# Optimized Transmission for Fading Multiple-Access and Broadcast Channels With Multiple Antennas

Mehdi Mohseni, *Student Member, IEEE*, Rui Zhang, *Student Member, IEEE*, and John M. Cioffi, *Fellow, IEEE*

***Abstract*—In mobile wireless networks, dynamic allocation of resources such as transmit powers, bit-rates, and antenna beams based on the channel state information of mobile users is known to be the general strategy to explore the time-varying nature of the mobile environment. This paper looks at the problem of optimal resource allocation in wireless networks from different information-theoretic points of view and under the assumption that the channel state is completely known at the transmitter and the receiver. In particular, the fading multiple-access channel (MAC) and the fading broadcast channel (BC) with additive Gaussian noise and multiple transmit and receive antennas are focused. The fading MAC is considered first and a complete characterization of its capacity region and power region are provided under various power and rate constraints. The derived results can be considered as nontrivial extensions of the work done by Tse and Hanly from the case of single transmit and receive antenna to the more general scenario with multiple transmit and receive antennas. Efficient numerical algorithms are proposed, which demonstrate the usefulness of the convex optimization techniques in characterizing the capacity and power regions. Analogous results are also obtained for the fading BC thanks to the duality theory between the Gaussian MAC and the Gaussian BC.**

***Index Terms*—Broadcast channel (BC), capacity region, convex optimization, delay constraint, fading, Gaussian noise, multiple-access channel (MAC), multiple antennas, multiple-input–multiple-output (MIMO), power region.**

## I. Introduction

INCREASING demand for higher data rates in mobile wireless systems with limited resources has motivated a great deal of valuable scholarly work to assess the information-theoretic limits of the channels that model the mobile environment. These channel models are often referred to as multiuser fading channels. This paper inscribes itself into this framework and its main goal is the derivation of the information-theoretic optimal resource allocations in the fading Gaussian multiple-access channel (MAC) and the fading Gaussian broadcast channel (BC). These derivations are under the assumption that the channel state information (CSI) is completely known at both the transmitter and the receiver. This paper considers the case where the base station and possibly each of the mobile users are equipped with multiple antennas—the so-called multiple-input–multiple-output (MIMO) channel—and demonstrates how convex optimization techniques can be utilized to characterize the information-theoretic limits of the fading Gaussian MIMO-MAC and MIMO-BC.

For single-user transmission systems, the information-theoretic limit of the fading channel is measured by the channel *capacity* which to date, has been well studied and is thoroughly known (see [1] and references therein). As an example, consider a *block-fading* additive white Gaussian noise (BF-AWGN) channel that is assumed to be constant during transmission of each code-block. This channel can be viewed as a collection of parallel deterministic AWGN channels each corresponding to a block transmission under a specific channel state. With full availability of CSI, the capacity of the BF channel can be found through optimal rate and power allocations into parallel AWGN channels based on their individual channel states.

Allocation of power and rate in fading channels is usually performed under some constraints imposed by practical considerations. For instance, the power allocation often needs to satisfy a *long-term* power constraint (LTPC) during the whole phase of transmission. In addition, some regulations intended for interference reduction may impose a *short-term* power constraint (STPC) that needs to be satisfied during transmission of each block [1]. There are also several well-adopted rate constraints which are applicable to different kinds of data traffic and are used to define various notions of the channel capacity. More specifically, when the transmission delay is not an issue and the data traffic admits *variable-rate* transmissions, the transmitter will be able to adjust the data rate of each block according to the channel state and attain the channel *ergodic capacity*. This notion of capacity measures the maximum long-term rate averaged over all the channel states. On the other hand, some data traffics like voice transmission and real-time video streaming may have strict delay constraints. One interesting notion of capacity for this scenario is the *delay-limited capacity* [2], defined as the maximum *constant-rate* that can be transmitted reliably over any possible channel state. The power and rate allocation schemes achieving the aforementioned capacities in a BF-AWGN channel are known (see [3]–[5]) as the variations of the well-known *water-filling* algorithm [6].

For multiuser transmission systems like the fading MAC and the fading BC, the information-theoretic limits can be characterized by the *capacity region*, that is defined as the convex-hull of the union of all achievable rates for the users given their individual power constraints. The same types of power and rate constraints considered in the single-user case are also applicable to the power and rate of each user, therefore, each of the capacity definitions in the single-user case can be generalized to define the corresponding capacity region in multiuser systems. Another interesting concept to characterize the information-theoretic limits of a multiuser system is its *power region*. The power region consists of all power-tuples with each, a given

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The authors are with the Department of Electrical Engineering, Stanford University, Stanford, CA 94305 USA (e-mail: mmohseni@stanford.edu; ee.ruizhang@stanford.edu; cioffi@stanford.edu).

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rate-tuple is achievable. Similar to capacity region, various notions of power region can be defined under different types of rate constraints.

The fading MAC can be considered as a collection of parallel deterministic MACs, each characterized by a joint channel state for all the mobile users. For the case of single transmit and receive antenna—also called single-input–single-output (SISO)—channel, the ergodic capacity region has been studied in [7] and completely established in [8] using the *polymatroid* structure of the capacity region for a deterministic MAC. Also, in [9], the delay-limited capacity region has been characterized for the fading Gaussian SISO-MAC, based on the *contra-polymatroid* structure [8] of the power region of a deterministic Gaussian SISO-MAC. Moreover, the case of transmission with nonzero outage probability has been considered in [10] for the fading Gaussian SISO-MAC.

The increasing importance of multiple transmit and receive antennas in wireless systems has been accompanied by a rapid pace of research on the information-theoretic characterization of the fading Gaussian MIMO-MAC (see [11] and references therein). Unlike the case of fading Gaussian SISO-MAC, in the MIMO case, except for the maximum sum-rate points considered in [12], generally, there is no closed-form analytic solution to the optimal power and rate allocation problem. Therefore, numerical optimization routines are usually needed to achieve this end. There is another important difference associated with the MIMO case as compared with the SISO case: although the polymatroid structure of the capacity region is still applicable to the MIMO case, the contra-polymatroid structure of the SISO-MAC power region is nonexistent. As a result, characterization of the power region for the Gaussian MIMO-MAC remains not yet fully understood.

For the fading Gaussian SISO-BC, characterization of the ergodic capacity region and the delay-limited capacity region have been done in [13]–[15] and [16], respectively. The information-theoretic characterization of a Gaussian MIMO-BC has been a research challenge for awhile due to its "nondegraded" nature [6]. However, recently, a successful progress has been made which proves that the "dirty paper coding" (DPC) rate region is indeed the capacity region (see [17] and references therein). It is worth remarking here that the above result combined with the duality theory between Gaussian MAC and BC [18]–[20] has one important implication: the information-theoretic characterization for a MIMO-BC now becomes a relatively easier task because it can be solved in its dual MIMO-MAC. For instance, determination of the minimum sum-power required to support a given rate-tuple for the Gaussian MIMO-BC has been attempted by several authors (see [21]–[24]). However, equipped with the duality theory, the solution to this problem can be easily attained if the characterization of the associated power region is available for the dual MIMO-MAC.

The remainder of this paper is organized as follows. Section II describes the channel model and defines the capacity and power regions. Section III studies the fading Gaussian MIMO-MAC and characterizes its power and capacity regions by formulating and solving a sequence of weighted sum-power minimization and weighted sum-rate maximization problems in Sections III-A and B, respectively. The *admission* problem for determining whether a rate-tuple in demand is achievable given a set of power constraints is considered and its solution is provided in Section III-C. It is also shown in Section III-C that the solution to the admission problem provides another effective means to characterize the power region and the capacity region via the power-profile vectors and the rate-profile vectors, respectively. Section IV proceeds to study the fading Gaussian MIMO-BC and extends the solutions to problems considered in the MAC case to analogous problems in the BC scenario by using the duality theory. Section V presents numerical examples which are used to demonstrate the usefulness of the proposed algorithms in solving practical problems and finally, a brief summary is given in Section VI.

*Notation*: This paper uses upper case boldface letters to denote matrices and lower case boldface letters to indicate vectors. For a square matrix $\boldsymbol{S}$, $|\boldsymbol{S}|$, $\boldsymbol{S}^{-1}$ and $\mathtt{Tr}(\boldsymbol{S})$ are its determinant, its inverse matrix and its trace, respectively. For any general matrix $\boldsymbol{M}$, $\boldsymbol{M}^\dagger$ denotes its conjugate transpose. $\boldsymbol{I}$ and $\boldsymbol{0}$ indicate the identity matrix and the matrix with all zero elements, respectively. $\mathbb{E}[\cdot]$ denotes statistical expectation. $\mathbb{C}^{x\times y}$ is the space of $x \times y$ matrices with complex number entries. $\mathbb{R}^M$ denotes the $M$-dimensional real Euclidean space and $\mathbb{R}_+^M$ is the non-negative orthant. The distribution of a Gaussian vector with the mean vector $\boldsymbol{x}$ and the covariance matrix $\boldsymbol{\Sigma}$ is denoted by $\mathcal{N}(\boldsymbol{x}, \boldsymbol{\Sigma})$, and $\sim$ means "distributed as." The sign $\succeq$ denotes the generalized inequality [25] and for a square matrix $\boldsymbol{S}$, $\boldsymbol{S} \succeq \boldsymbol{0}$ means $\boldsymbol{S}$ is positive semidefinite. $\mathbf{1}\{\mathcal{A}\}$ is the indicator function which takes the value of one if the event $\mathcal{A}$ is true, and zero otherwise.

## II. System Model

We first consider a fading MAC with the base station equipped with $r$ receive antennas and $K$ mobile users with $t_1, \ldots, t_K$ transmit antennas, respectively. Transmission from each mobile user to the base station is assumed to be synchronously on a time-block basis and the BF channel model is assumed for each individual user, i.e., the channel remains constant during each block transmission and possibly changes from one block to another. Also, we assume the space of fading states is discrete and finite and the fading process is stationary and ergodic. Thus, at each state $n$, the fading MAC can be considered as a discrete-time channel represented by

$$\boldsymbol{y}(n) = [\boldsymbol{H}_1(n) \cdots \boldsymbol{H}_K(n)] \begin{bmatrix} \boldsymbol{x}_1(n) \\ \vdots \\ \boldsymbol{x}_K(n) \end{bmatrix} + \boldsymbol{z}(n) \qquad (1)$$

where $\boldsymbol{y}(n) \in \mathbb{C}^{r\times 1}$ denotes the received signal vector. $\boldsymbol{x}_k(n) \in \mathbb{C}^{t_k\times 1}$ and $\boldsymbol{H}_k(n) \in \mathbb{C}^{r\times t_k}$ denote, respectively, the transmitted signal vector and the channel matrix of user $k$, $k = 1, \ldots, K$. $\boldsymbol{z}(n) \in \mathbb{C}^{r\times 1}$ denotes the additive Gaussian noise at the receiver and it is assumed that $\boldsymbol{z}(n) \sim \mathcal{N}(0, \boldsymbol{I})$. Furthermore, $n = 1, 2, \ldots, N$ is the index of channel fading state, which is assumed to be finite.

With full availability of instantaneous CSI, the transmission scheme can be adapted to the channel states. In this case, transmitters can dynamically allocate their available powers among different fading states to fully exploit the fading nature of the channel. Let the covariance matrix of the transmitted signal of user $k$ in state $n$ be $\boldsymbol{S}_k(n) = \mathbb{E}[\boldsymbol{x}_k(n)\boldsymbol{x}_k^\dagger(n)]$, where the expectation is taken over the code-book. Let $\boldsymbol{n}$ be a random variable taking the values of fading states, $n = 1, 2, \ldots, N$ and have the same probability distribution as the stationary probability distribution of fading states $f(n)$. Under the long-term power constraints (LTPC) considered in this paper, any codebook for user $k$ must satisfy

$$\mathbb{E}\left[\operatorname{Tr}\left(\boldsymbol{S}_k(\boldsymbol{n})\right)\right] = \sum_{n=1}^{N} \operatorname{Tr}\left(\boldsymbol{S}_k(n)\right) f(n) \le p_k^*$$

where $\mathbf{p}^* = (p_1^*, \ldots, p_K^*) \in \mathbb{R}_+^K$ is the vector of average-power constraints for all the users. Given a fixed set of transmit covariance matrices $\{\boldsymbol{S}_k(n)\}$ for $k = 1, \ldots, K$ and $n = 1, \ldots, N$, it was shown in [26] that all the rate-tuples in the set described below are achievable

$$\begin{aligned}\mathcal{C}_f\left(\{\boldsymbol{H}_k(n)\}, \{\boldsymbol{S}_k(n)\}\right) &= \Big\{\mathbf{r} \in \mathbb{R}_+^K : \\ &\sum_{k\in J} r_k \le \mathbb{E}\left[\frac{1}{2}\log\left|\sum_{k\in J} \boldsymbol{H}_k(\boldsymbol{n})\boldsymbol{S}_k(\boldsymbol{n})\boldsymbol{H}_k^\dagger(\boldsymbol{n}) + \boldsymbol{I}\right|\right], \\ &\forall J \subseteq \{1, \ldots, K\}\Big\}\end{aligned}$$

where the expectation is taken over the distribution of $\boldsymbol{n}$. Furthermore, with availability of CSI at the transmitters, the ergodic capacity region was shown in [8] to be

$$\mathcal{C}_{\text{MAC}}(\mathbf{p}^*) = \bigcup_{\substack{\{\boldsymbol{S}_k(n)\}:\ \forall k, \\ \mathbb{E}[\operatorname{Tr}(\boldsymbol{S}_k(\boldsymbol{n}))] \le p_k^*}} \mathcal{C}_f\left(\{\boldsymbol{H}_k(n)\}, \{\boldsymbol{S}_k(n)\}\right).$$

Although the proof given in [8] is for the case of SISO-MAC, it can be easily extended to the MIMO case with finite number of states. As an alternative characterization, we can also define the power region for the fading MAC under different rate constraints. For the average-rate constraints considered in this paper, the power region is defined as

$$\mathcal{P}_{\text{MAC}}(\mathbf{R}^*) = \left\{\mathbf{p} \in \mathbb{R}_+^K : \mathbf{R}^* \in \mathcal{C}_{\text{MAC}}(\mathbf{p})\right\} \qquad (2)$$

where $\mathbf{R}^* = (R_1^*, \cdots, R_K^*) \in \mathbb{R}_+^K$ denotes the vector of average-rate constraints. Fig. 1(a) illustrates the ergodic capacity region of a fading MAC. As is shown in the figure, this region is obtain by taking the union over all constituting sets $\mathcal{C}_f(\{\boldsymbol{H}_k(n)\}, \{\boldsymbol{S}_k(n)\})$. Fig. 1(b) shows the power region of a fading MAC for a given set of rate constraints.

To characterize the capacity and power regions, we exploit the fact that for a given set of transmit covariance matrices $\{\boldsymbol{S}_k(n)\}$ for all users over all states, the set $\mathcal{C}_f(\{\boldsymbol{H}_k(n)\}, \{\boldsymbol{S}_k(n)\})$ can be represented in terms of capacity regions of various deterministic MACs, each corresponding to a fading state. Recall that for a deterministic Gaussian MAC with fixed channel matrices $\{\boldsymbol{H}_k\}$ and given transmit covariance matrices $\{\boldsymbol{S}_k\}$, $k = 1, \ldots, K$, all achievable rate-tuples, belong to the following polyhedron [6]:

$$\mathcal{C}_g\left(\{\boldsymbol{H}_k\}, \{\boldsymbol{S}_k\}\right) = \Bigg\{\mathbf{r} \in \mathbb{R}_+^K : \sum_{k\in J} r_k \le \frac{1}{2}\log\left|\sum_{k\in J} \boldsymbol{H}_k\boldsymbol{S}_k\boldsymbol{H}_k^\dagger + \boldsymbol{I}\right|, \forall J \subseteq \{1, \ldots, K\}\Bigg\}. \qquad (3)$$

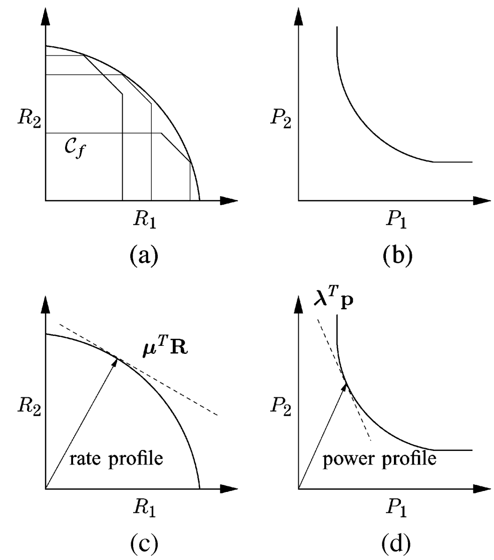


Fig. 1. (a) Ergodic capacity region of a general fading MAC or fading BC. (b) Power region of a fading MAC for a given set of rate constraints. (c) Characterization of the capacity region by weighted sum-rate maximization and rate profile. (d) Characterization of the power region by weighted sum-power minimization and power profile.

Using the *polymatroid* structure of the sets $\mathcal{C}_f$ and $\mathcal{C}_g$, in [8] it was shown that given any rate-tuple $(R_1, \ldots, R_K) \in \mathcal{C}_f(\{\boldsymbol{H}_k(n)\}, \{\boldsymbol{S}_k(n)\})$, for each $n = 1, \ldots, N$, there exists a rate-tuple $\mathbf{r}(n) \in \mathcal{C}_g(\{\boldsymbol{H}_k(n)\}, \{\boldsymbol{S}_k(n)\})$ such that

$$R_k = \mathbb{E}\left[r_k(\boldsymbol{n})\right] = \sum_{n=1}^{N} r_k(n) f(n), \quad k = 1, \ldots, K.$$

Alternatively, given any set of rate-tuples, $\mathbf{r}(n) \in \mathcal{C}_g(\{\boldsymbol{H}_k(n)\}, \{\boldsymbol{S}_k(n)\})$ for $n = 1, \ldots, N$, $\mathbb{E}[\mathbf{r}(\boldsymbol{n})]$ belongs to $\mathcal{C}_f(\{\boldsymbol{H}_k(n)\}, \{\boldsymbol{S}_k(n)\})$. Thus, for any rate-tuple $\mathbf{R} = (R_1, \ldots, R_K)$ in the ergodic capacity region, the rate of user $k$, $R_k$, can be expressed as $R_k = \mathbb{E}[r_k(\boldsymbol{n})]$, for some set of feasible transmit covariance matrices $\{\boldsymbol{S}_k(n)\}$ and rate-tuples $\mathbf{r}(n) \in \mathcal{C}_g(\{\boldsymbol{H}_k(n)\}, \{\boldsymbol{S}_k(n)\})$ for $n = 1, \ldots, N$. In other words, $R_k$ is the average of some achievable rates for user $k$ over all states. This observation enables us to express the ergodic capacity region of a fading MIMO-MAC alternatively as

$$\begin{aligned}\mathcal{C}_{\text{MAC}}(\mathbf{p}^*) = \bigcup_{\{\boldsymbol{S}_k(n)\}:\mathbb{E}[\operatorname{Tr}(\boldsymbol{S}_k(\boldsymbol{n}))] \le p_k^*} &\left\{\mathbf{R} : R_k = \mathbb{E}\left[r_k(\boldsymbol{n})\right],\right. \\ &\left.\mathbf{r}(n) \in \mathcal{C}_g\left(\{\boldsymbol{H}_k(n)\}, \{\boldsymbol{S}_k(n)\}\right) \text{ for all } n\right\}. \quad (4)\end{aligned}$$

Next, we consider the fading BC where the base station sends independent information to each of the mobile users. Without loss of generality, we consider a fading BC that is the *dual* channel [20] of the fading MAC in (1) and can be modeled as

$$\begin{bmatrix} \boldsymbol{v}_1(n) \\ \vdots \\ \boldsymbol{v}_K(n) \end{bmatrix} = \begin{bmatrix} \boldsymbol{H}_1^\dagger(n) \\ \vdots \\ \boldsymbol{H}_K^\dagger(n) \end{bmatrix} \boldsymbol{u}(n) + \begin{bmatrix} \boldsymbol{w}_1(n) \\ \vdots \\ \boldsymbol{w}_K(n) \end{bmatrix} \qquad (5)$$

where $\boldsymbol{u}(n) \in \mathbb{C}^{r\times 1}$ denotes the transmitted signal vector from the base station. $\boldsymbol{v}_k(n) \in \mathbb{C}^{t_k\times 1}$, $\boldsymbol{H}_k^\dagger(n) \in \mathbb{C}^{t_k\times r}$ and $\boldsymbol{w}_k(n) \in \mathbb{C}^{t_k\times 1}$ denote, respectively, the received signal vector, the channel matrix and the receiver AWGN associated with user $k$, where $\boldsymbol{w}_k(n) \sim \mathcal{N}(0, \boldsymbol{I})$. Let the covariance matrix of the transmitted signal during the fading state $n$ be $\boldsymbol{Z}(n) = \mathbb{E}[\boldsymbol{u}(n)\boldsymbol{u}^\dagger(n)]$ and the average transmit power constraint of the base station be $q^*$, i.e., $\sum_{n=1}^{N} \mathtt{Tr}(\boldsymbol{Z}(n))f(n) \leq q^*$. By the new capacity results in [17] combined with duality of [19], for a deterministic Gaussian BC with channel matrices $\{\boldsymbol{H}_k^\dagger\}$ and a total transmit power $\mathcal{E}$, all achievable rate-tuples belong to a convex set $\mathcal{C}_{\mathrm{sum}}(\{\boldsymbol{H}_k^\dagger\}, \mathcal{E})$ specified as below

$$\mathcal{C}_{\mathrm{sum}}\left(\left\{\boldsymbol{H}_k^\dagger\right\}, \mathcal{E}\right) = \bigcup_{\{\boldsymbol{S}_k\}: \sum_{k=1}^{K} \mathtt{Tr}(\boldsymbol{S}_k) \leq \mathcal{E}} \mathcal{C}_g\left(\{\boldsymbol{H}_k\}, \{\boldsymbol{S}_k\}\right). \tag{6}$$

Using this result, the ergodic capacity region for the fading BC in (5) can be expressed by

$$\mathcal{C}_{\mathrm{BC}}(q^*) = \bigcup_{\{\mathcal{E}(n)\}: \mathbb{E}[\mathcal{E}(\boldsymbol{n})] \leq q^*} \Big\{\mathbf{R} : R_k = \mathbb{E}\left[r_k(\boldsymbol{n})\right], \\ \mathbf{r}(n) \in \mathcal{C}_{\mathrm{sum}}\left(\left\{\boldsymbol{H}_k^\dagger(n)\right\}, \mathcal{E}(n)\right) \text{ for all } n\Big\}. \tag{7}$$

Since the BC has only one power constraint for the base station, characterization of its power region is equivalent to determination of the minimum power, $q_{\min}$, above which the average-rate constraints are satisfied, i.e.,

$$q_{\min}(\mathbf{R}^*) = \bigcap \left\{q \in \mathbb{R}_+ : \mathbf{R}^* \in \mathcal{C}_{\mathrm{BC}}(q)\right\}. \tag{8}$$

## III. Fading Gaussian MIMO-MAC

In this section, the capacity and power regions for the fading Gaussian MIMO-MAC are studied. Each region is characterized by solving a series of convex optimization problems for which, we provide efficient numerical algorithms.

### A. *Weighted Sum-Power Minimization*

In this section, we characterize the boundary points of the power region for the fading Gaussian MAC defined in (2) under average-rate constraints. This is done via solving a sequence of weighted sum-power minimization problems explained next. Since the power region is a convex set, each boundary point can be characterized by minimizing a weighted sum of the powers for some weights given by the *power-price* vector, $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_K) \in \mathbb{R}_+^K$. This is shown in Fig. 1(d) for a two-user case and formulated in the following.

*Problem 3.1:*

$$\mathtt{Minimize} \quad \sum_{k=1}^{K} \lambda_k \sum_{n=1}^{N} \mathtt{Tr}\left(\boldsymbol{S}_k(n)\right) f(n)$$

$$\mathtt{Subject\ to} \quad \sum_{n=1}^{N} r_k(n) f(n) \geq R_k^*, \quad \forall k \tag{9}$$

$$\boldsymbol{S}_k(n) \succeq \boldsymbol{0}, \ \forall k, n \tag{10}$$

$$\mathbf{r}(n) \in \mathcal{C}_g\left(\{\boldsymbol{H}_k(n)\}, \{\boldsymbol{S}_k(n)\}\right), \ \forall n. \tag{11}$$

In this problem, inequalities in (9) specify the average-rate constraints. Problem 3.1 is a convex optimization problem. The cost function is linear and the constraints in (9) are affine inequalities. Let $\mathcal{D}$ denote the region specified by the remaining constraints (10) and (11). The set $\mathcal{D}$ contains all admissible transmit covariance matrices $\boldsymbol{S}_k(n)$, and all achievable rates $r_k(n)$, for user $k = 1, \ldots, K$ over all fading states $n = 1, \ldots, N$. From concavity of the $\log|.|$ function, it is easy to verify that the set $\mathcal{D}$ is also a convex set. Therefore, standard convex optimization techniques can be employed to solve this problem.

In the following, we present an efficient algorithm to achieve this end. For simplicity purposes, assume uniform state distribution, $f(n) = (1/N)$ for all $n$. The Lagrangian [25] of Problem 3.1 with the vector of dual variables $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_K) \in \mathbb{R}_+^K$ associated with the inequality constraints in (9) is defined over domain $\mathcal{D}$ as

$$\mathcal{L}\left(\{\boldsymbol{S}_k(n)\}, \{r_k(n)\}, \boldsymbol{\mu}\right) = \sum_{k=1}^{K} \lambda_k \left(\frac{1}{N} \sum_{n=1}^{N} \mathtt{Tr}\left(\boldsymbol{S}_k(n)\right)\right) \\ - \sum_{k=1}^{K} \mu_k \left(\frac{1}{N} \sum_{n=1}^{N} r_k(n) - R_k^*\right). \tag{12}$$

Then, the Lagrange dual function defined as

$$g(\boldsymbol{\mu}) = \min_{\{\boldsymbol{S}_k(n), r_k(n)\} \in \mathcal{D}} \mathcal{L}\left(\{\boldsymbol{S}_k(n)\}, \{r_k(n)\}, \boldsymbol{\mu}\right) \tag{13}$$

serves as a lower bound on the optimal value of Problem 3.1 denoted by $p^*$, i.e., $\max_{\boldsymbol{\mu} \succeq 0} g(\boldsymbol{\mu}) \leq p^*$. However, for a convex problem where the Slater's condition[1] holds, the duality gap is zero [25]. Note that by using large enough powers, the sets $\mathcal{C}_f(\{\boldsymbol{H}_k(n)\}, \{\boldsymbol{S}_k(n)\})$ can be made arbitrary large to contain rate-tuple as an interior point. Thus, the Slater's condition holds and the duality gap is zero for the problem in hand. In other words, $p^* = \max_{\boldsymbol{\mu} \succeq 0} g(\boldsymbol{\mu}) \stackrel{\Delta}{=} g(\boldsymbol{\mu}^*)$, where $\boldsymbol{\mu}^*$ is a maximizer of the dual function and is not necessarily unique.

The above equality suggests that the optimal solution to Problem 3.1 can be found by first minimizing the Lagrangian to obtain the Lagrange dual function $g(\boldsymbol{\mu})$, and then maximizing $g(\boldsymbol{\mu})$ over all possible values of $\boldsymbol{\mu}$. Let $\{\boldsymbol{S}_k^*(n)\}$ and $\{r_k^*(n)\}$ be an optimal solution set for Problem 3.1 and $\boldsymbol{\mu}^*$ be a dual optimal solution (a dual function maximizer). Since the problem is convex and the duality gap is zero, Karush–Kuhn–Tacker (KKT) optimality conditions state that any primal optimal solution set minimizes $\mathcal{L}(\{\boldsymbol{S}_k(n)\}, \{r_k(n)\}, \boldsymbol{\mu}^*)$ and satisfies inequality constraints given in (9) simultaneously [25]. Hence, to solve this problem by exploiting the dual function, we initially need an efficient optimization algorithm to optimize the dual function $g(\boldsymbol{\mu})$ and find a $\boldsymbol{\mu}^*$. This dual optimal solution may not be unique, however, any dual optimal solution satisfies KKT conditions and suffices for our purpose. Then, we need to find a set of $\{\boldsymbol{S}_k(n)\}$ and $\{r_k(n)\}$ in $\mathcal{D}$ that minimizes the Lagrangian at $\boldsymbol{\mu}^*$ and satisfies the average rate constraints. By KKT conditions, this set will be an optimal solution set of Problem 3.1.

[1] Slater's condition requires the feasible set to have nonempty interior.

*1) Optimizing the Dual Function:* As the first step, we need to optimize $g(\boldsymbol{\mu})$. Since $g(\boldsymbol{\mu})$ is a concave function of $\boldsymbol{\mu}$, standard convex optimization techniques can be applied to achieve this end. In general, the dual function may not be differentiable or analytical expressions for its differentials may not exist. Hence, optimization algorithms that exploit the function's differentials, such as Newton method, cannot be employed. An appropriate choice here is the ellipsoid method [25] that is capable of handling nondifferentiable convex functions. A detailed description of this method can be found in Appendix I. As is explained in the appendix, this method is an iterative algorithm and on each iteration, it requires the dual function value and a subgradient at the corresponding value of $\boldsymbol{\mu} \in \mathbb{R}_+^K$. To compute the dual function, Lagrangian should be minimized. It is interesting to observe that the Lagrangian in (12) can be rearranged as

$$\mathcal{L}\left(\{\boldsymbol{S}_k(n)\},\{r_k(n)\},\boldsymbol{\mu}\right) = \sum_{k=1}^{K}\mu_k R_k^* + \frac{1}{N}\sum_{n=1}^{N}\left(\sum_{k=1}^{K}\lambda_k \mathtt{Tr}\left(\boldsymbol{S}_k(n)\right) - \sum_{k=1}^{K}\mu_k r_k(n)\right). \quad (14)$$

This expression implies that the minimization of $\mathcal{L}$ can be decomposed into $N$ independent optimization problems—also known as dual-decomposition method [27]—each given by

$$g_n'(\boldsymbol{\mu}) = \min_{\mathcal{D}_n}\sum_{k=1}^{K}\lambda_k \mathtt{Tr}\left(\boldsymbol{S}_k(n)\right) - \sum_{k=1}^{K}\mu_k r_k(n) \quad (15)$$

for $n = 1,\ldots,N$. Here, $\mathcal{D}_n$ denotes the subset of $\mathcal{D}$ relevant to a specific state $n$ defined by

$$\boldsymbol{S}_k(n) \succeq \mathbf{0}, \quad k = 1,\ldots,K$$
$$\mathbf{r}(n) \in \mathcal{C}_g\left(\{\boldsymbol{H}_k(n)\},\{\boldsymbol{S}_k(n)\}\right).$$

This decomposition reduces the complexity of finding the dual function by breaking the main problem into $N$ independent and smaller size problems. By comparing (13)–(15), the Lagrange dual function $g(\boldsymbol{\mu})$ can be rewritten as

$$g(\boldsymbol{\mu}) = \frac{1}{N}\sum_{n=1}^{N} g_n'(\boldsymbol{\mu}) + \sum_{k=1}^{K}\mu_k R_k^*. \quad (16)$$

Each optimization problem introduced in (15) can be cast as the following general problem:

$$\begin{aligned}
\mathtt{Maximize} \quad & \sum_{k=1}^{K}\mu_k r_k - \sum_{k=1}^{K}\lambda_k \mathtt{Tr}(\boldsymbol{S}_k) && (17)\\
\mathtt{Subject\ to} \quad & \mathbf{r} \in \mathcal{C}_g\left(\{\boldsymbol{H}_k\},\{\boldsymbol{S}_k\}\right) && (18)\\
& \boldsymbol{S}_k \succeq \mathbf{0}, \quad k = 1,\ldots,K. && (19)
\end{aligned}$$

To simplify the problem further, the following definition and lemma borrowed from [8] are utilized to remove the constraint in (18).

*Definition 3.1:* For a fixed set of transmit covariance matrices $\{\boldsymbol{S}_k\}$ and any permutation $\pi$ over $\{1,2,\ldots,K\}$, the rate-tuple $\mathbf{r}^{(\pi)}$ defined as

$$r_{\pi(k)}^{(\pi)} = \frac{1}{2}\log\frac{\left|\sum_{i=1}^{k}\boldsymbol{H}_{\pi(i)}\boldsymbol{S}_{\pi(i)}\boldsymbol{H}_{\pi(i)}^{\dagger} + \boldsymbol{I}\right|}{\left|\sum_{i=1}^{k-1}\boldsymbol{H}_{\pi(i)}\boldsymbol{S}_{\pi(i)}\boldsymbol{H}_{\pi(i)}^{\dagger} + \boldsymbol{I}\right|}$$

is a *Vertex* of the polymatroid $\mathcal{C}_g(\{\boldsymbol{H}_k\},\{\boldsymbol{S}_k\})$ in $\mathbb{R}_+^K$.

As can be seen from the definition, for a deterministic MAC with channel matrices $\{\boldsymbol{H}_k\}$, $\mathbf{r}^{(\pi)}$ is achievable by successive decoding scheme with decoding order determined by the permutation $\pi$. The message of user $\pi(K)$ is decoded first and the message for user $\pi(1)$ is decoded last [6].

*Lemma 3.1:* For any $\boldsymbol{\mu} \succeq 0$, the solution to the optimization problem

$$\mathtt{Maximize} \quad \sum_{k=1}^{K}\mu_k r_k \quad \mathtt{s.t.} \quad \mathbf{r} \in \mathcal{C}_g\left(\{\boldsymbol{H}_k\},\{\boldsymbol{S}_k\}\right)$$

is attained by a vertex $\mathbf{r}^{(\pi^*)}$, where $\pi^*$ is such that $\mu_{\pi^*(1)} \geq \mu_{\pi^*(2)} \geq \cdots \geq \mu_{\pi^*(K)}$.

*Proof:* Please refer to [8] and references therein. ∎

Results of this lemma simplifies the optimization problem in (17) to an optimization problem with twice continuously differentiable cost function and only positive semidefinite constraints given below

$$\begin{aligned}
\mathtt{Maximize} \quad & \sum_{k=1}^{K}\Big[-\lambda_k \mathtt{Tr}(\boldsymbol{S}_k) + \left(\mu_{\pi(k)} - \mu_{\pi(k+1)}\right) \times \frac{1}{2}\log\left|\sum_{i=1}^{k}\boldsymbol{H}_{\pi(i)}\boldsymbol{S}_{\pi(i)}\boldsymbol{H}_{\pi(i)}^{\dagger} + \boldsymbol{I}\right|\Big]\\
\mathtt{Subject\ to} \quad & \boldsymbol{S}_k \succeq \mathbf{0}, \quad k = 1,\ldots,K
\end{aligned}$$

where $\pi$ is a permutation such that $\mu_{\pi(1)} \geq \cdots \geq \mu_{\pi(K)} \geq \mu_{\pi(K+1)} = 0$. Since this problem is concave with twice differentiable objective function, we can solve it numerically by Interior-Point method to find the dual function at a given $\boldsymbol{\mu}$ [25]. To complete the ellipsoid method requirements and find a subgradient at this point, we rely on the following lemma.

*Lemma 3.2:* If $\{\boldsymbol{S}_k^*(n)\}$ and $\{r_k^*(n)\}$ minimize the Lagrangian over $\mathcal{D}$ at $\boldsymbol{\mu}$, i.e., $\mathcal{L}(\{\boldsymbol{S}_k^*(n)\},\{r_k^*(n)\},\boldsymbol{\mu}) = g(\boldsymbol{\mu})$, then the vector $\boldsymbol{\nu}$ defined as $\nu_k = R_k^* - (1/N)\sum_{n=1}^{N} r_k^*(n)$ for $k = 1,\ldots,K$ is a subgradient of $g$ at $\boldsymbol{\mu}$.

*Proof:* Since $\{\boldsymbol{S}_k^*(n)\}$ and $\{r_k^*(n)\}$ are already in $\mathcal{D}$, for any $\boldsymbol{\delta} \succeq 0$, we have

$$\begin{aligned}
g(\boldsymbol{\delta}) &\leq \mathcal{L}\left(\{\boldsymbol{S}_k^*(n)\},\{r_k^*(n)\},\boldsymbol{\delta}\right)\\
&= g(\boldsymbol{\mu}) + \sum_{k=1}^{K}(\delta_k - \mu_k)\left(R_k^* - \frac{1}{N}\sum_{n=1}^{N} r_k^*(n)\right).
\end{aligned}$$

∎

Note that any optimal solution of (13) corresponds to a subgradient of the dual function and can be used in the ellipsoid method.

All the steps in the proposed method to optimize the dual function are summarized in the following algorithm.

*Algorithm 3.1:*

- **Given** an ellipsoid $\mathcal{E}^{(0)} \subseteq \mathbb{R}^K$, centered at $\boldsymbol{\mu}^{(0)}$ and containing the optimal dual solution $\boldsymbol{\mu}^*$,[2]
- **Set** $i = 0$.
- **Repeat**
  1) Solve the optimization problems defined in (15) independently for each $n$ to obtain an optimal solution set $\{\boldsymbol{S}_k^*(n)\}$ and $\{r_k^*(n)\}$ that minimizes $\mathcal{L}(\{\boldsymbol{S}_k(n)\}, \{r_k(n)\}, \boldsymbol{\mu}^{(i)})$ over $\mathcal{D}$.
  2) Update the ellipsoid $\mathcal{E}^{(i+1)}$ based on $\mathcal{E}^{(i)}$ and the subgradient $\nu_k = R_k^* - (1/N)\sum_{n=1}^{N} r_k^*(n)$. Set $\boldsymbol{\mu}^{(i+1)}$ as the center of ellipsoid $\mathcal{E}^{(i+1)}$ (see Appendix I).
  3) Set $i \leftarrow i + 1$.
- **Until** the stopping criteria for the ellipsoid method is met.

*2) Finding the Primal Optimal Solution:* So far, we have proposed an algorithm to obtain an optimal dual solution $\boldsymbol{\mu}^*$. Having $\boldsymbol{\mu}^*$ in hand, it just remains to find a primal optimal solution, $\{\boldsymbol{S}_k^*(n)\}, \{r_k^*(n)\}$, that minimizes $\mathcal{L}(\{\boldsymbol{S}_k(n)\}, \{r_k(n)\}, \boldsymbol{\mu}^*)$ and also satisfies average-rate constraints given in (9). This problem seems like the same as finding the dual function at $\boldsymbol{\mu}^*$, however, the optimal solution obtained from the Interior-Point method may not satisfy the inequalities in (9). The reason is, in the Interior-Point method, depending on the order of elements of $\boldsymbol{\mu}^*$, we always choose the same vertices of polymatroids $\mathcal{C}_g(\{\boldsymbol{H}_k(n)\}, \{\boldsymbol{S}_k(n)\})$ as the optimal rates on every state $n$. Thus, their average will also generate the same vertex of the polymatroid $\mathcal{C}_f$ that corresponds to the minimum weighted sum-power. However, the target rate $\mathbf{R}^*$ may not be a vertex of this polymatroid. See Fig. 2 for an example of this situation. In the following, we address how to modify the optimal solutions of (13) to satisfy the average-rate constraints.

First, consider the case where all $\mu_k^*$s are different and positive. Recall that $\mu_k^*$ can be viewed as the local sensitivity of the optimal value of Problem 3.1 to perturbations made to the average-rate constraint of user $k$. In a MAC, generally increasing the target average rate of user $k$, $R_k^*$, requires more transmit power to support. Therefore, we can assume $\mu_k^*$s are positive without loss of generality. For this situation, the following lemma guarantees that the optimal rates, $\{r_k^*(n)\}$ of problem (15) for each state $n$ are unique and from the KKT conditions they automatically satisfy the average rate constraints in (9). This situation corresponds to the case where the target rate $\mathbf{R}^*$ lies on a vertex of the polymatroid $\mathcal{C}_f(\{\boldsymbol{H}_k(n)\}, \{\boldsymbol{S}_k^*(n)\})$ that has the minimum weighted sum-power.

*Lemma 3.3:* If $\mu_i^* \neq \mu_j^*$ for $i, j = 1, \ldots, K$ and $\mu_i^* > 0$ for all $i$, then $\{r_k^*\}$ and $\mathcal{C}_g(\{\boldsymbol{H}_k\}, \{\boldsymbol{S}_k^*\})$, the optimal rates and the optimal polymatroid solution of problem (17) are unique.

The proof of this lemma is given in Appendix II. Next, assume some of the $\mu_k^*$s are equal. As is shown in Fig. 2, this situation corresponds to the case where the target rate $\mathbf{R}^*$ lies on a surface (not necessarily a vertex) of the minimum weighted sum-power polymatroid $\mathcal{C}_f$. In this case, $\{r_k^*(n)\}$, the optimal rates of problem (17) for state $n$ are not necessarily unique and

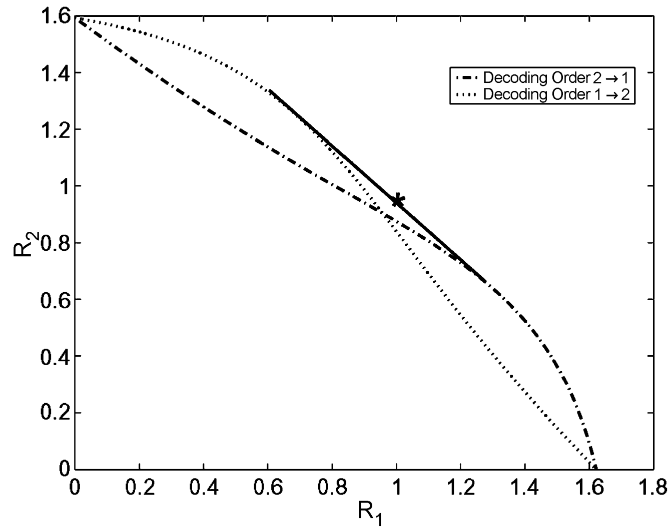


Fig. 2. Achievable rates for a two-user MAC with $N = 1$, $r = 4$, and $t_1 = t_2 = 1$. This region is plotted for $p_1^* + p_2^* \leq 1$ and each point on its boundary requires at least unit transmit sum-power to be achieved. Dashed lines show how two vertices of constructing polymatroids $\mathcal{C}_f$ sweep the region as power allocation varies. As indicated, the target rate shown by $*$ is not a vertex of any $\mathcal{C}_f$.

some might not satisfy the average-rate constraints in (9). However, the following lemma proves that under some conditions, for each $n$, the optimal polymatroid $\mathcal{C}_g(\{\boldsymbol{H}_k(n)\}, \{\boldsymbol{S}_k^*(n)\})$ of problem (17) is unique.

*Lemma 3.4:* Let $J_1, J_2, \ldots, J_l \subseteq \{1, 2, \ldots, K\}$ be $l$ disjoint subsets such that for each $1 \leq m \leq l$, $|J_m| \geq 2$ and for all $j \in J_m$, $\mu_j^*$s be equal. Then, for the single transmit and multiple receive antennas (SIMO) case and for the MIMO case with $r^2 \geq \sum_{k=1}^{K} t_k^2$, the optimal polymatroid solution of problem (17), $\mathcal{C}_g(\{\boldsymbol{H}_k\}, \{\boldsymbol{S}_k^*\})$, is unique.[3]

The proof of this lemma is given in Appendix III. As a corollary of this lemma, the minimum weighted sum-power polymatroid of Problem 3.1, $\mathcal{C}_f(\{\boldsymbol{H}_k(n)\}, \{\boldsymbol{S}_k^*(n)\})$, is also unique. Note that for problem (17) on state $n$, there are $\prod_{m=1}^{l} |J_m|!$ vertices of $\mathcal{C}_g(\{\boldsymbol{H}_k(n)\}, \{\boldsymbol{S}_k^*(n)\})$ that are optimal. These vertices are obtained by choosing different decoding orders on users of each set $J_m$. Furthermore, any optimal rate-tuple, $\mathbf{r}^*(n)$, including the one that satisfies the average-rate constraints lies on the convex-hull of these vertices and for $m = 1, \ldots, l$, satisfies the following equalities:

$$\sum_{k \in J_m} r_k^*(n) = \frac{1}{2} \log \left| \sum_{k \in J_m} \boldsymbol{H}_k(n) \boldsymbol{S}_k^*(n) \boldsymbol{H}_k(n)^\dagger + \boldsymbol{I} \right|.$$

Hence, the target rate-tuple $\mathbf{R}^*$ must also lie on the corresponding boundary portion of $\mathcal{C}_f(\{\boldsymbol{H}_k(n)\}, \{\boldsymbol{S}_k^*(n)\})$. This portion is the convex-hull of $\prod_{m=1}^{l} |J_m|!$ vertices that are obtained by averaging over the corresponding vertices in $\mathcal{C}_g$s. Also, the target rate-tuple satisfies

$$\sum_{k \in J_m} R_k^* = \sum_{n=1}^{N} \frac{1}{2N} \log \left| \sum_{k \in J_m} \boldsymbol{H}_k(n) \boldsymbol{S}_k^*(n) \boldsymbol{H}_k(n)^\dagger + \boldsymbol{I} \right|$$

[2]For a way to find this starting ellipsoid, see Appendix I.

[3]We believe that this lemma is true for a general MIMO-MAC, however, our proof is not easily extendable to the case where $\sum_{k=1}^{K} t_k^2 > r^2$.

for $m = 1, \ldots, l$. Thus, once $\{\boldsymbol{S}_k^*(n)\}$ are found by solving optimization problem (17) for every $n$, the aforementioned vertices of $\mathcal{C}_f(\{\boldsymbol{H}_k(n)\}, \{\boldsymbol{S}_k^*(n)\})$ can be characterized. Since $\mathbf{R}^*$ lies on the convex-hull of these vertices, it can be expressed as a convex combination of them with weights that can be obtained by solving a system of linear equations.

In practice with finite arithmetic precision, the approximate optimal dual solution obtained from the ellipsoid method may not have equal components even if some of the $\mu_k^*$s are equal. In this case, sufficiently small threshold values may be employed to detect the equal $\mu_k^*$s and sets $J_m$s. Also, least squares solution may be used (instead of system of linear equations) to find the approximate optimal rates.

To complete this section, we briefly discuss some important aspects of Algorithm 3.1.

*Delay Constraint*: Problem 3.1 considers average-rate constraints for each user over all fading states, and hence is applicable if transmission delay is not an issue. On the other hand, for applications that may impose a strict delay requirement for transmission, one useful measure of the information-theoretic limit is the *outage capacity* [4]. This notion of capacity is defined as the maximum constant-rate that can be reliably transmitted given a prescribed outage probability. Outage transmission for fading Gaussian SISO-MAC and the special case of zero outage probability—or the so-called delay-limited capacity—have been studied in [9] and [10],[4] respectively. Here, we show that Algorithm 3.1 can be easily modified to characterize the power region of the fading MIMO-MAC with delay-limited rate constraints. Let $P_{\text{out},k}$ denote the individual outage probability of user $k$ associated with the delay-limited rate constraint, $r_{\min,k}$, for $k = 1, \ldots, K$. The boundary points of the associated power region can be attained by solving the following problem.

*Problem 3.2:*

$$\begin{aligned} \texttt{Minimize} \quad & \sum_{k=1}^{K} \lambda_k \mathbb{E}\left[\texttt{Tr}\left(\boldsymbol{S}_k(\boldsymbol{n})\right)\right] \\ \texttt{Subject to} \quad & \mathbb{E}\left[\mathbf{1}\left\{r_k(\boldsymbol{n}) \geq r_{\min,k}\right\}\right] \geq 1 - P_{\text{out},k}, \forall k. \end{aligned}$$

For the special case of $P_{\text{out},k} = 0$ for $k = 1, \ldots, K$, Algorithm 3.1 can be employed to solve the above problem. This is done by considering Problem 3.1 with $N = 1$ and applying it to each state of Problem 3.2. The case of nonzero outage probability can be solved by combining the technique developed in [10, Sec. D] and Algorithm 3.1 and the details are omitted here.

*Decoding Orders*: For the case where $\mu_k^*$s are all different, on each state $n$, the optimal rate-tuple, $\mathbf{r}^*(n)$, is a vertex of $\mathcal{C}_g(\{\boldsymbol{H}_k(n)\}, \{\boldsymbol{S}_k^*(n)\})$ that is achievable by successive decoding scheme. From Lemma 3.1, the optimal decoding order among the users for state $n$ is determined by the order of the optimal dual variables $\mu_k^*$s. Since $\boldsymbol{\mu}^*$ is identical to all states, the decoding orders are the same for all $n$. This fact has an important implication on coding: users can use either the "multiple-codebook variable-rate" [3] scheme or the "single-codebook constant-rate" [28] scheme[5] to attain the same rate-tuple $\mathbf{R}^*$ over the $N$ states. When some of the $\mu_k^*$s are equal, as was mentioned earlier, the target rate-tuple lies on the convex-hull of some vertices each achievable by successive decoding scheme and can be obtained by timesharing among the codes achieving these vertices.

*Complexity*: Tse and Hanly [8, Sec. V] have considered Problem 3.1 for the SISO case and proposed an iterative algorithm [8, Alg. 4.2] for solving it. The algorithm in [8] that optimizes the dual function can be modified to work for our problem as well. However, Algorithm 3.1 introduced here has significantly less amount of computational complexity compared with the algorithm in [8], thanks to the ellipsoid method. Ellipsoid method converges in $\mathcal{O}(m^2)$ iterations, where $m$ is the number of variables. Hence, Algorithm 3.1 requires $\mathcal{O}(K^2)$ dual function computations or equivalently $\mathcal{O}(NK^2)$ runs of optimization problem (17) to compute $\boldsymbol{\mu}^*$. As can be seen, the complexity of the proposed algorithm is linear in $N$ since complexity of problem (17) is independent of $N$.

*Auxiliary Power Constraints*: From practical considerations, there may be an auxiliary transmit power constraint $\hat{p}_k$, imposed on transmission of user $k$ on each state, i.e., $\texttt{Tr}(\boldsymbol{S}_k(n)) \leq \hat{p}_k$ for $n = 1, \ldots, N$. Note that the set of auxiliary power constraints are applied separately to each fading state. Therefore, if they are added to Problem 3.1, Algorithm 3.1 only needs a modification made to the function $g_n'(\boldsymbol{\mu})$ in (15) by including the associated auxiliary power constraints for state $n$. However, this modification can be easily handled by the Interior-Point method. The only caveat here is the feasibility of Problem 3.1 after addition of these auxiliary power constraints.

### B. Weighted Sum-Rate Maximization

In this section, we are interested in characterizing the boundary points of the ergodic capacity region for the fading Gaussian MAC defined in (4). Since this region is convex, each boundary point can be found by maximizing a weighted sum of the rates for some *rate-rewards* $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_K) \in \mathbb{R}_+^K$ as illustrated in Fig. 1(c). This problem can be mathematically formulated as the following optimization problem.

*Problem 3.3:*

$$\begin{aligned} \texttt{Maximize} \quad & \sum_{k=1}^{K} \mu_k \sum_{n=1}^{N} r_k(n) f(n) \\ \texttt{Subject to} \quad & \sum_{n=1}^{N} \texttt{Tr}\left(\boldsymbol{S}_k(n)\right) f(n) \leq p_k^*, \quad \forall k \qquad (20) \end{aligned}$$

where inequalities in (20) specify the average-power constraints. All admissible values of $\boldsymbol{S}_k(n)$ and $r_k(n)$ are also contained in the domain $\mathcal{D}$, as specified in (10) and (11). Similar to the power minimization problem, this problem is also a convex optimization problem. Furthermore, it can be solved by

[4] Two notions of outage are defined in [10] including the *common outage* when all the users declare the outage simultaneously and the *individual outage* when each user declares the outage independently. In this paper, we adopt the notion of individual outage, though, similar development can be done for the case of common outage.

[5] Since the decoding order is the same for all states, the codeword of each user is able to span over all $N$ states when successive decoding is used at the receiver.

an algorithm similar to Algorithm 3.1, with the set of dual variables $\boldsymbol{\mu}$ associated with the average-rate constraints in (9) being replaced by $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_K) \in \mathbb{R}_+^K$, each corresponding to one of the power constraints in (20). Thus, the detailed descriptions of the algorithm for solving Problem 3.3 are omitted here except that $\nu_k = (1/N)\sum_{n=1}^N \mathtt{Tr}(\boldsymbol{S}_k^*(n)) - p_k^*$ is a subgradient of the dual function at $\boldsymbol{\lambda}$.

### C. Admission Problem

In preceding two sections, we have characterized the power region and the capacity region for the fading Gaussian MIMO-MAC through weighted sum-power minimization and weighted sum-rate maximization problems, respectively. In many practical situations, a more relevant question may be to determine whether a given set of individual rate demands is supportable for some given set of power constraints for the users. This is usually referred to as the *admission problem* [9] and it is equivalent to testify whether a chosen rate-tuple is within the capacity region associated with a given set of power constraints. For convenience, we assume again uniform state distribution, $f(n) = (1/N)$ for all $n$. We first formulate the above admission problem as the following feasibility problem.

*Problem 3.4:*

$$\begin{aligned} &\mathtt{Minimize} \quad 0 \\ &\mathtt{Subject\ to} \quad \frac{1}{N}\sum_{n=1}^{N} r_k(n) \geq R_k^*, \quad \forall k \qquad (21) \\ &\qquad\qquad\quad \frac{1}{N}\sum_{n=1}^{N} \mathtt{Tr}\left(\boldsymbol{S}_k(n)\right) \leq p_k^*, \quad \forall k \qquad (22) \end{aligned}$$

where $R_k^*$ and $p_k^*$ denote the average-rate demand and the average-power constraint for user $k$, respectively, and all admissible values of $\boldsymbol{S}_k(n)$ and $r_k(n)$ belong to the set $\mathcal{D}$ as specified before. The admission problem can be answered based on the solution set of Problem 3.4, if it is nonempty then the set of rate demands is supportable otherwise it is not. Clearly, the feasible set of Problem 3.4 is convex and in the following we will provide an efficient algorithm to determine whether this set is empty or not.

Similar to Section III-A, we introduce the vector of dual variables $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_K) \in \mathbb{R}_+^K$ corresponding to average-rate constraints in (21) and $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_K) \in \mathbb{R}_+^K$, corresponding to average-power constraints in (22). We then express the Lagrangian of Problem 3.4 as

$$\begin{aligned} &\mathcal{L}\left(\{\boldsymbol{S}_k(n)\}, \{r_k(n)\}, \boldsymbol{\mu}, \boldsymbol{\lambda}\right) \\ &\quad = \sum_{k=1}^{K} \lambda_k \left(\frac{1}{N}\sum_{n=1}^{N} \mathtt{Tr}\left(\boldsymbol{S}_k(n)\right) - p_k^*\right) \\ &\quad\quad - \sum_{k=1}^{K} \mu_k \left(\frac{1}{N}\sum_{n=1}^{N} r_k(n) - R_k^*\right) \end{aligned}$$

and the Lagrange dual function as

$$g(\boldsymbol{\mu}, \boldsymbol{\lambda}) = \min_{\{\boldsymbol{S}_k(n), r_k(n)\} \in \mathcal{D}} \mathcal{L}\left(\{\boldsymbol{S}_k(n)\}, \{r_k(n)\}, \boldsymbol{\mu}, \boldsymbol{\lambda}\right). \quad (23)$$

The following lemma enables us to determine whether Problem 3.4 is feasible by making use of its Lagrange dual function.

*Lemma 3.5:* Problem 3.4 is infeasible if and only if there exit $\boldsymbol{\mu}, \boldsymbol{\lambda} \in \mathbb{R}_+^K$ such that $g(\boldsymbol{\mu}, \boldsymbol{\lambda}) > 0$.

*Proof:* First, we prove the only if statement. Let $\{\boldsymbol{S}_k^0(n)\}$ and $\{r_k^0(n)\}$ be a feasible solution set, then for any $\boldsymbol{\mu}, \boldsymbol{\lambda} \succeq 0$

$$g(\boldsymbol{\mu}, \boldsymbol{\lambda}) \leq \mathcal{L}\left(\left\{\boldsymbol{S}_k^0(n)\right\}, \left\{r_k^0(n)\right\}, \boldsymbol{\mu}, \boldsymbol{\lambda}\right) \leq 0$$

or equivalently, $\max_{\boldsymbol{\mu}, \boldsymbol{\lambda} \succeq 0} g(\boldsymbol{\mu}, \boldsymbol{\lambda}) \leq 0$. Therefore, if there exist $\boldsymbol{\mu}, \boldsymbol{\lambda} \succeq 0$ such that $g(\boldsymbol{\mu}, \boldsymbol{\lambda}) > 0$, then Problem 3.4 is infeasible and $\boldsymbol{\mu}, \boldsymbol{\lambda}$ are one certificate of infeasibility. To prove the if statement, consider the power region of this MAC associated with the given average-rates in $\mathbf{R}^*$. For a given $\boldsymbol{\lambda} \succeq 0$, let $\{\boldsymbol{S}_k^{(\lambda)}(n)\}$ be the optimal transmit covariance matrices that minimize the weighted sum-power, $\sum_{k=1}^K \lambda_k (1/N) \sum_{n=1}^N \mathtt{Tr}(\boldsymbol{S}_k(n))$, and support the rate-tuple $\mathbf{R}^*$. In solving Problem 3.1, we showed that the primal-dual duality gap is zero. Therefore, for the given $\boldsymbol{\lambda}$, there exists $\boldsymbol{\mu}^* \succeq 0$ such that

$$\begin{aligned} g(\boldsymbol{\mu}^*, \boldsymbol{\lambda}) &= \min_{\{\boldsymbol{S}_k(n), r_k(n)\} \in \mathcal{D}} \mathcal{L}\left(\{\boldsymbol{S}_k(n)\}, \{r_k(n)\}, \boldsymbol{\mu}^*, \boldsymbol{\lambda}\right) \\ &= \sum_{k=1}^{K} \lambda_k \frac{1}{N}\sum_{n=1}^{N} \mathtt{Tr}\left(\boldsymbol{S}_k^{(\lambda)}(n)\right) - \sum_{k=1}^{K} \lambda_k p_k^*. \end{aligned}$$

However, if the problem is infeasible, the power vector $\mathbf{p}^*$ does not lie in the power region corresponding to target average rates $\mathbf{R}^*$, hence for some $\boldsymbol{\lambda}$, $\sum_{k=1}^K \lambda_k (1/N) \sum_{n=1}^N \mathtt{Tr}(\boldsymbol{S}_k^{(\lambda)}(n)) - \sum_{k=1}^K \lambda_k p_k^* > 0$. Equivalently, there exit $\boldsymbol{\mu}, \boldsymbol{\lambda} \in \mathbb{R}_+^K$ such that $g(\boldsymbol{\mu}, \boldsymbol{\lambda}) > 0$. ■

Similar as in (16), $g(\boldsymbol{\mu}, \boldsymbol{\lambda})$ can be written in terms of $N$ independent dual functions on each state and each of them can be found by solving optimization problem (17). Note that for any $\alpha > 0$, $g(\alpha\boldsymbol{\mu}, \alpha\boldsymbol{\lambda}) = \alpha g(\boldsymbol{\mu}, \boldsymbol{\lambda})$. Therefore, if the problem is infeasible, $g(\boldsymbol{\mu}, \boldsymbol{\lambda})$ is unbounded from above and if it is feasible $\max g(\boldsymbol{\mu}, \boldsymbol{\lambda}) = g(\boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) = 0$. Furthermore, in this case, $\alpha\boldsymbol{\mu}^*, \alpha\boldsymbol{\lambda}^*$ are also optimal for any $\alpha > 0$. This observation is very useful in choosing the starting ellipsoid in the following algorithm that is proposed to solve Problem 3.4. Let $\boldsymbol{\theta} = (\mu_1, \cdots, \mu_K, \lambda_1, \cdots, \lambda_K) \in \mathbb{R}_+^{2K}$.

*Algorithm 3.2:*

- **Given** an ellipsoid $\mathcal{E}^{(0)} \subseteq \mathbb{R}^{2K}$, centered at $\boldsymbol{\theta}^{(0)}$,
- **Set** $i = 0$.
- **Repeat**
  1) By solving (17) for each $n$, calculate the dual function at $\boldsymbol{\theta}^{(i)}$. Let $\{\boldsymbol{S}_k^*(n)\}$ and $\{r_k^*(n)\}$ minimize the Lagrangian.
  2) If $g(\boldsymbol{\theta}^{(i)}) > 0$, the problem is infeasible, exit the loop; else, go to the next step.
  3) Update the ellipsoid $\mathcal{E}^{(i+1)}$ based on $\mathcal{E}^{(i)}$, and the subgradient $\nu_k = R_k^* - (1/N)\sum_{n=1}^N r_k^*(n)$, $\nu_{K+k} = (1/N)\sum_{n=1}^N \mathtt{Tr}(\boldsymbol{S}_k^*(n)) - p_k^*$. Set $\boldsymbol{\theta}^{(i+1)}$ as the center for $\mathcal{E}^{(i+1)}$.
  4) Set $i \leftarrow i + 1$.
- **Until** the stopping criteria for the ellipsoid method is met.

If after running this algorithm, $g(\boldsymbol{\theta}^*) > 0$, the problem is infeasible, otherwise, it is feasible. Recall that the starting ellipsoid can be chosen as an arbitrary small ellipsoid covering $\mathbb{R}_+^{2K}$ in vicinity of the origin.

Having an efficient algorithm to solve Problem 3.4 enables us to study further the two following interesting problems.

*1) Characterization of the Power Region Based on the Power-Profile:* Suppose the rate demand vector $\mathbf{R}^*$, is in the capacity region of a given set of power constraints $\mathbf{p}^*$. It is usually desirable to find the power and rate allocation such that the consumed average-power of each user to transmit at its rate is proportional to its power constraint, i.e., a *proportionally fair* [9] power allocation among users. The above problem can be formulated as follows.

*Problem 3.5:*

$$\texttt{Minimize} \quad P \tag{24}$$

$$\texttt{Subject to} \quad \frac{1}{N}\sum_{n=1}^{N} r_k(n) \geq R_k^*, \quad \forall k \tag{25}$$

$$\frac{1}{N}\sum_{n=1}^{N} \texttt{Tr}\left(\boldsymbol{S}_k(n)\right) \leq \alpha_k P, \quad \forall k \tag{26}$$

where $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_K) \in \mathbb{R}_+^K$ is the *power-profile* vector chosen as $\alpha_k = p_k^* / \sum_{k'=1}^{K} p_{k'}^*$.

Problem 3.5 can be solved by using the Algorithm 3.2 together with a bisection search over $P$, as is explained in the following algorithm. Note that $\delta$ is an error tolerance on the optimal $P$.

*Algorithm 3.3:*

- **Initialize** $P_{\min} = 0$, $P_{\max} = \sum_{k=1}^{K} p_k^*$;
- **Repeat**
  1) $P = (1/2)(P_{\max} + P_{\min})$.
  2) Run Algorithm 3.2 with $\alpha_k P$ as the average-power constraints for $k = 1, \ldots, K$. If the problem is feasible, $P_{\max} \leftarrow P$, otherwise $P_{\min} \leftarrow P$.
- **Until** $(P_{\max} - P_{\min}) \leq \delta$.

Note that this algorithm can be also used to characterize each boundary point of the power region corresponding to a given power-profile vector $\boldsymbol{\alpha}$. Fig. 1(d) illustrates the characterization of the power region through power-profile vectors as compared with the characterization of this region through weighted sum-power minimization of Section III-A.

*2) Characterization of the Capacity Region Based on the Rate-Profile:* Similar to using the power-profile vector to characterize the power region of the fading MAC, we can also use the *rate-profile* vector to characterize the capacity region. Let $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_K) \in \mathbb{R}_+^K$ be the rate-profile vector. Then, each boundary point of the capacity region can be found by solving the following problem for some given rate-profile $\boldsymbol{\beta}$.

*Problem 3.6:*

$$\texttt{Maximize} \quad R \tag{27}$$

$$\texttt{Subject to} \quad \frac{1}{N}\sum_{n=1}^{N} r_k(n) \geq \beta_k R, \quad \forall k \tag{28}$$

$$\frac{1}{N}\sum_{n=1}^{N} \texttt{Tr}\left(\boldsymbol{S}_k(n)\right) \leq p_k^*, \quad \forall k. \tag{29}$$

By using an algorithm similar to Algorithm 3.3 with a bisection search on $R$, Problem 3.6 can be solved and the optimal solution characterizes a boundary point of the ergodic capacity region associated with the given rate-profile vector $\boldsymbol{\beta}$. This is depicted in Fig. 1(c).

## IV. Fading Gaussian MIMO-BC

In this section, we consider the fading BC defined in (5) and characterize the ergodic capacity region defined in (7) and the minimum average-power under the average-rate constraints defined in (8). Equipped with the duality theory [19], [20] between the Gaussian MAC and BC restated in (6), the information-theoretic characterization of the fading BC becomes a relatively easier task given all the results we have developed for the fading MAC in Section III. In the following paragraphs, we briefly go over the various problems considered for the fading MAC and address the modifications needed to make the developed algorithms work as well for the fading BC.

In Section III-A, we considered the weighted sum-power minimization problem for the fading MAC under long-term rate constraints. Based on duality, by setting the power prices, $\lambda_k = 1$, for $k = 1, \ldots, K$ in Problem 3.1, the optimal solution will correspond to the minimum average-power for the fading BC defined in (8). Analogous results on the delay constraint, the encoding orders and the auxiliary power constraints can also be obtained for the fading BC.

In Section III-B, we considered the weighted sum-rate maximization problem for the fading MAC to characterize the ergodic capacity region. By replacing the $K$ individual average-power constraints in (20) by a sum-power constraint, $(1/N)\sum_{k=1}^{K}\sum_{n=1}^{N} \texttt{Tr}(\boldsymbol{S}_k(n)) \leq q^*$, Problem 3.3 can be used to characterize the ergodic capacity region of the fading BC defined in (7).

In Section III-C, we formulated the admission problem for the fading MAC in Problem 3.4. By replacing the individual power constraints in (22) with the average sum-power constraint for the base station, Problem 3.4 can also be used for the fading BC. Similar modifications can be done in Problem 3.6 to make it work for characterizing the capacity region of the fading BC with any given rate-profile vector.

## V. Power Region Comparison With and Without Delay Constraint in Fading MIMO-MAC

In this section, we provide a numerical example to demonstrate the usefulness of the proposed algorithms in characterizing the information-theoretic limits of mobile wireless systems. We focus on a fading Gaussian MAC with a base station equipped with $r = 4$ receive antennas and two mobile users with $t_1 = t_2 = 2$ transmit antennas. It is assumed that the number of channel states $N = 50$ with equal probability and the channel matrix, $\boldsymbol{H}_k(n)$, associated with user $k$ and state $n$, is independently drawn from population of matrices with independent zero-mean and unit-variance circularly symmetric complex Gaussian (CSCG) elements. We are interested in comparing the

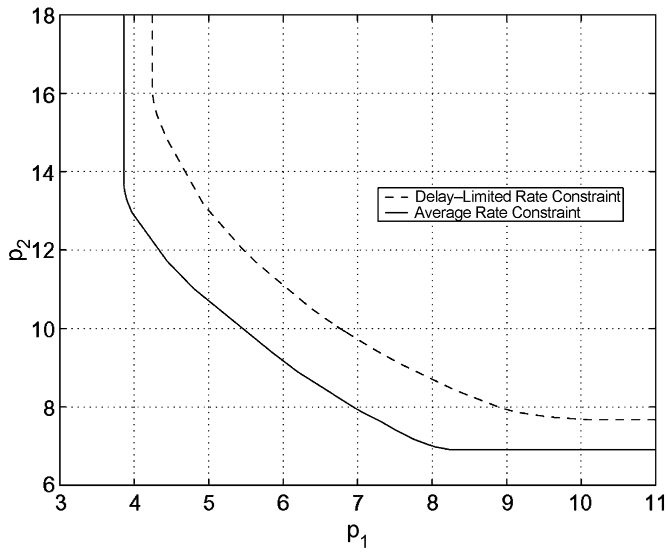


Fig. 3. Power region comparison for a two-user fading MIMO-MAC with $t_1 = t_2 = 2$ and $r = 4$ and under average-rate and delay-limited rate constraints. User 1 and user 2 have a rate of 2 and 2.5 nats/complex dimension, respectively.

power region of the two-user MAC with and without transmission delay constraint. For both cases, user 1 and user 2 have transmission rates of 2 and 2.5 nats per complex dimension, respectively. These transmission rates need to be achieved in average over all fading states for the case without delay constraint, while for the case with delay constraint, they need to be maintained for each fading state with probability one. Problem 3.1 and Problem 3.2 (with zero outage probability for all users) are solved for the case without and with delay constraint, respectively. The main goal of this example is to quantify the ultimate power saving attained by fully exploring the multiuser channel dynamics under a complete relaxation of transmission delay for both users. Fig. 3 shows the resultant power regions for these two cases. It is observed that relaxing the delay constraint only reduces the weighted sum of powers by about 7% on average. This observation may have an interesting consequence: when multiple antennas are used, the rich diversity in each user's channel may render the dynamic resource allocation less appealing in the energy/delay tradeoff as compared with the case of single transmit and receive antenna. It is noted that similar observations have been drawn in [29] and referred to as the so-called *channel-hardening* effect for single-user MIMO transmissions.

## VI. Summary

In this paper, optimal resource allocation problem is studied for the fading MAC and the fading BC from two major information-theoretic points of view. Especially, it is assumed that CSI is completely known at both transmitter and receiver that are equipped with multiple antennas. As the more general case, capacity and power region of a fading MAC under various power and rate constraints are explored. Efficient optimization algorithms are proposed to solve the convex optimization problems characterizing these regions. By using the duality theory between the Gaussian MAC and BC, analogous results for the fading BC are obtained.

## Appendix I
## Description of Ellipsoid Method

Ellipsoid method is an efficient optimization scheme whenever the cost and constraint functions are not continuously differentiable or analytical expressions for their differentials do not exist. Assume $f(x)$ is a convex function defined over $\mathbb{R}^n$ and has finite minimum. Also, assume the minimum is attained. To illustrate how ellipsoid method works, consider the following unconstrained optimization problem:

$$\texttt{Minimize} \quad f(x).$$

For simplicity purposes, constraint functions are not considered here, however, they can be included in the algorithm easily. The main idea in the ellipsoid method is to localize $x^*$, the optimal solution, in a sequence of ellipsoids $\mathcal{E}^{(k)}$ with vanishing volumes. Thus, the centers of these ellipsoids $x^{(k)}$, converge to $x^*$ [27].

This algorithm is an iterative algorithm starting from an initial ellipsoid $\mathcal{E}^{(0)}$ that contains $x^*$. At each iteration $k$, $x^{(k)}$ is chosen as the center of ellipsoid $\mathcal{E}^{(k)}$ and a subgradient of $f(x)$ at $x^{(k)}$, $g^{(k)}$, is determined. Since $g^{(k)}$ is a subgradient, $f(x^{(k)} + \Delta x) \geq f(x^{(k)}) + \Delta x^T g^{(k)}$ for any $\Delta x$, hence, $x^*$ must be in the half-ellipsoid

$$\mathcal{E}^{(k)} \cap \left\{ x : g^{(k)T} \left( x - x^{(k)} \right) \leq 0 \right\}.$$

For the next iteration, $\mathcal{E}^{(k+1)}$ is set as the minimum volume ellipsoid covering $\mathcal{E}^{(k)} \cap \{x : g^{(k)T}(x - x^{(k)}) \leq 0\}$. Suppose $A^{(k)}$ is the matrix describing $\mathcal{E}^{(k)}$ as

$$\mathcal{E}^{(k)} = \left\{ x : \left( x - x^{(k)} \right)^T A^{(k)^{-1}} \left( x - x^{(k)} \right) \leq 1 \right\}.$$

$x^{(k+1)}$ and $\mathcal{E}^{(k+1)}$ have the following simple formulas given $\mathcal{E}^{(k)}$ and $g^{(k)}$. Define $\tilde{g}^{(k)} = g^{(k)} / \sqrt{g^{(k)T} A^{(k)} g^{(k)}}$, then

$$x^{(k+1)} = x^{(k)} - \frac{1}{n+1} A^{(k)} \tilde{g}^{(k)}$$
$$A^{(k+1)} = \frac{n^2}{n^2 - 1} \left( A^{(k)} - \frac{2}{n+1} A^{(k)} \tilde{g}^{(k)} \tilde{g}^{(k)T} A^{(k)} \right).$$

It can be shown that volumes of these ellipsoids decrease exponentially, $\mathbf{Vol}(\mathcal{E}^{(k+1)}) < e^{-(1/2n)} \mathbf{Vol}(\mathcal{E}^{(k)})$, furthermore, the algorithm converges in $\mathcal{O}(n^2)$ iterations. As a stopping criteria, $\sqrt{g^{(k)T} A^{(k)} g^{(k)}} < \epsilon$ can be used.

Next, we will show how to choose $\mathcal{E}^{(0)}$ in Algorithm 3.1 to contain $\boldsymbol{\mu}^*$. For any given $j \in \{1, \ldots, K\}$, let $\{\boldsymbol{S}_k^{(j)}(n)\}, \{r_k^{(j)}(n)\} \in \mathcal{D}$ be chosen such that

$$\frac{1}{N} \sum_{n=1}^{N} r_k^{(j)}(n) = \begin{cases} R_k^*, & k \neq j \\ R_k^* + 1, & k = j \end{cases}. \tag{30}$$

One way to find $\{\boldsymbol{S}_k^{(j)}(n)\}, \{r_k^{(j)}(n)\}$ is to assume a fixed decoding order on users. Starting from the user decoded last, allocate power for that user among different states in a water-filling fashion to support its assigned rate, while considering interference from users with higher decoding orders as noise. Hence, power allocation is done sequentially from the user decoded last to the one decoded first. From the definition of the dual function, we have

$$g(\boldsymbol{\mu}^*) \le \mathcal{L}\left(\left\{\boldsymbol{S}_k^{(j)}(n)\right\}, \left\{r_k^{(j)}(n)\right\}, \boldsymbol{\mu}^*\right) = \frac{1}{N}\sum_{n=1}^{N}\sum_{k=1}^{K}\lambda_k \mathtt{Tr}\left(\boldsymbol{S}_k^{(j)}(n)\right) - \mu_j^*.$$

Also, $g(\boldsymbol{\mu}^*) \ge 0$, $\mu_j^*$ lies in the interval given below

$$0 \le \mu_j^* \le \frac{1}{N}\sum_{n=1}^{N}\sum_{k=1}^{K}\lambda_k \mathtt{Tr}\left(\boldsymbol{S}_k^{(j)}(n)\right).$$

Similar bounds can be found for all $j$ and $\mathcal{E}^{(0)}$ can be chosen to cover the resulting hypercube in $\mathbb{R}^K$.

## Appendix II
## Proof of Lemma 3.3

Without loss of generality, assume $\mu_1^* > \cdots > \mu_K^* > 0$. Let $\{\boldsymbol{S}_k^{(1)}\}$ and $\{\boldsymbol{S}_k^{(2)}\}$ be two optimal solutions of problem (17). Since $\mu_k^*$s are all different, from Lemma 3.1, for any set of optimal $\{\boldsymbol{S}_k^{(j)}\}$, $j=1,2$, the unique vertex $\mathbf{r}^{(j)}$ given below maximizes the cost function in (17)

$$r_k^{(j)} = \frac{1}{2}\log\frac{\left|\sum_{i=1}^{k}\boldsymbol{H}_i\boldsymbol{S}_i^{(j)}\boldsymbol{H}_i^\dagger + \boldsymbol{I}\right|}{\left|\sum_{i=1}^{k-1}\boldsymbol{H}_i\boldsymbol{S}_i^{(j)}\boldsymbol{H}_i^\dagger + \boldsymbol{I}\right|}.$$

Thus, the optimal value of (17), $p^*$, is given by

$$\sum_{k=1}^{K}\frac{\mu_k^* - \mu_{k+1}^*}{2}\log\left|\sum_{i=1}^{k}\boldsymbol{H}_i\boldsymbol{S}_i^{(j)}\boldsymbol{H}_i^\dagger + \boldsymbol{I}\right| - \sum_{k=1}^{K}\lambda_k\mathtt{Tr}\left(\boldsymbol{S}_k^{(j)}\right).$$

Since the problem is convex, $\boldsymbol{S}_k = \alpha\boldsymbol{S}_k^{(1)} + \bar{\alpha}\boldsymbol{S}_k^{(2)}$ is also an optimal solution for any $0 \le \alpha \le 1$ and $\bar{\alpha} = 1-\alpha$ and achieves $p^*$. Consequently

$$\begin{aligned}\sum_{k=1}^{K}\left(\mu_k^* - \mu_{k+1}^*\right)\frac{1}{2}\log\left|\sum_{i=1}^{k}\boldsymbol{H}_i\boldsymbol{S}_i\boldsymbol{H}_i^\dagger + \boldsymbol{I}\right| &= \alpha\sum_{k=1}^{K}\left(\mu_k^* - \mu_{k+1}^*\right)\frac{1}{2}\log\left|\sum_{i=1}^{k}\boldsymbol{H}_i\boldsymbol{S}_i^{(1)}\boldsymbol{H}_i^\dagger + \boldsymbol{I}\right| \\ &+ \bar{\alpha}\sum_{k=1}^{K}\left(\mu_k^* - \mu_{k+1}^*\right)\frac{1}{2}\log\left|\sum_{i=1}^{k}\boldsymbol{H}_i\boldsymbol{S}_i^{(2)}\boldsymbol{H}_i^\dagger + \boldsymbol{I}\right|.\end{aligned}$$

However, $\log|.|$ is a concave function and $(\mu_k^* - \mu_{k+1}^*) > 0$ for all $k$, therefore, the following equalities must hold for all $k = 1, \ldots, K$:

$$\log\left|\sum_{i=1}^{k}\boldsymbol{H}_i\boldsymbol{S}_i\boldsymbol{H}_i^\dagger + \boldsymbol{I}\right| = \alpha\log\left|\sum_{i=1}^{k}\boldsymbol{H}_i\boldsymbol{S}_i^{(1)}\boldsymbol{H}_i^\dagger + \boldsymbol{I}\right| + \bar{\alpha}\log\left|\sum_{i=1}^{k}\boldsymbol{H}_i\boldsymbol{S}_i^{(2)}\boldsymbol{H}_i^\dagger + \boldsymbol{I}\right|.$$

It is not hard to show that if for a given $A, B \succ \mathbf{0}$, $f(\alpha) = \log|\alpha A + \bar{\alpha}B| - \alpha\log|A| - \bar{\alpha}\log|B| = 0$ for all $0 \le \alpha \le 1$, then $A = B$. Note that $f(\alpha)$ is a twice continuously differentiable function and for $0 < \alpha < 1$

$$\frac{d^2}{d\alpha^2}f(\alpha) = -\mathtt{Tr}\left(\left((A-B)(\alpha A + \bar{\alpha}B)^{-1}\right)^2\right).$$

Since $f(\alpha) = 0$, $(d/d\alpha)f(\alpha) = (d^2/d\alpha^2)f(\alpha) = 0$ on the (0,1) interval. Therefore, the positive semidefinite matrix $((A-B)(\alpha A + \bar{\alpha}B)^{-1})^2$, has zero trace, and hence it must be zero. Equivalently, $(A-B)(\alpha A + \bar{\alpha}B)^{-1} = 0$, which results in $A = B$. Hence, $\sum_{i=1}^{k}\boldsymbol{H}_i\boldsymbol{S}_i^{(1)}\boldsymbol{H}_i^\dagger = \sum_{i=1}^{k}\boldsymbol{H}_i\boldsymbol{S}_i^{(2)}\boldsymbol{H}_i^\dagger$ for $k = 1, \ldots, K$ that yield

$$\boldsymbol{H}_k\boldsymbol{S}_k^{(1)}\boldsymbol{H}_k^\dagger = \boldsymbol{H}_k\boldsymbol{S}_k^{(2)}\boldsymbol{H}_k^\dagger, \quad \text{for } k = 1, \ldots, K.$$

Therefore, the optimal polymatroid $\mathcal{C}_g(\{\boldsymbol{H}_k\}, \{\boldsymbol{S}_k^*\})$ is unique and if we assume all $\boldsymbol{H}_k$s are full column rank matrices, the optimal set of $\{\boldsymbol{S}_k^*\}$ will be unique.

## Appendix III
## Proof of Lemma 3.4

Let $\{\boldsymbol{S}_k^{(1)}\}$ be an optimal solution to the optimization problem in (17). Following the same steps as in the proof of Lemma 3.3, for any other optimal solution $\{\boldsymbol{S}_k^*\}$, it can be shown that for $k \in \{1, \ldots, K\} \setminus \bigcup_{m=1}^{l} J_m$, $\boldsymbol{H}_k\boldsymbol{S}_k^{(1)}\boldsymbol{H}_k^\dagger = \boldsymbol{H}_k\boldsymbol{S}_k^*\boldsymbol{H}_k^\dagger$ and for $m = 1, \ldots, l$,

$$\sum_{k\in J_m}\boldsymbol{H}_k\boldsymbol{S}_k^{(1)}\boldsymbol{H}_k^\dagger = \sum_{k\in J_m}\boldsymbol{H}_k\boldsymbol{S}_k^*\boldsymbol{H}_k^\dagger. \tag{31}$$

Let $J$ be one of these subsets. Note that each linear matrix equality in (31) can be viewed as a set of $r^2$ linear equations in terms of $\sum_{k\in J} t_k^2$ variables that are elements of $\boldsymbol{S}_k^*$ for $k \in J$. Hence, if this set has a unique solution, $\boldsymbol{S}_k^*$ for $k \in J$ will be unique. Since in a MIMO system, users practically have independent channels, if $r^2 \ge \sum_{k=1}^{K} t_k^2$, then this set will correspond to a set of independent linear equations that has more equations than unknowns, $r^2 \ge \sum_{k\in J} t_k^2$. Hence, it will have a unique solution and $\boldsymbol{S}_k^{(1)} = \boldsymbol{S}_k^*$ for $k \in J$. Unfortunately, for the case of $r^2 < \sum_{k=1}^{K} t_k^2$, this simple argument does not work, and we need more sophisticated uniqueness proof. From (31), we can conclude that $\boldsymbol{S}_k^*$ for $k \in J$ should be an optimal solution to the following optimization problem:

$$\begin{aligned}\mathtt{Minimize}\quad & \sum_{k\in J}\lambda_k\mathtt{Tr}(\boldsymbol{S}_k) \\ \mathtt{Subject\ to}\quad & \sum_{k\in J}\boldsymbol{H}_k\boldsymbol{S}_k\boldsymbol{H}_k^\dagger = \sum_{k\in J}\boldsymbol{H}_k\boldsymbol{S}_k^{(1)}\boldsymbol{H}_k^\dagger, \\ & \boldsymbol{S}_k \succeq \mathbf{0}, \quad k \in J.\end{aligned}$$

Uniqueness of $\boldsymbol{S}_k^*$ for $k \in J$ in (17) is equivalent to uniqueness of the solution to the optimization problem given above. This problem is a Semi-Definite Program (SDP) that under certain conditions has a unique solution. These conditions if satisfied, guarantee the uniqueness of the optimal solution to problem (17). Characterizing all these conditions in general lies above the scope of this paper. However, for the case of SIMO, this SDP reduces to a linear program (LP) that under very general conditions has a unique optimal solution. Therefore, in the following, we provide a uniqueness proof for the SIMO case and for the MIMO case, we content ourselves only with the proof that was given for $r^2 \geq \sum_{k=1}^{K} t_k^2$.

Assume $t_1 = t_2 = \cdots = t_K = 1$ and let $s_k \in \mathbb{R}_+$ and $\mathbf{h}_k \in \mathbb{C}^{r\times 1}$ denote the transmit power and the channel matrix of user $k \in J$ for this SIMO case. Redefine $s_k^{(1)}$ as an optimal solution of (17) that has maximum number of nonzero elements in $J$. Let $s_k^{(1)} > 0$ for every $k \in I \subseteq J$ and $s_k^{(1)} = 0$ for $k \in J\backslash I$. Without loss of generality, we can restrict our attention to users in $I$ because any other optimal solution must be zero for $k \in J \setminus I$. Consider the following LP:

$$\begin{aligned} \texttt{Minimize} \quad & \sum_{k\in I} \lambda_k s_k \\ \texttt{Subject to} \quad & \sum_{k\in I} \mathbf{h}_k s_k \mathbf{h}_k^\dagger = \sum_{k\in I} \mathbf{h}_k s_k^{(1)} \mathbf{h}_k^\dagger, \\ & s_k \geq 0, \quad k \in I. \end{aligned}$$

This LP is feasible ($s_k^{(1)}$ is a feasible solution) and bounded, therefore its dual LP is also feasible and bounded with zero duality gap and the primal and dual optimal solutions satisfy the KKT optimality conditions given below

$$\lambda_k = \gamma_k + \mathbf{h}_k^\dagger \Psi \mathbf{h}_k, \quad k \in I \tag{32}$$

$$\sum_{k\in I} \mathbf{h}_k s_k \mathbf{h}_k^\dagger = \sum_{k\in I} \mathbf{h}_k s_k^{(1)} \mathbf{h}_k^\dagger \tag{33}$$

$$s_k \gamma_k = 0, \quad s_k, \gamma_k \geq 0, \quad k \in I \tag{34}$$

where the complex Hermitian matrix $\Psi$ is the dual variable associated with the equality constraint and $\gamma_k \in \mathbb{R}_+$ are dual variables associated with non-negativity constraints. $s_k^{(1)}$ is a solution to these optimality conditions. Equalities in (34) suggest that all $\gamma_k$s must be zero for this set of powers. Equalities in (32) introduce $|I|$ linear equations in terms of $r^2$ degrees of freedom (variables), $\Psi_{i,j}$. However, this set of linear equations has a solution for arbitrary $\boldsymbol{\lambda}$ if the number of equations is less than or equal to degrees of freedom, i.e., $|I| \leq r^2$. Assume these equations are linearly dependent, then there exists $\alpha_k \neq 0$ for $k \in I$ such that

$$\sum_{k\in I} \alpha_k \mathbf{h}_k \mathbf{h}_k^\dagger = \mathbf{0}.$$

Hence, from (32), we should have $\sum_{k\in I} \alpha_k \lambda_k = 0$. However, this is a very rare situation in practice since $\lambda_k$s are given weights and are independently chosen from channel matrices and problem data. Therefore, these equations are linearly independent and equality in (33) has only one solution, $s_k^* = s_k^{(1)}$ for all $k \in I$. Thus, the optimal solution is unique. The same argument can be used for other subsets $J_m$.

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**Mehdi Mohseni** (S'02) received the B.S. degree in electrical engineering from the Sharif University of Technology, Tehran, Iran, in 2001 and the M.S. degree in electrical engineering from Stanford University, Stanford, CA, in 2003. Since 2003, he has been working towards the Ph.D. degree at Stanford University.

His research interests include multiuser information theory, and wireless and broadband communications.

**Rui Zhang** (S'00) received the B.S. and M.S. degrees in electrical and computer engineering from the National University of Singapore in 2000 and 2001, respectively. Since 2002, he has been working towards the Ph.D. degree at the Department of Electrical Engineering, Stanford University, Stanford, CA.

His current research interests include digital transmission and coding, statistical signal processing, capacity of wireless channels, multiantenna systems, and wireless networks.

**John M. Cioffi** (S'77–M'78–SM'90–F'96) received the B.S. degree from the University of Illinois at Urbana–Champaign, Urbana, in 1978, and the Ph.D. degree from Stanford University, Stanford, CA, in 1984, both in electrical engineering.

He was with Bell Laboratories from 1978 to 1984, and IBM Research from 1984 to 1986. He has been a Professor of Electrical Engineering with Stanford University since 1986. He founded Amati Communications Corporation in 1991 (purchased by Texas Instruments in 1997), and was Officer/Director from 1991 to 1997. He is currently Chairman and founder of ASSIA Inc., a company responsible for the introduction and use of Dynamic Spectrum Management by several large telephone companies. He has served on the Board of Directors of public companies Amati, Marvell, and Integrated Telecom Express. He currently is on the Board of Directors of Teknovus, Teranetics, ClariPhy, and ASSIA. He is on the Advisory Boards of Wavion and Amicus. He has published over 280 papers and holds over 80 patents, most of which are widely licensed, including basic patents on DMT, VDSL, and vectored transmission. His specific interests are in the area of high-performance digital transmission.

Dr. Cioffi is a member of the National Academy of Engineering (2001). He has been the recipient of various awards: International Marconi Prize (2006), Holder of Hitachi America Professorship in Electrical Engineering at Stanford University (2002), the IEEE Kobayashi Medal (2001), the IEEE Third Millennium Medal (2000), IEE JJ Tomson Medal (2000), University of Illinois Outstanding Alumnus (1999), Committee T1 Outstanding Achievement Award of the ANSI (1995), Outstanding Achievement Award from the American National Standards Institute for "contributions to ADSL" (10/95), NSF Presidential Investigator (1987–1992), IEEE Communications Magazine Best Paper Award (1991), the IEEE ISSLS Best Paper Award (2004), and the Faculty Development Award from IBM Research (1986–1988).