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Optimization of synchronizability in multiplex networks by rewiring one layer

Sanjiv K. Dwivedi,[1] Murilo S. Baptista,[2] and Sarika Jalan[1,3]

[1]*Complex Systems Lab, Physics Discipline, Indian Institute of Technology Indore, Khandwa Road, Indore 453552, India*
[2]*Institute of Complex Systems and Mathematical Biology, SUPA, University of Aberdeen, Aberdeen AB24 3UE, United Kingdom*
[3]*Center for Biosciences and Biomedical Engineering, Indian Institute of Technology Indore, Khandwa Road, Indore 453552, India*

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The mathematical framework of multiplex networks has been increasingly realized as a more suitable framework for modeling real-world complex systems. In this work, we investigate the optimization of synchronizability in multiplex networks by evolving only one layer while keeping other layers fixed. Our main finding is to show the conditions under which the efficiency of convergence to the most optimal structure is almost as good as the case where both layers are rewired during an optimization process. In particular, interlayer coupling strength responsible for the integration between the layers turns out to be a crucial factor governing the efficiency of optimization even for the cases when the layer going through the evolution has nodes interacting much more weakly than those in the fixed layer. Additionally, we investigate the dependency of synchronizability on the rewiring probability which governs the network structure from a regular lattice to the random networks. The efficiency of the optimization process preceding evolution driven by the optimization process is maximum when the fixed layer has regular architecture, whereas the optimized network is more synchronizable for the fixed layer having the rewiring probability lying between the small-world transition and the random structure.

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Introduction. The framework for a single network has been extremely successful for predicting and understanding the behavior of complex systems [1]. However, recent studies of multiplex networks are providing new insights into the research in real-world complex systems by incorporating in the analysis the fact that more than one type of interaction may exist among the same units. Thus, multiplex networks are expected to provide a better understanding of the underlying structural and dynamical properties of real-world systems as compared to the traditional isolated networks approach [2]. For instance, diffusion processes taking place on multiplex networks have been shown to exhibit abrupt transitional behavior guided by interlayer coupling strength [3]. Entropy rates and information transmission was shown to be strongly regulated by the ratio between interconnectivity and the size of the single layer [4]. Similarly, cluster synchronization of a layer in multiplex networks has been demonstrated to be strongly affected by the network parameters of other layers [5,6]. Furthermore, endemic states in multiplex networks have been shown to crucially depend on the interconnectivity of the layers, not emerging in individual layers when considered in isolation [7,8]. The multiplex framework has been incorporated to explain patterns formation in a reaction-diffusion system [9].

Furthermore, synchronization phenomena or the collective behavior of coupled dynamical units has been a topic of intensive research [10]. The dynamical behavior of interacting units depends on the structural properties of interactions. One such relation between the structural property of a network and the synchronous dynamical behavior of units interacting via diffusive coupling is measured by the synchronizability of the network, defined by the ratio between the first nonzero and the largest eigenvalues of the corresponding Laplacian matrix [11,12]. The larger (smaller) the R values, the smaller (the larger) the coupling strength interval for which synchronization is observed. Furthermore, using a master stability function, various possibilities of synchronization such as inter- or intralayer have been discussed for multiplex networks [13].

The most optimized network in terms of synchronizability has been shown to exhibit homogeneity in its degree distribution and in the betweenness centrality of the nodes [14]. Optimization of synchronizability in networks with nodes connected by weighted strengths is a problem with an extra dimension of complexity. However, it has been shown that such networks can be successfully evolved to become optimally synchronizable [15–17]. Even more challenging is the optimization of multiplex networks, which would require optimization strategies involving several network parameters and larger dimensional systems. Take the brain as an example; it learns by rewiring its synaptic connections. If the brain were to adapt (optimize behavior) based on all its possible scenarios, that would be a fantastic complex optimization process. Rather, it is plausible to think that optimization in the brain (such as those driven by Hebbian learning rules) is driven by evolution rules applied locally. This Rapid Communication shows that indeed synchronizability of a whole multiplex network can be achieved by rewiring only one layer, thus showing that the computational complexity of optimization in multiplex networks can be drastically reduced.

More specifically, we study optimization of a layer in a multiplex network such that the entire network becomes more synchronizable. During the evolution, only one layer is rewired while keeping the other layer(s)'s topology fixed. Changing the network architecture of one layer affects the dynamical evolution of the other layers because of the interactions mediated by the interlayer couplings. We therefore investigate the efficiency of the optimization in terms of the interplay between the intralayer coupling strengths of the layer going through the evolution process and interlayer couplings. Furthermore, we investigate the impact of the network architecture of the fixed layer on the optimization efficiency. Our investigation reveals that the interlayer coupling strength plays a crucial role in determining the impact of the optimization process on the synchronization of the entire network. Interestingly, even if the layer going through the evolution has much weaker intralayer coupling strength as compared to that of the fixed layer, the efficiency of optimization is high if there is a strong interaction between the layers. Moreover, the optimization leads to the best synchronizable multiplex network when the network architecture of the fixed layer lies between a complete random architecture and the one observed at the small-world transition arising due to the combined impact of the degree homogeneity and the diameter.

Optimization of complex networks is behind the success of technological as well as natural adaptive processes. It is a current scientific challenge to understand natural optimization processes in order to reproduce them. For example, Ref. [18] introduced a secret-key exchange protocol based on the synchronization of two neural networks. The secret key would be formed by the final trained synaptical weights, adapted to promote full synchrony between the sender and receiver networks. Security of the method is based on the fact that an eavesdrop network would need more time to become synchronous with either network. This Rapid Communication shows that the ability of a whole network to synchronize can be optimized by only rewiring a single network layer. This result thus provides a way to improve on the security of this secret-key exchange protocol by creating a sender and a receiver network that could potentially become synchronous more quickly. Finally, deep learning machines change the internal structures of its neural network to optimize its logical outputs. Even though synchronization is not required to train a deep learning machine, if what we have shown in this work can be transported to deep learning training, i.e., only one or fewer hidden layers are trained, that would contribute to increasing efficiency in the training of these complex machines.

Theoretical framework. Let A and B be two adjacency matrices with dimension $N \times N$ representing layers of a multiplex network. The elements in the adjacency matrices (a_{ij} and b_{ij}) take values 1 and 0 depending upon whether or not there exists a connection between the i and j nodes. The weighted adjacency matrix of the multiplex networks can be written as

$$M = \begin{bmatrix} A & D_x\mathbb{I} \\ D_x\mathbb{I} & E_y B \end{bmatrix}, \tag{1}$$

where D_x and E_y represent inter- and intralayer coupling strength, respectively, and I (I^T) is the interlayer adjacency matrix representing the connections from B to A (A to B).

We optimize the eigenvalue ratio $(R) = \frac{\lambda^{\max}}{\lambda^2}$, inverse of synchronizability, where $\lambda^{\max}$ and λ^2 are the largest and the first nonzero eigenvalue of the Laplacian matrix of the multiplex network constructed from $\sum_{j=1}^{2N} M_{ij}\mathbb{I} - M$, where $\mathbb{I}$ represents the identity matrix. We use the simulated annealing technique [19] to perform the optimization. Our optimization aims at minimizing R, and thus, maximizing synchronizability. This optimization technique has several variations depending upon the problem at hand. For the current work, the method is explained as follows. We take an 2initial multiplex network with a given set of parameters. Next, we calculate the eigenvalue ratio R_1 of the corresponding Laplacian matrix of the initial multiplex network. Rewiring is performed only in one layer by keeping the second layer's architecture fixed throughout the evolution. We calculate the eigenvalue ratio R_2 of the multiplex network after performing a single rewiring. The initial multiplex network is replaced by the rewired multiplex network if the latter is more synchronizable and $R_2 \leqslant R_1$ otherwise replaced with the probability $p = \exp[(R_1 - R_2)/T]$. Whereas, the initial network is selected with the probability $1 - p$. T is a constant taken initially as 1.000. It is updated to the end of each generation by $0.999T$.

During the optimization process, the fixed layer introduces a limit to the synchronizability of the entire multiplex network. Nevertheless, the effect of the fixed layer varies depending upon inter- and intralayer coupling strengths. Naturally, if the layer going through the rewiring during evolution has stronger intralayer couplings as compared to that of the fixed layer, the optimization should be more efficient. Interestingly, we find that the interlayer coupling strength D_x has a more profound impact on the optimization. To observe the impact of E_y and D_x on the efficiency of the optimization process, we systematically investigate the following cases. In case I, the interlayer coupling strength is weak, i.e., $D_x = 1$ and the layer with weaker intralayer coupling strengths (layer A) is rewired resulting in evolution of this layer, whereas the architecture of the layer (B) with stronger intralayer coupling strengths is maintained throughout the evolution process. In case (II), the interlayer coupling strength is strong (D_x is large), and other parameters are the same as for case (I). In case (III), D_x is large and the layer with smaller intralayer coupling (layer A) is preserved during the evolution. The rewiring is performed only in the layer having larger intralayer coupling strength (layer B). To compare the results about the impact of change in only one layer on the synchronizability of the entire multiplex network with those obtained for changes in both the layers, we consider two more cases. In cases (IV) and (V), evolution is allowed in both the layers with case (IV) considering $D_x > 1$ and case (V) considering $D_x = 1$. In case (VI), $D_x = 1$ and the layer with weaker intralayer coupling strengths (layer A) is preserved, and the layer with stronger intralayer couplings is evolved. Furthermore, we measure the efficiency of synchronizability by $R_{\text{norm}} = \frac{R_{\text{opt}}}{R_{\text{ini}}}$, where R_{opt} and R_{ini} represent the value of R for the final optimized and the initial multiplex network, respectively. As R and the synchronizability of a network are inversely related, the lower the R_{norm} value, the better the efficiency of the synchronization.

Results. As the evolution progresses, the optimization attempts to bring the layer being rewired to a structure which is favorable for synchronization, whereas the fixed layer imposes a limit to the synchronizability or on the efficiency of the synchronization. Figure 1 demonstrates that for case (I), optimization does not succeed in producing a synchronizable network for any value of E_y we have considered. Whereas for case (II), the optimization succeeds in finding synchronizable networks for all the values of E_y considered here. Although the maximum efficiency corresponds to a value of E_y for which R_{norm} is minimal, the exact value of E_y for which efficiency is maximal depends on the size and average degree of the network. Further, a low value of D_x typically produces a low value of λ^2, whereas high values of D_x lead to high values of $\lambda^{\max}$ [20]. Both these factors contribute to an increase in

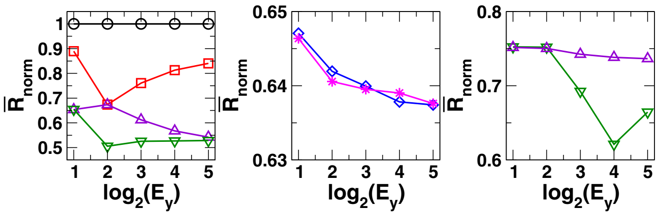

FIG. 1. $\bar{R}_{\text{norm}}$ against E_y for several optimization configurations. Left panel: R_{norm} for case (I) for $D_x = 1$ (circles), case (II) for D_x taking values from 2 to 32 (squares), case (III) corresponds to the scheme for $2 \leqslant D_x \leqslant 32$ and rewiring is done in the layer having stronger intralayer coupling (layer B) (upper triangles), and case (IV) (lower triangles). Middle panel: R_{norm} for case (V) which is similar to case (IV) except that the interlayer coupling is weak, i.e., $D_x = 1$ (stars) and case (VI) (diamonds). We consider $D_x = E_y$ for the cases having stronger interlayer coupling strengths. For the left panel and middle panel, $\langle k \rangle$ of each layer is 10 with $N = 500$. Right panel: R_{norm} for case (III) (upper triangles) and case (IV) (lower triangles). Network parameters are $\langle k \rangle = 20$ with size $N = 500$. For each case, optimization minimizes R for 200 000 iterations.

the R values, and for the model considered here, R can be determined as follows: For D_x being smaller with respect to E_y, referred to as the weaker D_x case, one can understand the behavior of R using the following approximation:

$$R \approx \frac{\max_{\alpha}[\lambda^{\max}(L^{\alpha}) + D_x]}{2D_x}, \tag{2}$$

where L^{α} is the Laplacian of the αth layer and $\lambda^{\max}(L^{\alpha})$ is the maximum eigenvalue of the Laplacian of the αth layer. For the model considered in Eq. (1), the α index represents the matrix A or matrix $E_y B$, and therefore $L^A = \sum_j A_{ij}\mathbf{I} - A$, and $L^B = \sum_j E_y B_{ij}\mathbf{I} - E_y B$.

For D_x being larger as compared to the intralayer coupling strength, referred to as the stronger D_x case, we have

$$R \approx \frac{2D_x + \sqrt{2}\lambda^{\max}(L^{\text{AV}})}{\lambda^2(L^{\text{AV}})}, \tag{3}$$

where L^{AV} is the average Laplacian of two layers.

For small D_x values, R is governed by Eq. (2). Since $\lambda^{\max}$ of the fixed layer having stronger intralayer coupling strength governs the numerator of Eq. (2) which leads to the same value of R throughout the optimization, this results in $R_{\text{norm}} \cong 1$. For larger D_x values, Eq. (3) starts to dominate over Eq. (2). The layer going through the evolution, even though having smaller intralayer couplings as compared to those of the fixed layer, contributes to R as it is the average value of the Laplacians of both the layers which appears in the denominator of Eq. (3). Furthermore, structural changes caused by the evolution process are capable of steering λ^2 of the evolved layer toward larger values, resulting in the smaller R values [Eq. (3)] and therefore, optimization is successful. For a further increase in D_x, Eq. (3) holds even better for the R values, and suddenly there is an increase in the efficiency of the optimization. However, the larger the values of D_x and E_y, the stronger the contribution of the fixed layer coupling strength in L^{AV} of Eq. (3). As a result, the efficiency again decreases for case (II). The efficiency for cases (V) and (VI), i.e., for smaller values of D_x, can be explained by Eq. (2) where $\lambda^{\max}$ comes from the rewired layer, which has stronger intralayer couplings and hence always dominates the numerator of Eq. (3). Interestingly, for smaller D_x values, rewiring in both the layers [case (V)] does not lead to an increase in the efficiency as compared to the rewiring in a single layer having stronger coupling strength [case (VI)] as illustrated in Fig. 1(b). For larger values of D_x and E_y, Eq. (3) controls the values of R where structural properties of both the layers are crucial to determine the spectral properties of the L^{AV} matrices. As a result, the efficiency is higher for case (IV) corresponding to rewiring performed in both the layers as compared to that of case (III), which corresponds to rewiring performed in only one layer. However, further increments in D_x, as well as in E_y (as $E_y = D_x$ for $D_x > 1$), lead to a domination of the contribution of stronger couplings in L^{AV} and as a result, the efficiency for case (IV) converges toward that of case (III). Figure 1(b) depicts that the efficiency of the optimization is the same for cases (V) and (VI), although there are huge differences in the computational cost for the optimization process. Case (V) considers rewiring performed in both layers and case (VI) has only one layer being rewired. Equation (2) explains this behavior since for both cases the R values depend on $\lambda^{\max}$ which is only determined by the layer having the stronger intralayer coupling strength going through rewiring for both the cases.

Furthermore, denser networks exhibit similar behavior for efficiency as the sparser networks. The one difference as compared to the sparser networks is that for cases (III) and (IV), the efficiency is equal for a larger value of D_x ($D_x \leqslant 4$) [Fig. 1 (right panel)]. As discussed earlier, that same efficiency (single layer vs both layer rewiring) is achievable if R is described by Eq. (2), since in the rewiring process only one layer having stonger intralayer couplings dominates the equation irrespective of whether both or a single layer get rewired. As already demonstrated in Ref. [21], with an increase in the average degree of networks, the value of D_x for which the best synchronizable networks are obtained shifts toward higher values and hence there exists an increment in the range of D_x toward the higher side, for which R is governed by Eq. (2). Consequently, for denser networks, the same efficiency for cases (III) and (IV) is observed.

To study the dependence of the optimization process on the topology of one fixed layer, we consider the initial fixed layer constructed by the small-world model with various rewiring probabilities p_r. The small-world transition [Fig. 2 (left panel)] for the Watts-Strogatz model is characterized by a clustering coefficient as high as that of the regular network and the characteristic path length being as small as that of the random networks. For an Erdös-Rényi (ER) network representing the layer going through the rewiring during the optimization process, and for small values of p_r typically smaller than the small-world (SW) transition, the initial and the optimized multiplex networks both have the same synchronizability [Fig. 2(b)]. For p_r larger than the value for the SW transition, the synchronizability of both the initial and the optimized multiplex networks starts increasing and attains its maximum value (the lowest R value) at a rewiring probability which is much higher than the critical parameter for the SW transition p_r, but much smaller than $p_r = 1$. Such a dependence of

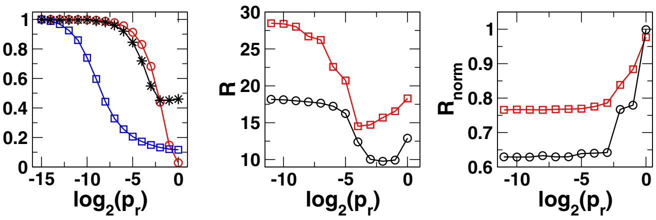

FIG. 2. Left panel: Clustering coefficient (circles), characteristic path length (squares), and normalized eigenvalue ratio (star) as a function of small-world rewiring probability (p_r) of an initial multiplex network having one layer represented by a SW network with p_r rewiring probability and other layers represented by an ER network. Middle panel: Impact of p_r on the optimized R value for the average degree of each layer taken as $\langle k \rangle = 10$ (circles) and $\langle k \rangle = 20$ (squares). The fixed layer is represented by a small-world network with p_r rewiring probability and the layer represented by an ER network is evolved through the optimization mechanism. Right panel: p_r vs R_{norm} for $\langle k \rangle = 10$ (circles) and $\langle k \rangle = 20$ (squares). Each layer of the multiplex networks has $N_1 = N_2 = 500$ and $D_x = 1$.

synchronizability on p_r is the result of an interplay between the degree homogeneity of the fixed layer and the layer going through the optimization. Initially for a p_r being smaller than the value for the SW transition, the diameter of the fixed layer is large resulting in a poor synchronizability of the entire multiplex network. For p_r being greater than the value for the SW transition, as long as the fixed layer still has a small degree of heterogeneity, the optimized multiplex networks possess the following topological characteristics contributing to better synchronizability: (1) degree homogeneity for both the fixed layer and the layer experiencing the rewiring (i.e., the distribution of degrees is not broad); (2) small values of both the average path length and the diameter of the entire multiplex network. For the fixed layer generated with $p_r = 1$ or close to 1, although the diameter and the average path length of the entire network are still small, the degree heterogeneity of the fixed layer is high enough that it does not get balanced by the rewiring of another layer during the optimization process, resulting in a smaller synchronizability of the optimized network. The value of p_r, corresponding to the maximally synchronizable network achieved through the evolution process, decreases as the average degree of the initial networks increases. This shift in p_r toward the lower values arises due to the fact that for denser networks, even very small rewiring probability values are sufficient to destroy the degree of homogeneity of the initial fixed layer, having a similar impact on the synchronizability of the final evolved network.

Moreover, optimization of denser networks leads to a less synchronizable evolved network than those achieved by optimizing sparser networks, since denser networks possess a larger amount of mismatch in the inter- and the intralayer connections [22]. For the sparser networks, the efficiency of synchronizability is high for a very large range of p_r. However, denser networks reflect comparatively a lesser efficiency of the optimization, i.e., smaller values of R_{norm} [Fig. 2 (right panel)], as the fixed layer restricts the value of R to decrease beyond a limit even though the second layer is rewired to enhance the synchronizability of the entire multiplex network.

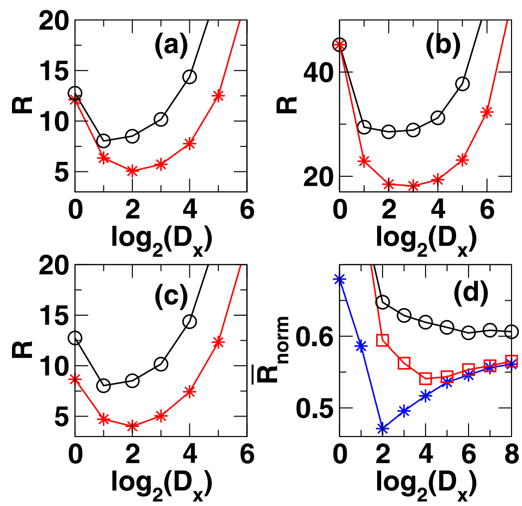

FIG. 3. The initial (circles) and optimized values (star) of the eigenvalue ratio (R) with an increase in the interlayer coupling strength D_x for (a) one layer having a fixed ER configuration and the other layer is rewired, (b) the fixed layer represented by a SF network and the other layer is rewired, and (c) both layers are initially represented by ER networks and both layers are rewired during the optimization. (d) depicts the efficiency of optimization (R_{norm}) when the fixed layer is represented by an ER network and the other layer is rewired (square). The case when the fixed layer is represented by the SF configuration and the other layer is rewired is depicted by circles. The case when both layers are represented by ER networks before optimization and both the layers are rewired during the optimization is depicted by stars. For all the cases, the network size in each layer is 500 with average degree 10.

Further, to study the impact of change in the structural properties of the fixed layer on the efficiency of optimization, we consider the fixed layer being represented by ER random and scale-free networks. Figure 3(a) depicts that there is a decrease in R with an initial increase in D_x. With a further increase in D_x, R starts increasing for the case of ER representing the fixed layer. For the fixed layer being represented by a scale-free network, R first decreases with an initial increase in the value of D_x, and after attaining a minimum value it remains almost constant for a further increase in D_x or for larger D_x values. As D_x increases further, R finally starts increasing. Again, similar to the previous case of a fixed layer represented by an ER network, the networks with lower D_x values are not optimizable [Fig. 3(b)]. This result is in contrast to the behavior exhibited for the unrestricted rewiring scheme. When both layers are rewired, the networks are optimizable for all the D_x values [Fig. 3(c)]. Figure 3(d) reflects that for the unrestricted rewiring, i.e., for rewiring taking place in both the layers, the efficiency of optimization is maximum for a certain value of D_x after which it again decreases. Interestingly, D_x for which efficiency is maximum is shifted toward a larger value for the case of a fixed layer being represented by ER random networks which also corresponds to the maximum efficiency. There is more shift toward a larger value for the case of a fixed layer represented by the scale-free (SF) networks. The reason behind this shift is that the local minima of R gets shifted toward a higher value of D_x for the layer having the scale-free architecture [12].

Moreover, irrespective of an initially considered mirror node correlation or the topology of the layer experiencing rewiring, the final evolved multiplex networks possess the same mirror node correlations as well as the intralayer configuration. Note that for various values of D_x, the mirror node correlations of the optimal multiplex networks have been shown to exhibit negative degree-degree correlations [21].

Conclusion. Our results show that there are several pathways to improve synchronizability of multiplex networks, either by altering parameters such as those that promote integration of the layers (increasing the interlayer coupling strength), or by evolving the network topology by rewiring edges within layers, under an optimization process. The surprising result is, however, that optimization of a single layer can achieve networks that are roughly as capable to synchronize as networks where all the layers are evolved under similar optimization criteria. This result is particularly relevant to works intended to improve synchronization of systems where only one layer is accessible or when one wants to optimize a system in a very cost-effective fashion. Having in mind that real-world systems are very large, complex, and composed of many layers, our work points out that optimization in such systems can indeed be carried out.

We have also studied the effectiveness of the optimization process, measured by the network synchronizability achieved through the evolution process, when the initial preevolved networks have different initial topologies. We found that the optimization leads to the maximum synchronizable multiplex networks when the fixed nonevolved layer has a topology lying in between a network with incipient small-world and fully random topologies.

Networks theory has proven its aptness in providing insights into controllability at a fundamental level. The controllability is desirable for dynamical behavior associated with the functionality of real-world systems. In traditional approaches, external inputs are imposed to affect the dynamics of few nodes which further causes a control of the entire system [23,24]. Our work might refine the concept of controllability by the addition of a system (one layer) that changes the dynamical evolution of the entire system (multiplex) to a desired behavior. Furthermore, our work might complement works on controllability by creating more synchronous evolved networks that could be more controllable.

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