

Optimization of SMES Coil by Using Virial Theorem

Hiroaki Tsutsui, Shinichi Nomura, and Ryuichi Shimada

Abstract—The coil for the superconducting magnetic energy storage (SMES) is optimized by use of the virial theorem with stored energy and stress. In this work, we show the theoretical limit of stored energy with the maximum stress. To achieve the ideal limit, we propose the toroidal coil with helical winding. It is a hybrid coil of a toroidal field (TF) coil and a solenoidal coil helically wound on a torus. The winding is modulated in such that the toroidal field is created in the torus whereas the poloidal field is only out of the torus. In this case, the electromagnetic force is represented by the difference in the poloidal and the toroidal magnetic pressure. The virial theorem in the magnet is the relation of the magnetic energy and the averaged stress, and shows that the best coil to store the magnetic energy under the weakest averaged stress requires equal averaged principal stresses in all directions, which determines the ratio of the poloidal and toroidal current of our toroidal coil. The coil increases the magnetic energy to 4 times the conventional TF coil with the same maximum stress.

Keywords—SMES, superconductivity, coil, virial theorem, magnetic field, stress, tensor, energy

I. INTRODUCTION

THE virial theorem shows the relation between long time average or constant value of the kinetic energy and the energy of the field. Famous examples are the relation of the gravity potential and the kinetic energy in the astrophysics, and the relation between the kinetic energy and potential energy of the charged particles in the electromagnetics. In the field of the superconducting magnetic energy storage (SMES), the relations between the mass of the structure and stored energy are called the virial theorem[1], [2].

Recently, we designed the SMES system with force balanced coil (FBC)[3]-[6] which is a helical type

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H. Tsutsui, S. Nomura and R. Shimada are with Research Laboratory for Nuclear Reactors, Tokyo Institute of Technology, 2-12-1 O-okayama, Meguro-ku, Tokyo 152-8550, Japan (telephone: +81-3-5734-3839, e-mail: htsutsui@nr.titech.ac.jp).

hybrid coil of the toroidal field (TF) coil and the solenoidal coil. The combination removes the net electromagnetic force in major radius direction[3] by canceling the centering force of the TF coil and the hoop force of the solenoidal coil. Furthermore, we showed the configuration without the tilting force, by giving poloidal dependence to the pitch angle of the helical winding[7]. Next, we modified our works and proposed the stress balanced coil (SBC)[5] which is able to vanish the electromagnetic force at the point where the magnetic field is strongest.

In this work, we extend and generalize our studies with the virial theorem, and show the way to minimize the stress working in the coil. Furthermore, we design the coil with minimum stress under the condition of the fixed magnetic energy. It means that this coil enable us to make a SMES system with less amount of the structure to support the coil, because the mass of the structure is proportional to the maximum stress. In the next section, we explain the virial theorem for the structure with electromagnetic force. In the section III, we apply the virial theorem to the thin toroidal shell, and show the optimal coil for stress. In the section IV, we investigate the toroidal effect, and summarize this work in section V.

II. VIRIAL THEOREM OF ELECTROMAGNETIC STRUCTURE

When electromagnetic forces and stress are balanced on the object Ω , the following equilibrium equations hold.

$$\mathbf{j} \times \mathbf{B} + \nabla \cdot \mathbf{S} = 0 \quad (1)$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \quad (2)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (3)$$

Here, \mathbf{j} is the current density, \mathbf{B} is the magnetic flux density, \mathbf{S} is the stress tensor. When we choose the coordinate system as the principal axis directions, the tensor \mathbf{S} is expressed by the diagonal matrix,

$$\mathbf{S} = \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix}. \quad (4)$$

These $\sigma_i (i = 1, 2, 3)$ are called the principal stress. From (1)–(3), the next equation is obtained.

$$\nabla \cdot (\mathbf{T} + \mathbf{S}) = 0. \quad (5)$$

Here \mathbf{T} is Maxwell's stress tensor defined by

$$\mathbf{T} = \frac{1}{\mu_0} (\mathbf{B}\mathbf{B} - \frac{B^2}{2}\mathbf{l}) \quad (6)$$

where \mathbf{l} is a unit tensor. Maxwell's stress tensor is also turned to be diagonal when we choose z axis parallel to the magnetic line of force.

$$\mathbf{T} = \frac{1}{2\mu_0} \begin{pmatrix} -B^2 & & \\ & -B^2 & \\ & & B^2 \end{pmatrix}. \quad (7)$$

Here we consider the case that the current is flowing only in the object Ω where the electromagnetic force is acting. From (5) with the Gauss's integral theory, the next equation holds

$$\int \text{Tr}(\mathbf{T} + \mathbf{S})dV = \int \text{Tr}(\mathbf{T})dV + \int \text{Tr}(\mathbf{S})dV = 0, \quad (8)$$

where $\text{Tr}()$ is a trace of tensor. When we pay attention to that the trace of tensor is a scalar and does not depend on the coordinate system, (8) with (4) and (7) is reduced to

$$\int \sum_i \sigma_i dV = \int \frac{B^2}{2\mu_0} dV = U_M. \quad (9)$$

This relation is called as the virial theorem[1]. Because the right hand side (the magnetic energy U_M) is positive, the positive stress (the tensile stress) is necessary to store the magnetic energy. Furthermore, to minimize the maximum stress under the condition of a fixed magnetic energy, they are effective to remove the negative stress (the compressive stress) and to have uniform tensile stress distribution. Next we define the normalized stress $\tilde{\sigma}$ as

$$\tilde{\sigma} \equiv \frac{V_\Omega}{U_M} \sigma, \quad (10)$$

$$\langle \sigma \rangle \equiv \frac{\int \sigma dV}{V_\Omega}, \quad (11)$$

where V_Ω is the volume of Ω . Using the above normalized stress, the virial theorem (9) is reduced to the simple form

$$\langle \sum_i \tilde{\sigma}_i \rangle = 1. \quad (12)$$

Hence it is clear that the uniform stress distribution is required in order to decrease the maximum stress

under the condition of a fixed magnetic energy and volume.

When we define the maximum stress in the object Ω as σ_{\max} , next relation is obtained[1].

$$M \geq \frac{\rho_m U_M}{3\sigma_{\max}}, \quad (13)$$

where M , ρ_m are the mass and the density of the object Ω , respectively. The equality in (13) holds only when three principal stresses have the same value and uniform in the object Ω . In the other case, stress distribution is necessary to evaluate the minimum of mass M precisely.

III. APPLICATION TO THIN TOROIDAL SHELL

Here we consider the axisymmetric toroidal shell with circular cross section whose thickness $\Delta\rho$ is much smaller than the major radius R and the minor radius a . The current distribution on the torus is determined so that the magnetic surface can coincide with the torus. Moreover, we assume that the material of torus is uniform and $A = R/a$ is much larger than unity. Under these assumptions, we can use the zero dimensional model, in which the physical quantities are uniform on the torus. In this work, we use semi-toroidal coordinate system (ρ, ϕ, θ) . By using the above assumptions, the principal stresses are σ_ρ , σ_ϕ , σ_θ , and next relation holds.

$$\sigma_\rho \ll \sigma_\phi, \quad \sigma_\theta. \quad (14)$$

Therefore, we put $\sigma_\rho = 0$ in this paper.

In the case that the aspect ratio $A = R/a$ is large enough, the magnetic energy U_M of the toroidal coil is given by

$$U_M = U_{\text{TF}} + U_{\text{PF}}, \quad (15)$$

$$U_{\text{TF}} = \frac{\mu_0 a^2}{4R} I_\theta^2, \quad (16)$$

$$U_{\text{PF}} = \frac{\mu_0 R}{2} \left(\log \frac{8R}{a} - 2 \right) I_\phi^2, \quad (17)$$

where U_{TF} is the energy of the toroidal magnetic field inside the torus, U_{PF} is the energy of the poloidal magnetic field outside the torus, and I_ϕ , I_θ are the toroidal and poloidal current. The net electromagnetic force F_R , F_a in the major and minor radius directions are obtained from the partial differentiations of magnetic energy U_M with respect to R and a .

$$F_R = \left. \frac{\partial U_M}{\partial R} \right|_{I=\text{const.}}$$

$$= \frac{\mu_0}{2} \left\{ (\log 8A - 1) I_\phi^2 - \frac{1}{2A^2} I_\theta^2 \right\}, \quad (18)$$

$$F_a = \left. \frac{\partial U_M}{\partial a} \right|_{I=\text{const.}}$$

$$= \frac{\mu_0}{2} \left(\frac{1}{A} I_\theta^2 - A I_\phi^2 \right). \quad (19)$$

According to the principle of virtual work, the stress is obtained as

$$\langle \sigma_\phi \rangle dV = \langle \sigma_\phi \rangle \Delta \rho 2\pi a 2\pi dR = F_R dR \quad (20)$$

$$\langle \sigma_\theta \rangle dV = \langle \sigma_\theta \rangle \Delta \rho 2\pi R 2\pi da = F_a da. \quad (21)$$

By use of the tensions for the unit length of toroidal and poloidal direction,

$$T_\theta = \sigma_\theta \Delta \rho, \quad T_\phi = \sigma_\phi \Delta \rho, \quad (22)$$

$\langle T_\phi \rangle$, $\langle T_\theta \rangle$ are obtained as follows:

$$\langle T_\phi \rangle = \frac{\mu_0}{8\pi^2 a} \left\{ (\log 8A - 1) I_\phi^2 - \frac{I_\theta^2}{2A^2} \right\}, \quad (23)$$

$$\langle T_\theta \rangle = \frac{\mu_0}{8\pi^2 R} \left(\frac{1}{A} I_\theta^2 - A I_\phi^2 \right). \quad (24)$$

Next, we can get the normalized stress defined by (10),

$$\langle \tilde{\sigma}_\phi \rangle = \frac{A^2 \log 8A - A^2 - \frac{N^2}{2}}{\frac{N^2}{2} + A^2 \log 8A - 2A^2}, \quad (25)$$

$$\langle \tilde{\sigma}_\theta \rangle = \frac{N^2 - A^2}{\frac{N^2}{2} + A^2 \log 8A - 2A^2}, \quad (26)$$

where

$$N = \frac{I_\theta}{I_\phi} \quad (27)$$

is the ratio of the toroidal current and the poloidal current which is called as the pitch number of a coil. The relationships between the pitch number and normalized stress are shown in Fig. 1. This figure shows that the stress in the toroidal direction changes from tension (positive) to pressure (negative) as the pitch number N increases. This characteristic is independent of the aspect ratio A . The principal stress in the poloidal direction reaches the value of 1/2 and becomes larger than that in the toroidal direction with increasing N . Finally, $\tilde{\sigma}_\phi = -1$, $\tilde{\sigma}_\theta = 2$ which are the values of the conventional TF coil in the limit of $N \rightarrow \infty$.

When we get the sum of the normalized stresses with (14), (25) and (26),

$$\langle \tilde{\sigma}_\phi \rangle + \langle \tilde{\sigma}_\theta \rangle = 1,$$

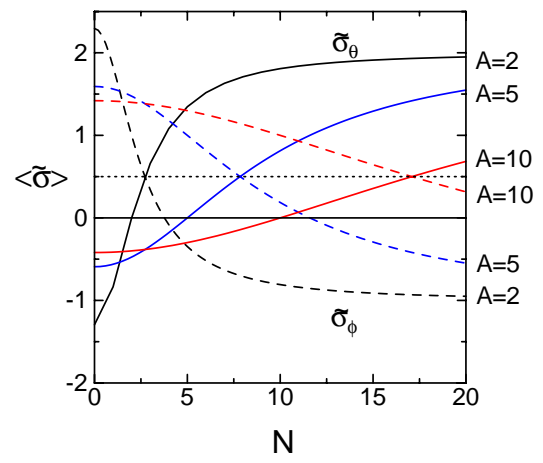


Fig. 1. Relations between pitch number N and averaged normalized stress $\langle \tilde{\sigma} \rangle$ for aspect ratio $A = 2, 5, 10$. Solid and dashed lines are $\tilde{\sigma}_\theta$ and $\tilde{\sigma}_\phi$, respectively.

the virial theorem (12) is satisfied. From (25) and (26), the average stress in toroidal direction or poloidal direction vanishes in the case of

$$N^2 = 2A^2(\log 8A - 1) \quad (\langle \sigma_\phi \rangle = 0), \quad (28)$$

$$N = A \quad (\langle \sigma_\theta \rangle = 0), \quad (29)$$

respectively. In particular, the coil with pitch number of (28) is called as the force-balanced coil (FBC)[3]. In the case that the material of the structure is not uniform and has the strength only in the particular direction, we had better use FBC. Because we assume the uniformity of the structure in this work, the configuration with the minimum stress is established when

$$\langle \tilde{\sigma}_\phi \rangle = \langle \tilde{\sigma}_\theta \rangle = \frac{1}{2}. \quad (30)$$

In other words, the configuration has the next pitch number,

$$N^2 = \frac{2}{3} A^2 \log 8A. \quad (31)$$

Figure 2 represents the relation of the aspect ratio and the pitch number, and shows that the configuration with minimum stress exists between FBC without averaged hoop force and SBC without averaged pinch force. In Fig. 3, the relations of the aspect ratio and the maximum stress are represented. Our virial coil reduces the maximum stress to 1/4 compared with TF coil, and to half compared with FBC. Since the mass of structure is proportional to the stress, our virial coil requires only 1/4 structure of TF coil.

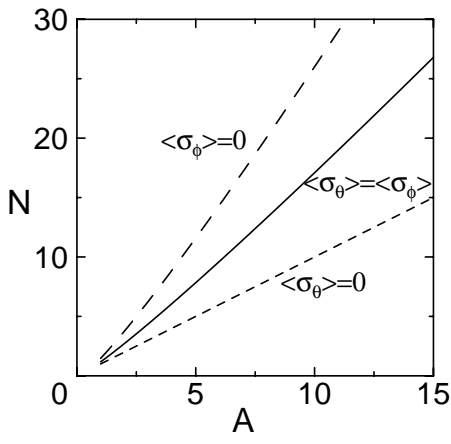


Fig. 2. Relations between aspect ratio A and pitch number N . Solid, dashed and dotted lines are in the case of $\langle \sigma_\theta \rangle = \langle \sigma_\phi \rangle$, $\langle \sigma_\phi \rangle = 0$ and $\langle \sigma_\theta \rangle = 0$, respectively.

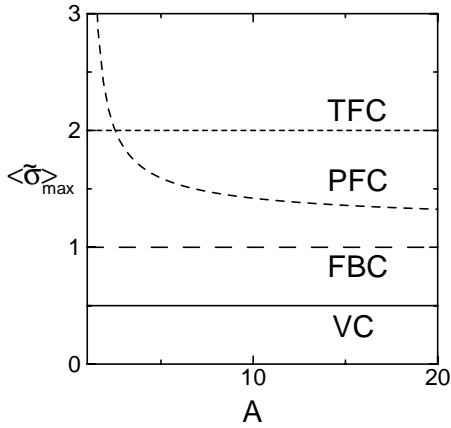


Fig. 3. Relations between maximum normalized stress $\tilde{\sigma}$ and aspect ratio A .

IV. TOROIDAL SHELL WITH CIRCULAR CROSS SECTION

In the previous section, we have investigated the case that the aspect ratio A is sufficiently large, where we are able to ignore the toroidal effect. In this section, we evaluate the poloidal distribution of stress to investigate the influence on the stress.

Supposing that the pressure inside the torus is $p(\theta)$ larger than that of outside, the force balance equations of the infinitesimal volume $ad\theta rd\phi\Delta\rho$ for the directions of ρ , θ are obtained. Using the tensions for the unit length T_θ , T_ϕ defined in (22), we get the next equations,

$$rT_\theta + (r - R)T_\phi = arp(r), \quad (32)$$

$$\frac{d}{dr}(rT_\theta) = T_\phi. \quad (33)$$

To solve this differential equation system, we define the function u ,

$$u(r) = a \int_R^r r' p(r') dr', \quad (34)$$

and get the solution as follows:

$$T_\theta = \frac{u}{(r - R)r}, \quad (35)$$

$$T_\phi = \frac{arp}{(r - R)} - \frac{u}{(r - R)^2}. \quad (36)$$

These equations enable us to calculate the stress distribution on a toroidal shell with a circular cross section whose current layer coincides with the magnetic surface. When we obtain the toroidal current distribution numerically, the pressure outside the torus is obtained from the toroidal surface current density j_ϕ . Hence the magnetic pressure acting on the torus is given by

$$p = \frac{\mu_0 J_\theta^2}{8\pi^2} \frac{1}{r^2} - \frac{\mu_0}{2} j_\phi^2. \quad (37)$$

Next, we calculate the stress distribution of the helical coil with $A = 2$ using (35)-(37). The results for some kinds of coils are shown in Fig. 4. To compare them in the condition of a fixed magnetic energy, they are normalized by the magnetic energy U_M . The poloidal angle θ is defined so as to take 0 at the outside of the torus and take π at the inside of the torus. Although the virial coil (solid line) is expected to reduce the maximum stress to 1/4 in comparison with TF coil (dashed line), it does not decrease as is expected in Fig.3. Therefore, in order to recover the reduction effect, we decide the pitch number N so that the principal stresses have the same value at the point of $\theta = \pi$ where the magnetic field is strongest, because the stress has the maximum value at the point. The stress distribution of the optimal coil is represented by dash-dotted line in Fig.4. This figure shows that the maximum stress is reduced to about 40% in comparison with that of TF coil. Therefore, our virial coil is effective to reduce the maximum stress even in the case of a small aspect ratio.

V. SUMMARY AND CONCLUSION

Using the virial theorem, the optimal coil to store the magnetic energy is obtained. The tension is available to store the magnetic energy while the compression does not work to store the magnetic energy and should be replaced to the tension under the condition of the constant maximum stress. Furthermore, in the case of the structure made of uniform material, the

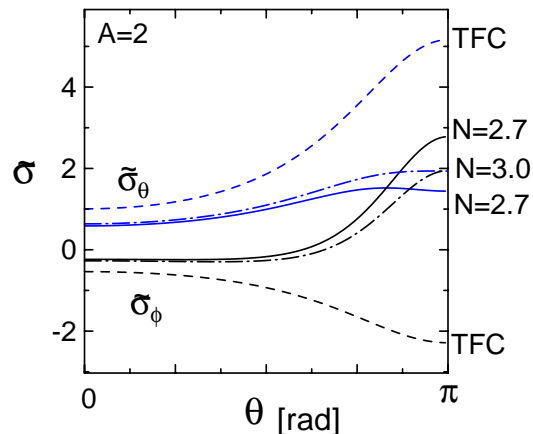


Fig. 4. Distribution of normalized stress $\tilde{\sigma}$ for $A = 2$. Solid and dashed lines are in the case of virial coil ($N = 2.7$) and TF coil, respectively. The optimized coil ($N = 3.0$), which minimize the maximum stress at $\theta = \pi$, is represented by dot-dashed line.

optimal configuration of stress is the uniform distribution of tension.

Next, we apply this result to the coil of the very thin toroidal shell with large aspect ratio approximation, and obtain the optimal pitch number of the coil which can store the maximum energy with the same maximum stress. The helical coil of the optimal pitch number can produce the magnetic energy 4 times greater than the conventional TF coil with any aspect ratio.

To investigate the toroidal effect, the distribution of stress is calculated. Although the reduction effect decreases with the aspect ratio, the maximum stress of the optimal coil is less than 40% that of TF coil in the case of small aspect ratio $A = 2$.

Finally, the stress has been evaluated only from the balance of force, since the direction of the principal stresses are obtained from the isotropic, continuous and axisymmetric nature in our model. However, when we design an actual coil, we have to calculate the stress from the displacement by the finite element method or other numerical methods. Even in such a case, the optimization shown in this work is applicable to store the large magnetic energy with the less structure material.

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