduce several dilogarithm identities and complete the proofs of Lemma 1.2 and Theorem 1.3 using these identities. In Appendix A.1, we show the potential function W defined in Section 3 can be obtained by the formal substitution of the colored Jones polynomial. Finally, in Appendix A.2, we investigate the necessary and sufficient condition for an essential solution of $\mathcal{H}_1$ (respectively, $\mathcal{H}_2$) to induce the inessential solution of $\mathcal{H}_2$ (respectively, $\mathcal{H}_1$).

2. Two ideal triangulations of the knot complement

In this section, we explain two ideal triangulations of the knot complement. One is Yokota triangulation corresponding to the Kashaev invariant in [18] and the other is Thurston triangulation corresponding to the colored Jones polynomial in [14]. A good reference of this section is [10], which contains wonderful pictures.

2.1. Yokota triangulation

Consider a hyperbolic knot K and its diagram D (see Figure 1(a)). We define *sides* of D as arcs connecting two adjacent crossing points. For example, Figure 1(a) has 16 sides.

Now split a side of D open so as to make a (1,1)-tangle diagram and label crossings with integers (see Figure 1(b)). Yokota assumed several conditions on this (1,1)-tangle diagram (for the exact statement, see **Assumption 1.1.– 1.4.** and **Assumption 2.2.** in [18]). The assumptions roughly mean that we remove all the crossing points that can be reduced trivially. Also, let the two open sides be I and J and consider the orientation from J to I. Assume I and J are in an over-bridge and in an under-bridge, respectively (over-bridge is a union of sides, following the orientation of the knot diagram, from one
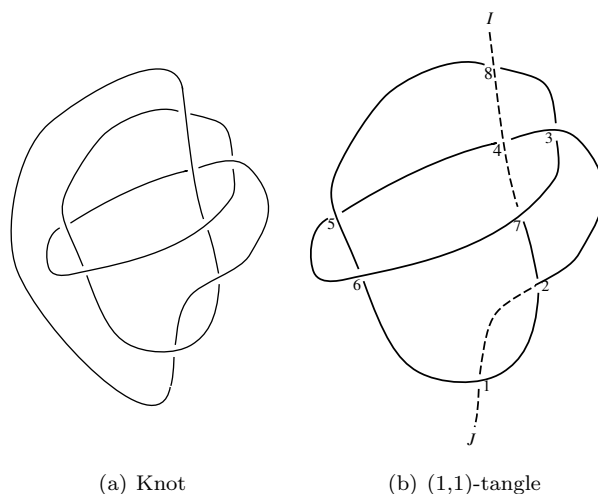
(a) Knot   (b) (1,1)-tangle

FIGURE 1. Example

over-crossing point to the next under-crossing point. Under-bridge is the one
from one under-crossing point to the next over-crossing point. The boundary
endpoints of I and J are considered over-crossing point and under-crossing
point, respectively. For example, in Figure 1(b), if we follow the diagram from
the below to the top, the first under-bridge containing J ends at the crossing
2, and the first over-bridge starts at the crossing 2 and ends at the crossing 4.
In total, it has 5 over-bridges and 5 under-bridges. Note that if we change the
orientation, the numbers of over-bridges and under-bridges change).

Now extend I and J so that, when following the orientation of the knot
diagram, non-boundary endpoints of I and J become the last under-crossing
point and the first over-crossing point, respectively, as in Figure 1(b). Then we
assume the two non-boundary endpoints of I and J do not coincide, because, if
they coincide, then we cut other side open and make a different tangle diagram.
Yokota proved in [18] that we can always make two non-boundary endpoints
different by cutting certain side open because, if not, then the diagram should
be that of a link or the trefoil knot (for details, see **Assumption 1.3.** and the
discussion that follows in [18]).

To obtain an ideal triangulation of the knot complement, we place an ideal
octahedron $A_nB_nC_nD_nE_nF_n$ on each crossing n as in Figure 2(a). We call
the edges A_nB_n, B_nC_n, C_nD_n and D_nA_n of the octahedron *horizontal edges*.
Figure 2(b) shows the positions of A_n, B_n, C_n, D_n and the horizontal edges.
We twist the octahedron by identifying the edges A_nE_n to C_nE_n and B_nF_n
to D_nF_n as in Figure 2(a) (the actual shape of the resulting diagram ap-
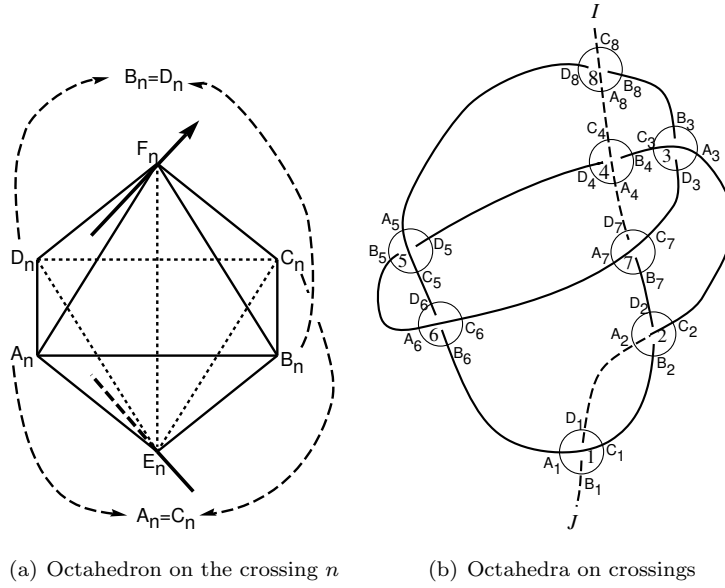pears in [10]). Then we glue the faces of the twisted octahedron following

(a) Octahedron on the crossing n (b) Octahedra on crossings

FIGURE 2. Example (continued)

the knot diagram. For example, in Figure 2(b), we glue $\triangle A_1E_1D_1 \cup \triangle C_1E_1D_1$
to $\triangle A_2F_2D_2 \cup \triangle A_2F_2B_2$, $\triangle C_2F_2D_2 \cup \triangle C_2F_2B_2$ to $\triangle A_3F_3D_3 \cup \triangle A_3F_3B_3$,
$\triangle C_3F_3D_3 \cup \triangle C_3F_3B_3$ to $\triangle A_4E_4B_4 \cup \triangle C_4E_4B_4$, $\triangle A_4E_4D_4 \cup \triangle C_4E_4D_4$ to
$\triangle C_5E_5D_5 \cup \triangle A_5E_5D_5$, and so on. Finally, we glue $\triangle D_8F_8C_8 \cup \triangle B_8F_8C_8$ to
$\triangle A_1E_1B_1 \cup \triangle C_1E_1B_1$. Note that, by gluing likewise, all A_n and C_n are iden-
tified to one point, all B_n and D_n are identified to another point, and all E_n
and F_n are identified to yet another point. Let these points be $-\infty$, ∞ and ℓ,
respectively. Then the regular neighborhoods of $-\infty$ and ∞ become 3-balls,
whereas that of ℓ becomes the tubular neighborhood of the knot K.

We split each octahedron $A_nB_nC_nD_nE_nF_n$ into four tetrahedra, $A_nB_nE_nF_n$,
$B_nC_nE_nF_n$, $C_nD_nE_nF_n$ and $D_nA_nE_nF_n$. Then we collapse faces that lie on
the split sides. For example, in Figure 2(b), we collapse the faces $\triangle A_1E_1B_1 \cup$
$\triangle C_1E_1B_1$ and $\triangle D_8F_8C_8 \cup \triangle B_8F_8C_8$ to different points. Note that this face
collapsing makes some edges on these faces into points. Actually, the non-
horizontal edges A_2F_2, B_4F_4, D_4F_4, D_7E_7, and the horizontal edges B_2C_2,
A_3B_3, A_5B_5, A_6B_6 in Figure 2(b) are collapsed to points because of the
face collapsing. This makes the tetrahedra $A_1B_1E_1F_1$, $B_1C_1E_1F_1$, $C_1D_1E_1F_1$,
$D_1A_1E_1F_1$, $A_2B_2E_2F_2$, $B_2C_2E_2F_2$, $D_2A_2E_2F_2$, $A_3B_3E_3F_3$, $A_4B_4E_4F_4$,
$B_4C_4E_4F_4$, $C_4D_4E_4F_4$, $D_4A_4E_4F_4$, $A_5B_5E_5F_5$, $A_6B_6E_6F_6$, $C_7D_7E_7F_7$,
$D_7A_7E_7F_7$, $A_8B_8E_8F_8$, $B_8C_8E_8F_8$, $C_8D_8E_8F_8$ and $D_8A_8E_8F_8$ be collapsed to
points or edges.
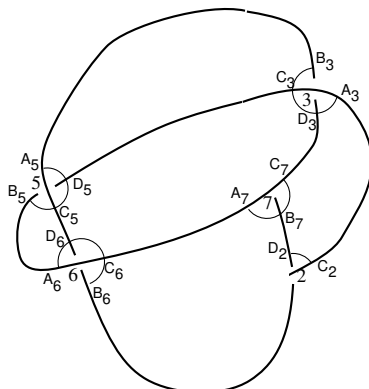
FIGURE 3. G with survived tetrahedra

The surviving tetrahedra after the collapsing can be depicted as follows. First, remove I and J on the tangle diagram and denote the result as G (see Figure 3). Note that, by removing $I \cup J$, some vertices are removed, two vertices become trivalent and some sides are glued together. In Figure 3, vertices 1, 4, 8 are removed, 2, 7 become trivalent and G has 9 sides (we consider the sides at the trivalent vertices are not glued together). Now we remove the horizontal edges on the removed vertices, the horizontal edges that are adjacent to $I \cup J$ and the horizontal edges in the unbounded region (see Figure 3 for the result). The surviving horizontal edges mean the surviving ideal tetrahedra after the collapsing. In the example, 12 tetrahedra survive.

The collapsing identifies the points ∞, $-\infty$, and ℓ to each other and connects the regular neighborhoods of them. Collapsing certain edges of a tetrahedron may change the topological type of ℓ, but Yokota excluded such cases by **Assumption 1.1.–1.3.** on the shape of the knot diagram (**Assumption 1.1.–1.2.** roughly means the diagram has no redundant crossings and **Assumption 1.3.** means the two non-boundary endpoints of I and J do not coincide). Therefore, the result of the collapsing makes the neighborhood of $\infty = -\infty = \ell$ to be the tubular neighborhood of the knot, and we obtain the ideal triangulation of the knot complement (see [18] for a complete discussion).

2.2. Thurston triangulation

Thurston triangulation, introduced in [14], uses the same octahedra and the same collapsing process, so it also induces an ideal triangulation of the knot complement. However, it uses a different subdivision of each octahedra. In Figure 2(a), Yokota triangulation subdivides each octahedron into four tetrahedra. However, Thurston triangulation subdivides it into five tetrahedra, $A_n B_n D_n F_n$, $B_n C_n D_n F_n$, $A_n B_n C_n D_n$, $A_n B_n C_n E_n$ and $A_n C_n D_n E_n$ (see the right-hand side

of Figure 4(a) for the shape of the subdivision). In this subdivision, if we apply the collapsing process, then some pair of tetrahedra shares the same four vertices (see the first case of (Case 2) in the proof of Observation 2.1 for an example). For the convenience of discussion, when this happens, we remove these two tetrahedra and call the result Thurston triangulation.

To see the relation between these two triangulations, we define *4-5 move* of an octahedron and *3-2 move* of a hexahedron as in Figure 4.
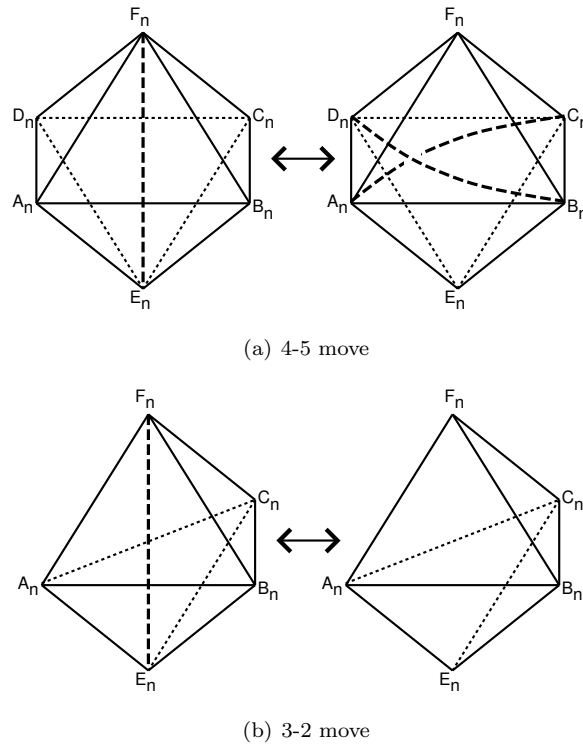
(a) 4-5 move

(b) 3-2 move

FIGURE 4. 4-5 and 3-2 moves

Before the collapsing process, two triangulations are related by only 4-5 moves on each crossings. However, the following observation shows they are actually related by 4-5 moves and also by 3-2 moves on some crossings after the collapsing.

Observation 2.1. *For a hyperbolic knot K with a fixed diagram, if the diagram satisfies* **Assumption 1.1.–1.4.** *and* **Assumption 2.2.** *in* [18], *then Yokota triangulation and Thurston triangulation are related by 3-2 moves and 4-5 moves on some crossings.*

Proof. First, for a non-trivalent vertex n of G, we show only one horizontal edge in Figure 2(a) can be collapsed. If any of two horizontal edges are collapsed, then the (1,1)-tangle diagram should be Figure 5(a) or Figure 5(b) for some tangle diagrams K_1 or K_2 because the collapsed edges should lie in the unbounded regions. However, Figure 5(a) is excluded because, if we close up the open side, then $K = K_1 \# K_2$ and K cannot be prime. We can also exclude Figure 5(b) because it violates **Assumption 1.1.** in [18]. Actually, in the later case, we can reduce the number of crossings as in Figure 5(b).

(a)   (b)
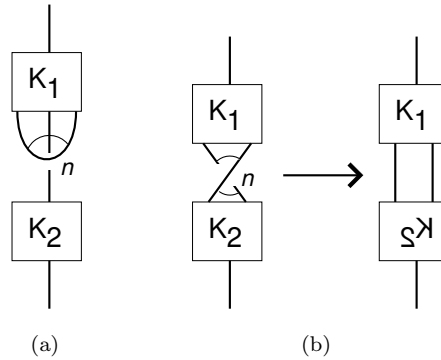
FIGURE 5. When two horizontal edges are collapsed

Because of this and Yokota's Assumptions, all possible cases of collapsing edges in Figure 2(a) are as follows:

(Case 1) if n is a non-trivalent vertex of G, then none or one of the horizontal edges is collapsed.

(Case 2) if n is a trivalent vertex of G, then

(1) D_nE_n is collapsed and none or one of A_nB_n, B_nC_n is collapsed,
(2) B_nE_n is collapsed and none or one of C_nD_n, D_nA_n is collapsed,
(3) A_nF_n is collapsed and none or one of B_nC_n, C_nD_n is collapsed.

(Case 1) is trivial, so we consider the first case of (Case 2).

If D_nE_n and A_nB_n are collapsed, then the survived tetrahedron is $B_nC_nE_nF_n$ in Yokota triangulation, and $B_nC_nD_nF_n$ in Thurston triangulation. They coincide because $D_n = E_n$ by the collapsing of D_nE_n.

If D_nE_n is collapsed and no others are, then the survived tetrahedra are $A_nB_nE_nF_n$ and $B_nC_nE_nF_n$ in Yokota triangulation, and $A_nB_nD_nF_n$, $B_nC_nD_nF_n$, $A_nB_nC_nD_n$ and $A_nB_nC_nE_n$ in Thurston triangulation. However, in Thurston triangulation, two tetrahedra $A_nB_nC_nD_n$ and $A_nB_nC_nE_n$ cancel each other because they share the same vertices A_n, B_n, C_n and $D_n = E_n$. The others coincide with the tetrahedra in Yokota triangulation because $D_n = E_n$.

Other cases of (Case 2) are the same as the first case, so the proof is completed.   □

3. Potential functions

3.1. The case of Kashaev invariant

In the case of Kashaev invariant, Yokota's potential function $V(z_1, \ldots, z_g)$ is defined by the following way.

For the graph G, we define *contributing sides* as sides of G which are not on the unbounded regions. For example, there are 5 contributing sides and 4 non-contributing sides in Figure 6. We assign complex variables $z_1, \ldots, z_g$ to contributing sides and real number 1 to non-contributing sides. Then we label each ideal tetrahedra with $IT_1, IT_2, \ldots, IT_s$ and assign t_l $(l = 1, \ldots, s)$ to the horizontal edge of IT_l as the shape parameter. We define t_l as the counterclockwise ratio of the complex variables $z_1, \ldots, z_g$.
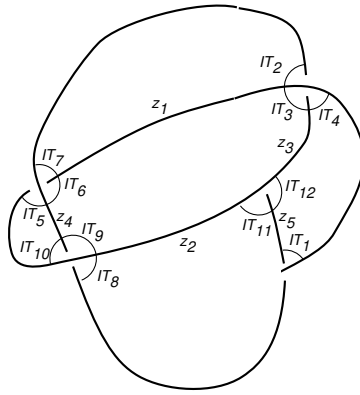
FIGURE 6. G with contributing sides

For example, in Figure 6,

$$t_1 = \frac{z_5}{1}, \ t_2 = \frac{z_1}{1}, \ t_3 = \frac{z_3}{z_1}, \ t_4 = \frac{1}{z_3}, \ t_5 = \frac{z_4}{1}, \ t_6 = \frac{z_1}{z_4},$$

$$t_7 = \frac{1}{z_1}, \ t_8 = \frac{z_2}{1}, \ t_9 = \frac{z_4}{z_2}, \ t_{10} = \frac{1}{z_4}, \ t_{11} = \frac{z_5}{z_2}, \ t_{12} = \frac{z_3}{z_5}.$$

For each tetrahedron IT_l, we assign dilogarithm function as in Figure 7. Then we define $V(z_1, \ldots, z_g)$ by the summation of all these dilogarithm functions. We also define the sign σ_l of IT_l by

$$\sigma_l = \begin{cases} 1 & \text{if } IT_l \text{ lies as in Figure 7(a)}, \\ -1 & \text{if } IT_l \text{ lies as in Figure 7(b)}. \end{cases}$$

Then $V(z_1, \ldots, z_g)$ is expressed by

$$V(z_1, \ldots, z_g) = \sum_{l=1}^{g} \sigma_l \left(\mathrm{Li}_2(t_l^{\sigma_l}) - \frac{\pi^2}{6} \right).$$

(a) Positive corner

$$\longrightarrow \ \mathrm{Li}_2(t_l) - \frac{\pi^2}{6}$$

(b) Negative corner

$$\longrightarrow \ \frac{\pi^2}{6} - \mathrm{Li}_2(\frac{1}{t_l})$$

FIGURE 7. Assigning dilogarithm functions to each tetrahedra

For example, in Figure 6,

$$\sigma_1 = \sigma_3 = \sigma_6 = \sigma_9 = \sigma_{11} = 1, \ \sigma_2 = \sigma_4 = \sigma_5 = \sigma_7 = \sigma_8 = \sigma_{10} = \sigma_{12} = -1,$$

and

$$V(z_1, \ldots, z_5) = \mathrm{Li}_2(z_5) - \mathrm{Li}_2(\frac{1}{z_1}) + \mathrm{Li}_2(\frac{z_3}{z_1}) - \mathrm{Li}_2(z_3) - \mathrm{Li}_2(\frac{1}{z_4})$$
$$+ \mathrm{Li}_2(\frac{z_1}{z_4}) - \mathrm{Li}_2(z_1) - \mathrm{Li}_2(\frac{1}{z_2}) + \mathrm{Li}_2(\frac{z_4}{z_2}) - \mathrm{Li}_2(z_4)$$
$$+ \mathrm{Li}_2(\frac{z_5}{z_2}) - \mathrm{Li}_2(\frac{z_5}{z_3}) + \frac{\pi^2}{3}.$$

It is shown in [17] that $V(z_1, \ldots, z_g)$ can be obtained by the formal substitution of the Kashaev invariant.[1]

3.2. The case of colored Jones polynomial

For each region of G, we choose one bounded region and assign 1 to it. Then we assign variables $w_1, \ldots, w_m$ to the remaining bounded regions, and 0 to the unbounded region (see Figure 8).

For each vertex of G, we assign the following functions according to the type of the vertex and the horizontal edges. For positive crossings:

$$: P_1(w_j, w_k, w_l, w_m) = -\mathrm{Li}_2(\frac{w_l}{w_m}) - \mathrm{Li}_2(\frac{w_l}{w_k}) + \mathrm{Li}_2(\frac{w_j w_l}{w_k w_m})$$
$$+ \mathrm{Li}_2(\frac{w_m}{w_j}) + \mathrm{Li}_2(\frac{w_k}{w_j}) - \frac{\pi^2}{6} + \log\frac{w_m}{w_j}\log\frac{w_k}{w_j},$$

---

[1] We remark that the Kashaev invariant of a knot K defined in [17] is the one of the mirror image $\overline{K}$ defined in [11]. This paper follows the definition of [17].
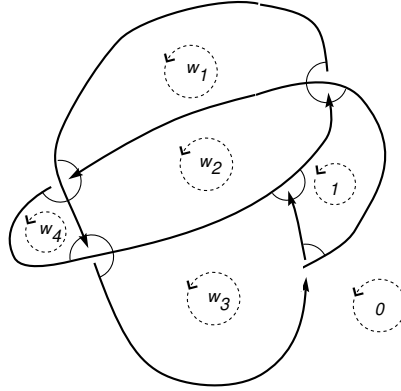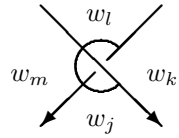
FIGURE 8. Assigning variables to each region

$$: P_2(w_j, w_k, w_l, w_m) = \text{Li}_2(\tfrac{w_m}{w_l}) - \text{Li}_2(\tfrac{w_l}{w_k}) - \text{Li}_2(\tfrac{w_k w_m}{w_j w_l})$$

$$+ \text{Li}_2(\tfrac{w_m}{w_j}) - \text{Li}_2(\tfrac{w_j}{w_k}) + \tfrac{\pi^2}{6} - \log \tfrac{w_k}{w_l} \log \tfrac{w_k}{w_j},$$

$$: P_3(w_j, w_k, w_l, w_m) = \text{Li}_2(\tfrac{w_m}{w_l}) + \text{Li}_2(\tfrac{w_k}{w_l}) + \text{Li}_2(\tfrac{w_j w_l}{w_k w_m})$$

$$- \text{Li}_2(\tfrac{w_j}{w_m}) - \text{Li}_2(\tfrac{w_j}{w_k}) - \tfrac{\pi^2}{6} + \log \tfrac{w_m}{w_l} \log \tfrac{w_k}{w_l},$$

$$: P_4(w_j, w_k, w_l, w_m) = -\text{Li}_2(\tfrac{w_l}{w_m}) + \text{Li}_2(\tfrac{w_k}{w_l}) - \text{Li}_2(\tfrac{w_k w_m}{w_j w_l})$$

$$- \text{Li}_2(\tfrac{w_j}{w_m}) + \text{Li}_2(\tfrac{w_k}{w_j}) + \tfrac{\pi^2}{6} - \log \tfrac{w_m}{w_l} \log \tfrac{w_m}{w_j}.$$
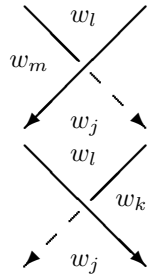
For negative crossings:

$$: N_1(w_j, w_k, w_l, w_m) = \text{Li}_2(\tfrac{w_l}{w_m}) + \text{Li}_2(\tfrac{w_l}{w_k}) - \text{Li}_2(\tfrac{w_j w_l}{w_k w_m})$$

$$- \text{Li}_2(\tfrac{w_m}{w_j}) - \text{Li}_2(\tfrac{w_k}{w_j}) + \tfrac{\pi^2}{6} - \log \tfrac{w_j}{w_m} \log \tfrac{w_j}{w_k},$$

$$: N_2(w_j, w_k, w_l, w_m) = -\text{Li}_2(\tfrac{w_m}{w_l}) + \text{Li}_2(\tfrac{w_l}{w_k}) + \text{Li}_2(\tfrac{w_k w_m}{w_j w_l})$$

$$- \text{Li}_2(\tfrac{w_m}{w_j}) + \text{Li}_2(\tfrac{w_j}{w_k}) - \tfrac{\pi^2}{6} + \log \tfrac{w_l}{w_k} \log \tfrac{w_j}{w_k},$$

$$: N_3(w_j, w_k, w_l, w_m) = -\text{Li}_2(\tfrac{w_m}{w_l}) - \text{Li}_2(\tfrac{w_k}{w_l}) - \text{Li}_2(\tfrac{w_j w_l}{w_k w_m})$$

$$+ \text{Li}_2(\tfrac{w_j}{w_m}) + \text{Li}_2(\tfrac{w_j}{w_k}) + \tfrac{\pi^2}{6} - \log \tfrac{w_l}{w_m} \log \tfrac{w_l}{w_k},$$

$$: N_4(w_j, w_k, w_l, w_m) = \mathrm{Li}_2(\tfrac{w_l}{w_m}) - \mathrm{Li}_2(\tfrac{w_k}{w_l}) + \mathrm{Li}_2(\tfrac{w_k w_m}{w_j w_l})$$
$$+ \mathrm{Li}_2(\tfrac{w_j}{w_m}) - \mathrm{Li}_2(\tfrac{w_k}{w_j}) - \tfrac{\pi^2}{6} + \log \tfrac{w_l}{w_m} \log \tfrac{w_j}{w_m}.$$
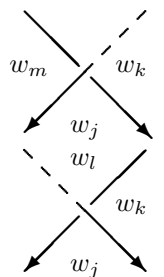
If no horizontal edge is collapsed at the positive nor the negative crossing, we assign any of $P_1, \ldots, P_4$ or $N_1, \ldots, N_4$ to the crossing, respectively. In Lemma 3.1, we will show this choice does not have any effect on the optimistic limit of the colored Jones polynomial.

For the endpoints of I and J, we use the same formula disregarding whether certain horizontal edge is collapsed or not. For the endpoint of I:

$$: P_1(w_j, w_j, w_l, w_m) = P_2(w_j, w_j, w_l, w_m)$$
$$= \mathrm{Li}_2(\tfrac{w_m}{w_j}) - \mathrm{Li}_2(\tfrac{w_l}{w_j}),$$

$$: N_1(w_j, w_k, w_l, w_j) = N_4(w_j, w_k, w_l, w_j)$$
$$= -\mathrm{Li}_2(\tfrac{w_k}{w_j}) + \mathrm{Li}_2(\tfrac{w_l}{w_j}).$$

For the endpoint of J:

$$: P_2(w_j, w_k, w_k, w_m) = P_3(w_j, w_k, w_k, w_m)$$
$$= \mathrm{Li}_2(\tfrac{w_m}{w_k}) - \mathrm{Li}_2(\tfrac{w_j}{w_k}),$$

$$: N_3(w_j, w_k, w_l, w_l) = N_4(w_j, w_k, w_l, w_l)$$
$$= -\mathrm{Li}_2(\tfrac{w_k}{w_l}) + \mathrm{Li}_2(\tfrac{w_j}{w_l}).$$

In Appendix, we show that the assigned functions above are, in fact, obtained by the formal substitution of certain forms of the R-matrix of the colored Jones polynomial.

Now we define the potential function $W(w_1, \ldots, w_m)$ of the knot diagram by the summation of all functions assigned to the vertices of G. For example,

the potential function $W(w_1, \ldots, w_4)$ of Figure 8 is

$$
\begin{aligned}
W(w_1, \ldots, w_4) = & -\operatorname{Li}_2(\frac{1}{w_3}) \\
& + \left\{ \operatorname{Li}_2(\frac{1}{w_2}) + \operatorname{Li}_2(\frac{w_1}{w_2}) - \frac{\pi^2}{6} + \log\frac{1}{w_2}\log\frac{w_1}{w_2} \right\} \\
& + \left\{ \operatorname{Li}_2(\frac{w_1}{w_2}) + \operatorname{Li}_2(\frac{w_4}{w_2}) - \frac{\pi^2}{6} + \log\frac{w_1}{w_2}\log\frac{w_4}{w_2} \right\} \\
& + \left\{ \operatorname{Li}_2(\frac{w_4}{w_2}) + \operatorname{Li}_2(\frac{w_3}{w_2}) - \frac{\pi^2}{6} + \log\frac{w_4}{w_2}\log\frac{w_3}{w_2} \right\} \\
& + \left\{ \operatorname{Li}_2(\frac{1}{w_2}) - \operatorname{Li}_2(\frac{w_3}{w_2}) \right\}.
\end{aligned}
\tag{5}
$$

We end this section with the invariance of the optimistic limit under the choice of the four different forms of the potential functions of a crossing.

Lemma 3.1. *For the functions $P_1, \ldots, P_4, N_1, \ldots, N_4$ defined above, let*

$$
P_{f0} := P_f - \sum_{a=j,k,l,m} \left(w_a \frac{\partial P_f}{\partial w_a} \right) \log w_a, \ N_{f0} := N_f - \sum_{a=j,k,l,m} \left(w_a \frac{\partial N_f}{\partial w_a} \right) \log w_a.
$$

Then

$$
P_{10} \equiv P_{20} \equiv P_{30} \equiv P_{40}, \ N_{10} \equiv N_{20} \equiv N_{30} \equiv N_{40} \pmod{4\pi^2},
$$

and for $a = j, k, l, m$,

$$
\exp\left(w_a \frac{\partial P_1}{\partial w_a} \right) = \exp\left(w_a \frac{\partial P_2}{\partial w_a} \right) = \exp\left(w_a \frac{\partial P_3}{\partial w_a} \right) = \exp\left(w_a \frac{\partial P_4}{\partial w_a} \right),
$$

$$
\exp\left(w_a \frac{\partial N_1}{\partial w_a} \right) = \exp\left(w_a \frac{\partial N_2}{\partial w_a} \right) = \exp\left(w_a \frac{\partial N_3}{\partial w_a} \right) = \exp\left(w_a \frac{\partial N_4}{\partial w_a} \right).
$$

Proof. For a given complex-valued function $F(w_j, w_k, w_l, w_m)$, let

$$
\widehat{F}(w_j, w_k, w_l, w_m) := F + \sum_{a=j,k,l,m} 2n_a\pi i \log w_a + 4n\pi^2
\tag{6}
$$

for some integer constants n_j, n_k, n_l, n_m, n. Then by a direct calculation,

$$
\widehat{F}_0 \equiv F_0 \pmod{4\pi^2}
$$

and

$$
\exp\left(w_a \frac{\partial F}{\partial w_a} \right) = \exp\left(w_a \frac{\partial \widehat{F}}{\partial w_a} \right).
$$

These show F and $\widehat{F}$ define the same optimistic limit, so we define an equivalence relation $\approx$ by $F \approx \widehat{F}$ for F and $\widehat{F}$ satisfying (6).

For

$$
P_1 = -\operatorname{Li}_2(\frac{w_l}{w_m}) - \operatorname{Li}_2(\frac{w_l}{w_k}) + \operatorname{Li}_2(\frac{w_j w_l}{w_k w_m}) + \operatorname{Li}_2(\frac{w_m}{w_j}) + \operatorname{Li}_2(\frac{w_k}{w_j})
$$

$$-\frac{\pi^2}{6} + \log\frac{w_m}{w_j}\log\frac{w_k}{w_j},$$

$$P_2 = \mathrm{Li}_2(\frac{w_m}{w_l}) - \mathrm{Li}_2(\frac{w_l}{w_k}) - \mathrm{Li}_2(\frac{w_k w_m}{w_j w_l}) + \mathrm{Li}_2(\frac{w_m}{w_j}) - \mathrm{Li}_2(\frac{w_j}{w_k}) + \frac{\pi^2}{6}$$

$$-\log\frac{w_k}{w_l}\log\frac{w_k}{w_j},$$

using the well-known identity $\mathrm{Li}_2(z) + \mathrm{Li}_2(\frac{1}{z}) \approx -\frac{\pi^2}{6} - \frac{1}{2}\log^2(-z)$ for $z \in \mathbb{C}$ in [6], we obtain

$$P_1 - P_2 = -\mathrm{Li}_2(\frac{w_l}{w_m}) - \mathrm{Li}_2(\frac{w_m}{w_l}) + \mathrm{Li}_2(\frac{w_j w_l}{w_k w_m}) + \mathrm{Li}_2(\frac{w_k w_m}{w_j w_l})$$

$$+ \mathrm{Li}_2(\frac{w_k}{w_j}) + \mathrm{Li}_2(\frac{w_j}{w_k}) - \frac{\pi^2}{3} + \left(\log\frac{w_m}{w_j} + \log\frac{w_k}{w_l}\right)\log\frac{w_k}{w_j}$$

$$\approx -\frac{\pi^2}{2} + \frac{1}{2}\log^2(-\frac{w_l}{w_m}) - \frac{1}{2}\log^2(-\frac{w_k w_m}{w_j w_l}) - \frac{1}{2}\log^2(-\frac{w_k}{w_j})$$

$$+ \left(\log\frac{w_m}{w_j} + \log\frac{w_k}{w_l}\right)\log\frac{w_k}{w_j}.$$

For any integer n, some integers $n_1, \ldots, n_4$ and indices $a, b \in \{i, j, k, l\}$, we have

$$2n\pi i\log\frac{w_a}{w_b} = 2n\pi i\left(\log w_a - \log w_b + 2n_1\pi i\right) \approx 0,$$

$$\frac{1}{2}\log^2(-\frac{w_k}{w_j}) = \frac{1}{2}\left\{\log\frac{w_k}{w_j} + (2n_2 - 1)\pi i\right\}^2$$

$$= \frac{1}{2}\log^2\frac{w_k}{w_j} + (2n_2 - 1)\pi i\log\frac{w_k}{w_j} - 2n_2(n_2 - 1)\pi^2 - \frac{\pi^2}{2}$$

$$\approx \frac{1}{2}\log^2\frac{w_k}{w_j} - \pi i\log\frac{w_k}{w_j} - \frac{\pi^2}{2}$$

and

$$\frac{1}{2}\left\{\log\frac{w_k}{w_j} - \log(-\frac{w_k w_m}{w_j w_l})\right\}^2$$

$$= \frac{1}{2}\left\{\log(-\frac{w_l}{w_m}) + 2n_3\pi i\right\}^2$$

$$= \frac{1}{2}\log^2(-\frac{w_l}{w_m}) + 2n_3\pi i\left\{\log\frac{w_l}{w_m} + (2n_4 + 1)\pi i\right\} - 2n_3^2\pi^2$$

$$\approx \frac{1}{2}\log^2(-\frac{w_l}{w_m}) - 2n_3(n_3 + 1)\pi^2 \approx \frac{1}{2}\log^2(-\frac{w_l}{w_m}).$$

Therefore, we obtain

$$P_1 - P_2 \approx \frac{1}{2}\log^2(-\frac{w_l}{w_m}) - \frac{1}{2}\log^2(-\frac{w_k w_m}{w_j w_l}) - \frac{1}{2}\log^2\frac{w_k}{w_j}$$

$$+ \pi i \log \frac{w_k}{w_j} + \log \frac{w_k w_m}{w_j w_l} \log \frac{w_k}{w_j}$$

$$\approx \frac{1}{2} \log^2(-\frac{w_l}{w_m}) - \frac{1}{2} \log^2(-\frac{w_k w_m}{w_j w_l}) - \frac{1}{2} \log^2 \frac{w_k}{w_j}$$

$$+ \log(-\frac{w_k w_m}{w_j w_l}) \log \frac{w_k}{w_j}$$

$$= \frac{1}{2} \log^2(-\frac{w_l}{w_m}) - \frac{1}{2} \left\{ \log \frac{w_k}{w_j} - \log(-\frac{w_k w_m}{w_j w_l}) \right\}^2$$

$$\approx \frac{1}{2} \log^2(-\frac{w_l}{w_m}) - \frac{1}{2} \log^2(-\frac{w_l}{w_m}) = 0.$$

Other equalities $P_2 \approx P_3 \approx P_4$ and $N_1 \approx N_2 \approx N_3 \approx N_4$ can be obtained by the same method or by the symmetry of the equations. $\qquad \square$

4. Geometric structures of the triangulations

For Yokota triangulation and Thurston triangulation, we assign complex variables to each tetrahedra and solve certain equations. Then one of the solutions gives the complete hyperbolic structure of the knot complement. We describe these procedures in this section.

First, consider the positive and negative crossings in Figure 9, where z_a, z_b, z_c, z_d are the variables assigned to the sides of G and w_j, w_k, w_l, w_m are the variables assigned to the regions of G. Note that z_a, z_b, z_c, z_d and w_j, w_k, w_l, w_m are used for defining the potential functions $V(z_1, \ldots, z_g)$ and $W(w_1, \ldots, w_m)$, respectively.
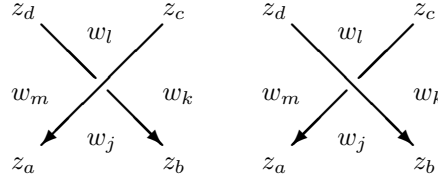

FIGURE 9. Assignment of variables

Then consider Figure 10. We assign $\frac{z_b}{z_a}, \frac{z_c}{z_b}, \frac{z_d}{z_c}, \frac{z_a}{z_d}$ to the horizontal edges $C_n D_n$, $D_n A_n$, $A_n B_n$, $B_n C_n$, respectively. This assignment determines the shape parameters of the tetrahedra of Yokota triangulation. Also, for the positive crossing, we assign $\left(\frac{w_j}{w_m}\right)^{-1}$, $\frac{w_k}{w_j}$, $\frac{w_k}{w_l}$, $\left(\frac{w_l}{w_m}\right)^{-1}$ to $C_n F_n$, $D_n E_n$, $A_n F_n$, $B_n E_n$, respectively, and assign $\left(\frac{w_k w_m}{w_j w_l}\right)^{-1}$ to $B_n D_n$ and $A_n C_n$ for the parameter of the tetrahedron $A_n B_n C_n D_n$. For the negative crossing, we assign $\frac{w_j}{w_m}$, $\left(\frac{w_k}{w_j}\right)^{-1}$, $\left(\frac{w_k}{w_l}\right)^{-1}$, $\frac{w_l}{w_m}$ to $B_n E_n$, $C_n F_n$, $D_n E_n$, $A_n F_n$, respectively, and assign $\left(\frac{w_j w_l}{w_k w_m}\right)^{-1}$ to $B_n D_n$ and $A_n C_n$ for the parameter of the tetrahedron

$A_nB_nC_nD_n$. These assignments determine the shape parameters of the tetrahedra of Thurston triangulation.

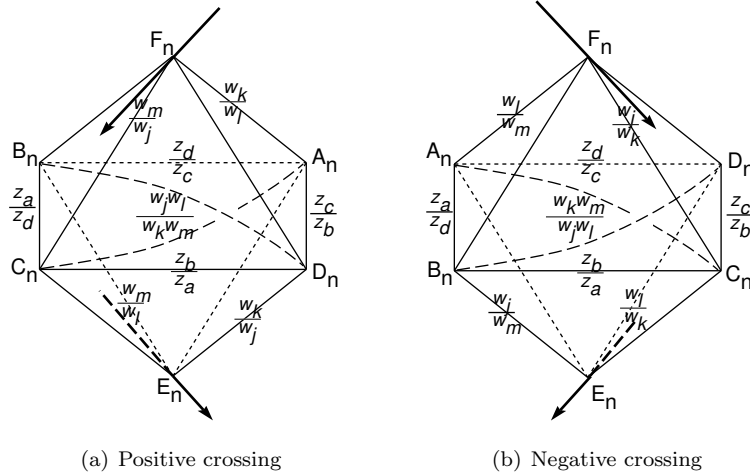

(a) Positive crossing   (b) Negative crossing

FIGURE 10. Assignment of shape parameters

We do not assign any shape parameters to the collapsed edges. Also, in the case of Thurston triangulation, we do not assign any shape parameters to the edges that contain the endpoints of the collapsed edges. For example, if C_nD_n is collapsed, then we do not assign any shape parameters to C_nF_n, D_nE_n nor B_nD_n. Also, if D_nE_n is collapsed in Figure 10(a), then we do not assign any shape parameters to B_nD_n, B_nE_n, C_nD_n nor D_nA_n.[2]

Yokota and Thurston triangulations are ideal triangulations, so by assigning shape parameters, we can determine all the shapes of the hyperbolic ideal tetrahedra of the triangulations. Note that if we assign a shape parameter $u \in \mathbb{C} - \{0, 1\}$ to an edge of an ideal tetrahedron, then other edges are also parametrized by $u, u' := \frac{1}{1-u}$ and $u'' := 1 - \frac{1}{u}$ as in Figure 11.

So as to get the hyperbolic structure, these shape parameters should satisfy the *edge relations* and the *cusp conditions*. The edge relations mean the product of all shape parameters assigned to each edge should be 1, and the cusp conditions mean the holonomies induced by the longitude and the meridian should be translations on the cusp. These two conditions can be expressed by a set of equations of the shape parameters, and we call this set of equations *hyperbolicity equations* (for details, see [15, Chapter 4]). We call a solution $(z_1, \ldots, z_g)$ of the hyperbolicity equations of Yokota triangulation *essential* if none of the shape parameters of the tetrahedra are one of $0, 1, \infty$. We also

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[2]The edges C_nD_n and D_nA_n are horizontal edges, but are identified to non-horizontal edges. When this happens, we do not assign shape parameters to these edges.
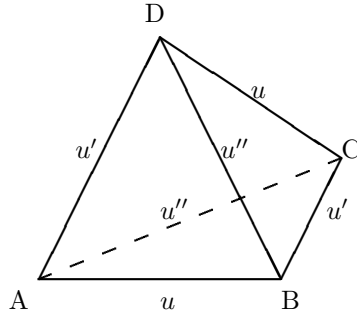
FIGURE 11. Parametrization of a hyperbolic ideal tetrahedron with shape parameter u

define an *essential solution* $(w_1, \ldots, w_m)$ of Thurston triangulation in the same way. It is a well-known fact that if the hyperbolicity equations have an essential solution, then they have the unique solution which gives the hyperbolic structure to the triangulation[3] (for details, see [16, Section 2.8]). We call this unique solution *the geometric solution*, and denote the geometric solution of Yokota triangulation by $\mathbf{z}^{(0)} = (z_1^{(0)}, \ldots, z_g^{(0)})$ and that of Thurston triangulation by $\mathbf{w}^{(0)} = (w_1^{(0)}, \ldots, w_m^{(0)})$. We remark that, in Theorem 1.3, we assumed the existence of the geometric solutions $\mathbf{z}^{(0)}$ and $\mathbf{w}^{(0)}$.

Yokota proved in [18] that, for the potential function V defined in Section 3.1, $\mathcal{H}_1 = \left\{ \exp\left(z_k \frac{\partial V}{\partial z_k} \right) = 1 \mid k = 1, \ldots, g \right\}$ becomes the hyperbolicity equations of Yokota triangulation. In other words, each element of $\mathcal{H}_1$ becomes an edge relation or a cusp condition for all $k = 1, \ldots, g$, and all other equations are trivially induced from the elements of $\mathcal{H}_1$.

Proposition 1.1 shows the same holds for the potential function W defined in Section 3.2 and $\mathcal{H}_2 = \left\{ \exp\left(w_l \frac{\partial W}{\partial w_l} \right) = 1 \mid l = 1, \ldots, m \right\}$. We prove this in this section.

Let $\mathcal{A}$ be the set of non-collapsed horizontal edges of Thurston triangulation of $S^3 - K$. Let $\mathcal{B}$ be the set of non-collapsed non-horizontal edges $A_n E_n$, $B_n E_n$, $C_n E_n$, $D_n E_n$, $A_n F_n$, $B_n F_n$, $C_n F_n$, $D_n F_n$ in Figure 10, which are not in $\mathcal{A}$.[4] Finally, let $\mathcal{C}$ be the set of edges $A_n C_n$, $B_n D_n$ in Figure 10, which are not in $\mathcal{A} \cup \mathcal{B}$.

For example, in Figure 3, $\mathcal{A} = \{$ $A_7 B_7 = B_6 C_6 = D_2 A_2 = D_2 F_2 = A_2 B_2 = B_2 F_2 = C_2 F_2 = A_3 F_3 = B_3 F_3 = D_3 F_3 = D_5 E_5$, $D_6 A_6 = B_5 C_5$, $C_6 D_6 = $

---

[3]Strictly speaking, we have unique values of shape parameters. However, these values uniquely determine the solutions $(z_1^{(0)}, \ldots, z_g^{(0)})$ and $(w_1^{(0)}, \ldots, w_m^{(0)})$. This was explained in [18] for Yokota triangulation, which will be discussed at the end of this section for Thurston triangulation.

[4]Collapsing may identify some horizontal edges to non-horizontal edges. In this case, we put these identified edges in $\mathcal{A}$.

$C_5D_5 = C_3D_3 = D_7A_7 = A_7E_7 = C_7D_7 = C_7E_7 = A_2E_2 = C_2E_2 = B_2E_2 = A_6E_6 = B_6E_6 = C_6E_6 = C_5F_5, D_5A_5 = B_3C_3, C_2D_2 = B_7C_7 = D_3A_3\}, \mathcal{B} = \{ D_3E_3 = B_7F_7 = D_7F_7 = A_6F_6 = B_6F_6 = D_6F_6 = B_5E_5 = C_5E_5 = A_5E_5 = C_3F_3, A_7F_7 = C_6F_6, D_6E_6 = B_5F_5 = D_5F_5 = A_5F_5 = A_3E_3 = B_3E_3 = C_3E_3 = C_7F_7, B_7E_7 = D_2E_2 \}$ and $\mathcal{C} = \emptyset$.

Lemma 4.1. *For a hyperbolic knot K with a fixed diagram, we assume the assumptions of Proposition 1.1. Then the edges in $\mathcal{B} \cup \mathcal{C}$ satisfy the edge relations trivially by the assigning rule of the shape parameters.*

Proof. If an edge A_nC_n or B_nD_n of Figure 10 is in $\mathcal{C}$, then the octahedron $A_nB_nC_nD_nE_nF_n$ does not have any collapsed edge. By the assigning rule of the shape parameters, all the edges in $\mathcal{C}$ satisfy edge relations trivially.

Now we show the case of $\mathcal{B}$. Consider the following four cases of two points n_1 and n_2 in Figure 12 and the two regions between the crossings parametrized by the variables w_a and w_b (for the positions of the points $A_{n_1}, B_{n_1}, \ldots, F_{n_2}$, see Figure 2). First, we assume no edges are collapsed in the tetrahedra $A_{n_1}B_{n_1}D_{n_1}F_{n_1}$ and $C_{n_2}B_{n_2}D_{n_2}F_{n_2}$. This means the two regions with w_a and w_b in Figure 12 are bounded.
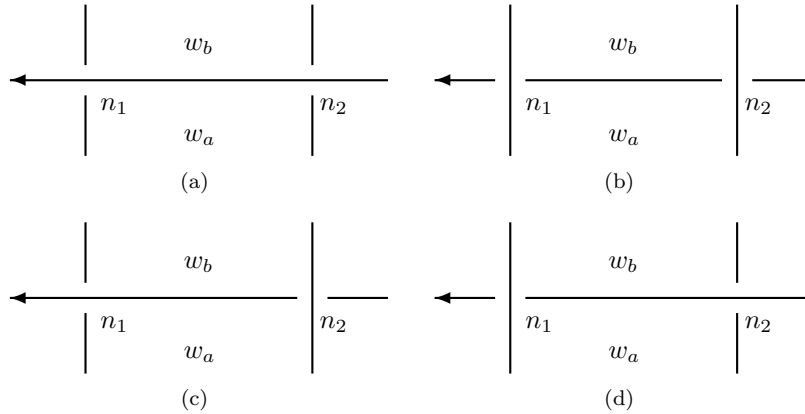

FIGURE 12. Four cases

In the case of Figure 12(a), we want to prove that the edge relation of the edge $A_{n_1}F_{n_1} = C_{n_2}F_{n_2} \in \mathcal{B}$ holds trivially. We draw a part of the cusp diagram in $A_{n_1}B_{n_1}D_{n_1}F_{n_1} \cup C_{n_2}B_{n_2}D_{n_2}F_{n_2}$ near $F_{n_1} = F_{n_2}$ as in Figure 13. Our tetrahedra are all ideal, so the triangles $\triangle\alpha_1\alpha_2\alpha_3$ and $\triangle\alpha_1\alpha_4\alpha_5$ are Euclidean. Note that $\alpha_1, \ldots, \alpha_5$ are points in the edges $A_{n_1}F_{n_1} = C_{n_2}F_{n_2}$, $B_{n_1}F_{n_1}$, $D_{n_1}F_{n_1}$,

$D_{n_2}F_{n_2}$, $B_{n_2}F_{n_2}$, respectively. Furthermore, edges $\alpha_1\alpha_2$ and $\alpha_1\alpha_3$ are identified to $\alpha_1\alpha_5$ and to $\alpha_1\alpha_4$, respectively.[5] On the edge $A_{n_1}F_{n_1} = C_{n_2}F_{n_2}$, two shape parameters w_a/w_b and w_b/w_a are assigned respectively by the assigning rule, so the edge relation of $A_{n_1}F_{n_1} = C_{n_2}F_{n_2} \in \mathcal{B}$ holds trivially.
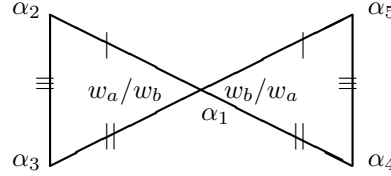

FIGURE 13. Part of the cusp diagram of Figure 12(a)

In the case of Figure 12(c), we want to prove that the edge relation of $A_{n_1}F_{n_1} \in \mathcal{B}$ holds trivially. If n_2 is a positive crossing, then we draw a part of the cusp diagram in $A_{n_1}B_{n_1}D_{n_1}F_{n_1} \cup A_{n_2}C_{n_2}D_{n_2}E_{n_2}$ near $F_{n_1} = E_{n_2}$, and if n_2 is a negative crossing, then we draw a part of the cusp diagram in $A_{n_1}B_{n_1}D_{n_1}F_{n_1} \cup A_{n_2}B_{n_2}C_{n_2}E_{n_2}$ near $F_{n_1} = E_{n_2}$ as in Figure 14.
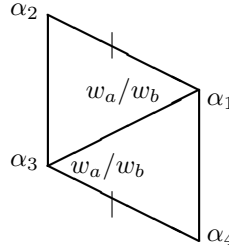

FIGURE 14. Part of the cusp diagram of Figure 12(c)

Note that if n_2 is a positive crossing, then $\alpha_1, \ldots, \alpha_4$ are points in the edges $A_{n_1}F_{n_1} = A_{n_2}E_{n_2}$, $B_{n_1}F_{n_1}$, $D_{n_1}F_{n_1} = D_{n_2}E_{n_2}$, $C_{n_2}E_{n_2}$, respectively, and if n_2 is a negative crossing, then $\alpha_1, \ldots, \alpha_4$ are points in the edges $A_{n_1}F_{n_1} = C_{n_2}E_{n_2}$, $B_{n_1}F_{n_1}$, $D_{n_1}F_{n_1} = B_{n_2}E_{n_2}$, $A_{n_2}E_{n_2}$, respectively. Furthermore, the edge $\alpha_2\alpha_1$ is identified to $\alpha_3\alpha_4$, so the diagram in Figure 14 becomes an annulus. The product of shape parameters around $\alpha_1 = \alpha_4$ in the annulus is $\frac{w_a}{w_b}\left(\frac{w_a}{w_b}\right)'\left(\frac{w_a}{w_b}\right)'' = -1$, and the one around $\alpha_2 = \alpha_3$ is also -1. Therefore, if we consider the previous annulus on the right of Figure 14, which shares the edge $\alpha_1\alpha_4$, then we obtain the edge relation of $A_{n_1}F_{n_1}$ trivially.

We remark that the previous annulus always exists because, when we follow the horizontal line in Figure 12(c) backwards, after meeting the under-crossing

---

[5]In fact, edges $\alpha_2\alpha_3$ and $\alpha_5\alpha_4$ are also identified, so the two triangles are cancelled by each other. This means the corresponding tetrahedra $A_{n_1}B_{n_1}D_{n_1}F_{n_1}$ and $C_{n_2}B_{n_2}D_{n_2}F_{n_2}$ are cancelled by each other.

point n_2, we let the next over-crossing point n_3 (see Figure 15) (If n_3 does not exist, then $A_{n_1}F_{n_1} \in \mathcal{A}$ but this violates our assumption). Then a part of the cusp diagram between n_2 and n_3 also forms an annulus, and this is the previous annulus.[6]
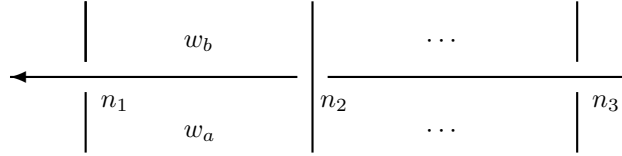

FIGURE 15. Previous annulus

The cases of Figure 12(b) and Figure 12(d) are the same as the cases of Figure 12(a) and Figure 12(c), respectively. Therefore, we find all the edges in $\mathcal{B}$ satisfy the edge relations trivially by the method of parametrizing edges.

Now we assume one of the regions parametrized by w_a or w_b in Figure 12 is an unbounded region. Then the cusp diagram in Figure 13 collapses to an edge $\alpha_2\alpha_3 = \alpha_5\alpha_4$ and the one in Figure 14 collapses to an edge $\alpha_2\alpha_3 = \alpha_1\alpha_4$. Therefore, our arguments for $\mathcal{B}$ still hold for the collapsed case.[7] $\square$

Proof of Proposition 1.1. Consider the function $P_1(w_j, w_k, w_l, w_m)$, which previously appeared in Section 3.2. By direct calculation, we obtain

$$(7) \qquad \exp\left(w_j \frac{\partial P_1}{\partial w_j}\right) = \left(\frac{w_j w_l}{w_k w_m}\right)' \left(\frac{w_m}{w_j}\right)'' \left(\frac{w_k}{w_j}\right)'' ,$$

$$(8) \qquad \exp\left(w_k \frac{\partial P_1}{\partial w_k}\right) = \left(\frac{w_j w_l}{w_k w_m}\right)'' \left(\frac{w_k}{w_l}\right)' \left(\frac{w_k}{w_j}\right)' ,$$

$$(9) \qquad \exp\left(w_l \frac{\partial P_1}{\partial w_l}\right) = \left(\frac{w_j w_l}{w_k w_m}\right)' \left(\frac{w_m}{w_l}\right)'' \left(\frac{w_k}{w_l}\right)'' ,$$

$$(10) \qquad \exp\left(w_m \frac{\partial P_1}{\partial w_m}\right) = \left(\frac{w_j w_l}{w_k w_m}\right)'' \left(\frac{w_m}{w_l}\right)' \left(\frac{w_m}{w_j}\right)' .$$

Note that (7), (8), (9) and (10) are the products of shape parameters assigned to the edges $C_n D_n$, $D_n A_n$, $A_n B_n$ and $B_n C_n$ of Figure 10(a), respectively.[8]

---

[6] As we have seen in the case of Figure 12(a), the crossing points between n_2 and n_3 do not have any effect on the part of the cusp diagram because the triangles in Figure 13 are cancelled by each other. Also, as explained below, the existence of the previous annulus still holds even if some regions between n_2 and n_3 are unbounded.

[7] What we need is to consider the next annuli on the left and right side, and do the same arguments.

[8] For example, consider equation (7) and Figure 10(a). The shape parameters assigned to the edge $C_n D_n$ are $\left(\frac{w_j w_l}{w_k w_m}\right)'$, $\left(\frac{w_m}{w_j}\right)''$ and $\left(\frac{w_k}{w_j}\right)''$, which come from the tetrahedra $C_n D_n A_n B_n$, $C_n D_n B_n F_n$ and $C_n D_n A_n E_n$, respectively.

Also, after evaluating $w_l = 0$ to P_1, we obtain

$$
(11) \qquad \exp\left(w_j \frac{\partial P_1(w_j, w_k, 0, w_m)}{\partial w_j} \right) = \left(\frac{w_m}{w_j} \right)'' \left(\frac{w_k}{w_j} \right)'',
$$

$$
(12) \qquad \exp\left(w_k \frac{\partial P_1(w_j, w_k, 0, w_m)}{\partial w_k} \right) = \frac{w_m}{w_j} \left(\frac{w_k}{w_j} \right)',
$$

$$
(13) \qquad \exp\left(w_m \frac{\partial P_1(w_j, w_k, 0, w_m)}{\partial w_m} \right) = \left(\frac{w_m}{w_j} \right)' \frac{w_k}{w_j}.
$$

Note that (11), (12) and (13) are the products of shape parameters assigned to the edges $C_n D_n$, $D_n A_n$ and $B_n C_n$ of Figure 10(a), respectively, after collapsing the edge $A_n B_n$. Direct calculation shows the same relations hold for P_2, P_3, P_4, N_1, N_2, N_3 and N_4.

Consider the first potential function for the endpoint of I in Section 3.2. Direct calculation shows

(14)
$$
\exp\left(w_l \frac{\partial P_1(w_j, w_j, w_l, w_m)}{\partial w_l} \right) = \exp\left(w_l \frac{\partial P_1(w_j, w_j, w_l, 0)}{\partial w_l} \right) = \left(\frac{w_j}{w_l} \right)'',
$$

(15)
$$
\exp\left(w_m \frac{\partial P_1(w_j, w_j, w_l, w_m)}{\partial w_m} \right) = \exp\left(w_m \frac{\partial P_1(w_j, w_j, 0, w_m)}{\partial w_m} \right) = \left(\frac{w_m}{w_j} \right)',
$$

(16)
$$
\exp\left(w_j \frac{\partial P_1(w_j, w_j, w_l, w_m)}{\partial w_j} \right) = \left(\frac{w_j}{w_m} \right)'' \left(\frac{w_l}{w_j} \right)' = \left(\frac{w_m}{w_j} \right)'' \left(\frac{w_j}{w_l} \right)' \frac{w_m}{w_l},
$$

(17)
$$
\exp\left(w_j \frac{\partial P_1(w_j, w_j, 0, w_m)}{\partial w_j} \right) = \left(\frac{w_j}{w_m} \right)'' = \left(\frac{w_m}{w_j} \right)'' \frac{w_m}{w_j} (-1),
$$

(18)
$$
\exp\left(w_j \frac{\partial P_1(w_j, w_j, w_l, 0)}{\partial w_j} \right) = \left(\frac{w_l}{w_j} \right)' = \left(\frac{w_j}{w_l} \right)' \frac{w_j}{w_l} (-1),
$$

where (14) and (15) are the products of shape parameters assigned to the edges $A_n B_n$ and $B_n C_n$ of Figure 10(a), respectively, after collapsing the edge $D_n E_n$ without or with the collapsing of a horizontal edge.

To explain that (16), (17) and (18) are still parts of edge relations, we need different arguments. First, consider Figure 16.

In Figure 16(a), the product of all shape parameters assigned to the edge expressed by dots is

$$
(19) \qquad \left(\frac{w_a}{w_b} \right)' \left(\frac{w_a}{w_b} \right)'' \left(\frac{w_b}{w_a} \right)' \left(\frac{w_b}{w_a} \right)'' = 1,
$$

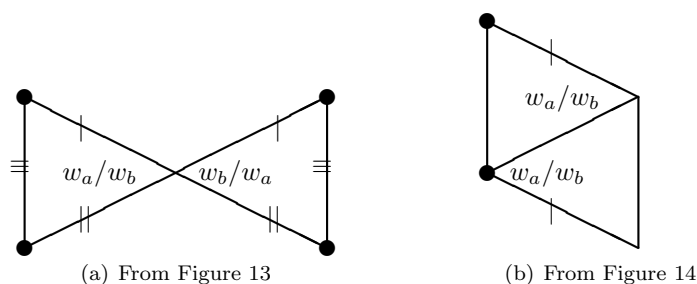(a) From Figure 13　　　　　　(b) From Figure 14

FIGURE 16. Parts of the cusp diagrams from Figure 13 and Figure 14

and in Figure 16(b), the product is

$$(20) \qquad \left(\frac{w_a}{w_b}\right)' \left(\frac{w_a}{w_b}\right)'' \frac{w_a}{w_b} = -1.$$

To see the meaning of (16), consider the following two cases in Figure 17, where n_1 is the endpoint of I and n_2 is the previous over-crossing point. Figure 17(a) means the case when there is no crossing point between n_1 and n_2, and Figure 17(b) means the other case.
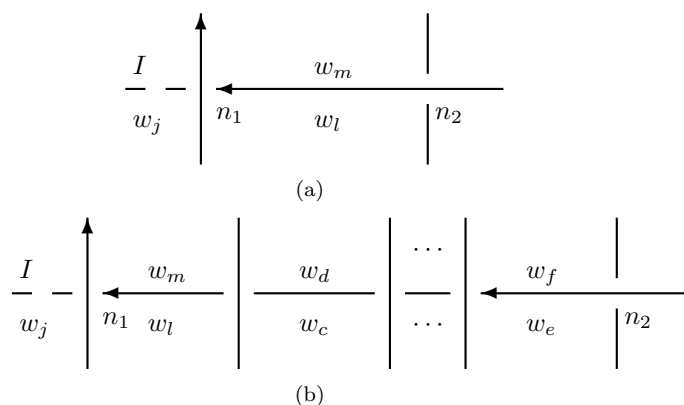

(a)

(b)

FIGURE 17. Two cases after the endpoint of I

Because n_1 is the endpoint of I, the edge $D_{n_1}E_{n_1}$ of the octahedron on n_1 in Figure 10(a) is collapsed to a point $D_{n_1} = E_{n_1}$ and becomes two tetrahedra as in Figure 18 (if one more horizontal edge is collapsed here, the result becomes one tetrahedron. This is the cases of equations (17) and (18)).

The part of the cusp diagrams of each case are in Figure 19 (see Figure 9 and Figure 10 for the assigning rule of the shape parameters).
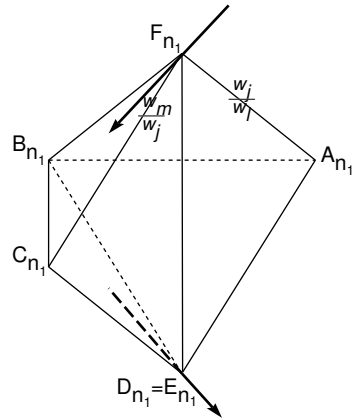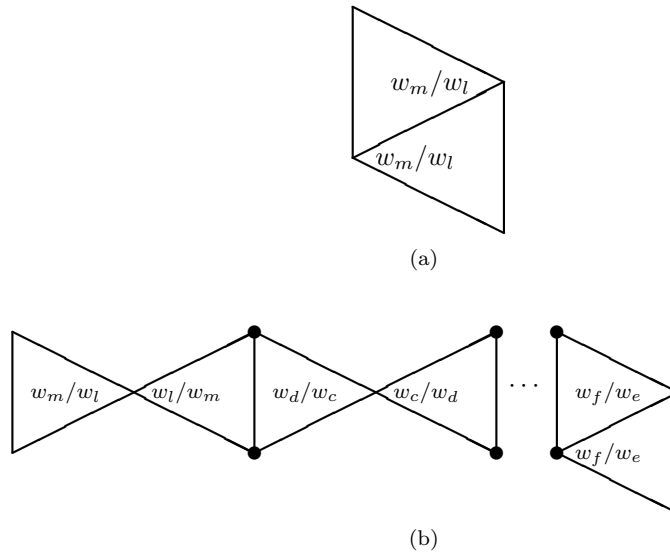
FIGURE 18. Figure 10(a) after collapsing the edge $D_{n_1}E_{n_1}$

(a)

(b)

FIGURE 19. The parts of the cusp diagram corresponding to Figure 17

In the case of Figure 17(a), the product of shape parameters assigned to the edges $C_{n_1}D_{n_1} = D_{n_1}A_{n_1}$ of Figure 18 is $\left(\frac{w_m}{w_j}\right)''\left(\frac{w_j}{w_l}\right)'$. These edges are identified to $C_{n_2}F_{n_2}$, and $\frac{w_l}{w_m}$ is assigned to this edge. This explains that (16)

is the product of shape parameters assigned to the edges $C_{n_1}D_{n_1} = D_{n_1}A_{n_1} = C_{n_2}F_{n_2}$.

In the case of Figure 17(b), the product of shape parameters assigned to the edges $C_{n_1}D_{n_1} = D_{n_1}A_{n_1}$ of Figure 18 is $\left(\frac{w_m}{w_j}\right)'' \left(\frac{w_j}{w_l}\right)'$. In Figure 19(b), these edges are identified to the edges drawn by the dots, and the product of shape parameters assigned to the edges is

$$\left(\frac{w_l}{w_m}\right)' \left(\frac{w_l}{w_m}\right)'' \times 1 \times \cdots \times (-1) = \frac{w_m}{w_l}$$

by (19) and (20). This also explains (16) is the product of shape parameters assigned to $C_{n_1}D_{n_1} = D_{n_1}A_{n_1}$ and some other edges identified to this. This fact is still true[9] even if some of the regions assigned by $w_c, w_d, \ldots, w_e, w_f$ are unbounded regions because the collapsing of the horizontal edges makes the cusp diagrams of Figure 13 and Figure 14 into edges. If the cusp diagram of Figure 13 becomes an edge, then ignoring the diagram is enough for our consideration, and if that of Figure 14 becomes an edge, then considering the previous annulus is enough. The previous annulus always exists because, by the same argument as in the proof of Lemma 4.1, if we choose the next over-crossing point n_3 by following the horizontal lines backwards, the cusp diagram between n_2 and n_3 becomes the previous annulus.[10]

Now we describe the meaning of (17). Let n_1 be the endpoint of I, n_2 be the previous over-crossing point and n_3 be the previous under-crossing point. Also, let $\tilde{n}$ be the previous point of n_1. Assume the edges $D_{n_1}E_{n_1}$ and $A_{n_1}B_{n_1}$ of Figure 10(a) are collapsed. Then $C_{n_1}D_{n_1} = B_{n_1}D_{n_1}$, and $\left(\frac{w_m}{w_j}\right)'' \frac{w_m}{w_j}$ is assigned to this edge. If $\tilde{n} = n_2$, then the edges identified to $C_{n_1}D_{n_1} = B_{n_1}D_{n_1}$ appear between the points $\tilde{n} = n_2$ and n_3 as the dots in Figure 16, and if $\tilde{n} \neq n_2$, then the edges appear between $\tilde{n}$ and n_2 in the same way. Particularly, Figure 16(a) may appear many times, but Figure 16(b) appears only one time at the points n_3 or n_2, respectively. By (19) and (20), the product of all shape parameters assigned to the dots is -1, so (17) is the product of shape parameters assigned to the edges $C_{n_1}D_{n_1} = B_{n_1}D_{n_1}$ and some others identified to these. This fact is still true when some of the horizontal edges or non-horizontal edges of the octahedra are collapsed because of the same reason explained above for the case of (16).

The same relations hold for (18) and the cases of other potential functions of the endpoints of I and J by the same arguments.

---

[9]Even if the endpoint of J lies between the crossings n_1 and n_2, this fact is still true because the collapsing of the non-horizontal edges does not change the part of the cusp diagram we are considering.

[10]There is a concern that the previous annulus is collapsed to an edge, and all the previous annuli, following the horizontal line, are collapsed to edges. However, this cannot happen because Thurston triangulation is a triangulation of the hyperbolic knot complement $S^3 - K$ and we assumed the existence of the geometric solution.

Therefore, we conclude that $\mathcal{H}_2$ becomes all the edge relations of $\mathcal{A}$ except the one horizontal edge whose region is assigned as 0 instead of the variables $w_1, \ldots, w_m$. For an ideal tetrahedron parametrized with $u \in \mathbb{C}$ as in Figure 11, the product of all shape parameters assigned to all edges in the tetrahedron is $(uu'u'')^2 = 1$. This implies the product of all edge relations becomes 1. On the other hand, from Lemma 4.1 and the above arguments, we found all but one edge relation by $\mathcal{H}_2$. Therefore, the remaining edge relation holds automatically.

Finally, we prove $\mathcal{H}_2$ contains the cusp condition. Note that edges $\alpha_1\alpha_4$ and $\alpha_2\alpha_3$ in Figure 14 are meridians of the cusp diagram. The same shape parameter $\frac{w_a}{w_b}$ is assigned to the corners $\angle\alpha_2\alpha_1\alpha_3$ and $\angle\alpha_1\alpha_3\alpha_4$, so one of the cusp conditions is trivially satisfied by the method of assigning shape parameters to edges. If we have all the edge relations and one cusp condition of a meridian, then we can obtain all remaining cusp conditions using these relations. Therefore, we conclude $\mathcal{H}_2$ are the hyperbolicity equations of Thurston triangulation of $S^3 - K$. □

We remark one technical fact. For Thurston triangulation, let the shape parameters of the ideal tetrahedra be $s_1, \ldots, s_h$. These parameters are defined by the ratios of a solution $w_1, \ldots, w_m$ of $\mathcal{H}_2$, so if the values of $w_1, \ldots, w_m$ are fixed, then the values of $s_1, \ldots, s_h$ are uniquely determined and satisfy the hyperbolicity equation. Likewise, if the values of $s_1, \ldots, s_h$ satisfying the hyperbolicity equations are fixed, then we can uniquely determine the solution of $w_1, \ldots, w_m$ of $\mathcal{H}_2$ as follows: First, we can determine some of the values of $w_1, \ldots, w_m$, which are assigned to the regions adjacent to the region assigned with the number 0. Once a value w_l of a region is determined, then all the values of the adjacent regions can be determined. Therefore, all $w_1, \ldots, w_m$ can be determined. Furthermore, those values are well-defined and become a solution of $\mathcal{H}_2$ because of the hyperbolicity equations.

In the next section, we will show the shape parameters of Yokota triangulation determines that of Thurston triangulation, and with certain restriction, vice versa. By the above discussion, this correspondence means each essential solution of $\mathcal{H}_1$ determines a unique solution of $\mathcal{H}_2$. Furthermore, if all the determined solutions of $\mathcal{H}_2$ are essential, then each essential solution of $\mathcal{H}_2$ determines a unique essential solution of $\mathcal{H}_1$.

5. Proof of Theorem 1.3

We start this section with the proof of Lemma 1.2.

Proof of Lemma 1.2. For a hyperbolic ideal octahedron in Figure 20, we assign shape parameters t_1, t_2, t_3, t_4, u_1, u_2, u_3 and u_4 to the edges CD, DA, AB, BC, CF, DE, AF and BE, respectively. Let $u_5 := \frac{1}{u_1u_3} = \frac{1}{u_2u_4}$, which is also a shape parameter assigned to the edges AC and BD of the tetrahedron ABCD.
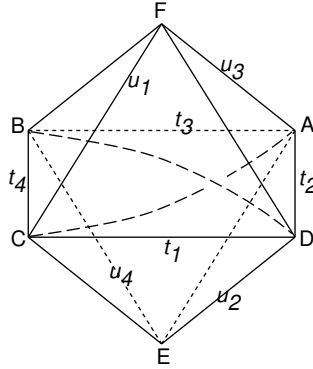
FIGURE 20. Assignment of shape parameters

Then we obtain the following relations.

$$(21) \quad \begin{cases} u_1 = t_1't_4'', \\ u_2 = t_1't_2'', \\ u_3 = t_3't_2'', \\ u_4 = t_3't_4'', \\ u_5 = \left(t_1't_2''t_3't_4''\right)^{-1}, \end{cases} \quad \begin{cases} t_1 = u_1''u_2''u_5', \\ t_2 = u_2'u_3'u_5'', \\ t_3 = u_3''u_4''u_5', \\ t_4 = u_4'u_1'u_5'', \\ t_1t_2t_3t_4 = 1. \end{cases}$$

Note that $t_1, \ldots, t_4$ and $u_1, \ldots, u_5$ are the shape parameters of the tetrahedra in Yokota triangulation and in Thurston triangulation, respectively. According to Observation 2.1, we know these two triangulations are related by 3-2 moves and 4-5 moves on collapsed octahedra and non-collapsed octahedra, respectively. Equation (21) shows the correspondence between the shape parameters under 4-5 moves, so if $t_1, \ldots, t_4 \notin \{0, 1, \infty\}$, then we can determine the values of $u_1, \ldots, u_5$ from the left side of (21). Also, the equation corresponding to 3-2 move can be obtained easily (see (40) for example). This implies that the shape parameters of Yokota triangulation determine that of Thurston triangulation. Furthermore, if all $u_1, \ldots, u_5 \notin \{0, 1, \infty\}$, then the shape parameters of Thurston triangulation recover that of Yokota triangulation by the right side of (21). This completes the proof.   □

Our goal of this section is to prove

$$V_0(z_1, \ldots, z_g) \equiv W_0(w_1, \ldots, w_m) \ (\mathrm{mod} \ 4\pi^2)$$

for any essential solution $(z_1, \ldots, z_g)$ of $\mathcal{H}_1$ and the corresponding essential solution $(w_1, \ldots, w_m)$ of $\mathcal{H}_2$. To prove this, we introduce the dilogarithm identities of an ideal octahedron in Lemma 5.1. Note that the functions $\mathrm{Li}_2(z)$ and $\log z$ are multi-valued functions. Therefore, to obtain well-defined values, we have to select a proper branch of the logarithm by choosing $\arg z$ and $\arg(1-z)$.

Let $D(z) := \text{Im}\,\text{Li}_2(z) + \log|z|\arg(1-z)$ be the Bloch-Wigner function for $z \in \mathbb{C} - \{0,1\}$. It is a well-known fact $D(z)$ is invariant under any choice of log-branch and that $D(z) = -D(\frac{1}{z}) = \text{vol}(T_z)$, where T_z is the hyperbolic ideal tetrahedron with the shape parameter z. Therefore, from Figure 20, we obtain

$$(22)\quad D(t_1)+D(t_2)+D(t_3)+D(t_4) = D(u_1)+D(u_2)+D(u_3)+D(u_4)+D(u_5).$$

Lemma 5.1. *Let $t_1, t_2, t_3, t_4, u_1, u_2, u_3, u_4, u_5 \notin \{0, 1, \infty\}$ be the shape parameters defined in the hyperbolic octahedron in Figure 20 satisfying (21) and (22). Then the following identities hold for any choice of log-branch.*

$$\text{Li}_2(t_1) - \text{Li}_2(\frac{1}{t_2}) + \text{Li}_2(t_3) - \text{Li}_2(\frac{1}{t_4})$$

$$(23)\quad \equiv \text{Li}_2(u_1) + \text{Li}_2(u_2) - \text{Li}_2(\frac{1}{u_3}) - \text{Li}_2(\frac{1}{u_4}) + \text{Li}_2(u_5)$$

$$- \frac{\pi^2}{6} + \log u_1 \log u_2 - \left(-\log(1-t_1) + \log(1-\frac{1}{t_4})\right)\log u_2$$

$$- \left(-\log(1-t_1) + \log(1-\frac{1}{t_2})\right)\log u_1$$

$$+ \left(-\log(1-t_1) + \log(1-\frac{1}{t_4})\right)\log(1-u_1)$$

$$+ \left(-\log(1-t_1) + \log(1-\frac{1}{t_2})\right)\log(1-u_2)$$

$$+ \left(-\log(1-t_3) + \log(1-\frac{1}{t_2})\right)\log(1-\frac{1}{u_3})$$

$$+ \left(-\log(1-t_3) + \log(1-\frac{1}{t_4})\right)\log(1-\frac{1}{u_4})$$

$$+ \left(\log(1-t_1) - \log(1-\frac{1}{t_2})\right.$$

$$\left. + \log(1-t_3) - \log(1-\frac{1}{t_4})\right)\log(1-u_5) \pmod{4\pi^2}$$

$$(24)\quad \equiv \text{Li}_2(u_1) - \text{Li}_2(\frac{1}{u_2}) - \text{Li}_2(\frac{1}{u_3}) + \text{Li}_2(u_4) - \text{Li}_2(\frac{1}{u_5})$$

$$+ \frac{\pi^2}{6} - \log u_2 \log u_3 + \left(-\log(1-t_3) + \log(1-\frac{1}{t_2})\right)\log u_2$$

$$+ \left(-\log(1-t_1) + \log(1-\frac{1}{t_2})\right)\log u_3$$

$$+ \left(-\log(1-t_1) + \log(1-\frac{1}{t_4})\right)\log(1-u_1)$$

$$+ \left(-\log(1-t_1) + \log(1-\frac{1}{t_2})\right)\log(1-\frac{1}{u_2})$$

$$+ \left(-\log(1-t_3) + \log(1 - \frac{1}{t_2}) \right) \log(1 - \frac{1}{u_3})$$

$$+ \left(-\log(1-t_3) + \log(1 - \frac{1}{t_4}) \right) \log(1 - u_4)$$

$$+ \left(\log(1-t_1) - \log(1 - \frac{1}{t_2}) \right.$$

$$\left. + \log(1-t_3) - \log(1 - \frac{1}{t_4}) \right) \log(1 - \frac{1}{u_5}) \pmod{4\pi^2}$$

$$(25) \quad \equiv -\operatorname{Li}_2(\frac{1}{u_1}) - \operatorname{Li}_2(\frac{1}{u_2}) + \operatorname{Li}_2(u_3) + \operatorname{Li}_2(u_4) + \operatorname{Li}_2(u_5)$$

$$- \frac{\pi^2}{6} + \log u_3 \log u_4 - \left(-\log(1-t_3) + \log(1 - \frac{1}{t_4}) \right) \log u_3$$

$$- \left(-\log(1-t_3) + \log(1 - \frac{1}{t_2}) \right) \log u_4$$

$$+ \left(-\log(1-t_1) + \log(1 - \frac{1}{t_4}) \right) \log(1 - \frac{1}{u_1})$$

$$+ \left(-\log(1-t_1) + \log(1 - \frac{1}{t_2}) \right) \log(1 - \frac{1}{u_2})$$

$$+ \left(-\log(1-t_3) + \log(1 - \frac{1}{t_2}) \right) \log(1 - u_3)$$

$$+ \left(-\log(1-t_3) + \log(1 - \frac{1}{t_4}) \right) \log(1 - u_4)$$

$$+ \left(\log(1-t_1) - \log(1 - \frac{1}{t_2}) \right.$$

$$\left. + \log(1-t_3) - \log(1 - \frac{1}{t_4}) \right) \log(1 - u_5) \pmod{4\pi^2}$$

$$(26) \quad \equiv -\operatorname{Li}_2(\frac{1}{u_1}) + \operatorname{Li}_2(u_2) + \operatorname{Li}_2(u_3) - \operatorname{Li}_2(\frac{1}{u_4}) - \operatorname{Li}_2(\frac{1}{u_5})$$

$$+ \frac{\pi^2}{6} - \log u_1 \log u_4 + \left(-\log(1-t_1) + \log(1 - \frac{1}{t_4}) \right) \log u_4$$

$$+ \left(-\log(1-t_3) + \log(1 - \frac{1}{t_4}) \right) \log u_1$$

$$+ \left(-\log(1-t_1) + \log(1 - \frac{1}{t_4}) \right) \log(1 - \frac{1}{u_1})$$

$$+ \left(-\log(1-t_1) + \log(1 - \frac{1}{t_2}) \right) \log(1 - u_2)$$

$$+ \left(-\log(1 - t_3) + \log(1 - \frac{1}{t_2}) \right) \log(1 - u_3)$$

$$+ \left(-\log(1 - t_3) + \log(1 - \frac{1}{t_4}) \right) \log(1 - \frac{1}{u_4})$$

$$+ \left(\log(1 - t_1) - \log(1 - \frac{1}{t_2}) \right.$$

$$\left. + \log(1 - t_3) - \log(1 - \frac{1}{t_4}) \right) \log(1 - \frac{1}{u_5}) \pmod{4\pi^2}.$$

Furthermore,

$$(27) \quad \mathrm{Li}_2(t_1) - \mathrm{Li}_2(\frac{1}{t_2}) - \mathrm{Li}_2(\frac{1}{t_4}) + \frac{\pi^2}{6}$$

$$\equiv \mathrm{Li}_2(u_1) + \mathrm{Li}_2(u_2) - \frac{\pi^2}{6} + \log u_1 \log u_2$$

$$+ \left(-\log(1 - t_1) + \log(1 - \frac{1}{t_4}) \right) \left(-\log u_2 + \log(1 - u_1) \right)$$

$$+ \left(-\log(1 - t_1) + \log(1 - \frac{1}{t_2}) \right) \left(-\log u_1 + \log(1 - u_2) \right) \pmod{4\pi^2}$$

when AB is collapsed to a point,

$$(28) \quad \mathrm{Li}_2(t_1) - \mathrm{Li}_2(\frac{1}{t_2}) + \mathrm{Li}_2(t_3) - \frac{\pi^2}{6}$$

$$\equiv -\mathrm{Li}_2(\frac{1}{u_2}) - \mathrm{Li}_2(\frac{1}{u_3}) + \frac{\pi^2}{6} - \log u_2 \log u_3$$

$$+ \left(-\log(1 - t_3) + \log(1 - \frac{1}{t_2}) \right) \left(\log u_2 + \log(1 - \frac{1}{u_3}) \right)$$

$$+ \left(-\log(1 - t_1) + \log(1 - \frac{1}{t_2}) \right) \left(\log u_3 + \log(1 - \frac{1}{u_2}) \right) \pmod{4\pi^2}$$

when BC is collapsed to a point,

$$(29) \quad -\mathrm{Li}_2(\frac{1}{t_2}) + \mathrm{Li}_2(t_3) - \mathrm{Li}_2(\frac{1}{t_4}) + \frac{\pi^2}{6}$$

$$\equiv \mathrm{Li}_2(u_3) + \mathrm{Li}_2(u_4) - \frac{\pi^2}{6} + \log u_3 \log u_4$$

$$+ \left(-\log(1 - t_3) + \log(1 - \frac{1}{t_4}) \right) \left(-\log u_3 + \log(1 - u_4) \right)$$

$$+ \left(-\log(1 - t_3) + \log(1 - \frac{1}{t_2}) \right) \left(-\log u_4 + \log(1 - u_3) \right) \pmod{4\pi^2}$$

when CD is collapsed to a point, and

$$(30) \quad \mathrm{Li}_2(t_1) + \mathrm{Li}_2(t_3) - \mathrm{Li}_2(\frac{1}{t_4}) - \frac{\pi^2}{6}$$

$$\equiv \ -\operatorname{Li}_2(\frac{1}{u_1}) - \operatorname{Li}_2(\frac{1}{u_4}) + \frac{\pi^2}{6} - \log u_1 \log u_4$$

$$+ \left(-\log(1 - t_1) + \log(1 - \frac{1}{t_4}) \right) \left(\log u_4 + \log(1 - \frac{1}{u_1}) \right)$$

$$+ \left(-\log(1 - t_3) + \log(1 - \frac{1}{t_4}) \right) \left(\log u_1 + \log(1 - \frac{1}{u_4}) \right) \ (\text{mod } 4\pi^2)$$

when DA is collapsed to a point.

Proof. For a function F consisting of dilogarithms and logarithms with certain fixed log-branch, we denote by F^* the same function with different log-branch corresponding to an analytic continuation of F. It is a well-known fact that

(31) $$\operatorname{Li}_2^*(z) \equiv \operatorname{Li}_2(z) + 2a\pi i \log z \ (\text{mod } 4\pi^2)$$

for certain integer a. Let $A := \operatorname{Li}_2(z) - \left(z \frac{\partial \operatorname{Li}_2(z)}{\partial z} \right) \log z$. Then using (31), we have

$$A^* = \operatorname{Li}_2^*(z) - \left(z \frac{\partial \operatorname{Li}_2^*(z)}{\partial z} \right) \log^* z$$

$$\equiv \operatorname{Li}_2(z) + 2a\pi i \log z - \left(z \frac{\partial \operatorname{Li}_2(z)}{\partial z} + 2a\pi i \right) \log^* z$$

$$\equiv \operatorname{Li}_2(z) - \left(z \frac{\partial \operatorname{Li}_2(z)}{\partial z} \right) \log^* z \ (\text{mod } 4\pi^2)$$

and

(32) $$A^* - A \equiv - \left(z \frac{\partial \operatorname{Li}_2(z)}{\partial z} \right) (\log^* z - \log z) \ (\text{mod } 4\pi^2).$$

Similarly, for $B := \operatorname{Li}_2(1/z) - \left(z \frac{\partial \operatorname{Li}_2(1/z)}{\partial z} \right) \log z$, we have

(33) $$B^* - B \equiv - \left(z \frac{\partial \operatorname{Li}_2(1/z)}{\partial z} \right) (\log^* z - \log z) \ (\text{mod } 4\pi^2).$$

Now, we consider (23). Let

$$X(t_1, \ldots, t_4) := \operatorname{Li}_2(t_1) - \operatorname{Li}_2(\frac{1}{t_2}) + \operatorname{Li}_2(t_3) - \operatorname{Li}_2(\frac{1}{t_4}),$$

$$X_0(t_1, \ldots, t_4) := X - \sum_{k=1}^{4} \left(t_k \frac{\partial X}{\partial t_k} \right) \log t_k,$$

$$Y(u_1, \ldots, u_5) := \operatorname{Li}_2(u_1) + \operatorname{Li}_2(u_2) - \operatorname{Li}_2(\frac{1}{u_3}) - \operatorname{Li}_2(\frac{1}{u_4}) + \operatorname{Li}_2(u_5)$$

$$- \frac{\pi^2}{6} + \log u_1 \log u_2,$$

$$Y_0(u_1, \ldots, u_5) := Y - \sum_{l=1}^{5} \left(u_l \frac{\partial Y}{\partial u_l} \right) \log u_l,$$

and
$$Z := (\text{right side of (23)}) - (\text{left side of (23)}).$$
Then by using (32), (33) and

$$\log^* u_1 \log^* u_2 - \log u_1 \log u_2$$
$$= \log^* u_1(\log^* u_2 - \log u_2 + \log u_2) - (\log^* u_1 - \log^* u_1 + \log u_1)\log u_2$$
$$\equiv \log u_1(\log^* u_2 - \log u_2) + \log u_2(\log^* u_1 - \log u_1) \ (\text{mod } 4\pi^2),$$

we obtain

$$(34) \quad (X_0^* - X_0) - (Y_0^* - Y_0)$$
$$\equiv -\sum_{k=1}^{4} t_k \frac{\partial X}{\partial t_k}(\log^* t_k - \log t_k) + \sum_{l=1}^{5} u_l \frac{\partial Y}{\partial u_l}(\log^* u_l - \log u_l) \ (\text{mod } 4\pi^2)$$

and

$$(35) \qquad\qquad u_l \frac{\partial Y^*}{\partial u_l} - u_l \frac{\partial Y}{\partial u_l} \equiv 0 \ (\text{mod } 2\pi i)$$

for $l = 1, \ldots, 5$.

First, we will prove Z is invariant modulo $4\pi^2$ for any choice of log-branch by showing

$$(36) \quad (Z + X_0 - Y_0)^* - (Z + X_0 - Y_0) \equiv (X_0^* - X_0) - (Y_0^* - Y_0) \ (\text{mod } 4\pi^2).$$

Note that

$$(37) \qquad Z + X_0 - Y_0$$
$$= \left(-\log(1 - t_1) + \log(1 - \frac{1}{t_4}) - \log u_1\right)\left(-u_1 \frac{\partial Y}{\partial u_1}\right)$$
$$+ \left(-\log(1 - t_1) + \log(1 - \frac{1}{t_2}) - \log u_2\right)\left(-u_2 \frac{\partial Y}{\partial u_2}\right)$$
$$+ \left(-\log(1 - t_3) + \log(1 - \frac{1}{t_2}) - \log u_3\right)\left(-u_3 \frac{\partial Y}{\partial u_3}\right)$$
$$+ \left(-\log(1 - t_3) + \log(1 - \frac{1}{t_4}) - \log u_4\right)\left(-u_4 \frac{\partial Y}{\partial u_4}\right)$$
$$+ \left(\log(1 - t_1) - \log(1 - \frac{1}{t_2}) + \log(1 - t_3)\right.$$
$$\left. - \log(1 - \frac{1}{t_4}) - \log u_5\right)\left(-u_5 \frac{\partial Y}{\partial u_5}\right) - \sum_{k=1}^{4} t_k \frac{\partial X}{\partial t_k} \log t_k.$$

From (21), we know

$$-\log(1 - t_1) + \log(1 - \frac{1}{t_4}) - \log u_1$$
$$\equiv -\log(1 - t_1) + \log(1 - \frac{1}{t_2}) - \log u_2$$

$$\equiv -\log(1 - t_3) + \log(1 - \frac{1}{t_2}) - \log u_3$$

$$\equiv -\log(1 - t_3) + \log(1 - \frac{1}{t_4}) - \log u_4$$

$$\equiv \log(1 - t_1) - \log(1 - \frac{1}{t_2}) + \log(1 - t_3) - \log(1 - \frac{1}{t_4}) - \log u_5$$

$$\equiv 0 \pmod{2\pi i}.$$

Therefore, from (37) and the above, we have

(38)

$$(Z + X_0 - Y_0)^*$$

$$\equiv \left(-\log^*(1 - t_1) + \log^*(1 - \frac{1}{t_4}) - \log^* u_1 \right) (-u_1 \frac{\partial Y}{\partial u_1})$$

$$+ \left(-\log^*(1 - t_1) + \log^*(1 - \frac{1}{t_2}) - \log^* u_2 \right) (-u_2 \frac{\partial Y}{\partial u_2})$$

$$+ \left(-\log^*(1 - t_3) + \log^*(1 - \frac{1}{t_2}) - \log^* u_3 \right) (-u_3 \frac{\partial Y}{\partial u_3})$$

$$+ \left(-\log^*(1 - t_3) + \log^*(1 - \frac{1}{t_4}) - \log^* u_4 \right) (-u_4 \frac{\partial Y}{\partial u_4})$$

$$+ \left(\log^*(1 - t_1) - \log^*(1 - \frac{1}{t_2}) + \log^*(1 - t_3) \right.$$

$$\left. - \log^*(1 - \frac{1}{t_4}) - \log^* u_5 \right) (-u_5 \frac{\partial Y}{\partial u_5}) - \sum_{k=1}^{4} t_k \frac{\partial X^*}{\partial t_k} \log^* t_k \pmod{4\pi^2}.$$

Combining (37) and (38), we obtain

(39) $(Z + X_0 - Y_0)^* - (Z + X_0 - Y_0)$

$$\equiv \sum_{l=1}^{5} u_l \frac{\partial Y}{\partial u_l} (\log^* u_l - \log u_l)$$

$$+ (\log^*(1 - t_1) - \log(1 - t_1))(u_1 \frac{\partial Y}{\partial u_1} + u_2 \frac{\partial Y}{\partial u_2} - u_5 \frac{\partial Y}{\partial u_5})$$

$$+ (\log^*(1 - \frac{1}{t_2}) - \log(1 - \frac{1}{t_2}))(-u_2 \frac{\partial Y}{\partial u_2} - u_3 \frac{\partial Y}{\partial u_3} + u_5 \frac{\partial Y}{\partial u_5})$$

$$+ (\log^*(1 - t_3) - \log(1 - t_3))(u_3 \frac{\partial Y}{\partial u_3} + u_4 \frac{\partial Y}{\partial u_4} - u_5 \frac{\partial Y}{\partial u_5})$$

$$+ (\log^*(1 - \frac{1}{t_4}) - \log(1 - \frac{1}{t_4}))(-u_1 \frac{\partial Y}{\partial u_1} - u_4 \frac{\partial Y}{\partial u_4} + u_5 \frac{\partial Y}{\partial u_5})$$

$$- \sum_{k=1}^{4} t_k \frac{\partial X^*}{\partial t_k} \log^* t_k + \sum_{k=1}^{4} t_k \frac{\partial X}{\partial t_k} \log t_k \pmod{4\pi^2}.$$

From (21), we know

$$u_1 \frac{\partial Y}{\partial u_1} + u_2 \frac{\partial Y}{\partial u_2} - u_5 \frac{\partial Y}{\partial u_5}$$
$$= -\log(1 - u_1) + \log u_2 - \log(1 - u_2) + \log u_2 + \log(1 - u_5)$$
$$\equiv -\log^* t_1 \pmod{2\pi i},$$

and

$$-u_2 \frac{\partial Y}{\partial u_2} - u_3 \frac{\partial Y}{\partial u_3} + u_5 \frac{\partial Y}{\partial u_5} \equiv -\log^* t_2 \pmod{2\pi i},$$
$$u_3 \frac{\partial Y}{\partial u_3} + u_4 \frac{\partial Y}{\partial u_4} - u_5 \frac{\partial Y}{\partial u_5} \equiv -\log^* t_3 \pmod{2\pi i},$$
$$-u_1 \frac{\partial Y}{\partial u_1} - u_4 \frac{\partial Y}{\partial u_4} + u_5 \frac{\partial Y}{\partial u_5} \equiv -\log^* t_4 \pmod{2\pi i}.$$

Applying (34) and (35) to (39), we obtain

$$(Z + X_0 - Y_0)^* - (Z + X_0 - Y_0)$$

$$\equiv \sum_{l=1}^{5} u_l \frac{\partial Y}{\partial u_l}(\log^* u_l - \log u_l)$$

$$- \sum_{k=1}^{4} (t_k \frac{\partial X^*}{\partial t_k} - t_k \frac{\partial X}{\partial t_k})(-\log^* t_k) - \sum_{k=1}^{4} t_k \frac{\partial X^*}{\partial t_k} \log^* t_k + \sum_{k=1}^{4} t_k \frac{\partial X}{\partial t_k} \log t_k$$

$$\equiv (X_0^* - X_0) - (Y_0^* - Y_0) \pmod{4\pi^2},$$

which shows (36).

Now we will prove $Z = 0$ for certain log-branch. Direct calculation shows the imaginary part of (23) becomes

$$D(t_1) - D(\frac{1}{t_2}) + D(t_3) - D(\frac{1}{t_4}) - \log|t_1| \arg(1 - t_1)$$

$$- \log|t_2| \arg(1 - \frac{1}{t_2}) - \log|t_3| \arg(1 - t_3) - \log|t_4| \arg(1 - \frac{1}{t_4})$$

$$= D(u_1) + D(u_2) - D(\frac{1}{u_3}) - D(\frac{1}{u_4}) + D(u_5) + \log|u_1| \arg u_2 + \arg u_1 \log|u_2|$$

$$- \log|u_1| \arg(1 - u_1) - \log|u_2| \arg(1 - u_2) - \log|u_3| \arg(1 - \frac{1}{u_3})$$

$$- \log|u_4| \arg(1 - \frac{1}{u_4}) - \log|u_5| \arg(1 - u_5)$$

$$- \log|u_1| \arg u_2 - \log|u_2| \arg u_1$$

$$+ \log|u_1| \arg(1 - u_1) + \log|u_2| \arg(1 - u_2) + \log|u_3| \arg(1 - \frac{1}{u_3})$$

$$+ \log|u_4| \arg(1 - \frac{1}{u_4}) + \log|u_5| \arg(1 - u_5)$$

$$- \arg(1 - t_1) \log \left| u_2^{-1} u_1^{-1} (1 - u_1)(1 - u_2)(1 - u_5)^{-1} \right|$$

$$- \arg(1 - \frac{1}{t_2}) \log \left| u_1 (1 - u_2)^{-1} (1 - \frac{1}{u_3})^{-1} (1 - u_5) \right|$$

$$- \arg(1 - t_3) \log \left| (1 - \frac{1}{u_3})(1 - \frac{1}{u_4})(1 - u_5)^{-1} \right|$$

$$- \arg(1 - \frac{1}{t_4}) \log \left| u_2 (1 - u_1)^{-1} (1 - \frac{1}{u_4})^{-1} (1 - u_5) \right|.$$

Using $u_5 = \frac{1}{u_1 u_3} = \frac{1}{u_2 u_4}$, we obtain

$$u_2^{-1} u_1^{-1} (1 - u_1)(1 - u_2)(1 - u_5)^{-1} = u_1'' u_2'' u_5' = t_1,$$

$$u_1 (1 - u_2)^{-1} (1 - \frac{1}{u_3})^{-1} (1 - u_5) = u_2' u_3' u_5'' = t_2,$$

$$(1 - \frac{1}{u_3})(1 - \frac{1}{u_4})(1 - u_5)^{-1} = u_3'' u_4'' u_5' = t_3,$$

$$u_2 (1 - u_1)^{-1} (1 - \frac{1}{u_4})^{-1} (1 - u_5) = u_4' u_1' u_5'' = t_4.$$

By applying these, we can verify the imaginary part of (23) is equivalent to

$$D(t_1) - D(\frac{1}{t_2}) + D(t_3) - D(\frac{1}{t_4}) = D(u_1) + D(u_2) - D(\frac{1}{u_3}) - D(\frac{1}{u_4}) + D(u_5),$$

which is also equivalent to (22). On the other hand, (23) is an analytic function on certain 3-dimensional open set, so the real part is some real constant. After evaluating (23) at $t_1 = t_2 = t_3 = t_4 = u_1 = u_2 = u_3 = u_4 = i$ and $u_5 = -1$,[11] we find the real constant is zero. Therefore, we complete the proof of (23).

The identity (24) can be obtained from (23) by substituting t_1, t_2, t_3, t_4 for $\frac{1}{t_2}, \frac{1}{t_3}, \frac{1}{t_4}, \frac{1}{t_1}$, respectively, and applying the following identity

$$\log \frac{1}{u_2} \log \frac{1}{u_3} + \left(-\log(1 - t_3) + \log(1 - \frac{1}{t_2}) \right) \log \frac{1}{u_2}$$

$$+ \left(-\log(1 - t_1) + \log(1 - \frac{1}{t_2}) \right) \log \frac{1}{u_3}$$

$$= \left(-\log(1 - t_3) + \log(1 - \frac{1}{t_2}) + \log \frac{1}{u_3} \right) \log \frac{1}{u_2}$$

$$+ \left(-\log(1 - t_1) + \log(1 - \frac{1}{t_2}) \right) \log \frac{1}{u_3}$$

$$\equiv - \left(-\log(1 - t_3) + \log(1 - \frac{1}{t_2}) + \log \frac{1}{u_3} \right) \log u_2$$

$$+ \left(-\log(1 - t_1) + \log(1 - \frac{1}{t_2}) \right) \log \frac{1}{u_3}$$

---

[11] Note that $\mathrm{Li}_2(-1) = -\frac{\pi^2}{12}$.

$$= - \left(-\log(1-t_3) + \log(1-\frac{1}{t_2}) \right) \log u_2$$

$$+ \left(-\log(1-t_1) + \log(1-\frac{1}{t_2}) - \log u_2 \right) \log \frac{1}{u_3}$$

$$\equiv \log u_2 \log u_3 - \left(-\log(1-t_3) + \log(1-\frac{1}{t_2}) \right) \log u_2$$

$$- \left(-\log(1-t_1) + \log(1-\frac{1}{t_2}) \right) \log u_3 \pmod{4\pi^2}.$$

The identities (25) and (26) are directly obtained from (23) and (24).

Now we assume the edge AB is collapsed to a point (see Figure 21). Then we obtain the following relations.

$$(40) \qquad \left\{ \begin{array}{l} u_1 = t_1' t_4'', \\ u_2 = t_1' t_2'', \end{array} \right. \qquad \left\{ \begin{array}{l} t_1 = u_1'' u_2'', \\ t_2 = u_1 u_2', \\ t_4 = u_1' u_2, \\ t_1 t_2 t_4 = 1. \end{array} \right.$$
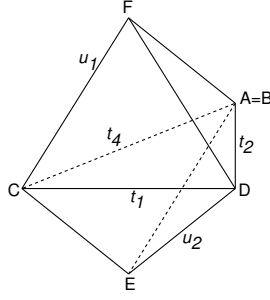

FIGURE 21. Assignment of shape parameters when the edge AB is collapsed

The identity (27) and the relation (40) can be obtained from (23) and (21) by sending $t_3 \to 1$ and using the following property

$$\lim_{t \to 1} \left(\log t \, \log(1-t) \right) = 0.$$

The identities (28), (29) and (30) can be obtained from (24), (25) and (26) by sending $t_4 \to 1$, $t_1 \to 1$ and $t_2 \to 1$, respectively. $\qquad \square$

Proof of Theorem 1.3. Now we prove the theorem by calculating the potential functions on each crossing n. First, consider the case in which no edge of the octahedron on the positive crossing n is collapsed. Let the variables assigned to the contributing sides be $z_a, \ldots, z_d$ as in Figure 9 and let $t_1 = \frac{z_b}{z_a}$, $t_2 = \frac{z_c}{z_b}$,

$t_3 = \frac{z_d}{z_c}$, $t_4 = \frac{z_a}{z_d}$ as in Figure 10(a). Then the Yokota potential function of the crossing becomes

$$X(z_a, \ldots, z_d) := \mathrm{Li}_2(t_1) - \mathrm{Li}_2(\frac{1}{t_2}) + \mathrm{Li}_2(t_3) - \mathrm{Li}_2(\frac{1}{t_4})$$

and

$$(41) \qquad X_0(z_a, \ldots, z_d) = \mathrm{Li}_2(t_1) - \mathrm{Li}_2(\frac{1}{t_2}) + \mathrm{Li}_2(t_3) - \mathrm{Li}_2(\frac{1}{t_4})$$

$$+ \left(-\log(1 - t_1) + \log(1 - \frac{1}{t_4}) \right) \log z_a$$

$$- \left(-\log(1 - t_1) + \log(1 - \frac{1}{t_2}) \right) \log z_b$$

$$+ \left(-\log(1 - t_3) + \log(1 - \frac{1}{t_2}) \right) \log z_c$$

$$- \left(-\log(1 - t_3) + \log(1 - \frac{1}{t_4}) \right) \log z_d.$$

Likewise, let the variables assigned to the regions be $w_j, \ldots, w_m$ as in Figure 9 and let $u_1 = \frac{w_m}{w_j}$, $u_2 = \frac{w_k}{w_j}$, $u_3 = \frac{w_k}{w_l}$, $u_4 = \frac{w_m}{w_l}$, $u_5 = \frac{w_j w_l}{w_k w_m}$ as in Figure 10(a). Then the potential function of the colored Jones polynomial of the crossing becomes P_f, which was defined in Lemma 3.1 for $f = 1, \ldots, 4$, and

(42)

$$P_{10} = \mathrm{Li}_2(u_1) + \mathrm{Li}_2(u_2) - \mathrm{Li}_2(\frac{1}{u_3}) - \mathrm{Li}_2(\frac{1}{u_4}) + \mathrm{Li}_2(u_5) - \frac{\pi^2}{6} + \log u_1 \log u_2$$

$$+ (-\log(1 - u_1) - \log(1 - u_2) + \log(1 - u_5) + \log u_1 + \log u_2) \log w_j$$

$$+ \left(\log(1 - u_2) + \log(1 - \frac{1}{u_3}) - \log(1 - u_5) - \log u_1 \right) \log w_k$$

$$+ \left(-\log(1 - \frac{1}{u_3}) - \log(1 - \frac{1}{u_4}) + \log(1 - u_5) \right) \log w_l$$

$$+ \left(\log(1 - u_1) + \log(1 - \frac{1}{u_4}) - \log(1 - u_5) - \log u_2 \right) \log w_m.$$

We define *the remaining term* Z_n by the difference of two potential functions $V_0 - W_0$ of the crossing n. In this case, $Z_n = X_0 - P_{10}$.

Assume $z_a, \ldots, z_d, w_j, \ldots, w_m$ satisfy the assumption of Lemma 5.1.[12] Let

$$\begin{cases} U_1 := -\log(1 - t_1) + \log(1 - \frac{1}{t_4}), \\ U_2 := -\log(1 - t_1) + \log(1 - \frac{1}{t_2}), \\ U_3 := -\log(1 - t_3) + \log(1 - \frac{1}{t_2}), \\ U_4 := -\log(1 - t_3) + \log(1 - \frac{1}{t_4}), \end{cases}$$

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[12]Any essential solution $(z_a, \ldots, z_d)$ of $\mathcal{H}_1$ and the corresponding essential solution $(w_j, \ldots, w_m)$ of $\mathcal{H}_2$ satisfy this assumption.

$$\begin{cases} T_1 := \log(1 - u_1) + \log(1 - u_2) - \log(1 - u_5) - \log u_1 - \log u_2, \\ T_2 := -\log(1 - u_2) - \log(1 - \frac{1}{u_3}) + \log(1 - u_5) + \log u_1, \\ T_3 := \log(1 - \frac{1}{u_3}) + \log(1 - \frac{1}{u_4}) - \log(1 - u_5), \\ T_4 := -\log(1 - u_1) - \log(1 - \frac{1}{u_4}) + \log(1 - u_5) + \log u_2. \end{cases}$$

Then by (21),

$$\begin{cases} U_1 \equiv \log u_1 \equiv \log w_m - \log w_j \pmod{2\pi i}, \\ U_2 \equiv \log u_2 \equiv \log w_k - \log w_j \pmod{2\pi i}, \\ U_3 \equiv \log u_3 \equiv \log w_k - \log w_l \pmod{2\pi i}, \\ U_4 \equiv \log u_4 \equiv \log w_m - \log w_l \pmod{2\pi i}, \end{cases}$$

$$\begin{cases} T_1 \equiv \log t_1 \equiv \log z_b - \log z_a \pmod{2\pi i}, \\ T_2 \equiv \log t_2 \equiv \log z_c - \log z_b \pmod{2\pi i}, \\ T_3 \equiv \log t_3 \equiv \log z_d - \log z_c \pmod{2\pi i}, \\ T_4 \equiv \log t_4 \equiv \log z_a - \log z_d \pmod{2\pi i}, \end{cases}$$

and $U_1 + U_3 = U_2 + U_4$, $T_1 + T_2 + T_3 + T_4 = 0$. Applying these and (23) to (41) and (42), we obtain the remaining term Z_n of the crossing n as follows.

$$\begin{aligned} Z_n &= X_0 - P_{10} \\ &\equiv U_1 \log z_a - U_2 \log z_b + U_3 \log z_c - U_4 \log z_d \\ &\quad + T_1 \log w_j + T_2 \log w_k + T_3 \log w_l + T_4 \log w_m - U_1 \log u_2 - U_2 \log u_1 \\ &\quad + U_1 \log(1 - u_1) + U_2 \log(1 - u_2) + U_3 \log(1 - \frac{1}{u_3}) \\ &\quad + U_4 \log(1 - \frac{1}{u_4}) - (U_1 + U_3) \log(1 - u_5) \\ &= T_1 \log w_j + T_2 \log w_k + T_3 \log w_l + T_4 \log w_m \\ &\quad + U_1 \left(\log z_a - \log z_d + \log(1 - u_1) + \log(1 - \frac{1}{u_4}) - \log(1 - u_5) - \log u_2 \right) \\ &\quad + U_2 \left(-\log z_b + \log z_d + \log(1 - u_2) - \log(1 - \frac{1}{u_4}) - \log u_1 \right) \\ &\quad + U_3 \left(\log z_c - \log z_d + \log(1 - \frac{1}{u_3}) + \log(1 - \frac{1}{u_4}) - \log(1 - u_5) \right) \\ &= T_2(\log w_k - \log w_j) + T_3(\log w_l - \log w_j) + T_4(\log w_m - \log w_j) \\ &\quad + U_1 (\log z_a - \log z_d - T_4) + U_2 (-\log z_b + \log z_d - T_2 - T_3) \\ &\quad + U_3 (\log z_c - \log z_d + T_3) \\ &\equiv T_2(\log w_k - \log w_j) + T_3(\log w_l - \log w_j) + T_4(\log w_m - \log w_j) \\ &\quad + (\log w_m - \log w_j) (\log z_a - \log z_d - T_4) \\ &\quad + (\log w_k - \log w_j) (-\log z_b + \log z_d - T_2 - T_3) \\ &\quad + (\log w_k - \log w_l) (\log z_c - \log z_d + T_3) \pmod{4\pi^2} \\ &= -(\log w_j - \log w_m) \log z_a - (\log w_k - \log w_j) \log z_b \end{aligned}$$

$$+ (\log w_k - \log w_l) \log z_c + (\log w_l - \log w_m) \log z_d.$$

By the same method, we can prove that the remaining term of the negative crossing in Figure 9 is the same as that of the positive crossing.

Now we consider the case in which only one horizontal edge is collapsed in an octahedron on a positive crossing n. Let the region assigned to r_l be the unbounded region and $z_c = z_d = 1$ in Figure 9. Also, let $t_1 = \frac{z_b}{z_a}$, $t_2 = \frac{1}{z_b}$, $t_4 = z_a$ and $u_1 = \frac{w_m}{w_j}$, $u_2 = \frac{w_k}{w_j}$. Then the Yokota potential function of the crossing becomes

$$X(z_a, z_b) := \mathrm{Li}_2(t_1) - \mathrm{Li}_2(\frac{1}{t_2}) - \mathrm{Li}_2(\frac{1}{t_4}) + \frac{\pi^2}{6}$$

and

$$(43) \qquad X_0(z_a, z_b) = \mathrm{Li}_2(t_1) - \mathrm{Li}_2(\frac{1}{t_2}) - \mathrm{Li}_2(\frac{1}{t_4}) + \frac{\pi^2}{6}$$
$$+ \left(-\log(1 - t_1) + \log(1 - \frac{1}{t_4}) \right) \log z_a$$
$$- \left(-\log(1 - t_1) + \log(1 - \frac{1}{t_2}) \right) \log z_b.$$

The potential function of the colored Jones polynomial of the crossing becomes

$$Y(w_j, w_k, w_m) := P_1(w_j, w_k, 0, w_m) = \mathrm{Li}_2(u_1) + \mathrm{Li}_2(u_2) - \frac{\pi^2}{6} + \log u_1 \log u_2$$

and

$$(44)$$

$$Y_0(w_j, w_k, w_m) = \mathrm{Li}_2(u_1) + \mathrm{Li}_2(u_2) - \frac{\pi^2}{6} + \log u_1 \log u_2$$
$$+ (-\log(1 - u_1) - \log(1 - u_2) + \log u_1 + \log u_2) \log w_j$$
$$+ (\log(1 - u_2) - \log u_1) \log w_k$$
$$+ (\log(1 - u_1) - \log u_2) \log w_m.$$

In this case, the remaining term is $Z_n = X_0 - Y_0$. Let

$$\begin{cases} U_1 := -\log(1 - t_1) + \log(1 - \frac{1}{t_4}), \\ U_2 := -\log(1 - t_1) + \log(1 - \frac{1}{t_2}), \end{cases}$$

$$\begin{cases} T_1 := \log(1 - u_1) + \log(1 - u_2) - \log u_1 - \log u_2, \\ T_2 := -\log(1 - u_2) + \log u_1, \\ T_4 := -\log(1 - u_1) + \log u_2. \end{cases}$$

Then by (40),

$$\begin{cases} U_1 \equiv \log u_1 \equiv \log w_m - \log w_j \pmod{2\pi i}, \\ U_2 \equiv \log u_2 \equiv \log w_k - \log w_j \pmod{2\pi i}, \end{cases}$$

$$\begin{cases} T_1 \equiv \log t_1 \equiv \log z_b - \log z_a \pmod{2\pi i}, \\ T_2 \equiv \log t_2 \equiv -\log z_b \pmod{2\pi i}, \\ T_4 \equiv \log t_4 \equiv \log z_a \pmod{2\pi i}, \end{cases}$$

and $T_1 + T_2 + T_4 = 0$. Applying these and (27) to (43) and (44), we obtain the remaining term Z_n as follows.

$$\begin{aligned} Z_n &= X_0 - Y_0 \\ &\equiv U_1 \log z_a - U_2 \log z_b + T_1 \log w_j + T_2 \log w_k + T_4 \log w_m - U_1 T_4 - U_2 T_2 \\ &= U_1 \log z_a - U_2 \log z_b + T_2(\log w_k - \log w_j - U_2) + T_4(\log w_m - \log w_j - U_1) \\ &\equiv U_1 \log z_a - U_2 \log z_b \\ &\quad - \log z_b(\log w_k - \log w_j - U_2) + \log z_a(\log w_m - \log w_j - U_1) \pmod{4\pi^2} \\ &= -(\log w_j - \log w_m) \log z_a - (\log w_k - \log w_j) \log z_b. \end{aligned}$$

By the same method, we can prove the remaining term of the negative crossing in this case is the same as that of the positive crossing. On the other hand, the remaining term becomes

$$Z_n = -(\log w_k - \log w_j) \log z_b + (\log w_k - \log w_l) \log z_c$$

when the region assigned to w_m is the unbounded region,

$$Z_n = (\log w_k - \log w_l) \log z_c + (\log w_l - \log w_m) \log z_d$$

when the region assigned to w_j is the unbounded region, and

$$Z_n = -(\log w_j - \log w_m) \log z_a + (\log w_l - \log w_m) \log z_d$$

when the region assigned to w_k is the unbounded region.

Now we consider the case when the crossing point n is the endpoint of I or J. There are four cases as in Figure 22. We only prove the case of Figure 22(a) because the others can be proved by the same method.

First, we assume all three regions in Figure 22(a) are bounded. Then in Figure 10(a), the edge $B_n E_n$ is collapsed to a point and $\frac{z_d}{z_c}$, $\frac{z_a}{z_d}$, $\frac{w_m}{w_j}$, $\frac{w_j}{w_l}$ are assigned to the edges $C_n D_n$, $D_n A_n$, $A_n F_n$, $C_n F_n$, respectively. Also, we obtain

$$(45) \qquad \frac{z_d}{z_c} = \left(\frac{w_j}{w_l}\right)'' = 1 - \frac{w_l}{w_j} \quad \text{and} \quad \frac{w_m}{w_j} = \left(\frac{z_a}{z_d}\right)'' = 1 - \frac{z_d}{z_a}.$$

Applying (45) to Yokota potential function $X(z_a, z_c, z_d) := \text{Li}_2(\frac{z_d}{z_c}) - \text{Li}_2(\frac{z_d}{z_a})$, we obtain

$$\begin{aligned} X_0 &= \text{Li}_2(\frac{z_d}{z_c}) - \text{Li}_2(\frac{z_d}{z_a}) + \log(1 - \frac{z_d}{z_a}) \log z_a - \log(1 - \frac{z_d}{z_c}) \log z_c \\ &\quad - \left(-\log(1 - \frac{z_d}{z_c}) + \log(1 - \frac{z_d}{z_a})\right) \log z_d \\ &= \text{Li}_2(\frac{z_d}{z_c}) - \text{Li}_2(1 - \frac{w_m}{w_j}) + \log \frac{w_m}{w_j}(\log z_a - \log z_d) + \log \frac{w_l}{w_j}(\log z_d - \log z_c). \end{aligned}$$
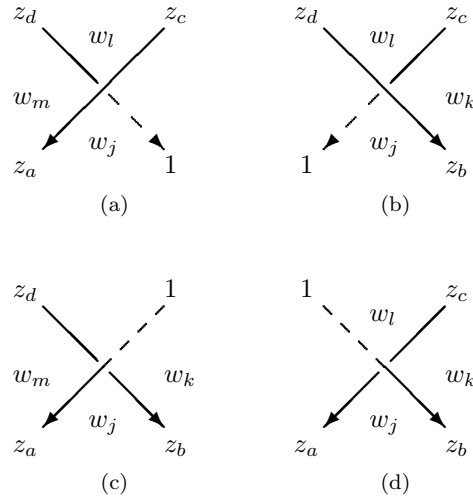
FIGURE 22. Four cases of the endpoint of I or J

Also, applying (45) to the potential function of the colored Jones polynomial
$Y(w_j, w_l, w_m) := P_1(w_j, w_j, w_l, w_m) = \mathrm{Li}_2(\frac{w_m}{w_j}) - \mathrm{Li}_2(\frac{w_l}{w_j})$, we obtain

$$Y_0 = \mathrm{Li}_2(\frac{w_m}{w_j}) - \mathrm{Li}_2(\frac{w_l}{w_j}) - \left(\log(1 - \frac{w_m}{w_j}) - \log(1 - \frac{w_l}{w_j})\right)\log w_j$$
$$- \log(1 - \frac{w_l}{w_j})\log w_l + \log(1 - \frac{w_m}{w_j})\log w_m$$
$$= \mathrm{Li}_2(\frac{w_m}{w_j}) - \mathrm{Li}_2(1 - \frac{z_d}{z_c}) - \log\frac{z_d}{z_a}(\log w_j - \log w_m) - \log\frac{z_d}{z_c}(\log w_l - \log w_j).$$

Using the well-known identity $\mathrm{Li}_2(z) + \mathrm{Li}_2(1 - z) = \frac{\pi^2}{6} - \log z \log(1 - z)$ for
$z \in \mathbb{C} - \{0, 1\}$ from [6], we obtain the remaining term

$$Z_n = X_0 - Y_0$$
$$\equiv -\log\frac{z_d}{z_c}\log\frac{w_l}{w_j} + \log\frac{w_m}{w_j}\log\frac{z_d}{z_a}$$
$$+ \log\frac{w_m}{w_j}(\log z_a - \log z_d) + \log\frac{w_l}{w_j}(\log z_d - \log z_c)$$
$$+ \log\frac{z_d}{z_a}(\log w_j - \log w_m) + \log\frac{z_d}{z_c}(\log w_l - \log w_j)$$
$$= \log\frac{w_l}{w_j}(-\log\frac{z_d}{z_c} + \log z_d - \log z_c) + \log\frac{w_m}{w_j}(\log\frac{z_d}{z_a} + \log z_a - \log z_d)$$
$$+ \log\frac{z_d}{z_a}(\log w_j - \log w_m) + \log\frac{z_d}{z_c}(\log w_l - \log w_j)$$

$$\equiv (\log w_l - \log w_j) \left(-\log \frac{z_d}{z_c} + \log z_d - \log z_c \right)$$

$$+ (\log w_m - \log w_j) \left(\log \frac{z_d}{z_a} + \log z_a - \log z_d \right)$$

$$+ \log \frac{z_d}{z_a} (\log w_j - \log w_m) + \log \frac{z_d}{z_c} (\log w_l - \log w_j) \ (\mathrm{mod} \ 4\pi^2)$$

$$= -(\log w_j - \log w_m) \log z_a + (\log w_j - \log w_l) \log z_c$$

$$+ (\log w_l - \log w_m) \log z_d.$$

Finally, we consider the case when the region assigned with w_l in Figure 22(a) is unbounded. Then the edges $\mathrm{B}_n\mathrm{E}_n$ and $\mathrm{C}_n\mathrm{D}_n$ are collapsed to points. Furthermore, $z_c = z_d = 1$ and $w_l = 0$, and z_a, $\frac{w_m}{w_j}$ are assigned to the edges $\mathrm{D}_n\mathrm{A}_n$, $\mathrm{A}_n\mathrm{F}_n$ in Figure 10(a), respectively. Applying

$$\frac{w_m}{w_j} = z_a'' = 1 - \frac{1}{z_a}$$

to Yokota potential function $X(z_a) := -\mathrm{Li}_2(\frac{1}{z_a}) + \frac{\pi^2}{6}$, we obtain

$$X_0 = -\mathrm{Li}_2(\frac{1}{z_a}) + \frac{\pi^2}{6} + \log(1 - \frac{1}{z_a}) \log z_a = -\mathrm{Li}_2(\frac{1}{z_a}) + \frac{\pi^2}{6} + \log \frac{w_m}{w_j} \log z_a,$$

and to the potential function of the colored Jones polynomial $Y(w_j, w_m) := P_1(w_j, w_j, 0, w_m) = \mathrm{Li}_2(\frac{w_m}{w_j})$, we obtain

$$Y_0 = \mathrm{Li}_2(\frac{w_m}{w_j}) + \log(1 - \frac{w_m}{w_j})(\log w_m - \log w_j)$$

$$= \mathrm{Li}_2(1 - \frac{1}{z_a}) + \log \frac{1}{z_a}(\log w_m - \log w_j).$$

Therefore, we obtain the remaining term

$$Z_n := X_0 - Y_0 \equiv \log \frac{1}{z_a} \log \frac{w_m}{w_j} + \log \frac{w_m}{w_j} \log z_a - \log \frac{1}{z_a}(\log w_m - \log w_j)$$

$$= \log \frac{1}{z_a}(\log \frac{w_m}{w_j} - \log w_m + \log w_j) + \log \frac{w_m}{w_j} \log z_a$$

$$\equiv -\log z_a(\log \frac{w_m}{w_j} - \log w_m + \log w_j) + \log \frac{w_m}{w_j} \log z_a \ (\mathrm{mod} \ 4\pi^2)$$

$$= -(\log w_j - \log w_m) \log z_a.$$

Likewise, we can show the remaining term becomes

$$Z_n = (\log w_l - \log w_m) \log z_d$$

when the region assigned to w_m in Figure 22(a) is unbounded. The remaining three cases in Figure 22 can be obtained by the same method.

We complete the proof by proving

$$\sum_{n \ : \ \mathrm{crossings \ of} \ G} Z_n = 0.$$

Note that we defined a contributing side of G in Section 3.1. Assume the side assigned by z_a in Figure 23 is a contributing side of G (This means that $z_a \neq 1$).
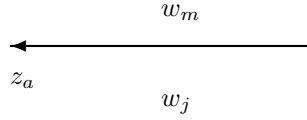

FIGURE 23. Contributing side assigned by z_a

If the side goes out of the crossing point n_1, then the coefficient of $\log z_a$ in Z_{n_1} is $-(\log w_j - \log w_l)$, and if the side goes into the crossing point n_2, then the coefficient of $\log z_a$ in Z_{n_2} is $(\log w_j - \log w_l)$. They are cancelled by each other, and this happens for all the contributing sides. $\qquad\square$

Appendix A. Appendix

A.1. Formal substitution of the colored Jones polynomial and the potential function

In this Appendix, we induce the potential function $W(w_1, \ldots, w_m)$ defined in Section 3.2 from the formal substitution (1) of the colored Jones polynomial.

The colored Jones polynomial is determined by the R-matrix and the local maxima/minima (see [8] for reference). However, as seen in (1), the local maxima/minima do not have an effect on the formal substitution. So we only consider the R-matrix of the colored Jones polynomial:

$$R^{j,k}_{l,m} = \delta_{m,j-h}\delta_{l,k+h} \frac{(q^{-1})_j (q^{-1})_k^{-1}}{(q^{-1})_h (q^{-1})_l^{-1} (q^{-1})_m} (-1)^{k+m+1} q^{-km-(k+m+1)/2},$$

$$(R^{-1})^{j,k}_{l,m} = \delta_{m,j+h}\delta_{l,k-h} \frac{(q)_j^{-1} (q)_k}{(q)_h (q)_l (q)_m^{-1}} (-1)^{j+l+1} q^{jl+(j+l+1)/2},$$

where $j, k, l, m, h \in \{0, 1, \ldots, N-1\}$ and $\delta_{j,k}$ is the Kronecker's delta. If $R^{j,k}_{l,m} \neq 0$, then h is uniquely determined by the formula $h = j - m = l - k$, and if $(R^{-1})^{j,k}_{l,m} \neq 0$, then $h = m - j = k - l$.

Note that this R-matrix is the inverse of the one in [8]. This implies the colored Jones polynomial of a knot K here is the one of the mirror image $\overline{K}$ in [8]. This choice is natural to [17] and Theorem 1.3.

Let K be the hyperbolic knot with a fixed diagram and G be the diagram defined in Section 2.1 with the orientation from J to I. We assign 0 to one bounded region of G, then assign variables $r_1, \ldots, r_m \in \{0, 1, \ldots, N-1\}$ to the remaining bounded regions of G and $r_{m+1} \in \{0, 1, \ldots, N-1\}$ to the unbounded region. We assign variables to each side according to the signed sum of variables of adjacent regions with orientations modulo N (see Figure 24 for an example).
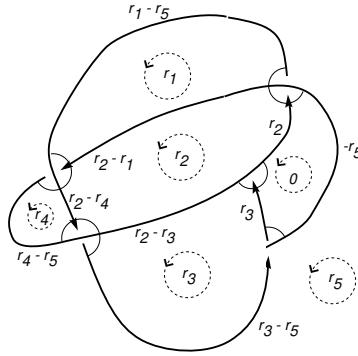
FIGURE 24. Assigning variables to each region and side

For each non-trivalent vertex of G, we assign the R-matrix to the positive crossing and the inverse to the negative crossing. Then we apply the formal substitution (1) to each R-matrix and substitute q^{r_n} to w_n as below. In the substitution process, if $r_n = 0$, then we put $w_n = 1$. Note that we apply the same R-matrix or its inverse in different forms according to the position of the collapsed horizontal edge. If none of the horizontal edges are collapsed in the octahedron, then we choose any formal substitution among the four possibilities. For positive crossings:

$$: \frac{(q)_{r_l-r_m}(q^{-1})_{r_k-r_l}^{-1}}{(q)_{r_j+r_l-r_k-r_m}(q^{-1})_{r_j-r_m}^{-1}(q)_{r_k-r_j}}(-1)^{r_l+r_j+1}$$
$$\times q^{(r_m-r_j)(r_k-r_j)-(2r_k-r_l-r_j+1)/2}$$

$$\sim \exp\left\{\frac{N}{2\pi i}\left(-\text{Li}_2(\tfrac{w_l}{w_m})-\text{Li}_2(\tfrac{w_l}{w_k})+\text{Li}_2(\tfrac{w_j w_l}{w_k w_m})+\text{Li}_2(\tfrac{w_m}{w_j})+\text{Li}_2(\tfrac{w_k}{w_j})-\tfrac{\pi^2}{6}+\log\tfrac{w_m}{w_j}\log\tfrac{w_k}{w_j}\right)\right\},$$

$$: \frac{(q^{-1})_{r_l-r_m}(q^{-1})_{r_k-r_l}^{-1}}{(q^{-1})_{r_j+r_l-r_k-r_m}(q^{-1})_{r_j-r_m}^{-1}(q^{-1})_{r_k-r_j}}(-1)^{r_l+r_j+1}$$
$$\times q^{-(r_k-r_l)(r_k-r_j)-(2r_k-r_l-r_j+1)/2}$$

$$\sim \exp\left\{\frac{N}{2\pi i}\left(\text{Li}_2(\tfrac{w_m}{w_l})-\text{Li}_2(\tfrac{w_l}{w_k})-\text{Li}_2(\tfrac{w_k w_m}{w_j w_l})+\text{Li}_2(\tfrac{w_m}{w_j})-\text{Li}_2(\tfrac{w_j}{w_k})+\tfrac{\pi^2}{6}-\log\tfrac{w_k}{w_l}\log\tfrac{w_k}{w_j}\right)\right\},$$

$$: \quad \frac{(q^{-1})_{r_l-r_m}(q)^{-1}_{r_k-r_l}}{(q)_{r_j+r_l-r_k-r_m}(q)^{-1}_{r_j-r_m}(q^{-1})_{r_k-r_j}}(-1)^{r_l+r_j+1}$$
$$\times q^{(r_m-r_l)(r_k-r_l)-(2r_k-r_l-r_j+1)/2}$$

$$\sim \exp\left\{\frac{N}{2\pi i}\left(\mathrm{Li}_2(\tfrac{w_m}{w_l}) + \mathrm{Li}_2(\tfrac{w_k}{w_l}) + \mathrm{Li}_2(\tfrac{w_j w_l}{w_k w_m}) - \mathrm{Li}_2(\tfrac{w_j}{w_m}) - \mathrm{Li}_2(\tfrac{w_j}{w_k}) - \tfrac{\pi^2}{6} + \log\tfrac{w_m}{w_l}\log\tfrac{w_k}{w_l}\right)\right\},$$

$$: \quad \frac{(q)_{r_l-r_m}(q)^{-1}_{r_k-r_l}}{(q^{-1})_{r_j+r_l-r_k-r_m}(q)^{-1}_{r_j-r_m}(q)_{r_k-r_j}}(-1)^{r_l+r_j+1}$$
$$\times q^{-(r_m-r_l)(r_m-r_j)-(r_l+r_j-2r_m+1)/2}$$

$$\sim \exp\left\{\frac{N}{2\pi i}\left(-\mathrm{Li}_2(\tfrac{w_l}{w_m}) + \mathrm{Li}_2(\tfrac{w_k}{w_l}) - \mathrm{Li}_2(\tfrac{w_k w_m}{w_j w_l}) - \mathrm{Li}_2(\tfrac{w_j}{w_m}) + \mathrm{Li}_2(\tfrac{w_k}{w_j}) + \tfrac{\pi^2}{6} - \log\tfrac{w_m}{w_l}\log\tfrac{w_m}{w_j}\right)\right\}.$$
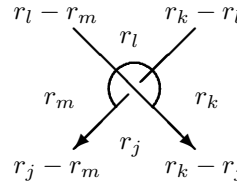
For negative crossings:

$$: \quad \frac{(q)^{-1}_{r_l-r_m}(q^{-1})_{r_k-r_l}}{(q^{-1})_{r_k+r_m-r_j-r_l}(q^{-1})_{r_j-r_m}(q)^{-1}_{r_k-r_j}}(-1)^{r_l+r_j+1}$$
$$\times q^{-(r_j-r_m)(r_j-r_k)+(r_l+r_j-2r_m+1)/2}$$

$$\sim \exp\left\{\frac{N}{2\pi i}\left(\mathrm{Li}_2(\tfrac{w_l}{w_m}) + \mathrm{Li}_2(\tfrac{w_l}{w_k}) - \mathrm{Li}_2(\tfrac{w_j w_l}{w_k w_m}) - \mathrm{Li}_2(\tfrac{w_m}{w_j}) - \mathrm{Li}_2(\tfrac{w_k}{w_j}) + \tfrac{\pi^2}{6} - \log\tfrac{w_j}{w_m}\log\tfrac{w_j}{w_k}\right)\right\},$$

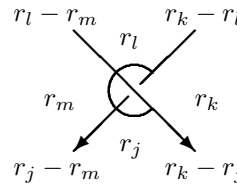$$: \quad \frac{(q^{-1})^{-1}_{r_l-r_m}(q^{-1})_{r_k-r_l}}{(q)_{r_k+r_m-r_j-r_l}(q^{-1})_{r_j-r_m}(q^{-1})^{-1}_{r_k-r_j}}(-1)^{r_l+r_j+1}$$
$$\times q^{(r_l-r_k)(r_j-r_k)+(2r_k-r_l-r_j+1)/2}$$

$$\sim \exp\left\{\frac{N}{2\pi i}\left(-\mathrm{Li}_2(\tfrac{w_m}{w_l}) + \mathrm{Li}_2(\tfrac{w_l}{w_k}) + \mathrm{Li}_2(\tfrac{w_k w_m}{w_j w_l}) - \mathrm{Li}_2(\tfrac{w_m}{w_j}) + \mathrm{Li}_2(\tfrac{w_j}{w_k}) - \tfrac{\pi^2}{6} + \log\tfrac{w_l}{w_k}\log\tfrac{w_j}{w_k}\right)\right\},$$

$r_l - r_m$ $r_k - r_l$

r_l

r_m r_k : $\dfrac{(q^{-1})^{-1}_{r_l-r_m}(q)_{r_k-r_l}}{(q^{-1})_{r_k+r_m-r_j-r_l}(q)_{r_j-r_m}(q^{-1})^{-1}_{r_k-r_j}}(-1)^{r_l+r_j+1}$

r_j

$r_j - r_m$ $r_k - r_j$ $\times q^{-(r_l-r_m)(r_l-r_k)+(r_l+r_j-2r_m+1)/2}$

$\sim \exp\left\{\frac{N}{2\pi i}\left(-\mathrm{Li}_2(\frac{w_m}{w_l}) - \mathrm{Li}_2(\frac{w_k}{w_l}) - \mathrm{Li}_2(\frac{w_j w_l}{w_k w_m}) + \mathrm{Li}_2(\frac{w_j}{w_m}) + \mathrm{Li}_2(\frac{w_j}{w_k}) + \frac{\pi^2}{6} - \log\frac{w_l}{w_m}\log\frac{w_l}{w_k}\right)\right\}$,
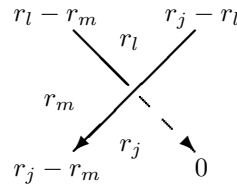
$r_l - r_m$ $r_k - r_l$

r_l

r_m r_k : $\dfrac{(q)^{-1}_{r_l-r_m}(q)_{r_k-r_l}}{(q)_{r_k+r_m-r_j-r_l}(q)_{r_j-r_m}(q)^{-1}_{r_k-r_j}}(-1)^{r_l+r_j+1}$

r_j

$r_j - r_m$ $r_k - r_j$ $\times q^{(r_l-r_m)(r_j-r_m)+(r_l+r_j-2r_m+1)/2}$
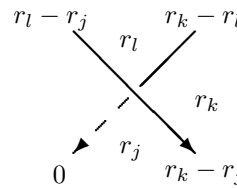
$\sim \exp\left\{\frac{N}{2\pi i}\left(\mathrm{Li}_2(\frac{w_l}{w_m}) - \mathrm{Li}_2(\frac{w_k}{w_l}) + \mathrm{Li}_2(\frac{w_k w_m}{w_j w_l}) + \mathrm{Li}_2(\frac{w_j}{w_m}) - \mathrm{Li}_2(\frac{w_k}{w_j}) - \frac{\pi^2}{6} + \log\frac{w_l}{w_m}\log\frac{w_j}{w_m}\right)\right\}$.

For the trivalent vertices of G, we assign 0 to the sides in I or J, then apply the same formal substitution to the R-matrix as follows (here, we use the same form of the R-matrix disregarding whether certain horizontal edge is collapsed or not).

For the endpoint of I:

$r_l - r_m$ $r_j - r_l$

r_l

r_m : $\dfrac{(q^{-1})_{r_j-r_m}}{(q^{-1})_{r_j-r_l}}(-1)^{r_l+r_j+1}q^{-(r_j-r_l+1)/2}$

r_j

$r_j - r_m$ 0 $\sim \exp\left\{\frac{N}{2\pi i}\left(\mathrm{Li}_2(\frac{w_m}{w_j}) - \mathrm{Li}_2(\frac{w_l}{w_j})\right)\right\}$,

$r_l - r_j$ $r_k - r_l$

r_l

                r_k : $\dfrac{(q)_{r_k-r_j}}{(q)_{r_l-r_j}}(-1)^{r_l+r_j+1}q^{(r_l-r_j-1)/2}$

r_j

0 $r_k - r_j$ $\sim \exp\left\{\frac{N}{2\pi i}\left(-\mathrm{Li}_2(\frac{w_k}{w_j}) + \mathrm{Li}_2(\frac{w_l}{w_j})\right)\right\}$.

For the endpoint of J:

$$: \frac{(q^{-1})_{r_k-r_m}}{(q^{-1})_{r_k-r_j}}(-1)^{r_k+r_j+1}q^{-(r_k-r_j+1)/2}$$

$$\sim \exp\left\{\frac{N}{2\pi i}\left(\mathrm{Li}_2(\tfrac{w_m}{w_k})-\mathrm{Li}_2(\tfrac{w_j}{w_k})\right)\right\},$$

$$: \frac{(q)_{r_k-r_l}}{(q)_{r_j-r_l}}(-1)^{r_l+r_j+1}q^{(r_j-r_l+1)/2}$$

$$\sim \exp\left\{\frac{N}{2\pi i}\left(-\mathrm{Li}_2(\tfrac{w_k}{w_l})+\mathrm{Li}_2(\tfrac{w_j}{w_l})\right)\right\}.$$

Note that the colored Jones polynomial is expressed by the products of various forms of the R-matrices of crossings or trivalent vertices of G (with slight modification by the local maxima/minima) and summed over all the possible indices $r_1,\ldots,r_{m+1}$ (see [8] for the calculation of the colored Jones polynomial; the description in [8] may look slightly different from ours, but removing the sides of the tangle diagram assigned with 0 in [8] gives the diagram G). Now we define a potential function $\widetilde{W}(w_1,\ldots,w_{m+1})$ of the knot diagram by letting the product of all formal substitutions of G to be $\exp\left\{\frac{N}{2\pi i}\widetilde{W}(w_1,\ldots,w_{m+1})\right\}$. One important property of $\widetilde{W}$ is that the variable w_{m+1} assigned to the unbounded region appears only in the numerator. Therefore, we can define another potential function $W(w_1,\ldots,w_m) := \widetilde{W}(w_1,\ldots,w_m,0)$,[13] which coincides with the potential function $W(w_1,\ldots,w_m)$ defined in Section 3.2.

For example, $\widetilde{W}$ and W of Figure 24 become

$$\widetilde{W}(w_1,\ldots,w_5) = \left\{\mathrm{Li}_2(\frac{1}{w_2})-\mathrm{Li}_2(\frac{w_3}{w_2})\right\}+\left\{\mathrm{Li}_2(\frac{w_5}{w_3})-\mathrm{Li}_2(\frac{1}{w_3})\right\}$$

$$+\left\{-\mathrm{Li}_2(\frac{w_5}{w_4})+\mathrm{Li}_2(\frac{w_4}{w_2})+\mathrm{Li}_2(\frac{w_5 w_2}{w_4 w_3})-\mathrm{Li}_2(\frac{w_5}{w_3})\right.$$

$$\left.+\mathrm{Li}_2(\frac{w_3}{w_2})-\frac{\pi^2}{6}+\log\frac{w_4}{w_2}\log\frac{w_3}{w_2}\right\}$$

$$+\left\{-\mathrm{Li}_2(\frac{w_5}{w_1})+\mathrm{Li}_2(\frac{w_1}{w_2})+\mathrm{Li}_2(\frac{w_5 w_2}{w_1 w_4})-\mathrm{Li}_2(\frac{w_5}{w_4})\right.$$

$$\left.+\mathrm{Li}_2(\frac{w_4}{w_2})-\frac{\pi^2}{6}+\log\frac{w_1}{w_2}\log\frac{w_4}{w_2}\right\}$$

$$+\left\{-\mathrm{Li}_2(w_5)+\mathrm{Li}_2(\frac{1}{w_2})+\mathrm{Li}_2(\frac{w_5 w_2}{w_1})-\mathrm{Li}_2(\frac{w_5}{w_1})\right.$$

---

[13]Note that $\mathrm{Li}_2(0)=0$.

$$+ \mathrm{Li}_2(\frac{w_1}{w_2}) - \frac{\pi^2}{6} + \log \frac{1}{w_2} \log \frac{w_1}{w_2} \Big\},$$

and

$$W(w_1, \ldots, w_4) = 2 \left\{ \mathrm{Li}_2(\frac{1}{w_2}) + \mathrm{Li}_2(\frac{w_4}{w_2}) + \mathrm{Li}_2(\frac{w_1}{w_2}) \right\} - \mathrm{Li}_2(\frac{1}{w_3}) - \frac{\pi^2}{2}$$
$$+ \log \frac{w_4}{w_2} \log \frac{w_3}{w_2} + \log \frac{w_1}{w_2} \log \frac{w_4}{w_2} + \log \frac{1}{w_2} \log \frac{w_1}{w_2}.$$

This potential function $W(w_1, \ldots, w_4)$ coincides with the one defined previously in (5).

Note that using W instead of $\widetilde{W}$ does not violate the formulation of the optimistic limit because, for a solution $(w_1^{(0)}, \ldots, w_m^{(0)})$ of

$$\mathcal{H}_2 = \left\{ \exp\left(w_l \frac{\partial W}{\partial w_l} \right) = 1 \,|\, l = 1, \ldots, m \right\},$$

$(w_1^{(0)}, \ldots, w_m^{(0)}, 0)$ becomes a solution of

$$\widetilde{\mathcal{H}}_2 := \left\{ \exp\left(w_l \frac{\partial \widetilde{W}}{\partial w_l} \right) = 1 \,|\, l = 1, \ldots, m+1 \right\}.$$

We are considering only the solutions of $\widetilde{\mathcal{H}}_2$ with the condition $w_{m+1} = 0$ because this condition corresponds to the collapsing process of tetrahedra of Thurston triangulation in Section 2.2 and the solutions correspond to the triangulation. However, other solutions with the condition $w_{m+1} \neq 0$ also have good geometric meanings and this will be discussed in later papers.

A.2. Inessential solutions induced by essential solutions

Let $\mathbf{z}$ and $\mathbf{w}$ be the solutions in Lemma 1.2. In this Appendix, we determine the condition when an essential solution induces an inessential solution. Note that solutions $\mathbf{z}$ and $\mathbf{w}$ uniquely determine shape parameters of ideal tetrahedra in Yokota triangulation and in Thurston triangulation, respectively, and that, by definition, essential solution determines the shape parameters with none of them belonging to $\{0, 1, \infty\}$. Therefore, we focus on the shape parameters of each triangulation. We call the set of shape parameters of ideal tetrahedra *essential* when no elements of it belongs to $\{0, 1, \infty\}$.

Note that the shape parameters of two triangulations are determined by the local picture at each crossings and that, from Observation 2.1, what we have to consider are 3-2 moves and 4-5 moves at the crossings. Consider the two cases of Figure 20 and Figure 21, which correspond to 4-5 move and 3-2 move, respectively, and for which we have the determining relations of shape parameters in (21) and in (40), respectively.

Lemma A.1.   (1) *In Figure 21, if $\{t_1, t_2, t_4\}$ is essential, then $\{u_1, u_2\}$ is essential. Conversely, if $\{u_1, u_2\}$ is essential, then $\{t_1, t_2, t_4\}$ is essential if and only if*

$$(46) \qquad\qquad u_1 + u_2 = 1.$$

(2) *In Figure 20, if $\{t_1, t_2, t_3, t_4\}$ is essential, then $\{u_1, u_2, u_3, u_4, u_5\}$ is essential if and only if*

$$(47)\quad t_1 - \frac{1}{t_2} \neq 0,\ \frac{1}{t_2} - t_3 \neq 0,\ t_3 - \frac{1}{t_4} \neq 0,\ \frac{1}{t_4} - t_1 \neq 0,\ t_1 - \frac{1}{t_2} + t_3 - \frac{1}{t_4} \neq 0$$

(Note that $u_5 = \frac{1}{u_1 u_3} = \frac{1}{u_2 u_4}$). Conversely, if $\{u_1, u_2, u_3, u_4, u_5\}$ is essential, then $\{t_1, t_2, t_3, t_4\}$ is essential if and only if

$$(48) \qquad \begin{cases} \frac{1}{u_1} + \frac{1}{u_2} - \frac{1}{u_1 u_2} \neq u_5, \\ u_2 + u_3 - u_2 u_3 \neq \frac{1}{u_5}, \\ \frac{1}{u_3} + \frac{1}{u_4} - \frac{1}{u_3 u_4} \neq u_5, \\ u_4 + u_1 - u_4 u_1 \neq \frac{1}{u_5}. \end{cases}$$

Proof. From the relations (40) and (21), if one of the sets $\{t_1, t_2, t_3, t_4\}$ and $\{u_1, u_2, u_3, u_4, u_5\}$ is essential, then the shape parameters of the other set are expressed by products of nonzero and non-infinity numbers. This implies any shape parameter in the other set cannot be zero nor infinity. Therefore, what we have to check is the case when $t_k = 1$ or $u_l = 1$ for some k, l.

Consider Figure 21. Assume $\{t_1, t_2, t_4\}$ is essential and $u_1 = 1$. Then from $u_1 = t_1' t_4'' = 1$, we obtain $t_1 t_4 = 1$. Using $t_1 t_2 t_4 = 1$, this induces $t_2 = 1$, which contradicts the essentiality of $\{t_1, t_2, t_4\}$. The case when $u_2 = 1$ is the same.

Conversely, assume $\{u_1, u_2\}$ is essential. By direct calculation from (40), we obtain

$$t_1 = u_1'' u_2'' = 1 \iff u_1 + u_2 = 1 \iff t_2 = u_1 u_2' = 1 \iff t_4 = u_1' u_2 = 1.$$

Now consider Figure 20. Assume $\{t_1, t_2, t_3, t_4\}$ is essential. Then direct calculation from (21) shows (47) is equivalent to

$$u_2 \neq 1,\ u_3 \neq 1,\ u_4 \neq 1,\ u_1 \neq 1,\ u_5 \neq 1.$$

For example, using $t_1 t_3 = \frac{1}{t_2 t_4}$, we have

$$u_5 = (t_1' t_2'' t_3' t_4'')^{-1} = 1 \iff (1 - \frac{1}{t_2})(1 - \frac{1}{t_4}) = (1 - t_1)(1 - t_3)$$

$$\iff t_1 - \frac{1}{t_2} + t_3 - \frac{1}{t_4} = 0.$$

Conversely, assume $\{u_1, u_2, u_3, u_4, u_5\}$ is essential. Then direct calculation from (21) shows (48) is equivalent to

$$t_1 \neq 1,\ t_2 \neq 1,\ t_3 \neq 1,\ t_4 \neq 1. \qquad\qquad \square$$

From the above, if the essential solution **z** in Lemma 1.2 determines the shape parameters of Yokota triangulation that satisfy the conditions (47) in Lemma A.1, then the corresponding solution **w** is also essential. Conversely, the essential solution **w** in Lemma 1.2 determines the shape parameters of Thurston triangulation that satisfy the conditions (46) and (48) in Lemma A.1, then the corresponding solution **z** is also essential. We expect these conditions hold for almost all cases. For example, the essential solutions **z** and **w** of twist knots in [3] and [2], and the geometric solutions **w** of the two-bridge knots in [13] satisfy these conditions. Furthermore, if every octahedron in the Yokota triangulation has a collapsed horizontal edge, then the essential solution **z** always satisfies the condition. Therefore, essential solutions **z** coming from the standard diagrams of 2-bridge knots in [13] always induce the essential solutions **w**.

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