Optimization of globally convex functions — Source link

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OPTIMIZATION OF GLOBALLY CONVEX FUNCTIONS*

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DEDICATED TO PROFESSOR E.J. MCSHANE
ON THE OCCASION OF HIS EIGHTY–FIFTH BIRTHDAY

Abstract. Convex functions have nice properties with respect to both minimization and maximization. Similar properties are established here for functions that are permitted to have bad local behavior but are globally convex in the sense that they behave "convexly" on triples of collinear points that are widely dispersed. The results illustrate a development that seems desirable in the interest of more realistic mathematical modeling: the "globalization" of important function properties. In connection with the maximization of globally convex functions over convex bodies in a given finite-dimensional normed space E, there is interest in estimating the maximum, for points c of bodies $C \subset E$, of the ratio between two measures of how close c comes to being an extreme points of C. Good estimates are obtained for the cases in which E is Euclidean or has the "max" norm.

Key words. convex, quasiconvex, maximum, minimum, optimization, extreme points, norm

AMS(MOS) subject classifications. 52A20, 52A40, 90C25, 46B20, 46C05

Introduction. In the mathematical modeling of practical optimization problems, the following assumptions are often made:

(a) the feasible region C is a convex subset of a Minkowski space E (a normed finite-dimensional real vector space);

(b) the objective function φ is convex–that is, φ is a real-valued function on C such that

$$\varphi(y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(z) \tag{1}$$

for all

$$x, z \in C \quad \text{with } x \neq z, \qquad 0 < \lambda < 1, \qquad y = \lambda x + (1-\lambda)z. \tag{2}$$

Because of the following well-known facts, convexity is useful in connection with both minimization and maximization:

(P1) Each local minimum of a convex function is a global minimum.

(P2) If a convex function attains a maximum, then (under mild restrictions on the domain) it does so at an extreme point of its domain.

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Each of the properties (P1) and (P2) serves to narrow the search for the extreme values of a convex function, and each is the basis of algorithms for finding or approximating these values and the points of the domain C where they are attained. (There are many references for convex minimization. See [PR] for a survey of approaches to convex maximization.) However, in a number of situations it seems that assumption (a) is fully justified while assumption (b) is dictated as much by mathematical convenience as by realism. Even when the real objective function appears to be convex when viewed globally, it is likely to exhibit small "blips" in local behavior that cause it to deviate from the mathematical ideal expressed in (b). Similar statements apply to other important function properties, such as linearity and monotoneity. The main purpose of the present paper is to suggest the desirability of "globalizing" various function properties that are important in optimization–that is, of formulating definitions which permit bad local behavior while preserving the function property in some global sense–and of studying the consequences of these definitions. The resulting mathematical framework may turn out to be especially appropriate for some of the many practical optimization problems in which rapid solution is more important than precise solution, so that the the practical needs can be met by any algorithm that is sufficiently fast and comes sufficiently close to finding the optimum. (This is the consideration, for example, that has led to the popularity of simulated annealing as an algorithmic tool for solving problems in VLSI design and other areas [KGV] [VA].)

It is not clear which generalizations of the notion of convex function will prove to be most useful in modeling. That can be determined only by extensive computational practice in conjunction with development of the underlying theory. However, in the previous generalizations with which we are familiar ([Po] [Gi] [DAZ] [ADSZ] [SZ]), local behavior is restricted in ways that exclude the sort of blips that we want to permit. To illustrate the sort of mathematical developments that we have in mind, we here formulate some notions of global convexity or global quasiconvexity that depend on a nonnegative parameter δ. They reduce to the usual notions when $\delta = 0$, but for $\delta > 0$ they do permit wild local behavior and they lead to "δ-versions" of (P1) and (P2).

Our main δ-versions of (P1) are Theorems 2.1 and 2.2, which are straightforward and easy to prove. However, the search for quantitative precision in the case of (P2) leads to some difficult problems concerning the relationship between two measures of how close a nonextreme point c of a body C comes to being extreme. (The term *body* is used to mean a finite-dimensional line-free closed convex set, so the existence of extreme points is guaranteed.) The obvious measure is $\xi(C, c)$, the minimum distance of c from C's set of extreme points, and we want to find a φ-maximizing point c for which this distance is not too large. The search for such a point turns out to involve $\mu(C, c)$, half the length of a longest segment in C that has c as its midpoint. Our main δ-version of (P2) is Theorem 4.3. which implies that if the objective function φ is δ-convex or δ-quasiconvex in an appropriate sense, and if φ attains a maximum on C, then φ attains a maximum at a point $q \in C$ such that $\xi(C,q) \leq \delta\rho(C)$, where

$$\rho(c) = \sup\{\xi(C,c)/\mu(C,c) : \text{ nonextreme } c \in C\}.$$

By the way of illustration, suppose that X is a Euclidean disk of unit radius centered at a point p. When an interior point x of X is at distance r from p, it is true that $\xi(X,x) = 1 - r, \mu(X,x) = (1-r^2)^{1/2}$, and $\rho(X,x) = \left((1-r)/(1+r)\right)^{1/2}$; hence $\rho(X) = \rho(X,p) = 1$. If Y is an equilateral triangle inscribed in X, then $\rho(Y) = \rho(Y,p) = \sqrt{3}$. However, the supremum $\rho(C)$ appears to be difficult to compute for most choices of C and of the underlying Minkowski space E, and because of ρ's role in the extension of (P2) this leads to interest in estimating the quantity

$$\rho(E) = \sup\{\rho(C) : \text{ body } C \subset E\}$$

for various choices of E. It is proved here that $\rho(E) \leq d$ for all d-dimensional E (this bound is attained), while $d - 1 \leq \rho(E) \leq d$ when E has the "max" norm and

$$\sqrt{d} \leq \rho(E) \leq \frac{d}{d-1}\sqrt{5d}$$

when E is Euclidean. Also,

$$\rho(\text{Euclidean plane}) = \sqrt{3}.$$

Our section headings are as follows: 1. Some global versions of convexity; 2. Minima; 3. Qualitative properties of $\rho(C,c)$; 4. Maxima; 5. Estimation of $\rho(E)$.

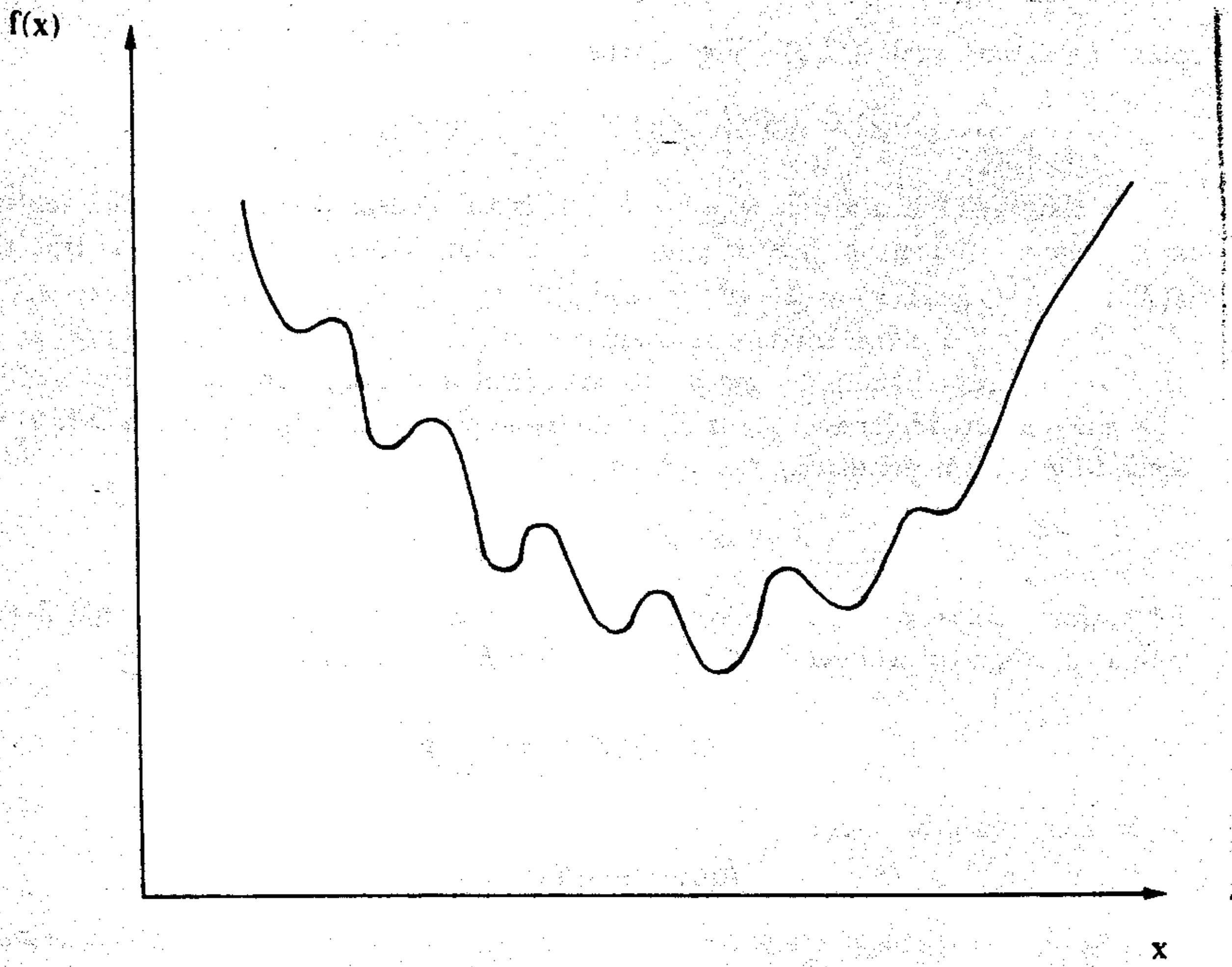
1. Some global versions of convexity. With respect to optimization, some important properties of convex functions extend to quasiconvex functions. A function φ is *quasiconvex* if its domain is a convex set and

$$\varphi(y) \leq \max\{\varphi(x), \varphi(z)\} \tag{3}$$

whenever (2) holds. We "globalize" the notions of convexity and quasiconvexity by saying (for $\delta \geq 0$) that a function φ is *δ-convex* $\langle$resp. *δ-quasiconvex*$\rangle$ if it is real-valued, its domain is a convex set C in a normed (real) vector space, and the inequality (1) $\langle$resp.(3)$\rangle$ holds for all x, y, z, and λ that satisfy both (2) and

$$\|x - y\| \geq \frac{\delta}{2} \quad \text{and} \quad \|z - y\| \geq \frac{\delta}{2}. \tag{4}$$

In other words, the function φ behaves "convexly" or "quasiconvexly" on collinear triples that are sufficiently dispersed. Note that the 0-convex functions are precisely those that are convex in the usual sense; similarly for the 0-quasiconvex functions. However, as is suggested by the figure, small blips are permitted in the function's local behavior when $\delta > 0$.



A function φ is *strictly δ-convex*(resp. *strictly δ-quasiconvex*) if it is real-valued, its domain is convex, and strict inequality holds in (1) (resp. (3)) for all x, y, z and λ that satisfy both (2) and (4). Actually, it suffices for present purposes to require that (1) or (3) holds for all x, y and z that satisfy (2) and (4) with $\lambda = \frac{1}{2}$. This amounts to requiring that

$$\varphi\left(\frac{1}{2}x + \frac{1}{2}z\right) \leq \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(z) \tag{5}$$

or

$$\varphi\left(\frac{1}{2}x + \frac{1}{2}z\right) \leq \max\{\varphi(x), \varphi(z)\} \tag{6}$$

whenever $\|x - z\| \geq \delta$. these weak notions are called *midpoint δ-convexity* and *midpoint δ-quasiconvexity* respectively, and the related strict notions are defined in the obvious way.

The following remark provides some insight into the notions just defined.

1.1 THEOREM. *For a convex subset C of a normed linear space, and for $\delta > 0$, let C^δ*(resp. C_δ) *denote the set of all points $c \in C$ such that c is not the midpoint* (resp. *not* *an endpoint) of any line-segment of length δ in C. Suppose that ψ and ω are real-valued functions on C such that*

$$\psi(c) \geq 0 \text{ for all } c \in C^\delta, \psi(c) = 0 \text{ for all } c \notin C^\delta;$$
$$\omega(c) \geq 0 \text{ for all } c \in C_\delta, \omega(c) = 0 \text{ for all } c \notin C_\delta.$$

Then whenever a function φ on C has any of the δ-convexity properties defined above, the function $\varphi + \psi - \omega$ has the same property.

Proof. If $c \in C^\delta$, then (because of (4)) c cannot appear as the y in any of the defining inequalities (1) or (3). When c appears as the x or z in any of these inequalities, increasing the value of $\varphi(c)$ (by adding $\psi(c)$ when $\psi(c) > 0$) merely serves to strengthen the inequality. Similarly, if $c \in C_\delta$ then c cannot appear as x or z, and when c appears as y, reducing the value of $\varphi(c)$ (by subtracting $\omega(c)$ when $\omega(c) > 0$) strengthens the inequality □

When the domain C is of diameter less than δ, every real-valued function on C is strictly δ-convex because no triple (x, y, z) of points of C satisfies condition (4). Thus it is clear that for the δ-convexity of an objective function to be useful in practice, the value of δ must be appropriately related to the geometry of the domain C. Even when the domain is in all senses large with respect to δ, the above δ-notions permit much wilder behavior (at least near extreme points of the domain) than will be encountered in practice. Nevertheless, we are able to establish sharp δ-versions of (P1) and (P2) for functions that are midpoint δ-convex or midpoint δ-quasiconvex. It does not seem that sharper conclusions could be obtained by adding local smoothing conditions such as continuity, and in any case we would not want to assume continuity because it is lacking in some important applications such as the fixed-charge problem of Hirsch and Dantzig [HD].

If one wanted to augment the δ-requirement by a weak global smoothing condition in order to limit the wildest behavior of the function φ to the extreme points of its domain, the following assumption might be appropriate, where $]x, z[$ (resp. $[x, z]$) denotes the open (resp. closed) line segment whose ends are x and z.

(7) For each triple of distinct points x, y and z of C such that $y \in]x, z[$, the segment $[x, z]$ is contained in a segment $[x', z']$ in C for which $\psi(y) \leq \max\{\varphi(x'), \varphi(z')\}$.

We shall see that the Property (P2) follows from (7) alone, without the intervention of any δ-requirement. However, (7) has no effect on (P1).

Note that if φ (with convex domain C) satisfies condition (7), then without destroying this property the function φ may be redefined at all extreme points of C by assigning arbitrary new values that are not less than the original values. However, we don't want to rule out this aspect, because the same is true of the usual notion of convexity.

If a function φ has any of the convexity or quasiconvexity properties defined above, the same is true of each positive multiple of φ. The convexity properties are also preserved under addition of functions, but that is not true of the quasiconvexity properties or of (7). For example, let φ and ζ both have [0, 1] as domain, with

$$\varphi(x) = \zeta(x) = 0 \quad \text{for} \quad 0 < x < 1$$
$$\varphi(0) = \zeta(1) = 1, \quad \varphi(1) = \zeta(0) = -2$$

Then φ and ζ are both quasiconvex and hence also satisfy condition (7). However, the function $\varphi + \zeta$ does not satisfy (7) and it is δ-quasiconvex only for $\delta < \frac{1}{2}$. This example is easily modified to produce one in which the functions are continuous.

The next two theorems deal with an example that was chosen to illustrate the extent to which midpoint δ-convexity and midpoint δ-quasiconvexity are weaker than the usual notions of convexity and quasiconvexity.

1.2 THEOREM. *Suppose that η is a positive constant, and let*

$$\varphi(t) = \eta t^2 + \cos t \qquad \textit{for all } t \in \mathbb{R}.$$

Then the following three conditions are equivalent: φ is convex; φ is quasiconvex; $\eta \geq \frac{1}{2}$.

Proof. For equivalence of the first and third condition, observe that $\varphi''(t) = 2\eta - \cos t$, whence the second derivative φ'' is everywhere nonnegative if and only if $\eta \geq \frac{1}{2}$. To complete the proof, note that the following is a corollary of the next result: if φ is midpoint quasiconvex, then $\eta \geq \frac{1}{2}$. □

To simplify the notation in the next statement and proof, we define

$$\psi(\delta) = \frac{1}{2}\frac{\sin^2 \delta}{\delta^2} \qquad \text{for } 0 < \delta < 2\pi,$$

whence of course $\psi(\delta) \to \frac{1}{2}$ as $\delta \to 0$. It follows from L'Hôpital's rule that as $\delta \to 0$,

$$\frac{\psi(\delta)}{\frac{1}{2} - \frac{1}{6}\delta^2} \to 1$$

and hence

$$\psi(\frac{1}{2}\delta) \sim \frac{1}{2} - \frac{1}{24}\delta^2 \qquad \text{and} \qquad \psi(\frac{1}{4}\delta) \sim \frac{1}{2} - \frac{1}{96}\,\delta^2.$$

1.3 THEOREM. *Suppose that η is a positive constant, and let*

$$\varphi(t) = \eta t^2 + \cos t \qquad \textit{for all } t \in \mathbb{R}.$$

Suppose that $0 < \delta < 2\pi$. Then:

(i) *φ is midpoint δ-convex if and only if $\eta \geq \psi(\frac{1}{4}\delta)$;*

(ii) *φ is midpoint δ-quasiconvex if and only if $\eta \geq \psi(\frac{1}{2}\delta)$.*

Proof. Let us first consider (i). The function φ is midpoint δ-convex if and only if (5) holds whenever $x + \delta \le z$. Writing $h = z - x \ge 0$ and setting

$$g(x,h) = 2\cos(x + \frac{1}{2}h) - \cos x - \cos(x+h),$$

we see that the desired condition is

$$g(x,h) \le \frac{\eta}{2}h^2 \qquad \text{for } h \ge \delta. \tag{8}$$

To obtain a more useful expression for $g(x,h)$, apply the trigonometric addition formulas to see that

$$g(x,h) = \cos x[2(\cos\frac{1}{2}h - 1) + 1 - \cos h] + \sin x(\sin h - 2\sin\frac{1}{2}h),$$

then apply the half-angle formulas and regroup to obtain

$$g(x,h) = 4\left(\sin\frac{1}{4}h\right)^2 \cos(x + \frac{1}{2}h).$$

Thus in terms of the function ψ defined earlier, the desired condition (8) becomes

$$\psi(\frac{1}{4}\delta)\cos(x + \frac{1}{2}h) \le \eta \qquad \text{for } h \ge \delta. \tag{9}$$

Now recall that in the range $0 < \theta \le \pi/2$, $\sin\theta/\theta$ is a decreasing function of θ and hence

$$\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1.$$

Since

$$\frac{\sin\frac{1}{4}h}{\frac{1}{4}h} \le \frac{2}{\pi} \qquad \text{when } h \ge 2\pi,$$

it follows that the maximum value of $\varphi(\frac{1}{4}h)$ for $|h| \ge \delta$ is attained when $h = \delta$. Hence the condition

$$\eta \ge \psi(\frac{1}{4}\delta) \tag{10}$$

is sufficient for (9). On the other hand, for each given h there exists an x for which $\cos(x + \frac{1}{2}h) = 1$, and hence the condition (10) is necessary as well as sufficient for the desired midpoint δ-convexity.

We turn now to (ii). The function φ is midpoint δ-quasiconvex if and only if

$$\varphi(x) \le \max\{\varphi(x-h), \varphi(x+h)\} \qquad \text{for all } h \ge \delta.$$

(and we are concerned here only with $0 \leq \delta \leq \pi$). Thus the desired condition is that, with $+$ or $-$, it should be true that

$$\eta x^2 + \cos x \leq \eta(x \pm h)^2 + \cos(x \pm h),$$

or, equivalently,

$$(11\pm) \qquad \pm 2 \sin\frac{h}{2} \sin(x \pm \frac{h}{2}) \leq \pm \eta h(2x \pm h).$$

Now let us suppose (without loss of generality) that $x \geq 0$ and $h \geq 0$, and consider the inequalities

$$(12\pm) \qquad \frac{1}{2}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\frac{\sin(\frac{h}{2} \pm x)}{\frac{h}{2} \pm x} \geq \eta.$$

Elementary manipulation shows that (11+) is equivalent to (12+), while (11–) is equivalent to (12–) when $2x - h < 0$ and to

$$(13) \qquad \frac{1}{2}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\frac{\sin(\frac{h}{2} \pm x)}{\frac{h}{2} \pm x} \geq \eta.$$

when $2x - h > 0$.

To deal with the case in which $x < h/2$, note that $(\sin\theta)/\theta$ is a concave function on $[-\pi, \pi]$, whence

$$\frac{\sin(\frac{h}{2} - x)}{\frac{h}{2} - x} + \frac{\sin(\frac{h}{2} + x)}{\frac{h}{2} + x} \leq 2 \sin\frac{h}{2},$$

and thus at least one of the summands is at most

$$\frac{\sin(h/2)}{h/2}.$$

Hence requiring that

$$\eta \geq \frac{1}{2}\frac{\sin(\delta/2)}{\delta/2}$$

will guarantee that

$$\eta \geq \frac{1}{2} \max_{h=\delta} \left(\frac{\sin(h/2)}{h/2}\right)^2$$

and thus assure that (12±) holds. Further, by taking $x = 0$ and $h = \delta$ we see that this is the best possible lower bound for η when $x < h/2$.

Only the case in which $x > h/2$ remains. For this case, (12–) or (13) must be established, and we show in fact that (12–) holds. Indeed, if the number y is defined by the condition that

$$x + \frac{h}{2} = y + \frac{\delta}{2}$$

then (12–) becomes

$$\frac{\sin(y + \delta/2)}{y + \delta/2} \leq \frac{\sin(\delta/2)}{\delta/2},$$

and this holds for all $y \geq 0$. □

2. Minima. When $\delta > 0$ and φ is a function with domain C, we say that φ attains a *δ-local minimum* (resp. *strict δ-local minimum*) at a point $q \in C$ if $\varphi(q) \leq \varphi(c)$(resp. $\varphi(q) < \varphi(c)$) for all points $c \in C \setminus \{q\}$ at distance less than δ from the point q.

2.1 THEOREM. *If $\delta > 0$ and φ is a midpoint δ-convex or strictly midpoint δ-quasiconvex function that attains a δ-local minimum at a point q of its domain, then the global minimum of φ is attained at q.*

2.2 THEOREM. *If $\delta > 0$ and φ is a midpoint δ-quasiconvex function that attains a δ-local strict minimum at a point q of its domain, the strict global minimum of φ is attained at q.*

Proofs. As stated, the above theorems fail when $\delta = 0$ and thus they do not cover the classical cases of convex and quasiconvex functions. In order to cover these cases as well, simply replace the δ-local minima by ϵ-local minima for some positive $\epsilon \geq \delta$. The proofs below are phrased in terms of this ϵ.

If the conclusion of 2.1 (resp. 2.2) fails, there is a point $z_1 \in C\setminus\{q\}$ such that $\varphi(z_1) < \varphi(q)$(resp. $\leq \varphi(q)$). Of course, $\|z_1 - q\| \geq \epsilon$, and hence with $z_2 = \frac{1}{2}(z_1 + q)$ it is true that $\|z_1 - z_2\| \geq \epsilon/2$ and $\|q - z_2\| \geq \epsilon/2$. For 2.1, it follows from the midpoint δ-convexity of φ that

$$\varphi(z_2) \leq \frac{\delta}{2}\varphi(z_1) + \frac{\delta}{2}\varphi(q) < \varphi(q),$$

or from the strict midpoint δ-quasiconvexity of φ that

$$\varphi(z_2) < \max\{\varphi(z_1), \varphi(q)\} \leq \varphi(q).$$

For 2.2 it follows from the midpoint δ-quasiconvexity of φ that

$$\varphi(z_2) \leq \max\{\varphi(z_1), \varphi(q)\} \leq \varphi(q).$$

But then $\|z_2 - q\| \geq \epsilon$, so with $z_3 = \frac{1}{2}(z_2 + q)$ we have $\varphi(z_3) < \varphi(q)$ in 2.1 and $\varphi(z_3) \leq \varphi(q)$ in 2.2. Continuing in this manner, we form a sequence $z_1, z_2, \ldots$ in $C\setminus\{q\}$ such that always $\varphi(z_i) < \varphi(q)$(resp. $\leq \varphi(q)$). Since the sequence converges to q, we have reached a contradiction that completes the proof. □

Theorems 2.1 and 2.2 are sharp in the following two senses:

(a) In 2.1, the conditions on φ cannot be replaced by δ-quasiconvexity.

(b) In 2.1 and 2.2, the conclusions may fail if φ is δ-convex but the δ-local conditions are replaced by δ'-local conditions with $\delta' < \delta$.

To obtain examples in support of (a) and (b), suppose that $\alpha < \beta < \gamma$. For (a), let φ be constant with value σ on the closed interval $[\alpha, \beta]$ and constant with value $\eta > \sigma$ on the half-open interval $]\beta,\gamma]$. Then φ is quasiconvex and each point of its domain $[\alpha,\gamma]$ provides a local minimum for φ. However, the points of $]\beta,\gamma]$ actually provide global maxima rather than global minima. For (b), note that if $\gamma-\alpha<\delta$ then every real-valued function on $[\alpha,\gamma]$ is (trivially) δ-convex. If, in addition, $\gamma-\beta=\delta'<\delta$, there are many functions on $[\alpha,\gamma]$ for which γ provides a δ'-local strict minimum but either there is no global minimum or the global minimum is attained only in $[\alpha,\beta]$.

For another example in support of (b), let $C=[-1,2[$ and let the function φ on C be such that

$$\varphi(-1)=-1, \quad \varphi(t)=0 \text{ for } -1<t<1, \quad \varphi(t)\geq t \text{ for } 1\leq t<2.$$

Then φ is strictly 2-convex, and for $0<\delta'<2$ it has a δ'-local minimum at the point $\delta'-1$ but does not have a global minimum there. If φ is modified to give it the value $-\frac{1}{2}$ at the point $\delta'-1$, then φ is still strictly 2-convex and now has a δ'-local strict minimum at $\delta'-1$.

3. Qualitative properties of $\rho(C,c)$. For the results of Section 2, the convex domain C need not be closed and it may lie in an arbitrary normed linear space. However, from now on it is convenient to work with the

STANDING HYPOTHESES: E is a finite-dimensional normed vector space and C is a *body* in E, meaning that C is closed, convex, has nonempty interior, and does not contain any line. The sets of extreme and nonextreme points of C are denoted respectively by C_e and C_m.

It is known that C_e is nonempty, and in fact C is the convex hull of the union of its extreme points and extreme rays [K1]. For each $c\in C$, let

$$\xi(C,c)=\inf\{\|c-p\|:p\in C_e\},$$

the distance from the point c to the set C_e. This is a measure of how close the point c comes to being an extreme point of C. Another such measure is given by $\mu(C,c)$, defined as 0 when $c\in C_e$ and defined for $c\in C_m$ as half the length of a longest segment in C that has c as its midpoint. (The existence of such a segment follows from an easy argument using the compactness of E's unit sphere, the closedness of C, and the fact that C contains no line.) Note that $\mu(C,c)$ is the smallest number σ such that each line L in E through c includes an endpoint of $L\cap C$ at distance at most σ from c.

When C is fixed, the function $\xi(C,c)|_{c\in C}$ will be denoted by $\xi(C,.)$; similarly for the functions $\mu(C,c)|_{c\in C}$ and $\rho(C,c)|_{c\in C_m}$, where

$$\rho(C,c)=\frac{\xi(C,c)}{\mu(C,c)}.$$

Of course, $\rho(C,.) \equiv 1$ when C is 1-dimensional, but in general the two measures of near extremeness are different and the behavior of the function $\rho(C,.)$ is of interest. As was explained in the Introduction, the extreme value

$$\rho(C) = \sup\{\rho(C,c) : c \in C_m\}$$

plays an essential role in Theorem 4.3's extension of property (P2) to δ-convex functions. Hence the extreme value

$$\rho(E) = \sup\{\rho(C) : \text{ body } C \subset E\}.$$

is also of interest, and it is estimated in Section 5.

It would be of interest under various assumptions concerning the body C, to know the complexity of computing $\xi(C,c), \mu(C,c), \rho(C,c)$, and $\rho(C)$, and also, when C is bounded, the complexity of computing

$$\xi(C) = \sup\{\xi(C,c) : c \in C\} \quad \text{and} \quad \mu(C) = \sup\{\mu(C,c) : c \in C\}.$$

Of course, $\xi(C) \leq \frac{1}{2}\delta(C)$ and $\mu(C) \leq \frac{1}{2}\delta(C)$, where $\delta(C)$ is C's diameter. When C is a polytope presented in terms of its vertices, $\delta(C)$ and $\xi(C,c)$ are easy to compute but computation of the other numbers appears to be difficult. For a polytope presented in terms of its bounding hyperplanes, the computation of $\delta(C)$ is NP-hard in l^p-spaces with $1 \leq p < \infty$ [GK], and the same may well be true of the other numbers.

The remainder of the present section is devoted to some qualitative properties of the functions $\xi(C,.), \mu(C,.)$ and $\rho(C,.)$. For these properties, it suffices to deal with the usual Euclidean norm for $\mathbb{R}^d$, since each norm for $\mathbb{R}^d$ is caught between two positive multiples of the Euclidean norm. However, the quantitative details in Sections 4 and 5 depend on the choice of norm.

3.1 THEOREM. *For each body C, the function $\xi(C,.)$ is everywhere continuous and the function $\mu(C,.)$ is everywhere upper semicontinuous.*

Proof. Routine use of the triangle inequality shows that $|\xi(C,x) - \xi(C,y)| \leq \|x-y\|$, whence of course $\xi(C,.)$ is continuous. The upper semicontinuity of $\mu(C,.)$ follows from a simple argument based on the closedness of C and the compactness of the unit sphere in the containing space. □

For each $d \geq 3$, there exists a compact body $C \subset \mathbb{R}^d$ whose set C_e of extreme points is not closed. (For example, let C be the convex hull of the union of a $(d-1)$-dimensional spherical ball B and a segment S that is orthogonal to B's hyperplane and is centered at a point p in the boundary A of B. Then $p \in C_m$ but $A \backslash \{p\} \subset C_e$.) Clearly the function $\mu(C,.)$ is discontinuous at each point p of C_m that belongs to the closure of C_e, and in fact the restriction of $\mu(C,.)$ to $C_m \cup \{p\}$ is also discontinuous at p.

Although the function $\mu(C,.)$ need not be everywhere continuous, continuity at certain points can be established. The following lemma is useful for that purpose. (As the terms are used here, a *polyhedron* is a set that is the intersection of a finite number of closed halfspaces; a *polytope* is a bounded polyhedron.)

3.2 LEMMA. *Suppose that C is a body in $\mathbf{R}^d$, the origin 0 is the midpoint of an open segment $]-q,q[$ in C, and H is the hyperplane through 0 orthogonal to the segment. If C is a polyhedron or the origin is interior to C, then for each $\lambda \in]0,1[$ there is a neighborhood U of 0 in $C \cap H$ such that the vector sum*

$$U + \lambda]-q,q[= \{u + \lambda q : \ u \in U, \ |\tau| < \lambda\}$$

is a neighborhood of 0 in C.

Proof. The case in which $0 \in \text{int } C$ is left to the reader. In the remaining case, there is a representation of C in the form $C = \bigcap_{i=1}^{n} K_i$, where each K_i is a closed halfspace with bounding hyperplane J_i and where, for some $m \geq 1, 0 \in \bigcap_{i=1}^{m} J_i$ and $0 \in \text{int } \bigcap_{i=m+1}^{n} K_i$. By a well-known theorem ([Go] [K2]), the polyhedron $K = \bigcap_{i=1}^{m} K_i$ is the direct sum of the subspace $J = \bigcap_{i=1}^{m} J_i$ and the polyhedral cone $K \cap J^{\perp}$, where $J^{\perp}$ is the orthogonal complement of J. Of course, $[-q,q] \subset J$, whence $J^{\perp} \subset H$. Since the segment $[-\lambda q, \ \lambda q]$ is interior to the intersection $\bigcap_{i=m+1}^{n} K_i$, there is a positive η such that this intersection contains the vector sum of the segment $[-\lambda q, \ \lambda q]$ and the η-neighborhood V of the origin in R^d. Then the set $U = V \cap C \cap H$ is the desired neighborhood of 0 in $C \cap H$. □

3.3 THEOREM. *The function $\mu(C,.)$ is continuous at each interior point of C, and if C is polyhedral then $\mu(C,.)$ is continuous everywhere.*

Proof. By Theorem 3.1, the function $\mu(C,.)$ is everywhere upper semicontinuous. To complete the proof, we show that if p is interior to C, or $p \in C$ and C is polyhedral, then $\mu(C,.)$ is lower semicontinuous at p. This is obvious when $\mu(C,p) = 0$, for the function μ is nonnegative. Suppose, then, that $\mu(C,p) > 0$, assume without loss of generality that $p = 0$, and let $[-q,q]$ be a longest segment in C that has 0 as midpoint. Now consider an arbitrary $\lambda \in]0,1[$, and let U be as in Lemma 3.2. Then for each $\epsilon \in]0,\lambda[$, the set $W = U + \epsilon]-q,q[$ is a neighborhood of 0 in C, and for each $w \in W$ it is true that

$$w + (\lambda - \epsilon)]-q,q[\subset U + \lambda]-q,q[\subset C$$

and hence $\mu(C,w) \geq (\lambda - \epsilon)\|q\|$. This shows that the function $\mu(C,.)$ is lower semicontinuos at p. □

3.4 LEMMA. *If the body C is a pointed cone, then the function $\rho(C,.)$ attains a positive minimum on C. If C is pointed polyhedral cone, then $\rho(C,.)$ also attains a maximum on C.*

Proof. We assume without loss of generality that the origin is the apex of C, whence there is a hyperplane H that misses the origin and there is a compact body B in H such that $C = [0, \infty[B$. Since C is a cone, $C_m = C\backslash\{0\}$, and for each $c \in C_m$ and $\lambda > 0$ it is true that $\xi(C, \lambda c) = \lambda\xi(C, c)$ and $\mu(C, \lambda c) = \lambda\mu(C, c)$. Hence the range of the function $\rho(C, .)$ on C_m is equal to its range on B.

On the set B, the functions $\xi(C, .)$ and $\mu(C, .)$ are both positive. By Theorem 3.1, $\xi(C, .)$ is continuous and $\mu(C, .)$ is upper semicontinuous. Hence their quotient $\rho(C, .)$, being lower semicontinuous and positive on the compact set B, attains a positive minimum on B.

If C is polyhedral then, by Theorem 3.3, the function $\mu(C, .)$ is actually continuous on C, whence $\rho(C, .)$ is continuous and hence attains a maximum on the compact set B. □

3.5 LEMMA. *If the body C is unbounded and its set C_e of extreme points is bounded, then*

$$\lim_{n\to\infty} \sup\{\rho(C, c) : c \in C_m, \|c\| \geq n\} = 1$$

Proof. We may assume without loss of generality that the origin 0 belongs to the bounded set C_e. Let K denote the union of all rays that issue from 0 and are contained in C, and let B denote the closed convex hull of C_e. Then K is a pointed closed convex cone, B is a compact convex set, and $C = B + K$. Let $\beta = \sup\{\|b\| : b \in B\}$. Since $0 \in C_e$, it is true for each point $c \in C$ that $\xi(C, c) \leq \|c\|$. Each point $c \in C\backslash B$ has at least one representation in the form $c = b + k$ with $b \in B$ and $k \in K\backslash\{0\}$. For each such representation, it is true that $b + 2k \in C$, whence $c \in C_m$ with $\mu(C, c) \geq \|k\|$ and hence

$$\rho(C, c) \leq (\|k\| + \beta)/\|k\|.$$

From this it follows that

$$\lim_{n\longrightarrow\infty} \sup\{\rho(C, c) : \ c \in C_m, \ \|c\| \geq n\} \leq 1.$$

When the point $c \in K\backslash B$ belongs to R, $[0, 2c]$ is the unique longest segment in C that has c as its midpoint, and hence

$$\rho(C, c) \geq (\|k\| - \beta))/\|c\| = 1.$$

This implies that

$$\lim_{n\longrightarrow\infty} \sup\{\rho(C, c)c) : \ c \in C_m, \ \|c\| \geq n\} \geq 1. \quad □$$

3.6 THEOREM. *If the body C is a polyhedron then the function $\rho(C, .)$ attains a maximum $\rho(C) \geq 1$.*

Proof. Consider an arbitrary edge or extreme ray R of C, and an endpoint p of R. Then $p \in C_e$ and the set C_e is finite, so for each point $c \in R\backslash\{p\}$ sufficiently close to p it is true that $c \in C_m$ and

$$\xi(C, c) = \|c - p\| = \mu(C, c).$$

This implies that the function $\rho(C,.)$ attains the value 1.

Now suppose that the function $\rho(C,.)$ does not attain a maximum on C_m, and let $c_1, c_2, \ldots$ be a sequence in C_m such that

$$\rho(C, c_n) \longrightarrow \rho(C) \quad \text{as} \quad n \longrightarrow \infty.$$

If $\|c_n\| \longrightarrow \infty$, then $\rho(C) \leq 1$ by Lemma 3.5, and the desired conclusion follows from the preceding paragraph.

In the remaining case, we may assume that the sequence $c_1, c_2, \ldots$ converges to a point $p \in C$. If $p \in C_m$ then $\rho(C, c) = \rho(C)$ by Theorem 3.3. If $p \in C_m$, then $p \in C_e$. It is not hard to verify that the values of $\rho(C, c)$ attained when the point $c \in C_m$ belongs to a sufficiently small neighborhood of the extreme point p are precisely the values attained by the function $\rho(K,.)$ where K is the cone consisting of all rays that issue from p and pass through the various points of C. When C is a polyhedron, this cone K is also polyhedral and the function $\rho(K,.)$ attains a maximum by Lemma 3.4. □

To end this section, we note that for each 2-dimensional body C, the following four conditions are equivalent: (a) C is a polyhedron; (b) the set C_e is finite; (c) the function $\rho(C,.)$ attains a minimum; (d) the infimum of $\rho(C,.)$ is positive. Of course, (a) implies (b) & (c) & (d) in any dimension. However, for bodies in $\mathbf{R}^3$, (b)&(c)&(d) does not imply (a), (c) does not imply (b) or (d), and some other questions about implications among these conditions are unsettled. In particular, we do not know whether (d) implies (c) nor whether (d) implies (b).

4. Maxima. The following remark illustrates the manner in which the quantity $\rho(c)$ enters into the δ-version of (P2).

4.1 REMARK. If φ is a midpoint strictly δ-quasiconvex function on a body C, and q is a point of C at which φ attains its maximum, then there is an extreme point x of C such that $\|q - x\| \leq \delta\rho(C)$.

Proof. It suffices to show that $\mu(C, q) \leq \delta$, for then

$$\xi(C, q) = \mu(C, q)\rho(C, q) < \delta\rho(C, q).$$

In the contrary case, q is the midpoint of a segment $[p, r]$ in C of length exceeding δ. But then, by the definition of midpoint strict δ-quasiconvexity, $\varphi(p) < \varphi(q)$ or $\varphi(r) > \varphi(q)$. This contradicts the assumed maximizing property of q. □

Our main tool for dealing with maxima is the following lemma from [K3].

4.2 LEMMA. *Suppose that K is a convex set and the partial ordering $\prec$ on K is defined as follows: $v \prec w$ if and only if, for the line L through v and w, it is true that v is an* *inner point and w is an endpoint of the intersection $L \cap K$. (Intuitively, w, can see beyond v in K, but v cannot see beyond w.) With respect to this ordering, each linearly ordered subset of K is affinely independent.*

This lemma yields a quick proof of the theorem of Hirsch and Hoffman [HH] (essentially (P2) of the Introduction) asserting that if a convex function attains its maximum on a finite-dimensional line-free closed convex set, then it does so at an extreme point. As we now show, it also yields extensions of this result to functions that are globally convex or quasiconvex.

4.3 THEOREM. *Suppose that φ is a real-valued function defined on the body C. Suppose also that φ is midpoint δ-quasiconvex and C is compact, or that φ is midpoint δ-convex and is bounded above on each ray in C. Then for each value α of φ there are points q and x of C such that $\varphi(q) \geq \alpha, x$ is an extreme point of C, and $\|q - x\| < \delta\rho(C)$.*

Proof. In terms of the ordering $\prec$ described in Lemma 4.2, we define a second ordering $\ll$ by saying that $v \ll w$ provided that $v \prec w$ and $\varphi(v) \leq \varphi(w)$. Every set that is linearly ordered by $\ll$ is also linearly ordered by $\prec$, and hence by Lemma 4.2 is of cardinality at most $d + 1$ where d is the dimension of C. Now consider an arbitrary point $c \in C$ such that $\varphi(c) = \alpha$, and let

$$c = c_0 \ll c_1 \ll \cdots \ll c_k \tag{8}$$

be a $\ll$-ordered sequence that starts with the point c and cannot be extended. Set $q = c_k$, whence of course $\varphi(q) \geq \alpha$. To complete the proof it suffices to show that $\mu(C, q) < \delta$, for then $q \in C_e$ or we have

$$\xi(C, q) = \frac{\xi(C, q)}{\mu(C, q)}\mu(C, q) = \rho(C, q)\mu(C, q) < \rho(C)\delta.$$

Suppose that $\mu(C, q) \geq \delta$, let p and r be the endpoints of a longest segment in C that has q as a midpoint, and let L denote the line through p and r. At least one of p and r must be an endpoint of the intersection $L \cap C$, and we may assume that p is such an endpoint. If the intersection $L \cap C$ is a segment $[p, s]$ (where of course $r \in [p, s]$), it follows from the midpoint δ-quasiconvexity of φ that $\varphi(p) \geq \varphi(q)$ or $\varphi(s) \geq \varphi(q)$. Then the sequence (8) can be extended by adding p or s at the end, and the assumed maximality of the sequence is contradicted.

In the remaining case, the intersection $L \cap C$ is a ray R that issues from p and passes through r. This ray includes the point $r_k = q + k(q - p)$ for each integer $k \geq -1$. From the midpoint d-convexity of φ it follows that

$$\varphi(r_{k+1}) \geq \varphi(r_k) + (\varphi(r_k) - \varphi(r_{k-1})),$$

for each $k \geq 1$, and this leads to the conclusion that

$$\varphi(r_k) \geq \varphi(q) + k(\varphi(q) - \varphi(p)).$$

Since, by hypothesis, φ is bounded above on the ray R, it follows that $\varphi(p) \geq \varphi(q)$. But then the sequence (8) can be extended by adding p, and the contradiction completes the proof. □

Now suppose that the function $\rho(C,.)$ attains a maximum on C at a point $c \in C_m$. (By Theorem 3.5, this occurs whenever C is a polytope.) Let $\delta = \mu(C,c)$, whence of course $\xi(C,c) = \delta\rho(C)$. Then Theorem 4.3 is sharp for C in the following sense:

For each $\eta > \delta$ there exists a continuous η-convex function φ on such C such that the maximum of φ is attained only at c. Hence δ is the largest "modulus of global convexity" ϵ such that the ϵ-convexity of a continuous φ insures the existence of an extreme point of C within $\delta\rho(C)$ of c.

To construct the desired function φ, start from an arbitrary continuous convex function ψ that is bounded above on C. Then let $\varphi = \psi + \zeta$, where

$$\zeta(x) = 1 - \frac{\|x-c\|}{\eta - \delta} \quad \text{for } \|x-c\| < \eta - \delta,$$

and $\zeta = 0$ elsewhere.

In the introduction, we mentioned a possible smoothing condition (7). A weakened form of this condition appears as a hypothesis in the following theorem.

4.4 THEOREM. *Suppose that the body C is compact and the function φ on C satisfies the following condition: each point $y \in C_m$ is an inner point of a segment $[x,z]$ in C such that $\varphi(y) \leq \max\{\varphi(x), \varphi(z)\}$. Then for each value α of φ there is an extreme point x of C such that $\varphi(x) \geq \alpha$. In particular, if φ attains a maximum on C then it does so at an extreme point.*

Proof. Follow the first two paragraphs of the proof of Theorem 4.3. □

5. Estimation of $\rho(E)$.

5.1 LEMMA. *For each $c \in C_m$ there exists a ball B centered at c such that $\rho(C,c) = \rho(C \cap B, c)$.*

Proof. Let x be an extreme point of C, and let $[p,r]$ have maximum length among the segments that are centered at c and contained in C. Let B be any ball that is centered at c and contains x, p and r. Then of course $\mu(C \cap B, c) = \mu(C,c)$. Also, $\xi(C \cap B, c) = \xi(C,c)$, because each point of $(C \cap B)_e \backslash C_e$ belongs to the boundary of B and hence is at distance at least $\|x - c\|$ from C. □

5.2 LEMMA. *If C is bounded and $c \in C_m$, there exists a d-simplex S such that $c \in S$ and the vertices of S are extreme points of C. For each such S, it is true that $c \in S_m$ and $\rho(C,c) \le \rho(S,c)$.*

Proof. Since $C = \text{con } C_e$, the existence of S follows from Carathéodory's theorem [Ca]. It is obvious that $c \in S_m, \xi(C,c) \le \xi(S,c)$, and $\mu(C,c) \ge \mu(S,c)$. □

5.3 THEOREM. *For each d-dimensional $E, \rho(E) \le d$.*

Proof. In view of Lemmas 5.1 and 5.2, it suffices to show that if the origin 0 belongs to a d-simplex S with vertices $v_0, \ldots, v_d$, and if $\|v_i\| \ge 1$ for all i, then S contains a segment of length at least $2/d$ centered at the origin.

Since $0 \in S$, there are numbers $\lambda_i \ge 0$ such that

$$\sum_{i=0}^{d} \lambda_i = 1 \qquad \text{and} \qquad \sum_{i=0}^{d} \lambda_i v_i = 0.$$

Then $\lambda_i \ge 1/(d+1)$ for at least one value of i, and we may assume without loss of generality that this is λ_0. From the fact that $-\lambda_0 v_0 = \sum_{i=1}^{d} \lambda_i v_i$, it follows that the point

$$p = -\frac{\lambda_0}{1-\lambda_0} v_0$$

is a convex combination of $\{v_1, \ldots, v_d\}$ and hence $p \in S$. But then S contains the segment $[p, v_0]$, and hence contains the segment $[-p, p]$ centered at 0. Since $\lambda_0 \ge 1/(d+1)$, it is true that

$$\|p\| \ge \frac{\lambda_0}{1-\lambda_0}\|v_0\| \ge \frac{\frac{1}{d+1}}{1-\frac{1}{d+1}} = \frac{1}{d}. \qquad \square$$

For each positive integer d, and for $1 \le p < \infty$, let $\mathbb{R}_p^d$ denote the space $\mathbb{R}^d$ with the norm of

$$x = (x_1, \ldots, x_d) \in \mathbb{R}^d$$

given by

$$\|x\| = \left(\sum_{i=1}^{d} |x_i|^p\right)^{\frac{1}{p}} \qquad (1 \le p < \infty).$$

Let $\mathbb{R}_\infty^d$ denote the space $\mathbb{R}^d$ with the norm

$$\|x\| = \max\{|x_1|, \ldots, |x_d|\}.$$

5.4 THEOREM. *If E is the d-dimensional subspace of $\mathbb{R}_\infty^{d+1}$ consisting of all points for which the sum of the coordinates is 0, then $\rho(E) = d$.*

Proof. Let S denote the d-simplex in $\mathbb{R}_\infty^{d+1}$ consisting of all points $x = (x_1, \ldots, x_{d+1})$ such that $\sum_{i=1}^{d+1} x_i = 1$ and $x_i \ge 0$ for all i. Then $\rho(S) \le d$ by Theorem 5.3. To complete the proof of Theorem 5.4, we show that $\rho(S,c) \geq d$ for the centroid c of S. Since the distance from c to each extreme point of S is $d/(d+1)$, we want to show that whenever p is a point of the space such that $c+p \in S$ and $c-p \in S$, then $\|p\| \leq 1/(d+1)$. But that is obvious, for if some coordinate of p exceeds $1/(d+1)$ in absolute value, then $c+p$ or $c-p$ has a negative coordinate and hence does not belong to S. □

In the d-dimensional space E of Theorem 5.4, the unit ball is a hyperplane section of a $(d+1)$-cube. When $d=2$, it is a regular hexagon. In the general case, it is of the form $T \cap -T$, where T is a d-simplex whose centroid is the origin. Theorem 5.4 establishes the sharpness of Theorem 5.3 in the sense that for each d there exists a d-dimensional E for which $\rho(E) = d$. However, it is also of interest to determine (or at least find sharp estimates for) $\rho(E)$ for the "standard" spaces $\mathbf{R}_p^d (1 \leq p \leq \infty)$. The following lemma is useful in dealing with the case $p = \infty$.

5.5 Lemma. *If E is a subspace of F, then $\rho(E) \leq \rho(F)$.*

Proof. It suffices to consider the case in which E is a hyperplane through the origin in F. Let $p \in F \backslash E$. For each $\eta < \rho(E)$, there exists a compact convex body C in E and a point $c \in C_m$ such that $\rho(C,c) > \eta$. For each $\lambda > 0$, let $C_\lambda = C + [-\lambda, \lambda]p$, a compact convex body in F. Since clearly

$$\lim_{\lambda \to 0} \rho(C_\lambda, c) = \rho(C,c),$$

the desired conclusion follows. □

5.6 Corollary. *For each d,*

$$d - 1 \leq \rho(\mathbf{R}_\infty^d) \leq d.$$

Proof. Use Theorems 5.3 and 5.4, and Lemma 5.5. □

Before turning to the case of Euclidean d-space $\mathbf{R}_2^d$, we establish one more geometric lemma that applies to all spaces and is of some interest in itself.

5.7 Lemma. *For each $\eta < \rho(E)$, there exists in E a d-simplex X such that the origin 0 is interior to S, each vertex of S is at distance 1 from the origin, and $\eta < \rho(S,0)$.*

Proof. By the definition of $\rho(E)$ in conjunction with Lemma 5.1 and 5.2, there exists a d-simplex $T \subset E$ and a point $t \in T_m$ such that $\rho(T,p) > \eta$. By an easy continuity argument, $\rho(T,s) > \eta$ for each interior point s of T sufficiently close to t. Now with s fixed, let x be an extreme point of T closest to s. We may assume without loss of generality that $s = 0$ and $\|x\| = 1$. Then for each vertex v of T, let

$$\overline{v} = \frac{v}{\|v\|} \in T,$$

and let S be the simplex whose vertices are the $\overline{v}$'s. It is clear that the pair $(S,0)$ has the stated properties. □

5.8 LEMMA. *If S is a regular Euclidean d-simplex and s is the centroid of S, then*

$$\rho(S,s) = \begin{cases} \sqrt{d} & \text{when } d \text{ is odd} \\ \sqrt{(d+1)} & \text{when } d \text{ is even.} \end{cases}$$

Proof. With $u_0, \ldots, u_d$ denoting the standard unit basis vectors for $\mathbf{R}_2^{d+1}$ let

$$c = \frac{1}{d+1}\sum_{i=0}^{d} u_i \qquad \text{and} \qquad S = \text{ con } \{u_0 - c, \ldots, u_d - c\}.$$

Then S is a regular Euclidean d-simplex with centroid $s = 0$, and

$$\xi(S,s) = \left(\frac{d}{d+1}\right)^{1/2}$$

We want to compute $\mu(S,s)$, which is half of the maximum of the norm on the set $S \cap -S$.

The points x of $S \cap -S$ are characterized by the existence of nonnegative numbers $\lambda_0, \ldots, \lambda_d$ with sum 1 and nonnegative numbers $\eta_0, \ldots, \eta_d$ with sum 1 such that

$$\sum_{i=0}^{d} \lambda_i(u_i - c) = x = \sum_{i=0}^{d} \eta_i(c - u_i).$$

From this it follows that

$$\sum_{i=0}^{d}(\lambda_i + \eta_i)u_i = 2c$$

and by linear independence of the u_i's that

$$\lambda_0 + \eta_0 = \lambda_1 + \eta_1 = \ldots \lambda_d + \eta_d = \frac{2}{d+1}.$$

Note also that since the origin is the orthogonal projection of the point c onto the hyperplane aff S, maximizing x over $x \in S \cap -S$ is equivalent to maximizing $\|x - c\|^2$ over $x \in S \cap -S$. And for $x = \sum_{i=0}^{d} \nu_i u_i$ as described,

$$\|x - c\|^2 = \sum_{i=0}^{d} \eta_i^2.$$

We claim that for any x that maximizes $\|x\|^2$ over $S \cap -S$, there is at most one index i for which the numbers λ_i and η_i are both positive. For suppose there are two such indices, say 1 and 2 with $\eta_1 \geq \eta_2$. Then for a sufficiently small positive ϵ we may increase η_i, and λ_2 by ϵ and decrease η_2 and λ_i by ϵ without violating the above conditions, and since

$$(n_1+\epsilon)^2+(\eta_2-\epsilon)^2+\sum_{i=3}^{d}\eta_i^2=\sum_{i=0}^{d}\eta_i^2+2\delta(\eta_1-\eta_2+\epsilon)>\sum_{i=0}^{d}\eta_i^2=x-c^2,$$

the maximizing property of x is contradicted. It follows, therefore, that for each index i with at most exception, either $\lambda_i = 0$ and $\eta_i = 2/(d+1)$ or $\lambda_i = 2/(d+1)$ and $\eta_i = 0$. And since $\sum_{i=0}^{d}\eta_i = 1$, no more than half of the $d+1$ η_i's can be equal to $2/(d+1)$.

From the above observations we see that when d is odd —say $d = 2n+1$—the maximum of $\|v\|$ for $v \in S \cap -S$ is attained by setting

$$v=\frac{1}{d+1}\left(\sum_{i=0}^{n}u_i\right)-c.$$

Then $\|v\| = 1/\sqrt{2n+2} = \sqrt{d+1}$ and $\rho(S,s) = \xi(S,s)/\|v\| = \sqrt{d}$.

When d is even — say $d = 2n$— there is a maximizing x such that for some $r \leq n$.

$$\eta_i=\frac{2}{d+1}\quad \text{for } 0\leq i<r \text{ and } \eta_i=0 \text{ for } r<i\leq d.$$

From the facts that

$$\frac{2r}{d+1}+\eta_r=1 \qquad \text{and} \qquad 0\leq\eta_r\leq\frac{2}{d+1},$$

it follows that $2r \geq d-1$, whence $r = n$ and $\eta_r = 1/d$. Hence the point

$$w=\left(\frac{2}{d+1}\sum_{i=0}^{n-1}u_i+\frac{1}{d+1}u_n\right)-c$$

is a point of $S \cap -S$ farthest from 0, and we have

$$\|w\|=\frac{\sqrt{2n}}{2n+1}=\frac{\sqrt{d}}{d+1}\quad \text{and}\quad \rho(S,s)=\frac{\xi(S,s)}{\|w\|}=\sqrt{d+1}.\quad \square$$

For each dimension d, Lemma 5.8 provides a lower bound for $\rho(\mathbb{R}_2^d)$ and it may be that this bound is sharp. We are able to show that it is sharp for $d \leq 2$ (i.e., $\rho(\mathbb{R}_2^2) = \sqrt{3}$), and for $d \geq 3$ to establish an upper bound that is no more than $\sqrt{5}$ times the lower bound. It follows from Lemma 5.7 that in seeking an upper bound for $\rho(\mathbb{R}_2^d)$, we may confine our attention to the ratio $\rho(Z,0)$, where Z satisfies the conditions of the following lemma. Under these conditions, $\xi(Z,0) = 1$ and we seek a lower bound on $\mu(Z,0)$.

5.9 LEMMA. *With $d \geq 2$, Suppose that the origin in $\mathbf{R}_2^d$ is interior to a d-simplex Z whose vertices $z_0, \ldots, z_d$ are all of unit norm, and that the positive numbers λ_i are such that*

$$\sum_{i=0}^{d} \lambda_i = 1 \quad \text{and} \quad \sum_{i=0}^{d} \lambda_i z_i = 0.$$

Let

$$S = \text{ con } R\{z_1, \ldots, z_d\} \quad \text{and} \quad s = -\frac{\lambda_0}{1-\lambda_0} z_0 \in S.$$

The the following statements are true:

(i) *S is the facet of Z opposite z_0;*
(ii) *s belongs to the interior of S relative to the hyperplane aff S;*
(iii) *for each i, $\lambda_i < \frac{1}{2}$;*
(iv) $S \cap -Z = (1-2\lambda_0)(S-s) \cap (s-S)) + s$.

(Thus the intersection $S \cap -Z$ contains a contraction in the ratio $1-2\lambda_0$ of the reflection of the set S in the point s.)

Proof. The assertion (i) is obvious, and (ii) follows from the fact that

$$s = -\frac{1}{1-\lambda_0}(\lambda_0 z_0) = -\frac{1}{1-\lambda_0}\,(-\sum_{i=1}^{d} \lambda_i z_i) = \sum_{i=1}^{d} \frac{\lambda_i}{1-\lambda_0}\; z_i.$$

For (iii), note that if $\langle \;,\; \rangle$ denotes the usual inner product, then

$$\lambda_0 = \lambda_0 \langle z_0, z_0 \rangle = \langle z_0, \lambda_0 z_0 \rangle = \langle z_0, -\sum_{i=1}^{d} \lambda_i z_i \rangle = \sum_{i=1}^{d} \lambda_i(-\langle z_0, z_1 \rangle)$$

$$\leq \sum_{i=1}^{d} \lambda_i |\langle z_0, z_i \rangle| \leq \sum_{i=1}^{d} \lambda_i = 1 - \lambda_0.$$

From this it follows that $\lambda_0 \leq \frac{1}{2}$, with $\lambda_0 = \frac{1}{2}$ only if each of the points $z_1, \ldots, z_d$ is equal to $-z_0$. Equality is excluded by the assumption that the origin is interior to Z.

Now let $T = S - s$. To establish (iv), we observe that

$$\begin{aligned} S \cap -Z &\supseteq (T+s) \cap ((1-2\lambda_0)(-T-s+z_0) - z_0) \\ &= (T+s) \cap ((1-2\lambda_0)(-T) + s). \\ &\supseteq T_0 \cap ((1-2\lambda_0)(-T)) + s \\ &\supseteq ((1-2\lambda_0)T) \cap ((1-2\lambda_0)(-T)) + s \\ &\supseteq (1-2\lambda_0)(T0 - T) + s. \\ &= (1-2\lambda_0)(S-s) \cap (s-S) + s. \end{aligned}$$

The first $\supseteq$ or $=$ follows from the fact that $S = T + s$ and

$$\begin{aligned}(1-2\lambda_0)(-T-s+z_0)-z_0 &= (1-2\lambda_0)(-S)+(2\lambda_0)(-z_0)\\ &\subset \text{ con } ((-S)\cap\{-z_0\}) = -Z.\end{aligned}$$

(Actually,

$$S\cap -Z == (T+s)\cap((1-2\lambda_0)(-T)+s),$$

but we do not use this fact.) □

5.10 THEOREM. $\rho(\mathsf{R}_2^2) = \sqrt{3}$.

Proof. Let the notation be as in Lemma 5.9. With p denoting the foot of the perpendicular from the origin to the line aff S through z_1 and z_2, let the z_i's and λ_i's be relabeled so as to obtain $\|p\| \geq \frac{1}{2}$. (It is easy to see that this is possible.) With

$$\delta = (1-\|p\|^2)^{1/2} - \|s-p\|,$$

the situation is as shown in the next figure, and it follows from (iv) of 5.9 that there exists $q \in S\cap -Z$ such that

$$\|q-p\| \geq \|s-p\| + (1-2\lambda_0)\delta.$$

Let us define $y = \|s-p\|$ and $r = \|p\|$, whence $\|s\| = (y^2+r^2)^{1/2}$ and it follows that

$$\lambda_0 = \frac{(y^2+r^2)^{1/2}}{1+(y^2+r^2)^{1/2}}.$$

Hence the lower bound on $\|q-p\|$ may be written as

$$\begin{aligned}g(y) &= y + \left(1 - \frac{2(y^2+r^2)^{1/2}}{1+(y^2+r^2)^{1/2}}\right)\left((1-r^2)^{1/2} - y\right)\\ &= \frac{2(y^2+r^2)^{1/2}y}{1+(y^2+r^2)^{12}} + \left(1 - \frac{2(y^2+r^2)^{1/2}y}{1+(y^2+r^2)^{1/2}}\right)(1-r^2)^{1/2}\end{aligned}$$

We claim that for each fixed $r \geq \frac{1}{2}$, the value of $g(y)$ is minimized by setting $y = 0$. Since

$$\frac{1}{2}(g(y)-g(0)) = \frac{y(y^2+r^2)^{1/2}(1+r) + (r-(y^2+r^2)^{1/2})(1-r^2)^{1/2}}{(1+r)(1+(y^2+r^2)^{1/2})},$$

it will suffice to show that $f(y) \geq f(0)$ for all $y \geq 0$, where

$$f(y) = \left(y(1+r) - (1-r^2)^{1/2}\right)(y^2+r^2)^{1/2}.$$

Now

$$f'(y) = (1+r)(y^2+r^2)^{1/2} + y(y^2+r^2)^{-1/2}\left(y(1+r) - (1-r^2)^{1/2}\right)$$
$$= \left(2(1+r)y^2 - y(1-r^2)^{1/2} + (1+r)r^2\right)(y^2+r^2)^{-1/2},$$

and since the discriminant $(1-r^2) - 8(1+r)^2r^2$ is positive when $r \geq \frac{1}{\lambda}$ it follows that $f'(y) \geq 0$ for all $y \geq 0$. But then $f(y) \geq 0$, and from this it follows that

$$g(y) \geq g(0) = \frac{(1-r)^{3/2}}{1+r)^{1/2}}$$

Recalling the relevant definitions, we see that there is a point q of $S \cap -Z$ such that $\|q-p\| \geq g(0)$, whence by the Pythagorean theorem the squared norm of q is at least

$$h(r) = \frac{(1-r)^3}{1+r} + r^2.$$

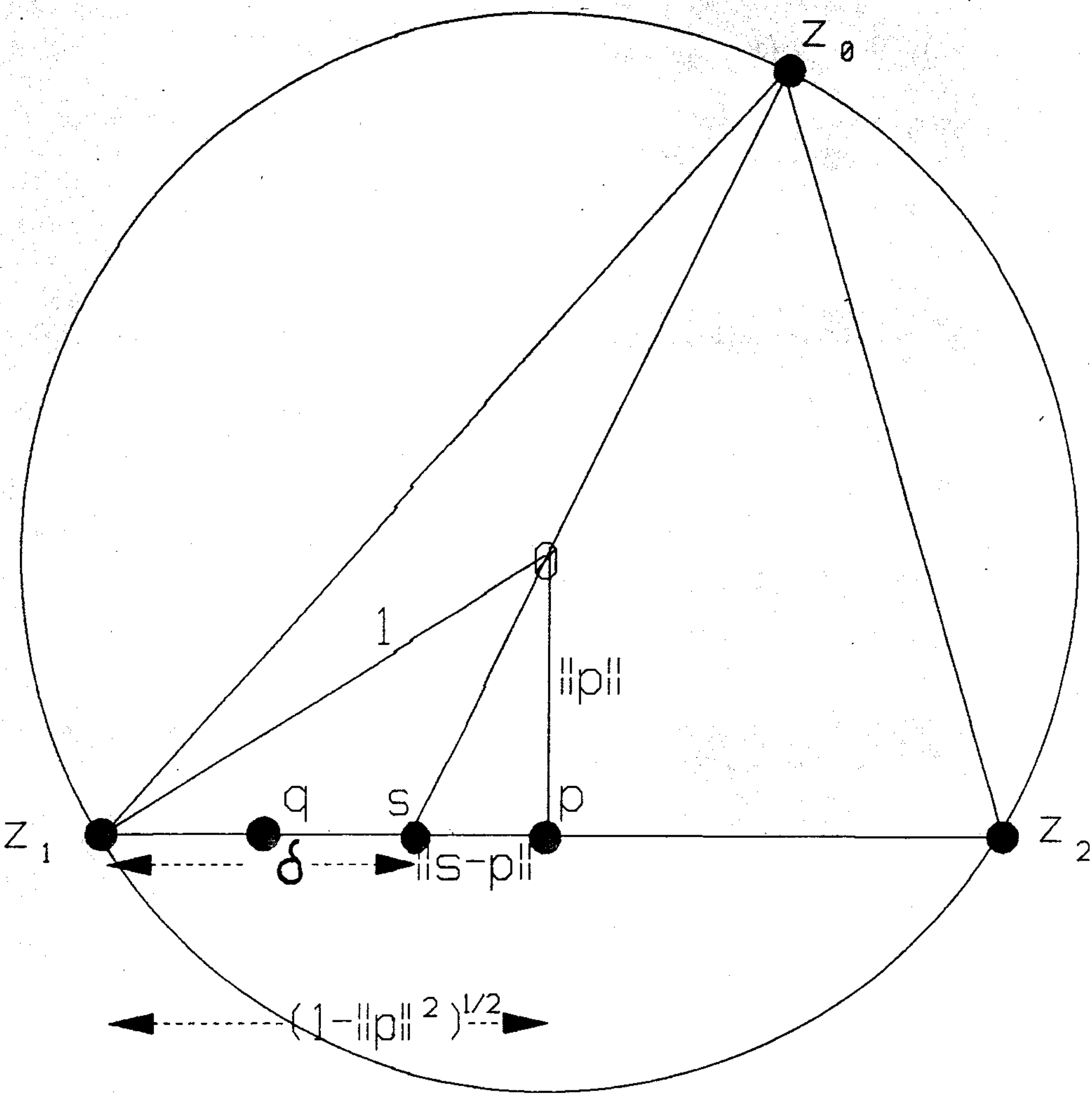
Now

$$h'(r) = \frac{4}{(1+r)^2}(r^2+2r-1),$$

and this is positive when $r \geq \frac{1}{2}$. Hence for $r \geq \frac{1}{2}$,

$$h(r) \geq \frac{(1/2)^3}{3/2} + \frac{1}{4} = \frac{1}{3},$$

whence the half-length of the segment $[-q,q] \subset Z \cap -Z$ is at least $1/\sqrt{3}$. This implies that $\rho(Z,0) \leq \sqrt{3}$, whence $\rho(\mathbf{R}_2^2) \leq \sqrt{3}$. The reverse inequality appears in Lemma 5.8.


Z 0
0
1
llpll
q
s
p
Z 1
Z 2
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lls-pll
(1-llpll 2)1/2

In preparation for the next result, we require a computational lemma.

5.11 LEMMA. *If the sequence* $\gamma_1, \gamma_2, \ldots$ *is defined by the condition that* $\gamma_1 = 1$ *and*

$$\gamma_d = \frac{1}{d^2} + \frac{(d-1)^4}{d^2(d+1)^2}\gamma_{d-1},$$

for $d \geq 2$*, then*

$$\gamma_d = \frac{1}{15d^3(d+1)^2}\ (3d^4 + 15d^3 + 2d^2 + 15d + 2) \geq \frac{1}{5d}\left(1 - \frac{1}{d}\right)^2.$$

Proof. Let $a_d = d^2\gamma_d$, so that $a_1 = 1$ and

$$\begin{aligned}
\alpha_d &= 1 + \left(\frac{d-1}{d+1}\right)^2 \alpha_{d-1} = 1 + \left(\frac{d-1}{d+1}\right)^2 \left(1 + \left(\frac{d-2}{d}\right)^2 \alpha_{d-2}\right) \\
&= 1 + \left(\frac{d-1}{d+1}\right)^2 + \left(\frac{d-1}{d+1}\right)^2 \left(\frac{d-2}{d}\right)^2 \left(1 + \left(\frac{d-3}{d-1}\right)^2 \alpha_{d-3}\right) \\
&= \frac{1}{d^2(d+1)^2}\ (2d^2(d^2+1) + (d-1)^2(d-2)^2 + (d-2)^2(d-3)^2\alpha_{d-3}) \\
&= \frac{1}{d^2(d+1)^2}\ \left(2d^2(d^2+1) + \sum_{i=2}^{d-2} i^2(i+1)^2\right)
\end{aligned}$$

A straightforward induction shows that

$$\sum_{i=1}^{n} i^2(i+1)^2 = \frac{1}{15}\ n(n+1)(n+2)(3n^2+6n+1).$$

with $d = n - 2$ this yields

$$\alpha_d = \frac{1}{15d(d+1)^2}\ (3d^4 + 15d^3 + 2d^2 + 15d + 2)$$

and hence

$$\begin{aligned}
\gamma_d &= \frac{1}{d^2}\ \alpha > \frac{1}{15d^3(d+1)^2}\ (3d^4 - 6d^2 + 3) = \frac{3}{15d^3(d+1)^2}\ (d+1)^2(d-1)^2 \\
&= \frac{1}{5d^3}(d-1)^2 = \frac{1}{5d}\left(1 - \frac{1}{d}\right)^2.
\end{aligned}$$

5.12 THEOREM. *If $d \geq 2$ then*

$$\frac{d}{d-1}\sqrt{5d} \geq \rho(\mathbf{R}_2^d) \geq \begin{cases} \sqrt{d} & \text{for odd } d \\ \sqrt{d+1} & \text{for even } d. \end{cases}$$

Proof. The lower bounds $\sqrt{d}$ and $\sqrt{d+1}$ come from Lemma 5.8. To justify the upper bound, it is more convenient to work with $\beta_d = 1/\rho(\mathbf{R}_2^d)$. We know already that $\beta_1 = 1$ and $\beta_2 = 1/\sqrt{3}$. To prove

$$\beta_d \geq \frac{d-1}{d\sqrt{5d}}$$

for a given $d \geq 2$, it will suffice, in view of Lemma 5.11, to show that

$$\beta_k^2 \geq \frac{1}{k^2} + \frac{(k-1)^4}{k^2(k+1)^2}\beta_{k-1}$$

for all $k \leq d$. This will be accomplished by induction on d.

With the notation as in Lemma 5.9, let the z_i's and λ_i's be relabeled so as to obtain $\lambda_0 \geq 1/(d+1)$. Let p denote the orthogonal projection of the origin on the hyperplane aff S and let

$$\delta = (1 - \|p\|^2)^{1/2} - \|s - p\|.$$

Of course,

$$\frac{\mu(S,s)}{\xi(S,s)} \geq \beta_{d-1}.$$

and with the aid of (iv) of 5.9 we see that

$$\mu(S \cap -Z, s) \geq (1 - 2\lambda_0)\mu(S,s).$$

For $1 \leq j \leq d$ it follows from the Pythagorean theorem that

$$\|p - v_j\|^2 + \|p\|^2 = \|v_j\|^2 = 1$$

and from the triangle inequality that

$$\|p - v_j\| \leq \|p - s\| + \|s - v_j\|.$$

Hence $\xi(S,s) \geq \delta$ and we have

$$\mu(S \cap -Z, s) \geq (1 - 2\lambda_0)\mu(S,s) \geq (1 - 2\lambda_0)\beta_{d-1}\delta.$$

That is, the set $S \cap -Z$ contains a segment which is centered at s and has half-length at least $(1 - 2\lambda_s)\beta_{d-1}\delta$. For at least one end of this segment, the squared distance from p is at least

$$\|p - s\|^2 + (1 - 2\lambda_0)^2\beta_{d-1}^2\delta^2$$

and from the origin is at least

$$Q = \|p - s\|^2 + (1 - 2\lambda_0)^2 \beta_{d-1}^2 \delta^2 + \|p\|^2.$$

To prove Theorem 5.12 it will suffice, in view of Lemmas 5.7 and 5.11 and the fact that $\xi(Z, 0) = 1$, to show that

$$Q \geq \frac{1}{d^2} + \frac{(d-1)^4}{d^2(d+1)^2} \beta_{d-1}.$$

For notational convenience, we write

$$y = \|s\| = \frac{\lambda_0}{1 - \lambda_0} \geq \frac{1}{d} \quad \text{and} \quad r = \|p\| \leq y \leq 1.$$

Then $\lambda_0 = y/(1+y)$, and Q becomes

$$\begin{aligned} h(y) &= y_2 + \left(1 - \frac{2y}{1+y}\right)^2 \beta_{d-1}^2 \left((1-r^2)^{1/2} - (y^2 - r^2)^{1/2}\right)^2 \\ &= y_2 + (1-y)^4 \beta_{d-1}^2 \left((1-r^2)^{1/2} + (y^2 - r^2)^{1/2}\right)^{-2} \end{aligned}$$

For each fixed y, the value of $h(y)$ decreases with decreasing $|r|$. Hence $h(y) \geq f(y)$, where

$$f(y) = y^2 + \frac{(1-y)^4}{(1+y)^2} \beta_{d-1}^2$$

and

$$\begin{aligned} f'(y) &= (1+y)^{-3} \left(2y(1-y)^3 - 4(1-y)^3(1+y)\beta_{d-1}^2 - 2(1-y)^4 \beta_{d-1}^2\right) \\ &\geq (1+y)^{-3} \left(2y(1+y)^3 - 6\beta_{d-1}^2\right) \quad \text{when} \quad \frac{1}{d} \leq y \leq 1. \end{aligned}$$

With $\beta_{d-1} \leq 1/\sqrt{d-1}$ and $y \geq 1/d$, we have $f'(y) \geq 0$ and hence

$$f(y) \geq (f(1/d))^2 = \frac{1}{d^2} + \frac{(1-\frac{1}{d})^4}{(1+\frac{1}{d})^2} \beta_{d-1}. \quad \square$$

For bodies C lying in certain Minkowski spaces E, Theorems 5.3, 5.4, 5.10 and 5.12 bound the number $\rho(C)$ from above. Because of the relevance of $\rho(C)$ to the maximization of δ-convex functions, it is also of interest to know how *small* $\rho(C)$ can be for bodies $C \subset E$. It is easy to see that whenever $\dim E \geq 2$, there are pointed polyhedral cones $K \subset E$ that have arbitrarily small values for $\rho(K)$. (Simply let K have a "large opening".) However, the situation for bounded bodies is much less obvious, and hence the following observation seems worth including.

5.13 THEOREM. *For each $d \geq 2$, the set $\{\rho(C)$: bounded body $C \subset \mathbf{R}_2^d\}$ is equal to the interval $]0, \rho(\mathbf{R}_2^d)]$. In particular, when $d = 2$, it is the interval $]0, \sqrt{3}]$.*

Proof. We first show that if $\epsilon > 0$, and if C_ϵ is the lens in the xy-plane formed by intersecting the disks of radius $R = (1+\epsilon^2)/(2\epsilon)$ centered at the points $(-R+\epsilon, 0)$ and $(R-\epsilon, 0)$, then $\rho(C_\epsilon) \leq \epsilon$. Consider an arbitrary point $p = (x, y) \in C_\epsilon$ with $x, y \geq 0$. The point q of ∂C_ϵ nearest to p is the radial extension to ∂C_ϵ of p from the point $(-R+\epsilon, 0)$. At $q, \partial C$ has a tangent line L, and we consider the cap D that contains the point q and is cut from C_ϵ by the line M parallel to L and passing through the point $z = (0, 1)$. Let r denote the point at which M intersects the segment joining the point $(-R+\epsilon, 0)$ to q. We consider separately the two possibilities: $p \in [r, q]; r \in [p, q]$.

Suppose first that $p \in [r, q]$, and let $\eta = \|p - q\|$. Then there is a chord of C_ϵ that passes through p, is centered at p, and is of length $2\sqrt{2\eta R - \eta^2}$. Hence

$$\rho(C_\epsilon, p) \leq \frac{\eta}{\sqrt{2\eta R - \eta^2}} \leq \epsilon.$$

Now suppose that $r \in [p, q]$, and note that if the lens subtends an angle θ at z, then $\tan\theta = 1/(-\epsilon)$. Consequently, if η is as before, then $\eta = \psi + \varphi$ where $\psi = \|p - r\|, \varphi = \|r - q\|$, and

$$\|p - r\| = \psi \leq \|z - r\|(\tan\theta) = (R\varphi - \varphi^2)^{1/2}/(R - \varphi).$$

Now C_ϵ contains the chord $[z, p - z]$ centered at p, and the square of half the length of this chord is given by

$$\|p - r\|^2 + \|z - r\|^2 \geq \|z - \|^2 = 2R\varphi - \varphi^2.$$

As $\eta = \varphi + \varphi$, we have

$$\eta \leq \varphi + (R\varphi - \varphi^2)^{1/2}/(R - \varphi)$$

For small ϵ,

$$\varphi \geq (R\varphi - \varphi^2)^{1/2}/(R - \varphi)$$

and consequently $\varphi \geq \eta/2$. Thus C_ϵ contains a chord centered at p of half-length at least $R\eta - \frac{1}{4}\eta^2$, and it follows that

$$\rho(C_\epsilon) \leq \eta(R\eta - \frac{1}{4}\eta^2)^{1/2} \leq \epsilon.$$

That completes the discussion of the case $d = 2$.

Now for $d \geq 3$, write $\mathbf{R}_2^d = \mathbf{R}_2^2 \times \mathbf{R}_2^{d-2}$ in the usual way, so that each point $p \in \mathbf{R}_2^d$ can be written as $p = (p', p'')$ with $p' \in \mathbf{R}_2^2$ and $p'' \in \mathbf{R}_2^{d-2}$. Let μ be the gauge-function of the body $C_\epsilon \subset \mathbf{R}^2$, let $\| \quad \|$ denote the Euclidean norm for $\mathbf{R}^{d-2}$, and define

$$K_\epsilon = \{p : (\mu(p')^2 + \|p''\|^2 \leq 1\}.$$

Then K_ϵ is a body in $\mathbf{R}^d$, and it is not hard to verify that $\rho(K_\epsilon) \leq \epsilon$. (When $d = 3$, K_ϵ is obtained by rotating the set C_ϵ about its axis of symmetry.)

To complete the proof, note that by a simple continuity argument, it is true for each space E that the set

$$\{\rho(C): \quad \text{bounded body } C \subset E\}$$

is a connected subset of $\mathbf{R}$. □

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