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Optimization algorithm of parameters in the Tank model

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Abstract

This paper proposes the optimization technique to identify the parameters in Sugawara's Tank model that has been widely employed for long-term runoff analysis. Researchers have developed some mathematical optimization methods to identify unknown parameters, notably, the Newton method, the Powell method and the Davidon-Fletcher-Powell method. Computational burden is too stringent in application of the Powell method to parameter optimization in the Tank model, because a great number of runoff computations are required.
In this paper, the unknown parameters are identified essentially by the Newton method. The efficiency of optimization performance resulting from use of the Newton method clearly depends on how effectively the sensitivity coefficients can be derived. An important feature of the proposed approach is the theoretical derivation of sensitivity coefficients which can directly be used in the optimization scheme of the Newton method.
A vector differential equation in terms of storages in the tanks is numerically integrated using the transition matrix which is computed by expanding the matrix exponential. The elements of this transition matrix play a significant role in eliminating additional computations involved in the solution of sensitivity coefficients. On the basis of simulation results, the Newton method coupled with sensitivity coefficients appears to have potential performance advantages for optimizing the parameters in the Tank model.

1 Runoff model

A Tank model adopted in this study is well known as excellent model because this model includes the mechanism of rainfall loss.

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The four-cascade Tank model is shown in Figure 1. The variables are the discharge Q and the storage depth in each tank, which are denoted by S_1, S_2, S_3, and S_4, respectively. The unknown parameters are the coefficients of lateral runoff orifice a_1, a_2, a_3, a_4, and a_5, and the coefficients of permeable orifice b_1, b_2, and b_3. The others are the heights of each orifice z_1, z_2, and z_3. The continuity equation of the Tank model is shown as follows:

$$\begin{aligned} \frac{dS_1}{dt} &= r - q_1 - q_2 - p_1 \\ \frac{dS_2}{dt} &= p_1 - q_3 - p_2 \\ \frac{dS_3}{dt} &= p_2 - q_4 - p_3 \\ \frac{dS_4}{dt} &= p_3 - q_3 \end{aligned} \qquad (1)$$

where

$$q_j = a_j \times (S_i - z_j) \times Y(S_i, z_j), \quad i = 1,2,3, \quad j = 1,2,3,4$$

$$q_5 = a_5 \times S_4 \qquad (2)$$

$$p_j = b_j \times S_j, \quad j = 1,2,3$$

$$Y(S_i, z_j) = \frac{1}{\pi}\left\{\tan^{-1}\frac{S_i - z_j}{\delta} + \frac{\pi}{2}\right\} \qquad (3)$$

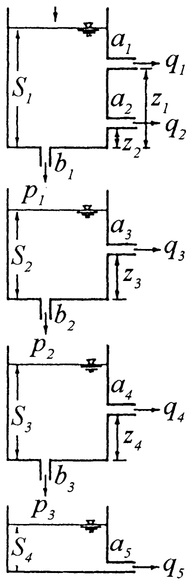

Figure 1: Tank model

in which r is rainfall intensity and q_j is discharge from the lateral orifice. Equation (3) is a Heaviside function and $Y(S_i,z_j)$ is denoted by $Y_{i,j}$ hereafter. An appropriate value for δ is 10^{-6}.

By introducing the special function, $Y_{i,j}$, into the continuity equation, calculation becomes considerably simple as there is no need to consider the relationship between the runoff orifice height and the storage depth.

The discharge is calculated as follows:

$$Q(t) = q_1 Y_{1,1} + q_2 Y_{1,2} + q_3 Y_{2,3} + q_4 Y_{3,4} + q_5 \qquad (4)$$

Equations (2) and (3) are substituted into (1) and the resulting expressions are rearranged to yield

$$\frac{d\mathbf{S}}{d\mathbf{t}} = \mathbf{AS} + \mathbf{B} \qquad (5)$$

where $\mathbf{S} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix}$ $\mathbf{B} = \begin{bmatrix} a_1 z_1 Y_{1,1} + a_2 z_2 Y_{1,2} + r \\ a_3 z_3 Y_{2,3} \\ a_4 z_4 Y_{3,4} \\ 0 \end{bmatrix}$

$$\mathbf{A} = \begin{bmatrix} -a_1 Y_{1,1} - a_2 Y_{1,2} - b_1 & 0 & 0 & 0 \\ b_1 & -a_3 Y_{2,3} - b_2 & 0 & 0 \\ 0 & b_2 & -a_4 Y_{3,4} - b_3 & 0 \\ 0 & 0 & b_3 & -a_5 \end{bmatrix}$$

Equation (5) can be transformed into a discrete equation as follows:

$$\mathbf{S}_{k+1} = \mathbf{\Phi S}_k + \mathbf{\Gamma B} \tag{6}$$

where $\mathbf{\Phi} = \mathbf{exp(AT)}$ $\quad \Gamma = (\mathbf{exp(AT) - I)A}^{-1}$

$\mathbf{T}$ is the time increment, $\mathbf{I}$ is the identity matrix and k is the discrete time instant. The solution of eq. (6) can easily be obtained and is equivalent to the solution of eq. (5).

2 Parameter optimization method

The objective of optimization technique is to find the optimal value of parameters in such a way that the sum of sequences of difference between the observed and computed hydrographs is minimized, which is called the Newton-Raphson method.

2.1 Sensitivity equation

The Newton-Raphson method herein for identifying the model parameters requires the partial derivatives of Q with respect to each parameter, i.e.,sensitivity coefficients. The sensitivity coefficients are analytically derived by differentiating eq. (1) with respect to a_i(i=1,2,3,4,5), b_i (i=1,2,3), and z_i (i=1,2,3,4), in turn. The resultant system of equation is represented as follows:

$$\frac{d}{dt}\begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \end{bmatrix} = \begin{bmatrix} -(a_1 Y_{1,1} + a_2 Y_{1,2} + b_2)\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ b_1 \mathbf{I} & -(a_3 Y_{2,3} + b_2)\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -(a_4 Y_{3,4} + b_3)\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -a_5 \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \\ \mathbf{B}_4 \end{bmatrix} \tag{7}$$

$$\mathbf{u}_1 = \begin{bmatrix} \frac{\partial S_1}{\partial a_1} \\ \frac{\partial S_1}{\partial a_2} \\ \bullet \\ \bullet \\ \frac{\partial S_1}{\partial z_4} \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} \frac{\partial S_2}{\partial a_1} \\ \frac{\partial S_2}{\partial a_2} \\ \bullet \\ \bullet \\ \frac{\partial S_2}{\partial z_4} \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} \frac{\partial S_3}{\partial a_1} \\ \frac{\partial S_3}{\partial a_2} \\ \bullet \\ \bullet \\ \frac{\partial S_3}{\partial z_4} \end{bmatrix} \quad \mathbf{u}_4 = \begin{bmatrix} \frac{\partial S_4}{\partial a_1} \\ \frac{\partial S_4}{\partial a_2} \\ \bullet \\ \bullet \\ \frac{\partial S_4}{\partial z_4} \end{bmatrix}$$

$$\mathbf{B}_1 = [(z_1 - S_1)Y_{1,1} \quad (z_2 - S_1)Y_{1,2} \quad 0 \;\, 0 \;\, 0 \quad -S_1 \quad 0 \;\, 0 \quad a_1 Y_{1,1} \quad a_2 Y_{1,2} \quad 0 \; 0]^T$$

$$\mathbf{B}_2 = [0 \;\, 0 \quad (z_3 - S_2)Y_{2,3} \quad 0 \;\, 0 \quad S_1 \quad -S_2 \quad 0 \;\, 0 \;\, 0 \quad a_3 Y_{2,3} \quad 0]^T$$

$$\mathbf{B}_3 = [0 \;\, 0 \;\, 0 \quad (z_4 - S_3)Y_{3,4} \quad 0 \;\, 0 \quad S_2 \quad -S_3 \quad 0 \;\, 0 \;\, 0 \quad a_4 Y_{3,4}]^T$$

$$\mathbf{B}_4 = [0 \;\, 0 \;\, 0 \;\, 0 \quad -S_4 \quad 0 \;\, 0 \quad S_3 \quad 0 \;\, 0 \;\, 0 \quad 0]^T$$

where **I** is the identity matrix; **0** is the null matrix; T denotes the transpose of matrix as superscript. The vectors $\mathbf{u}_i$ represent the sensitivity coefficients of S_i. Equation (7) can be rewritten as a discrete form as follows:

$$\begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \end{bmatrix}_{k+1} = \begin{bmatrix} \phi_{11}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \phi_{21}\mathbf{I} & \phi_{22}\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \phi_{31}\mathbf{I} & \phi_{32}\mathbf{I} & \phi_{33}\mathbf{I} & \mathbf{0} \\ \phi_{41}\mathbf{I} & \phi_{42}\mathbf{I} & \phi_{43}\mathbf{I} & \phi_{44}\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \mathbf{u}_4 \end{bmatrix}_k + \begin{bmatrix} \gamma_{11}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \gamma_{21}\mathbf{I} & \gamma_{22}\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \gamma_{31}\mathbf{I} & \gamma_{32}\mathbf{I} & \gamma_{33}\mathbf{I} & \mathbf{0} \\ \gamma_{41}\mathbf{I} & \gamma_{42}\mathbf{I} & \gamma_{43}\mathbf{I} & \gamma_{44}\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \\ \mathbf{B}_3 \\ \mathbf{B}_4 \end{bmatrix} \tag{8}$$

where the components, ϕ_{ij} , γ_{ij} are the same coefficients of eq. (6).

2.2 Objective function

The objective function is defined by

$$J = Min \Sigma E_j^2 \qquad j = 1, 2, 3, \cdot \cdot \cdot, N \tag{9}$$
$$E_j = \frac{q_j^* - q_j}{\sqrt{q_j^*}}$$

where q_j^* is the observed discharge at discrete time, j, q_j is the corresponding computed discharge, and N is the sample size of discharge data.

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Given the input data, r_j and prescribed parameter values, the differential equation (1) can be numerically solved for output sequence, q_j. The parameters are then adjusted in an objective manner until the objective function is minimized. The above condition is accomplished by the following iterative scheme:

$$\mathbf{P}^{m+1} = \mathbf{P}^m + \Delta\mathbf{P}^m \qquad \mathbf{P}^m = \begin{bmatrix} a_1^m \\ a_2^m \\ a_3^m \\ \bullet \\ z_4^m \end{bmatrix} \qquad \Delta\mathbf{P}^m = \begin{bmatrix} \Delta a_1^m \\ \Delta a_2^m \\ \Delta a_3^m \\ \bullet \\ \Delta z_4^m \end{bmatrix}$$

where $\mathbf{P}^{m+1}$ represents the parameter vector at $(m+1)$th iteration, and $\Delta\mathbf{P}^m$ is the correction vector of parameters. The iteration is continued until the absolute error $|\Delta\mathbf{P}^m/\mathbf{P}^m|$ is less than the preassigned tolerance ε, i.e., the termination criterion is

$$\left|\frac{\Delta\mathbf{P}^m}{\mathbf{P}^m}\right| \leq \varepsilon \qquad (10)$$

A value for ε is 0.01% in this study. The correction vector, $\Delta\mathbf{P}^m$ can be determined by a least-squares method as follows:

$$\Delta\mathbf{P}^m = [\mathbf{X}^T\mathbf{X}]^{-1}[\mathbf{X}^T\mathbf{E}] \qquad (11)$$

$$\mathbf{E} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ \bullet \\ E_N \end{bmatrix} \qquad \mathbf{X} = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \bullet & \bullet & \bullet & x_{1,12} \\ x_{2,1} & x_{2,2} & x_{2,3} & \bullet & \bullet & \bullet & x_{2,12} \\ x_{3,1} & x_{3,2} & x_{3,3} & \bullet & \bullet & \bullet & x_{3,12} \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ x_{N,1} & x_{N,2} & x_{N,3} & \bullet & \bullet & \bullet & x_{N,12} \end{bmatrix}$$

The quantity of E_j is the error between the observed and computed values for the discharge, and $x_{i,j}$ included the sensitivity coefficients is expressed as follows:

$$x_{i,1} = \frac{1}{\sqrt{q_i^*}}\frac{\partial q}{\partial a_1}, \quad x_{i,2} = \frac{1}{\sqrt{q_i^*}}\frac{\partial q}{\partial a_2}, \quad x_{i,3} = \frac{1}{\sqrt{q_i^*}}\frac{\partial q}{\partial a_3} \quad \cdots\cdots$$

If the eq. (10) is satisfied, the value $\mathbf{P}^{m+1}$ is taken as optimal estimates of the model parameters.

Two considerations should be laid on applying a least-squares approach to eq. (11); the first problem is such that the order of magnitude in columns of **X** is greatly different and second is such that strong correlation between columns of **X** are anticipated to exist. Poor performance of optimization might take place in such circumstances, because the response surface of the objective function becomes flat. The first difficulty can conveniently be overcome by the normalization technique in columns in such a way that the diagonal elements of $[\mathbf{X}^T\mathbf{X}]$ become unity. The second problem is also surmounted by use of least-squares differential correction coupled with components regression in such that all the $\mathbf{X}$ columns are decomposed into independent columns via the technique of lower triangular Cholesky factorization on which the error vector $\mathbf{E}$ is regressed. These two strategies in solving eq. (11) for $\Delta\mathbf{P}^m$ were found to be extremely powerful to search the optimized values in the multidimensional space.

3 Conclusion

A simulation method was implemented to verify the usefulness of the proposed approach under the assumption that the true values of the parameters were known. The discharge resulting from the use of known values of parameters was assumed to be the observed data. The initial values of the unknown were set up as being different from the values among +30% and -30%.
Table 1 shows the true and initial values of each parameter and the number of iterations for the convergence. Consequently, the estimates of unknown parameters in the Tank model converged to the true values in less than ten iterations.

Table 1: Result of the simulation method

parameter	a_1	a_2	a_3	a_4	a_5	b_1
true value	0.2	0.1	0.05	0.01	0.002	0.1
20%	0.24	0.12	0.06	0.012	0.0024	0.08
20%	0.16	0.08	0.04	0.008	0.0016	0.12
30%	0.26	0.13	0.065	0.013	0.0026	0.07
30%	0.14	0.07	0.035	0.007	0.0014	0.13

b_2	b_3	z_1 (cm)	z_2 (cm)	z_3 (cm)	z_4 (cm)	number of iterations
0.08	0.03	10	5	3	1	
0.064	0.024	12	6	3.6	1.2	7
0.096	0.036	8	4	2.4	0.8	10
0.056	0.021	13	6.5	3.9	1.3	9
0.104	0.039	7	3.5	2.1	0.7	6

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