OPTIMIZATION OF STRUCTURAL ELEMENTS

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Abstract

A numerical method for the solution of structural optimization problems involving ordinary differential equations is presented for a simple situation where the constraint is of an aeroelastic nature. The method is adapted from optimal control theory and has proven successful in a number of structural optimization problems. Its extension to two dimensional structures is outlined ; limitation to situations involving plates, however, is emphasized. It is assumed that the instability exhibited by the optimality condition is related to the fact that plates cannot in general achieve global extrema. Suggestions for further research in this area are presented.

1. INTRODUCTION

The general theory of optimization finds a privileged field of application within the framework of Mechanics. Optimization of structural design, in particular, has raised considerable interest in recent years. The available literature concerned with this subject is vast. A comprehensive and organized review up to 1972 can be found in an article by Niordson and Pedersen (1972). Another excellent survey, concerned with constraints of a dynamic nature only, has been provided by Pierson (1972). Since then the flow of new research publications has continued steadily. A new phase seems now to have been reached with the apparition of monographs by Moe and Gisvold (1971), Hemp (1973), Gallagher and Zienkiewicz (1973), Distefano (1974), Majid (1974), among others. Research in structural optimization mainly follows two separate directions. The first approach, sometimes termed structural synthesis, is concerned with the optimization of discrete or discretized structures ; it is cast as a mathematical programming problem of extremizing an objective function subject to several equality or inequality constraints.

For a more complete description, the reader is referred to the papers
by Schmit (1971) or Fox (1973), as well as to the texts by Moe and
Gisvold (1971) or Gallagher and Zienkiewicz (1973) already mentioned.
Although relatively complex structures can be considered, the number
of design variables remains, unfortunately, very limited for practical
reasons. The second approach is concerned with the optimal design of
continuous structural elements, the properties of which are amenable to
description by ordinary (one-dimensional structures) or partial (two-
and three - dimensional structures) differential equations. A consider-
able literature exists in this field, although it is systematically
ignored by those advocating the mathematical programming approach. This
is rather unfortunate, since the continuous approach has the merit of
raising basic questions such as the existence, uniqueness and character-
ization of the solution, questions which cannot be answered by mathema-
tical programming techniques alone. Moreover, the constitution of a
library of optimal structural members under a variety of constraints
and boundary conditions would provide the practicing designer with
efficient ideas in situations where intuition alone is of little help.

The present paper is concerned with the minimum-mass design of elastic
structures whose behaviour is described by ordinary differential equa-
tions (no mention will be made of so-called plastic design, which is
quite a different subject altogether). Despite a large existing litera-
ture, very few numerical solutions for structural elements have been
presented, and the majority of those are of academic interest only. We
present in what follows a numerical solution procedure borrowed from
optimal control theory, which has proven quite useful in situations
where the conventional variational approach leads to a set of complex
nonlinear equations.

2. OPTIMAL DESIGN OF CONTINUOUS STRUCTURAL ELEMENTS

Application of the classical variational calculus to structural opti-
mization dates as far back as Lagrange (1770-1773) who investigated
the ideal shape that a column should possess in order to sustain its
own weight. In this approach, the resulting equations are unfortunately
quite complex and usually nonlinear. It is extremely rare to find a
closed-form solution. A relative exception is the class of so-called
linear stiffness structures, for which both stiffness and weight are
linear in the design variable (cross-sectional area or thickness) ;

this is the case of sandwich construction in particular. The paper by
Prager and Taylor (1968) contains an exhaustive treatment of this sub-
ject. A tutorial presentation can be found in Dym (1974). Generally,
however, the investigator is left with a complex two-point boundary
value problem (TPBVP) to solve in the case of one-dimensional elements,
or a system of nonlinear partial differential equations in the case of
two- or three-dimensional situations.

Application of optimal control theory to structural optimization was
independently suggested by Dixon (1967), Ashley and McIntosh (1968) and
Haug (1969). It has up to now hardly been realized that the designer
has at his disposal the powerful tools which have been developed for
the solution of similar problems arising in control theory, described
in the book by Bryson and Ho (1969), among others. The idea is to cast
the optimization problem in a form amenable to conventional treatment
through the introduction of state and design (control) variables. The
optimality conditions are then derived through an extension of classic-
al variational methods (a particular case of the more general Pontrya-
gins's Maximum Principle). The resulting system of equations is then
solved by an iterative procedure.

A systematic presentation of the above points as well as an evaluation
of the various numerical schemes will be found in a paper by Armand
(1973), who also presents a review of the field. McIntosh (1974) also
described the foundations of the method in a review paper in which
comparisons with discrete solutions can be found. The approach presented
here will be best described through a simple example where the constraint
is of an aeroelastic nature, first suggested by Ashley and McIntosh
(1968) and treated in detail by Armand and Vitte (1970), in which res-
ults of computation can be compared with the exact analytical solution.

3. MINIMUM - MASS DESIGN OF A RECTANGULAR WING FOR GIVEN TORSIONAL
   DIVERGENCE SPEED

   3.1 Statement of the problem

   Consider a cantilever straight wing with elastic axis perpendicular
   to the free stream. The wing profile is assumed to be constant along
   the span and characterized by a lift-coefficient slope a_o. The other
   parameters and variables are defined in Fig. 1. The thickness T of
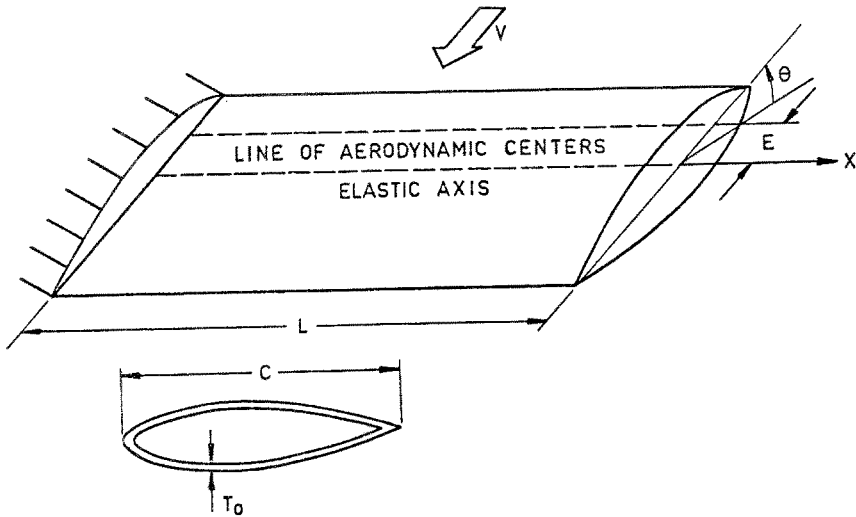
the skin varies along the span.

Figure 1 : Unswept cantilever wing with constant chord.

The torsional divergence speed V for this wing is obtained by
solving the eigenvalue problem :

$$\frac{d}{dX}(GJ \frac{d\theta}{dX}) + \frac{1}{2}\rho V^2 CEa_o \theta = 0 \qquad (1)$$

$$\theta(0) = 0,$$

$$GJ\frac{d\theta}{dX}\Big|_{X = L} = 0 \qquad (1')$$

ρ is the free stream density. The reader is referred to Bispling-
hoff, Ashley and Halfman (1955) for details.

If we assume that the torsional stiffness of the wing is dominated
by the contribution from the skin, then the torsion constant J is
directly proportional to the skin thickness (Bredt's formula),

$$J = KT$$

It is possible to rewrite eqs. (1) and (1') in a convenient, non-
dimensional form if we introduce that particular wing of constant
skin thickness T_0 with identical cross-sectional profile and the
same torsional divergence speed -the so-called reference wing. We
define the dimensionless quantities

$$x = \frac{X}{L} , \quad t = \frac{T}{T_0} ,$$

and denote by ()' the differentiation with respect to x ; the eigen-
value problem (1), (1') is then rewritten as :

$$(t\theta')' + \omega^2\theta = 0 , \quad \omega = \frac{\pi}{2} \tag{2}$$

$$\theta(0) = 0$$

$$t\theta' \Big|_{x = 1} = 0 \tag{2'}$$

The optimization problem is now stated as follows :
Minimize the functional

$$m = \int_0^1 t(x)dx$$

representing the dimensionless skin mass, subject to the constraints
(2) and (2').

Eq. (2) is split into a system of two first order differential equa-
tions after introducing a new variable s, as follows :

$$\theta' = \frac{s}{t}$$

$$\tag{3}$$

$$s' = -\omega^2\theta$$

with boundary conditions :

$$\theta(0) = 0 \qquad s(1) = 0 \tag{3'}$$

We introduce unknown Lagrange multipliers λ_θ and λ_s and form the Hamiltonian

$$H = t + \lambda_\theta \frac{s}{t} - \lambda_s \omega^2 \theta \tag{4}$$

Necessary conditions for an extremum of m are now derived. The optimality condition is :

$$\frac{\partial H}{\partial t} \equiv 1 - \frac{\lambda_\theta s}{t^2} = 0 \tag{5}$$

λ_θ, λ_s are solution of the system of Euler-Lagrange equations :

$$\lambda_\theta' \equiv - \frac{\partial H}{\partial \theta} = \omega^2 \lambda_s$$

$$\lambda_s' \equiv - \frac{\partial H}{\partial s} = - \frac{\lambda_\theta}{t} \tag{6}$$

Transversality conditions furnish two additional boundary conditions:

$$\lambda_\theta(1) = 0 \qquad \lambda_s(0) = 0 \tag{6'}$$

Equations (3) and (6) together with eq. (5) and boundary conditions (3') and (6') form a system of 4 first order differential equations in the four unknowns θ, s, λ_θ, λ_s. This system will be solved by resorting to the so-called neighboring extremal methods ; the idea is to construct a nominal solution satisfying the governing differential equations (3), the optimality condition (5) and the Euler-Lagrange equations (6), but not all the boundary conditions (3') and (6').

3.2 <u>The numerical procedure : a transition-matrix algorithm (Newton's method)</u>

The idea is to find the relation between unspecified boundary conditions at one end and specified boundary conditions at the other end. We start by "guessing" the unknown initial conditions $s(0)$ and $\lambda_\theta(0)$, then integrate (3) and (6) forward from 0 to 1, t being determined through (5). We obtain values for $s(1)$ and $\lambda_\theta(1)$, which in general will differ from zero. A "measure" of the variations of the final values of s and λ_θ when the initial boundary conditions are perturbed is provided by the matrix T, the so-called transition matrix

$$T = \begin{bmatrix} \dfrac{\partial s(1)}{\partial s(0)} & \dfrac{\partial s(1)}{\partial \lambda_\theta(0)} \\[4mm] \dfrac{\partial \lambda_\theta(1)}{\partial s(0)} & \dfrac{\partial \lambda_\theta(1)}{\partial \lambda_\theta(0)} \end{bmatrix} \tag{7}$$

linking small variations of s and λ_θ at 1 to small variations of the same quantities at 0

$$\begin{bmatrix} \delta s(1) \\[2mm] \delta \lambda_\theta(1) \end{bmatrix} = T \begin{bmatrix} \delta s(0) \\[2mm] \delta \lambda_\theta(0) \end{bmatrix} \tag{8}$$

To determine T, we begin by computing the first variations $\delta\theta$, δs, $\delta\lambda_\theta$, $\delta\lambda_s$ of the 4 quantities θ, s, λ_θ, λ_s from the system of differential equations defining them ; we obtain :

$$\delta\theta' = \frac{1}{2} \sqrt{\frac{s}{\lambda_\theta}} \left(\frac{\delta s}{s} - \frac{\delta\lambda_\theta}{\lambda_\theta} \right)$$

$$\delta s' = -\omega^2 \delta\theta$$

$$\delta\lambda_\theta' = \omega^2 \delta\lambda_s$$

$$\delta\lambda_s' = \frac{1}{2}\sqrt{\frac{\lambda_\theta}{s}} \left(\frac{\delta s}{s} - \frac{\delta\lambda_\theta}{\lambda_\theta} \right) \tag{9}$$

If $\delta s(0)$ is set equal to unity, all other perturbations being init-
ially set to zero, integration of this system (9) together with the
initial system (3), (3'), (5), (6), (6') on $[0,1]$ will yield values
for $\delta s(1)$ and $\delta\lambda_\theta(1)$ which constitute the first column of T. Similar-
ly, the second column of T is obtained by integrating both systems
with, as initial conditions, $\delta\lambda_\theta(0)$ set equal to unity, all other
perturbations equal zero, (3') and (5').

If the initial conditions $s(0)$ and $\lambda_\theta(0)$ had been correctly
chosen, $s(1)$ and $\lambda_\theta(1)$ would both vanish. We therefore define
variations of the final values $\Delta s(1)$ and $\Delta\lambda_\theta(1)$ which will hope-
fully bring the next solution closer to the desired values of $s(1)$
and $\lambda_\theta(1)$, defined as :

$$\Delta s(1) = -\varepsilon s(1)$$

$$\Delta\lambda_\theta(1) = -\varepsilon\lambda_\theta(1) \qquad\qquad 0\le\varepsilon\le 1$$

Corresponding variations $\Delta s(0)$ and $\Delta\lambda_\theta(0)$ of initial values are
obtained from (8) :

$$\begin{bmatrix} \Delta s(0) \\ \Delta\lambda_\theta(0) \end{bmatrix} = T^{-1} \begin{bmatrix} \Delta s(1) \\ \Delta\lambda_\theta(1) \end{bmatrix}$$

We now take as a new guess :

$$s(0)_{new} = s(0)_{old} + \Delta s(0),$$

$$\lambda_\theta(0)_{new} = \lambda_\theta(0)_{old} + \Delta\lambda_\theta(0)$$

and start the whole process again until $s(1)$ and $\lambda_\theta(1)$ have the
specified value zero to the desired accuracy.

With values of $s(0)$ and $\lambda_\theta(0)$ initially equal to 0.3 and 6, respec-
tively, and $\varepsilon = 1$, convergence was very smoothly obtained in 6 iter-
ations with a relative error of 0.04 % on the value of t at the root
(Fig. 2). The profile obtained is that given by the exact analytical
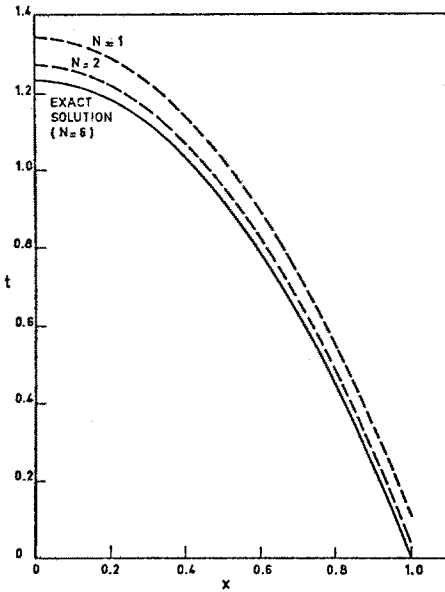solution (Ashley and McIntosh (1968)).

Fig. 2 : A transition-matrix procedure-
no minimal thickness constraint
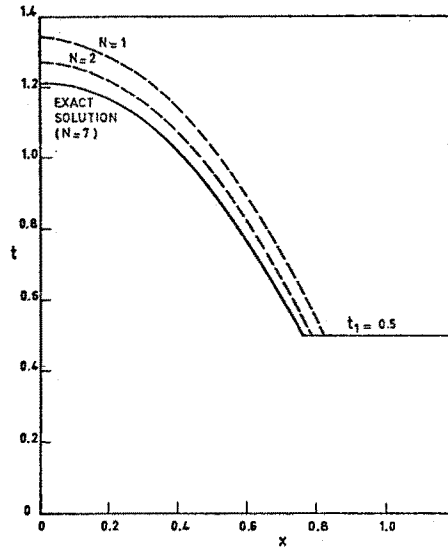(N = number of iterations).

Fig. 3 : A transition-matrix procedure-
minimum thickness constraint

3.3 Optimization with a minimum thickness constraint

The more realistic case of a constraint on the value of the thick-
ness is handled in the easiest fashion. A single statement needs to
be added to the computer program, setting t equal to the prescribed
value t_{min} everytime its computed value $t = \sqrt{\lambda_\theta s}$ is smaller than
t_{min}. Results of the iteration procedure are shown in Fig. 3 for
$t_{min} = 0.5$ where they are compared with the exact analytical solu-
tion given by Armand and Vitte (1970). The actual distribution was
obtained after 7 integrations with a relative error of 0.01 % on
t(0).

The problem just described is governed by fairly simple equations,
which explains the relative ease with which the numerical solution
was iteratively found. Other successful applications of this proce-
dure can be found in the work by Armand and Vitte (1970), or Weiss-
haar (1970).

A slighty more complex case, for which no closed-form solution
exists, is that of a vibrating simply-supported beam, first treated

by Niordson (1965), and independently extended by Haug (1969) and
Armand (1973) to handle minimum thickness constraint. The procedure
described above has proven very powerful, both for the constrained
and unconstrained case. Good estimates for the unknown boundary
conditions were, in that case, obtained from the solution of the
system of equations under consideration corresponding to the uniform
reference case (therefore ignoring the optimality condition).

4. TWO-DIMENSIONAL STRUCTURAL ELEMENTS

The behaviour of two-dimensional structural elements such as plates and
shells is described by partial differential equations in two independent
variables. Optimization of such elements ot satisfy given requirements
can be formulated as a problem in optimal control theory of distributed-
parameter systems, extension to a spatial domain D of optimal control
theory in which only one independent variable, the time t, is consi-
dered.

The domain D is the planar image of the spatial structure considered,
and is referred to a set of orthonormal axes Ox, Oy.

State and design variables are defined in the same fashion as before ;
the deflection (in-plane or out-of-plane) and its derivatives will play
the role of state variables, whereas the thickness will in some cases
represent a design variable.

For a minimum-mass problem, the functional to be minimized is the sur-
face integral of the thickness over the domain D ; the constraints are
in the form of partial differential equations to be satisfied inside
the domain, together with equality and/or inequality constraints on
functions of the state and/or design variables.

A problem of this type is a special case of a general Mayer-Bolza pro-
blem for multiple integrals, which received authoritative treatment
from Lurie (1963) in a paper which also hints at possible applications
to structural optimization.

A statement of the general optimization problem as well as a derivation
of the necessary conditions will be found in Armand (1972).

Applications of the theory to structural optimization problems has
been up to now very limited. The plate optimization problem, in parti-
cular, has not yet been solved in a satisfactory manner, although many
researchers have considered it : reference is made to McIntosh (1974)
for a comprehensive review. It shoud be mentioned at this point,
however, that numerical problems are not characteristic of two-dimen-
sional plates but also apply to axi-symmetric (one-dimensional) plates
as well ; numerical difficulties encountered at the solution stage for
situations involving plates, however, undoubtly correspond to problems
associated with the non-existence of a global optimum.

The above observations are in accordance with the works of Olhoff (1973)
and Mroz (1973), who independently pointed out that the optimization
problem in the case of a vibrating plate does not seem to possess a
global optimum solution, but instead an infinity of local extrema.

Further research should therefore be pursued in this direction with the
tools of control theory, since it is of particular interest to investi-
gate existence and possible uniqueness of solutions to those problems
for which an analytical formulation is still possible.

Numerical methods developed in the frame of optimal control of distri-
buted-parameter systems theory and similar to methods used to solve the
TPBVP's of classical optimal control have hardly been used yet and will
without doubt prove extremely helpful.

A very promising approach is that described by Pierson and Genalo (1974),
who applied a gradient projection algorithm to obtain the optimal
design of a rectangular panel subject of a flutter speed constraint.

6. ACKNOWLEDGEMENTS

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