Optimization and approximation problems related to polynomial system solving

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Introduction

Many problems in mathematics, computer science, engineering closely related to

polynomial systems

Typical questions are concerned with

- solvability
- good estimates for number of solutions
- computing solutions, f.e. numerically

Important from computer science point of view

complexity of dealing with above tasks

Some issues related to polynomial systems treated here

- example from mechanism design
- (non-) existence of approximation algorithms in combinatorial optimization
- probabilistically checkable proofs PCPs

Framework:

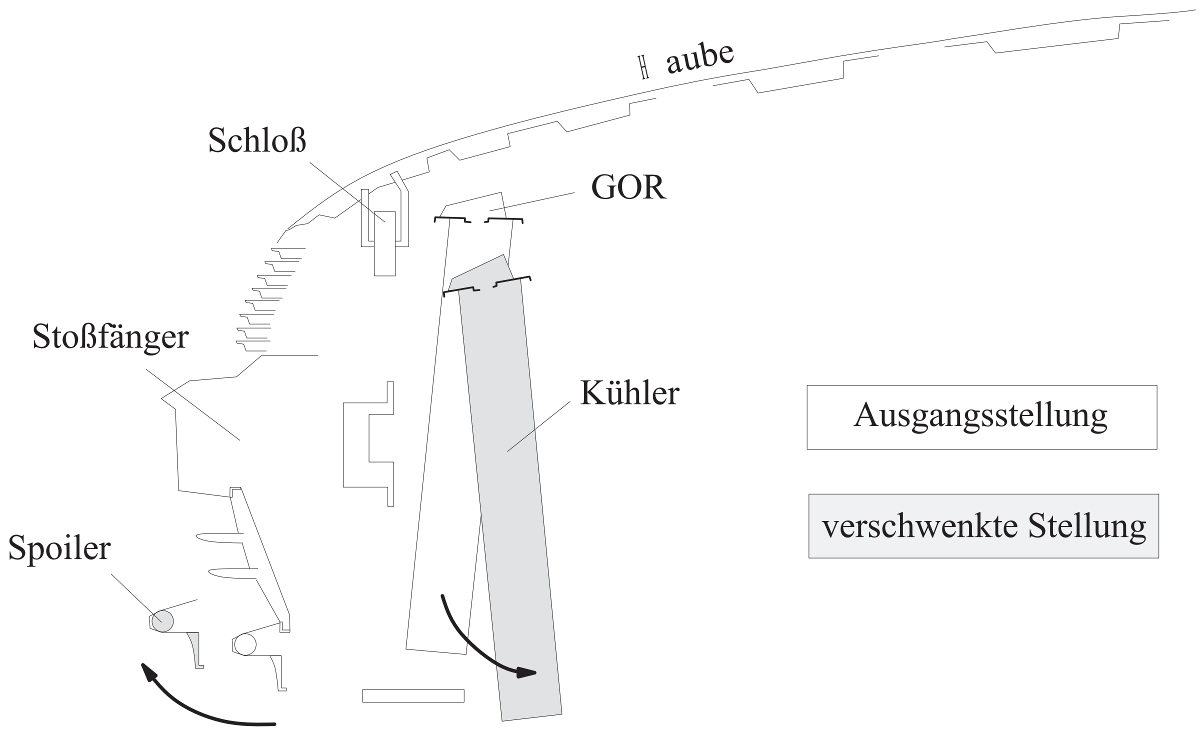
Both classical (Turing) and real number (Blum-Shub-Smale) complexity theory

1. Motivating example: Motion synthesis in robotics

Many practical problems within computational geometry result in question, whether a polynomial system is solvable

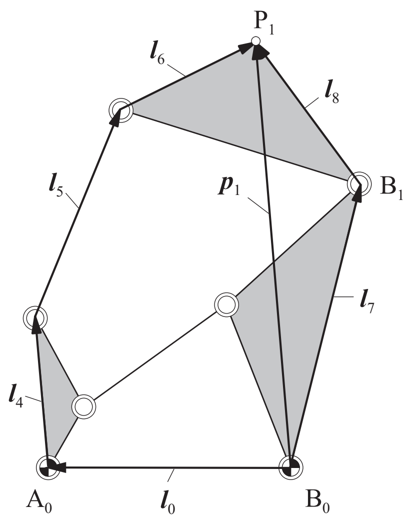
Here: Design of certain mechanisms in mechanical engineering

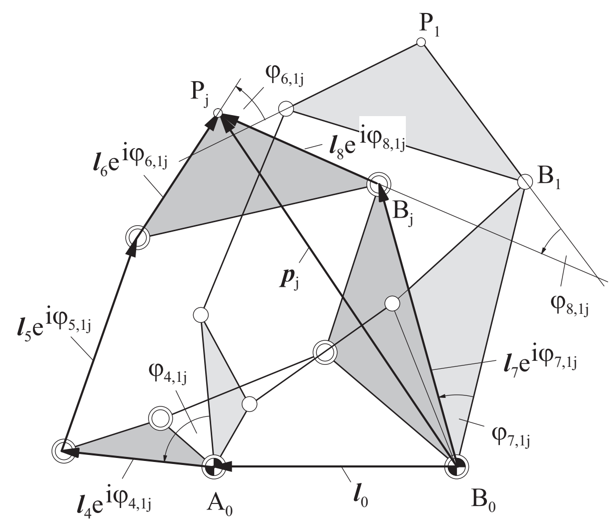
Example: Protection of pedestrians in traffic



Required motion of cooler and spoiler

Problem in kinematics: Design gearing mechanism satisfying certain demands





Stephenson gear

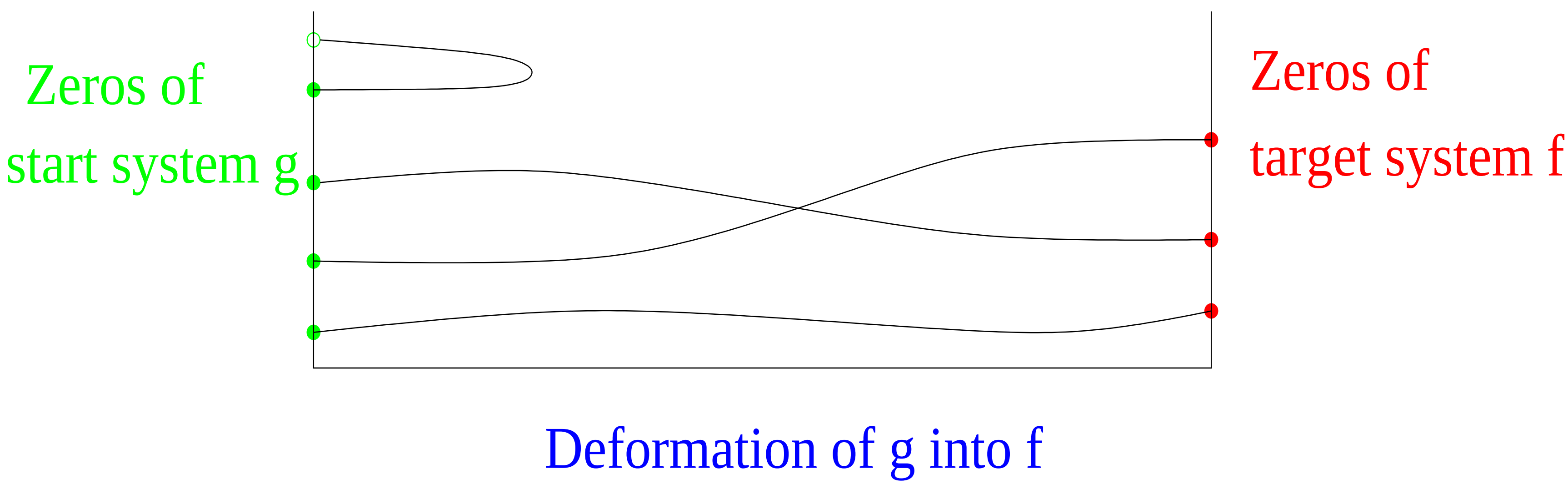
Example of a required motion:

Move point P through certain positions

Typically leads to problem of solving a polynomial system with real or complex coefficients

Difficulty: Already few variables and low degrees can result in a system out of range of current solution methods!

Homotopy methods: Deform an easy to handle start system into the target system; follow numerically zeros of start system into those of target system



Complete motion synthesis for so called Stephenson mechanisms can be performed efficiently by homotopy methods

M.& Schmitt & Schreiber, 2002

Efficiency of homotopy methods relies on existence and number of zeros (paths)

$\rightsquigarrow$ analysis needs more theory

2. Approximation algorithms

Several (deep) mathematical methods for bounding number of zeros for polynomial systems $f : \mathbb{C}^n \mapsto \mathbb{C}^n$:

Bézout number	generalizes fundamental theorem of algebra, easy to compute, too a large bound
Mixed Volumes	Minkowski sum of Newton polytopes, hard to compute, (generically) correct bound
multi-homogeneous Bézout numbers	partitioning of variables, then Bézout for each group; mainly used in practice; so far no complexity results

Example: Eigenpairs

Find eigenpairs $(\lambda, v) \in \mathbb{C}^{n+1}$ of $M \in \mathbb{C}^{n\times n}$:

$$M \cdot v - \lambda \cdot v = 0 \ , \ v_n - 1 = 0$$

Has (generically) n solutions, but Bézout number 2^n.

Multi-homogeneous Bézout numbers: Group variables as

$$M \cdot v - \lambda \cdot v = 0 \ , \ v_n - 1 = 0$$

and homogenize w.r.t. both groups

$$\lambda_0 \cdot M \cdot v - v \cdot \lambda = 0 \ , \ v_n - v_0 = 0$$

Then the number of isolated roots in $(\mathbb{C})^n$ is bounded by the 2-homogeneous Bézout number, which here is n.

Theorem (Malajovich & M., 2005):

a) Given a polynomial system $f : \mathbb{C}^n \to \mathbb{C}^n$ there is no efficient algorithm that computes the minimal multi-homogeneous Bézout number (unless P = NP).

b) The same holds with respect to efficiently approximating the minimal such number within an arbitrary constant factor.

Sketch of proof:

Relate problem to 3-coloring problem for graphs

Establish multiplicative structure of multi-hom. Bézout numbers

In practice: Balance whether additional effort for constructing start system pays out

Remark. MHBN important in analysis of central path in interior point methods (Shub et al.)

3. Complexity theory over $\mathbb{R}$

Above systems particular in that solutions generically exist

Related interesting questions:

- deciding existence of solutions for general polnomials
- exact counting of number of solutions in general

Theorem (Blum & Shub & Smale '89):

Deciding solvability of real polynomial systems is $\mathrm{NP}_{\mathbb{R}}$-complete over $\mathbb{R}$; similarly for complex systems and $\mathrm{NP}_{\mathbb{C}}$.

Remark. All problems in $\mathrm{NP}_{\mathbb{R}}$ can be decided in simple exponential time in the real number model. Similarly over $\mathbb{C}$.

Corollary (Bürgisser & Cucker '05):

Counting the number of solutions is $\#\mathrm{P}_{\mathbb{R}}$-complete and $\#\mathrm{P}_{\mathbb{C}}$-complete, respectively.

Many other recent completeness results for counting problems in algebraic geometry by Bürgisser & Cucker & Lotz

Clear: also the following optimization problem is hard

MAX-Quadratic Polynomial Systems MAX-QPS:

Input: $n, m \in \mathbb{N}$, real polynomials in n variables $p_1, \ldots, p_m \in \mathbb{R}[x_1, \ldots, x_n]$ of degree at most 2; find maximal number of p_i's that have a common real root

But what's about approximating this maximum?

Theorem (M., 2005): If total number of non-vanishing coefficients in system is $O(m^2)$ there is no $\mathrm{APX}_{\mathbb{R}}$ algorithm for MAX-QPS unless $\mathrm{P}_{\mathbb{R}} = \mathrm{NP}_{\mathbb{R}}$.

Close relation also over the reals to PCPs

Example. $\mathrm{NP}_{\mathbb{R}}$ verification for solvability of system

$$p_1(x) = 0 \ , \ \ldots, p_m(x) = 0$$

guesses solution $x^* \in \mathbb{R}^n$ and plugs it into all p_i's ; obviously all components of x^* have to be seen

Question. Can we rewrite $\mathrm{NP}_{\mathbb{R}}$-verification proofs in such a way that seeing only constantly many positions of the proof suffices to detect faults with high probability?

Formalization using verifiers gives family of complexity classes

$$\mathrm{PCP}_{\mathbb{R}}(r, q)$$

Theorem (M., 2005)

$\mathrm{NP}_{\mathbb{R}}$ has long, transparent proofs:

$$\mathrm{NP}_{\mathbb{R}} \subseteq \mathrm{PCP}_{\mathbb{R}}(poly, const)$$

Example: Each faulty proof claiming that a polynomial system is solvable can be rewritten such that a fault occurs with high probability in each part of the proof having constant length

Proof: New techniques for self-testing and self-correction of functions on real domains

Main challenge: Are there as well short transparent proofs, i.e. is

$$\mathrm{NP}_{\mathbb{R}} \subseteq \mathrm{PCP}_{\mathbb{R}}(log, const) \ ?$$

Close relation to approximation algorithms for real number optimization problems; a positive answer would yield

Theorem (M., 2006)

If $\mathrm{NP}_{\mathbb{R}}$ has short transparent proofs there is no $\mathrm{FPTAS}_{\mathbb{R}}$ approximation scheme for instances of MAX-QPS with $O(n)$ many non-zero coefficients.

Another open question: Are there any fixed-constant approximation algorithms at all for MAX-QPS? (Flarup & M. '06)

Summary

Analysis of polynomial systems results in interesting problems in many different areas including

- robotics
- combinatorial optimization
- classical and real number complexity theory

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2. Definition of multi-hom. Bézout number

Consider $n \in \mathbb{N}$, a finite $A \subset \mathbb{N}^n$ and a polynomial system

$$\begin{cases} f_1(z) = \sum_{\boldsymbol{\alpha} \in A} f_{1\boldsymbol{\alpha}} z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \\ \vdots \\ f_n(z) = \sum_{\boldsymbol{\alpha} \in A} f_{n\boldsymbol{\alpha}} z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n} \end{cases}$$

where the $f_{i\alpha}$ are non-zero complex coefficients.

Thus, all f_i have the same support A

A multi-homogeneous structure: partition of $\{1, \ldots, n\}$ into k subsets

$$(I_1, \ldots, I_k) \ , \ I_j \subseteq \{1, \ldots, n\}$$

Define for each partition $(I_1, \ldots, I_k)$:

- block of variables related to I_j : $Z_j = \{z_i | i \in I_j\}$
- corresponding degree of f_i with respect to $\mathbb{Z}_j$: $d_j := \sum_{l \in I_j} \; \alpha_l$ (the same for all polynomials f_i because of same support)

Definition: a) The multi-hom. Bézout number w.r.t. partition $(I_1, \ldots, I_k)$ is the coefficient of $\prod_{j=1}^{k} \zeta_j^{|I_k|}$ in the formal polynomial $(\zeta_1 + \cdots + \zeta_k)^n$ (if each group not yet homogeneous)

$$\text{Béz}(A, I_1, \ldots, I_k) = \begin{pmatrix} & & n & \\ |I_1| & |I_2| & \cdots & |I_k| \end{pmatrix} \prod_{j=1}^{k} d_j^{|I_j|}$$

b) Minimal multi-hom. Bézout number:

$$\min_{\mathbf{I} \text{ partition}} \text{Béz}(A, \mathbf{I})$$