### Optimizing dividends and capital injections limited by bankruptcy, and practical approximations for the Cramér-Lundberg process

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### Abstract

The recent papers Gajek-Kucinsky(2017) and Avram-Goreac-Li-Wu(2020) investigated the control problem of optimizing dividends when limiting capital injections stopped upon bankruptcy. The first paper works under the spectrally negative Lévy model; the second works under the Cramér-Lundberg model with exponential jumps, where the results are considerably more explicit. The current paper has three purposes. First, it illustrates the fact that quite reasonable approximations of the general problem may be obtained using the particular exponential case studied in Avram-Goreac-Li-Wu(2020). Secondly, it extends the results to the case when a final penalty P is taken into consideration as well besides a proportional cost k > 1 for capital injections. This requires amending the "scale and Gerber-Shiu functions" already introduced in Gajek-Kucinsky(2017). Thirdly, in the exponential case, the results will be made even more explicit by employing the Lambert-W function. This tool has particular importance in computational aspects and can be employed in theoretical aspects such as asymptotics.

**Keywords:** dividend problem, capital injections, penalty at default, scale functions, Lambert-W function, De Vylder-type approximations, rational Laplace transform

### Contents

1	Introduction	2
2	The cost function of $(-a,0,b)$ policies, for the spectrally negative Lévy case 2.1 The HJB System	ţ
3	Explicit determination of $a^*, b^*$ when $F(x) := 1 - e^{-\mu x}, P > -\frac{c}{q}$ 3.1 The simplified cost function and optimality equations	
4	Which exponential approximation?	13
	4.1 Three de Vylder-type exponential approximations for the ruin probability	13
	4.2 Three two point Padé approximations of the Laplace transform $\widehat{W}_q$ of scale function	1

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	5.1 A Cramér-Lundberg process with hyperexponential claims of order 2	17
	5.2 A Cramér-Lundberg process with hyperexponential claims of order 3	19
	5.3 A Cramér-Lundberg process with oscillating density and scale function	22
6	The maximal error of exponential approximations $J_0$ along one parameter families of Cramér-Lundberg processes	23
	The profit function when the claims are distributed according to a matrix exponential jumps density	25

Examples of computations involving scale function and dividend value approximations 16

### 1 Introduction

This paper concerns the approximate optimization of a new type of boundary mechanism, which emerged recently in the actuarial literature [APY18, GK17, AGLW20], in the context of the optimal control of dividends and capital injections.

The model. Consider the spectrally negative Lévy risk model:

$$X_t = x + ct - \bar{\xi}_t, c \ge 0,$$

where  $\bar{\xi}_t$  is a spectrally positive Lévy process, with Lévy measure  $\nu(x)dx$ . The classic example is that of the perturbed Cramèr-Lundberg risk model with

$$\bar{\xi}_t = \sum_{i=1}^{N_t} \xi_i + \sigma B_t,$$

where  $B_t$  is a Brownian motion, where  $N_t$  is an independent Poisson process of intensity  $\lambda > 0$ , and  $(\xi_i)_{i \geq 1}$  is a family of i.i.d.r.v. whose distribution, density and moments are denoted respectively by  $F, f, m_i, i \in \{1, ...\}$ . Furthermore,

• the process is modified by **dividends and capital injection**:

$$\pi := (D, I) \Rightarrow X_t^{\pi} := X_t - D_t + I_t,$$

where D, I are adapted, non-decreasing and càdlàg processes, with  $D_{0-} = I_{0-} = 0$ ;

- the first time when we do not bail-out to positive reserves  $\sigma_0^{\pi} := \inf \{t > 0 : X_{t-}^{\pi} \triangle \bar{\xi}_t + \triangle I_t < 0 \}$  is called **bankruptcy/absolute ruin**;
- prior to bankruptcy, dividends are limited by the available reserves:  $\triangle D_t := D_t D_{t-} \le X_{t-}^{\pi} \triangle \bar{\xi}_t + \triangle I_t$ , where  $\bar{\xi}_t := \sum_{i=1}^{N_t} \xi_i$ . The set of "admissible" policies satisfying this constraint will be denoted by  $\tilde{\Pi}(x)$ .

The objective is to maximize the **profit**:

$$J_x^{\pi} := \mathbb{E}_x \left[ \int_{[0,\sigma_0^{\pi}]} e^{-qt} \left( dD_s - k dI_s \right) - P e^{-q\sigma_0^{\pi}} \right], k \ge 1, P \ge 0.$$

The value function is

$$V(x) := \sup_{\pi \in \tilde{\Pi}(x)} J_x^{\pi}, \ x \in \mathbb{R}.$$

**Motivation.** The recent papers [GK17, AGLW20] investigated the above control problem of optimizing dividends and capital injections for processes with jumps, when bankruptcy is allowed as well. The second paper works under the Cramér-Lundberg model with exponential jumps, while the first works under the spectrally negative Lévy model, allowing also for the presence of Brownian motion and infinite activity jumps. It turns out that the optimal policy belongs to the class of  $(-a, 0, b), a > 0, b \ge 0$  "bounded buffer policies", which consist in allowing only capital injections smaller than a given a and declaring bankruptcy at the first time when the size of the overshoot below 0 exceeds a, and in paying dividends when the reserve

reaches an upper barrier b. These will briefly be described as (-a,0,b) policies from now on. Furthermore, the optimal  $(a^*,b^*)$  are the roots of one variable equations with explicit solutions related to the Lambert-W(right) function (ProductLog in Mathematica).

Below, our goal is to show numerically that exponential approximations provide quite reasonable results (as the de Vylder approximation provides for the ruin problem). We will focus in our examples on the case of matrix exponential jumps (which are known to be dense in the class of general nonnegative jumps, with even error bounds for completely monotone jumps being available [VAVZ14]), for two reasons. One is in order to highlight certain exact equations which are similar to their exponential versions, and which may at their turn be used to produce even more accurate approximations in the future, and, secondly, since numerical Laplace inversion for this class may easily tuned to have arbitrarily small errors.

**History of the problem**: The case of no capital injections (also characterized by  $k = \infty$  or absorption below 0) is the dividend problem posed by De Finetti [DF57, Ger69] where dividends are paid above barrier  $b^*$  and  $a^* = 0$  is imposed. "The challenge is to find the right compromise between paying early in view of the discounting or paying late in order not to reach ruin too early and thus profit from the positive safety loading for a longer time" [AAM20].

Forced injections and no bankruptcy at 0 (also characterized by a reflection at 0) is studied in Shreve [SLG84] where dividends are paid above barrier  $b^*$  and  $a^* = \infty$  is imposed.

From Lokka, Zervos, [LZ08] we know that in the Brownian motion case, it is optimal to either always inject, if  $k \le k_c$ , for some critical cost  $k_c$  (i.e. use Shreve), or, stop at 0 (use De Finetti). We propose to call this the **Lokka**, **Zervos alternative**. The "proof" of this alternative starts by largely assuming it via a heuristically justified border Ansatz [LZ08, (5.2)]:  $\max\{-V(0), V'(0) - k\} = 0 \implies$  either V(0) = 0 or V'(0) = k.

Extensive literature on SLG forced bailouts (no bankruptcy) can be found at Avram et al., (2007) [APP07], Kulenko and Schmidli, (2008) [KS08], Eisenberg and Schmidli, (2011) [ES11], Pérez et al., (2018) [PYB18], Lindensjo, Lindskog (2019) [LL19], Noba et al., 2020 [NPY20].

Articles [GK17, AGLW20] are the only papers which relate declaring bankruptcy to the size of jumps, with general and exponential jumps, respectively. [GK17] deals also with the presence of Brownian motion and infinite activity jumps, by conditioning at the first draw-down time; the optimality proof is quite involved.

In [AGLW20], it is also shown that neither V(0) = 0 nor V'(0) = k are possible: the Lokka-Zervos alternative disappears, but another interesting alternative holds. Above a certain critical  $k_c$  the optimal dividends barrier switches from strictly positive to 0, and  $k_c$  is related in (26) to the Lambert-W function.

The results of [GK17, AGLW20] may be divided in three parts:

1. Compute the value of bounded buffer policies. The key result is (5) below, an explicit determination of the objective  $J_0 = J_0^{a,b}$ , which allows optimizing it numerically.

**REMARK 1.** Computing the value function is considerably simplified by the use of first passage recipes available for spectrally negative Lévy processes [AKP04, Kyp14, KKR13, AGVA19], which are built around two ingredients: the  $W_q$  and  $Z_q$  q-scale functions, defined respectively for  $x \geq 0, q \geq 0$  as:

- (a) the inverse Laplace transform of  $\frac{1}{\kappa(s)-q}$ , where  $\kappa(s)$  is the Laplace exponent (which characterizes a Lévy process) and
- (b)  $Z_q(x) = 1 + q \int_0^x W_q(y) dy$

- see the papers [Sup76, Ber98, AKP04] for the first appearance of these functions. The name q-scale/harmonic functions is justified by the fact that these functions are harmonic for the process X killed upon entering  $(-\infty,0)$ , in the sense that

$$\{e^{-q\min[t,T_0]}\ W_q(X_{\min[t,T_0]}), e^{-q\min[t,T_0]}\ Z_q(X_{\min[t,T_0]})\}, t\geq 0$$

are martingales, as shown in [Pis04, Prop. 3] (in the case of  $Z_q$ , there is also a **penalty** of 1 at ruin, generalizing to other penalties produces the so-called Gerber-Shiu function).

- 2. Equations determining candidates for the optimal  $a^*, b^*$  are obtained by differentiating the objective (which is expressed in terms of the scale functions  $W_q, Z_q$ ), and the optimal pair  $(a^*, b^*)$  is identified. As a result, the critical  $k_c$  is related in (26) to the Lambert-W function.
- 3. The optimality of the  $(-a^*, 0, b^*)$  policy is established.

Note that the last step is quite non-trivial and is achieved by different methods in [GK17] and [AGLW20]. The latter paper starts by formulating a (new) HJB equation associated to this stochastic control problem – see (8).

**REMARK 2.** The objective may be optimized numerically using the first step only (the equation (10) for  $J_0^{a,b}$ ).

Exponential approximations may also be used, which are similar in spirit with the de Vylder-type approximations. Recall that the philosophy of the de Vylder approximation is to approximate a Cramèr-Lundberg process by a simpler process with exponential jumps, with cleverly chosen exponential rate  $\mu$ , and the parameters  $\lambda, c$  may also be modified, if one desires to make the approximations exact at x=0—see for example [AHPS19] for more details)

The efficiency of the de Vylder approximation for approximating ruin probabilities is well documented [DV78]. The natural question of whether this type of techniques may work for other objectives, like for example for optimizing dividends and/or reinsurance was already discussed in [Høj02, DD05, BDW07, GSS08, AHPS19]. In this paper, following on previous works [ACFH11, AP14, ABH18], we draw first the attention to the fact that we have not one, but three de Vylder-type approximations for  $W_q(x)$  (as for the ruin probability). The best approximation in our experiments when the loading coefficient  $\theta$  is large turn out to be the classic de Vylder approximation. However, for approximating near the origin, the two point Padé approximation which fixes both the values  $W_q(0) = \frac{1}{c}$ ,  $W'_q(0) = \frac{q+\lambda}{c^2}$  works better. The end result here is simply replacing the inverse rate  $\mu^{-1}$  by  $m_1$ , in the formula for the scale function of the Cramèr-Lundberg process with exponential jumps. In between x=0 and  $x\to\infty$ , the winner is sometimes the "Renyi approximation" which replaces the inverse rate by  $\frac{m_2}{2m_1}$ , and modifies  $\lambda$  as well (for the de Vylder approximation,  $\mu^{-1}$  is replaced by  $\frac{m_3}{3m_2}$ , and both  $\lambda, c$  are modified).

We end this introduction by highlighting in figure 1 the fact that for exponential jumps, the limited capital injections objective function  $J_0$  given by (14) for arbitrary b but optimal a = s(b) (via a complicated formula) improves the value function with respect to de Finetti and Shreve, Lehoczky and Gaver, for any b.

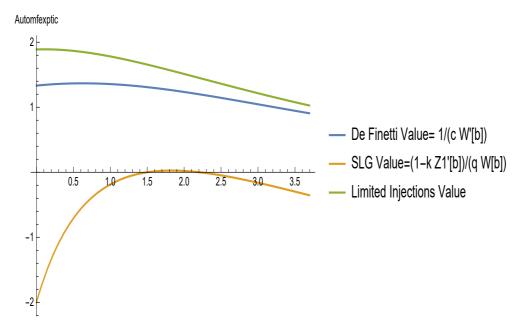


Figure 1: The value function  $J_0$  given by (14), for arbitrary b but optimal a. The inequality observed is a consequence of the properties of the Lambert function. The improvement with respect to de Finetti is considerable, of 0.382292% (the SLG approach is not competitive in this case). Note also that the optimal barrier b=0.109023 is smaller than the de Finetti and SLG optima of 0.626672, 1.82726 respectively.

Contents and contributions. Section 2 offers a conjectured profit formula for (-a, 0, b) policies, where we include also a final penalty P. The theoretical result of the section 1 revisits [GK17, AGLW20] by linking the two formulations together and emphasizing the impact of the bankruptcy penalty P (via the scale function G). Its proof is beyond the scope of the present (already lengthy enough) paper and it can be inferred from either one of [GK17] and [AGLW20] through a three step argument:

- 1. express the cost by conditioning on the reserve  $(J_x)$  starting from  $0 \le x \le b$  hitting either 0 or b;
- 2. get a further relationship on costs  $J_b$  and  $J_0$  by conditioning on the first claim;
- 3. finally, mix these conditions together in order to obtain the explicit formula for  $J_x$ .

We also wish to point out (in section 2.1) the link to an appropriate HJB variational inequality equation8 and the definition 1 specifying the action regions and their computation starting from the regimes in the HJB system. To the best of our knowledge, this is new and the preliminary studies conducted on more complicated problems (involving reinsurance and reserve-dependent premium) seem to reinforce the relevance of this tool.

In section 7 we provide an alternative matrix exponential form of the exact cost, in the case of matrix exponential jumps.

An explicit determination of  $a^*, b^*$  and an equity cost dichotomy when dealing with exponential jumps are given in section 3, taking also advantage of properties of the Lambert-W function, which were not exploited before. The two main novelties of the section are:

- emphasizing the computations of the optimal buffer/barrier (from [AGLW20]) in relation with the scale-like quantities appearing in [GK17];
- making explicit use of the (computation-ready) Lambert-W function to describe the dependency of optimal  $a^*b$  (in equation 14) and of the dichotomy-triggering cost  $k_c$  in equation 26.

Again, a further novelty is the presence of the bankruptcy cost P.

Section 4 reviews, for completeness, the de Vylder approximation-type approximations. Section 4.1 recalls, for warm-up, some of the oldest exponential approximations for ruin probabilities. Section 4.2 recalls in Proposition 5, following [AP14, AHPS19] three approximations of the scale function  $W_q(x)$   $^{\S}$ , obtained by approximating its Laplace transform. These amount finally to replacing our process by one with exponential jumps and cleverly crafted parameters based on the first three moments of the claims.

In section 5, we consider particular examples and obtain very good approximations for two fundamental objects of interest: the growth exponent  $\Phi_q$  of the scale function  $W_q(x)$ , and the (last) global minimum of  $W'_q(x)$ , which is fundamental in the de Finetti barrier problem. Proceeding afterwards to the problem of dividends and limited capital injections, concepts in section 3 are used to compute a straightforward exponential approximation based on an exponential approximation of the claim density, and a new "correct ingredients approximation" which consists of plugging into the objective function (10) for exponential claims the exact "non-exponential ingredients" (scale functions and, survival and mean functions) of the non-exponential densities. Both methods are observed to yield reasonable values in approximating the objective.

This leads us to our conclusion that from a practical point of view, exponential approximations are typically sufficient in the problems discussed in this paper.

# 2 The cost function of (-a, 0, b) policies, for the spectrally negative Lévy case

In this section, we allow  $\bar{\xi}_t$  to be a spectrally positive Lévy process, with a Lévy measure admitting a density  $\nu(dy) = \nu'(y)dy$ . The simplest example is that of the perturbed Cramèr-Lundberg risk model with

$$\bar{\xi}_t = \sum_{i=1}^{N_t} \xi_i + \sigma B_t,$$

where  $N_t$  is a Poisson process of intensity  $\lambda > 0$ ,  $(\xi_i)_{i \geq 1}$  is an independent family of i.i.d.r.v. with density f(y), and  $B_t$  is an independent Brownian motion.

We revisit here the problem of optimizing the value of "bounded buffer (-a, 0, b) policies", following [GK17, AGLW20] (in order to relate the results, one needs to replace  $\gamma$  in the objective of [GK17] by 1/k), while taking into account also the bankruptcy penalty P.

An important role in the results will be played by the expected scale after a jump

$$C(x) = \int_0^x W_q(x - y)\overline{\nu}(y)dy = cW_q(x) - Z_q(x) + \frac{\sigma^2}{2}W_q'(x),$$
 (1)

<sup>§</sup>essentially, this is the "dividend function with fixed barrier", which had been also extensively studied in previous literature before the introduction of  $W_q(x)$ 

where  $\overline{\nu}(y) = \int_y^\infty \nu(u) du$  is the tail of the Lévy measure and  $\sigma$  is the Brownian volatility (the identity above follows easily from the q-harmonicity of  $Z_q$ , after an integration by parts of the convolution term and a division by q).

The problem of limited reflection requires introducing a new "scale function  $S_a(x)$  and Gerber-Shiu function  $G_{a,\sigma}(x)$ " – see Remark 4 for further comments on this terminology:

$$\begin{cases} S_a(x) = Z_q(x) + C_a(x), C_a(x) = \int_0^x W_q(x - y) \, \overline{\nu}(a + y) \, dy \\ G_{a,\sigma}(x) = G_a(x) + k \frac{\sigma^2}{2} W_q(x) \end{cases}$$
 (2)

where

$$G_a(x) = \int_0^x W_q(x - y) (k \ m_a(y) + P\overline{\nu}(a + y)) dy := kM_a(x) + PC_a(x),$$
  
$$m_a(y) = \int_0^a z\nu(y + z) dz.$$

**EXAMPLE 1.** With exponential jumps and possibly  $\sigma > 0$ , using the identities

$$\overline{\nu}(y+a) = e^{-\mu a}\overline{\nu}(y), m_a(y) = \lambda e^{-\mu y}m(a), m(a) = \int_0^a y \ \mu e^{-\mu y}dy = \frac{1 - e^{-\mu a}}{\mu} - ae^{-\mu a},$$

we find that the functions (2) are expressible as products of C(x) and the survival or mean function of the jumps:

$$\begin{cases}
C_a(x) = C(x)e^{-\mu a} = C(x)\overline{F}(a), \ \overline{F}(a) = 1 - F(a) \\
S_a(x) = Z_q(x) + e^{-\mu a}C(x) \\
G_a(x) = (km(a) + P\overline{F}(a))C(x)
\end{cases}$$
(3)

([GK17] use  $s_c, r_c$ , instead of  $M_a(x) := \int_0^x W_q(x-y) \ m_a(y) \ dy, C_a(x)$ , respectively). When  $P = 0 = \sigma$ , these reduce to quantities in [AGLW20].

The formulas above will be used below as a heuristic approximation in non-exponential cases.

### REMARK 3. Note that

$$C_a(0) = 0, G_a(0) = 0, S_a(0) = 1, C(0) = 0, C'(0) = \begin{cases} \frac{\lambda}{c} & \sigma = 0\\ 0 & \sigma > 0 \end{cases}$$
 (4)

and that C(x),  $G_a(x)$ ,  $S_a(x)$  are increasing functions in x.

We state now a generalization of [GK17, Thm. 4] for the value function  $J_0^{a,b}$  of (-a,0,b) policies, in terms of  $S_a(x)$ ,  $G_a(x)$ . In the Cramèr-Lundberg case illustrated below, the proof is straightforward, following [AGLW20]. In the other case, one needs to adapt the proof of [GK17].

**THEOREM 1. Cost function for** (a,b) **policies** For a spectrally negative Lévy processes, let

$$J_x = J^{a,b}(x) := \mathbb{E}_x \left[ \int_0^{T_{-a}} e^{-qt} \left( dD_t - k \, dI_t \right) - Pe^{-qT_{-a}} \right]$$

denote the expected discounted dividends minus capital injections associated to policies consisting in paying capital injections with proportional cost  $k \ge 1$ , provided that the severity of ruin is smaller than a > 0, and paying dividends as soon as the process reaches some upper level b. Put

$$G_{a,\sigma}(x) = G_a(x) + k \frac{\sigma^2}{2} W_q(x).$$

Then, it holds that

$$J_{x} = \begin{cases} G_{a,\sigma}(x) + J_{0}^{a,b} S_{a}(x) = G_{a,\sigma}(x) + \frac{1 - G'_{a,\sigma}(b)}{S'_{a}(b)} S_{a}(x), & x \in [0, b] \\ kx + J_{0}^{a,b} & x \in [-a, 0] \\ 0 & x \le -a \end{cases}$$
 (5)

**REMARK 4.** The first equality in (5) will be easily obtained by applying the strong Markov property at the stopping time  $T = \min[T_{0,-}, T_{b,+}]$ , but it still contains the unknown  $J_0$ .

This relation suggests a definition of the scale  $S_a$  and the Gerber-Shiu function  $G_{a,\sigma}$ , as the coefficient of  $J_0$  and the part independent of  $J_0$ , respectively.

This equality is also equivalent to

$$J_0^{a,b} = \frac{J_x - G_{a,\sigma}(x)}{S_a(x)} = \frac{1 - G'_{a,\sigma}(b)}{S'_a(b)},\tag{6}$$

which suggests another analytic definition of the scale and Gerber-Shiu function corresponding to an objective  $J_x$  which involves reflection at b.

The functions  $S_a(x)$ ,  $G_a(x)$  may be shown to stay the same for problems which require only modifying the boundary condition at b, like the problem of capital injections for the process reflected at b, or the problem of dividends for the process reflected at b, with proportional retention  $k_D$  (this is in coherence with previously studied problems).

**COROLLARY 2.** Let us consider the Cramèr-Lundberg setting without diffusion (i.e.  $\sigma = 0$ ), For fixed  $k \geq 1, b \geq 0$ , the optimality equation  $\frac{\partial}{\partial a}J_0^{a,b} = 0$  may be written as

$$J_0^{a,b} = ka - P \Leftrightarrow J_{-a}^{a,b} = -P. \tag{7}$$

**REMARK 5.** The first equality in (7) provides a relation between the objective  $J_0$  and the variable a; the second recognizes this as the smooth fit equation  $J_{-a} = 0$ .

**Proof:** Recalling the expressions of  $J_0^{a,b}$ ,  $G_a(x)$ , in (6), in (2), and from [GK17, Lem. A.4]

$$M_{a'}(x) = -aC_{a'}(x),$$

where  $C_{a'}(x)$ ,  $M_{a'}(x)$  denote derivatives with respect to the subscript a, Whenever b > 0, if a achieves the maximum in  $J_0^{a,b}$ , it is straightforward (think of the economic interpretation) that a achieves the maximum of  $a \mapsto J_x^{a,b}$  for every  $x \in [0,b]$ . Therefore, we find

$$\begin{split} \frac{\partial}{\partial a}J_0^{a,b} &= 0 \Leftrightarrow J_0^{a,b} = \frac{-G_{a'}(x)}{C_{a'}(x)} = \frac{-kM_{a'}(x) - PC_{a'}(x)}{C_{a'}(x)} = ka - P \\ \Leftrightarrow J_{-a}^{a,b} &= J_0^{a,b} - ka \Leftrightarrow J_{-a}^{a,b} = -P. \end{split}$$

### 2.1 The HJB System

The optimality proof in [AGLW20] is based on showing that the function  $J_x$  (5) with  $a^*, b^*$  defined in (18), (19) is the minimal AC-supersolution of the HJB system

$$\begin{cases}
\max\{H(x, V, V'(x)), 1 - V'(x), V'(x) - k\} = 0, \ \forall x \in \mathbb{R}_+ \\
\max\{V'(x) - k, -P - V(x)\} = 0, \ \forall x \in \mathbb{R}_-
\end{cases}$$
(8)

where the Hamiltonian H is given by

$$H(x,\phi,v) := cv + \lambda \int_{\mathbb{R}_+} \phi(x-y)\mu e^{-\mu y} - (q+\lambda)\phi(x). \tag{9}$$

To discuss (8), it is useful to introduce the concept of **dividend-limited injections strategies** and **barrier strategies**. The following are also valid for its generalizations to mixed singular/continuous controls taking into account reinsurance:

**DEFINITION 1.** Dividend-limited injections strategies are stationary strategies where the dividends are paid according to a partition of the state space  $\mathbb{R}$  in five sets  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{C}_0, \mathcal{D}$  as follows:

1. If the surplus is in A (absolute ruin), bankruptcy is declared and a penalty P is paid;

- 2. If the surplus is in  $\mathcal B$  bailouts/capital injections are used for bringing the surplus to the closest point of  $\mathcal C$ .
- 3. If the surplus is in the open set C (continuation/no action set), no controls are used.
- 4. If the current surplus is in  $C_0 \subset \mathcal{D}$  (these are upper-accumulation points of C), dividends are paid at a positive rate, in order to keep the surplus process from moving.
- 5. If the current surplus is in  $\mathcal{D}$ , a positive amount of money is paid as dividends in order to bring the surplus process to  $\mathcal{C}_0$ .

Barrier strategies are stationary strategies for which A, B, C, D are four consecutive intervals.

**REMARK 6.** The four sets A, B, C, D correspond to the cases when equality in the HJB equation (8) is attained by at least one of the operators  $-V - P, V' - k, (\mathcal{G} - q)V$ , and 1 - V', respectively. Note this generalizes [AM14, Ch. 5.3], where only the last two operators are considered.

**REMARK 7.** One may conjecture that dividend-limited injections strategies are of a (recursive) multiband nature. In the case of exponential jumps, [AGLW20] show that the four sets  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$  in the optimal solution are intervals, denoted respectively by  $(-\infty, -a), [-a, 0], (0, b), [b, \infty)$ .

Cheap equity corresponds the case when  $C = \emptyset$ , and the partition reduces to three sets.

## 3 Explicit determination of $a^*, b^*$ when $F(x) := 1 - e^{-\mu x}, P > -\frac{c}{g}$

In this section we turn to the exponential case, where explicit formulas for the optimizers  $a^*$ ,  $b^*$  are available. In particular, we will take advantage of properties of the Lambert-W function, which were not exploited in [AGLW20]. Subsequently, in sections 5, 6 we will show that exponential approximations work typically excellently in the general case. Although these results have already been established in [AGLW20], the present formulations have two achievements:

- 1. allow an unified formulation of [AGLW20] and [GK17] (via the previously introduced scale functions);
- 2. make use of a numerical tool (Lambert-W function) to express the optimal quantities of interest  $a^*, b^*$ .

### 3.1 The simplified cost function and optimality equations

PROPOSITION 3. Cost function and optimality equations in the exponential case

1.

$$J_0^{a,b} = \frac{1 - C'(b) \left( k \ m(a) + P\overline{F}(a) \right)}{(\overline{F}(a))C'(b) + aW_c(b)} = \frac{\gamma(b) - k \ m(a) - P\overline{F}(a)}{a\theta(b) + \overline{F}(a)},\tag{10}$$

where we put

$$\gamma(b) = \frac{1}{C'(b)}, \theta(b) = \frac{W_q(b)}{C'(b)}.$$

2. Put

$$j(b) := \frac{\gamma'(b)}{q\theta'(b)}. (11)$$

For fixed  $a \ge 0$ , the optimality equation  $\frac{\partial}{\partial b} J_0^{a,b} = 0$  may be written as

$$J_0^{a,b} = j(b). (12)$$

3. For fixed  $k \ge 1$  and  $b \ge 0$ , at critical points with  $a(b) = a^{(k,P)}(b) \ne 0$  satisfies  $\frac{\partial}{\partial a} J_0^{a(b),b} = 0$  we must have

$$\[J_0^{a,b} - (ka - P)\]_{a=a(b)} = 0.$$

Explicitly,

$$0 = \eta(b, a) := \frac{\gamma(b)}{\theta(b)} - \frac{k}{\mu\theta(b)} F(a) - q(ka - P). \tag{13}$$

4. When  $P \ge -\frac{c}{q}$  and  $b \ge 0$  is fixed, the solution of (13) may be expressed in terms of the principal value of the "Lambert-W(right)" function (an inverse of  $L(z) = ze^z$ )

$$[-e^{-1}, \infty) \ni L_0(z), z \in [-1, \infty)$$

[CGH<sup>+</sup>96, Boy98, BFS08, Pak15, VLSHGG<sup>+</sup>19] (this observation is missing in [AGLW20]).

$$(0,\infty)\ni a(b)=\mu^{-1}\left(-h(b)+L_0\left(\frac{e^{h(b)}}{q\theta(b)}\right)\right), \quad h(b)=h(b,P)=\frac{1}{q\theta(b)}-\frac{\mu}{k}\left(\frac{\gamma(b)}{q\theta(b)}+P\right)$$
(14)

It follows that

$$J_0^{a(b),b} = \frac{k}{\mu} \left( -h(b) + L_0 \left( \frac{1}{q\theta(b)} e^{h(b)} \right) \right) - P.$$
 (15)

5. In the special case b = 0, (13) implies that  $a = a^{(k,P)} = a^{(k,P)}(0)$  satisfies the simpler equation

$$0 = \delta_{k,P}(a) := \lambda \eta(0,a) = \tilde{c} - k \left( aq + \frac{\lambda}{\mu} (1 - e^{-\mu a}) \right), \ \tilde{c} = c + qP > 0, \tag{16}$$

with solution

$$\mu \ a^{(k,P)} = -g + L_0 \left(\frac{\lambda}{q} e^g\right) > 0, \ g = h(0) = \frac{\lambda}{q} - \frac{\mu}{kq} \tilde{c}.$$
 (17)

6. At a critical point  $(a^*,b^*)$ ,  $a^*>0$ ,  $b^*>0$ , we must have both  $J_0^{a^*,b^*}=j(b^*)=ka^*-P\Longrightarrow 0$ 

$$a^* = s(b^*), s(b) := \frac{j(b) + P}{k},$$
 (18)

and

$$0 = \eta(b^*), \quad \eta(b) := \eta(b, s(b)) = \frac{\gamma(b)}{\theta(b)} - qj(b) - \frac{k}{\mu\theta(b)} F\left(\frac{j(b) + P}{k}\right) = 0.$$
 (19)

7. The equation  $0 = \eta(b)$  may be solved explicitly for P, yielding

$$P = -\frac{k}{\mu} \log \left[ 1 + \frac{q\theta(b)j(b) - \gamma(b)}{\frac{k}{\mu}} \right] - j(b).$$
 (20)

**Proof:** 1. follows from Theorem 1.1.

2. Let M(b), N(b) denote the numerator an denominator of  $J_0^{a,b} := \frac{M(b)}{N(b)}$  in (10). The optimality equation  $\frac{\partial}{\partial b} J_0^{a,b} = \frac{N'(b)}{N(b)} \left( \frac{M'(b)}{N'(b)} - J_0^{a,b} \right) = 0$  simplifies to

$$J_0^{a,b} = \frac{M'(b)}{N'(b)} = \frac{\gamma'(b)}{q\theta'(b)} = j(b).$$

- 3. (13) is a consequence of 1 and of the smooth fit result Corollary 2.
- 4. See the proof of the particular case 5;  $a \in (0, \infty)$  holds since  $P \ge -\frac{c}{q} \Longrightarrow h(b) < \frac{1}{q\theta(b)}$ .
- 5. (16) follows from  $W_q(0) = \frac{1}{c}$ ,  $\theta(0) = \lambda^{-1}$ . To get (17), rewrite the equation (16) as  $ze^z = \frac{\lambda}{q}e^g$ ,  $z = \mu a + g$ ;  $a \in (0, \infty)$  holds since  $P \ge -\frac{c}{q} \Longrightarrow g < \frac{\lambda}{q}$ .
  - 6. follows from 2. and .3.
  - 7. is straightforward.

**REMARK 8.** Note that the de Finetti and Shreve, Lehoczky and Gaver solutions  $a^* = a(b^*) = \begin{cases} 0 \\ \infty \end{cases}$  are always non-optimal, when  $P > -\frac{c}{2}$  (see (14)).

are always non-optimal, when  $P \ge -\frac{c}{q}$  (see (14)). However, as  $k \to \infty$ ,  $h(b) \to \frac{1}{q\theta(b)} \ne 0$  and,  $a(b) = \mu^{-1} \left( -h(b) + L_0 \left( h(b) e^{h(b)} \right) \right) = 0$ . This suffices to infer that you get de Finetti case. On the other hand,

$$P \to \infty \Longrightarrow h(b) \to -\infty \Longrightarrow a(b) \to \infty \Longrightarrow \begin{cases} J_0^{a(b),b} \to \frac{\gamma(b) - k/\mu}{q\theta(b)} = \frac{1 - kC'(b)/\mu}{qW_q(b)} = J_{0,SLG}(b) \\ \eta(b) \to q \left(J_{0,k}^{SLG}(b) - j(b)\right) \end{cases}, \forall b > 0.$$

$$(21)$$

Thus, these regimes can be recovered asymptotically. Let now  $b_k^{*,S}$ ,  $b_P^{*,D}$  denote the unique roots of  $\eta(b) = 0$  in the two asymptotic cases, which coincide with the classic Shreve, Lehoczky and Gaver and de Finetti barriers.

Then, it may be checked that  $b^* \leq \min[b_k^{*,S}, b_P^{*,D}]$ .

### **3.2** Existence of the roots of the equations $\eta(b) = 0, \delta_{k,P} = 0$

The following (new) result discusses the existence of the roots of the equations  $\eta(b) = 0$ ,  $\delta_{k,P} = 0$  introduced in proposition (3) and relates them to the Lambert-W function.

**PROPOSITION 4.** 1.  $\theta$  increases from  $\theta(0) = \frac{1/c}{\lambda/c} = \frac{1}{\lambda}$  to  $\theta(\infty) = \frac{1}{c\Phi_q - q}$ , as we see it in the figure below.

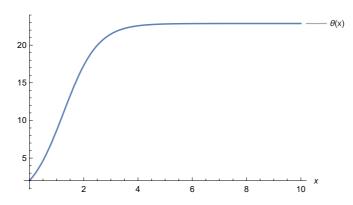


Figure 2: Plot of  $\theta$  with  $\theta(0) = 2$  and  $\theta(\infty) = 22.8743$ , for  $\mu = 2$ , c = 3/4,  $\lambda = 1/2$ , q = 1/10, P = 1 and k = 3/2.

 $\gamma$  is increasing-decreasing (from  $\frac{c}{\lambda}$  to 0), with a maximum at the unique root of C''(x) = 0 given by

$$\bar{b} := \frac{1}{\Phi_q - \rho_-} \log \left( \frac{\rho_-^2}{\Phi_a^2} \right), \tag{22}$$

where  $\Phi_q$ ,  $\rho_-$  denote the positive and negative roots of the Cramèr-Lundberg equation  $\kappa(s) = 0$ . The figure below illustrates the plot of the function  $\gamma$  and j(b) in which the  $\bar{b}$  is represented by the black point.

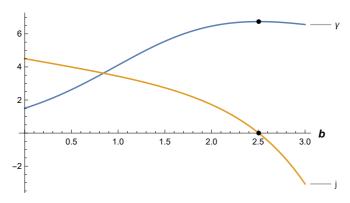


Figure 3: Plots of j(b) and  $\gamma(b)$  with  $\bar{b} = 2.5046$  and j(0) = 4.5, for  $\mu = 2$ , c = 3/4,  $\lambda = 1/2$ , q = 1/10, P = 1 and k = 3/2.

If  $c\mu - (q + \lambda) > 0$ , then  $\bar{b} > 0$  defined in (22) is the unique positive root of j(b) and  $\eta(\bar{b}) = \frac{1}{W_q(\bar{b})} > 0$ . See the figure (4)

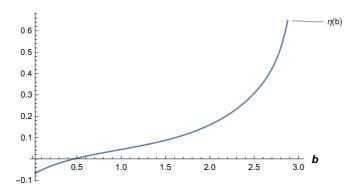


Figure 4: For  $\mu=2,\ c=3/4,\ \lambda=1/2,\ q=1/10,\ P=1$  and k=3/2, the root of  $\eta(b)=0$  is at b=0.469843

The function  $j(b) = \frac{\gamma'(b)}{q\theta'(b)}$  is nonnegative and decreasing to 0 on  $[0,\bar{b}]$ , with

$$j(0) = \frac{\lambda}{\mu q} \frac{-C''(0)}{(C'(0))^2} = \frac{c\mu - (q+\lambda)}{\mu q}.$$
 (23)

See the figure (3).

2. Put

$$\delta_{k,P} := \delta_{k,P}(a^{(k,P)}) = \delta_{k,P}(\frac{j(0) + P}{k}) = \delta_{k,P}(\frac{\tilde{c}\mu - (q+\lambda)}{k\mu q}) = \frac{\lambda + q - \lambda k \left(1 - e^{-\frac{\tilde{c}\mu - \lambda - q}{qk}}\right)}{\mu}, \quad (24)$$

and assume

$$\lim_{k \to \infty} \delta_{k,P} = \frac{\lambda + q}{\mu} - \frac{\lambda \left(\mu \widetilde{c} - (\lambda + q)\right)}{q\mu} = \frac{\left(\lambda + q\right)^2 - \lambda \mu \widetilde{c}}{q\mu} < 0 \Leftrightarrow \widetilde{c}\mu > \lambda^{-1} \left(\lambda + q\right)^2. \tag{25}$$

Then,  $\forall P > -\frac{c}{q}$ , the function  $\delta_{k,P}$  is decreasing in k with  $\delta_{1,P} > 0$ , and has a unique root

$$k_c = k_c(P) := \frac{q + \lambda}{\lambda} \frac{f}{f + L_0(-fe^{-f})} > \frac{q + \lambda}{\lambda}, \tag{26}$$

where

$$f := \frac{\lambda}{q+\lambda} \frac{\widetilde{c}\mu - (\lambda+q)}{q} > 1 \Leftrightarrow \widetilde{c}\mu > \lambda^{-1} (\lambda+q)^2 \Leftrightarrow P > P_l := q^{-1} \left(\mu^{-1}\lambda^{-1} (\lambda+q)^2 - c\right)$$
(27)

(note that the denominator  $f + L_0(-fe^{-f})$  does not equal 0 since f > 1 and  $L_0$  takes always values bigger than -1; or, note that  $-f = L_{-1}(L(-f))$ , where  $L_{-1}$  is the other real branch of the Lambert function).

Furthermore,

$$\delta_{k,P} < 0 \Leftrightarrow k > k_c(P). \tag{28}$$

3. It follows that  $\eta(b) = 0$  has at least one solution of in  $(0, \bar{b}]$  iff

$$\eta(0) = \frac{c}{\lambda} - \frac{1}{\lambda} \left( c - \frac{q + \lambda}{\mu} \right) - \frac{k}{\mu} F\left( a^{(k,P)} \right) = \frac{1}{\lambda \mu} \left( \lambda + q - \lambda k F\left( a^{(k,P)} \right) \right) = \frac{1}{\lambda} \delta_{k,P} < 0 \Leftrightarrow k > k_c. \tag{29}$$

The first such solution will be denoted by  $b^*$ .

**Proof:** For 1. see [AGLW20, Proof of Theorem 11, A2].

2. By using the assumption  $\tilde{c}\mu > \lambda^{-1}(\lambda + q)^2$  we get  $\tilde{c}\mu \geq \lambda + q$ , and  $k \in [1, \infty) \to \delta_{k,P}$  is decreasing. Put  $d = \frac{\tilde{c}\mu - (\lambda + q)}{q}$ . The inequality  $\delta_{k,P} < 0$  (see (29)) may be reduced to

$$e^{-\frac{d}{k}} < 1 - \frac{q + \lambda}{\lambda k} \Leftrightarrow 1 < e^{\frac{d}{k}} \left( 1 - \frac{(q + \lambda)}{d\lambda} \frac{d}{k} \right) := e^z \left( 1 - z/f \right).$$

Rewriting the latter as  $-f > e^z (z - f)$  we recognize, by putting z = y + f, an inequality reducible to  $ye^y < -fe^{-f}$ . The solution is

$$y < L_0 \left( -f e^{-f} \right),$$

where  $L_0$  is the principal branch of the Lambert-W function.

The final solution is (28), where we may note that the variables k, P have been separated.

3. is straightforward.

**REMARK 9.** The function  $[1,\infty) \ni f \mapsto \frac{f}{f+L_0(-fe^{-f})} \in (1,\infty)$  blows up at f=1, and converges to 1 when  $f \to \infty$  (or when either  $\mu$  or  $\tilde{c}=c+qP$  are large enough) as may be noticed in the figure below, which blows up at the value  $P_l := -4/5$ . Note also that when f (or one of  $c, P, \mu$  are large enough),  $k_c$  given by (26) stabilizes to the equilibrium  $\frac{\lambda+q}{\lambda}=6/5$ ; this is related to [APP07, Lemma 2], [KS08, Lemma 7], who obtain the same condition for  $b^*=0$  (without buffering capital injections). Intuitively, under these conditions, buffering is not crucial.

At the other end, as f tends to its lower limit and to the regime B, the notion of equity expensiveness vanishes, and  $k_c \to \infty$ .

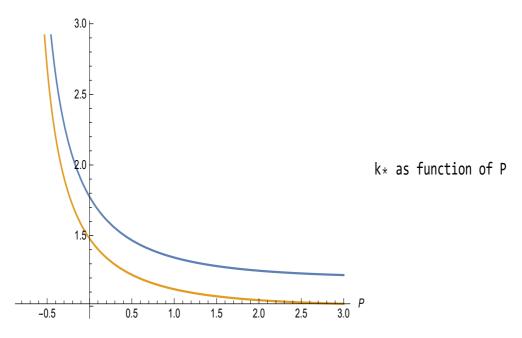
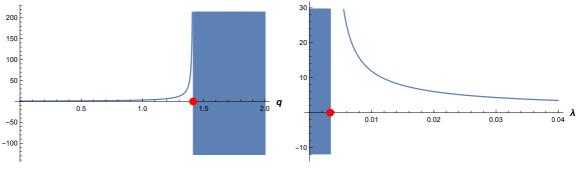


Figure 5:  $k_c$  as function of P, for several values of c, with the vertical asymptote at  $P_l$  fixed.

The next two figures illustrate how  $k_c$  blows up at the critical values  $q_l := (1/2)(-2\lambda + P\lambda\mu + \sqrt{\lambda}\sqrt{\mu}\sqrt{4}c - 4P\lambda + P^2\lambda\mu)$  and  $\lambda_l := \left(c\mu - \sqrt{(-c\mu + \mu(-P)q + 2q)^2 - 4q^2} + \mu Pq - 2q\right)$  (represented by red points in the figures below). The dark (blue) parts correspond to the regime A.



(a)  $k_c$  defined in (26)as function of q, for  $\mu=2$ , c= (b)  $k_c$  as function of  $\lambda$ , for  $\mu=2$ , c=3/2, q=1/10 and P=1;  $q_l=\sqrt{2}$ .

Figure 6:  $k_c$  as a function of q,  $\lambda$ 

### 4 Which exponential approximation?

### 4.1 Three de Vylder-type exponential approximations for the ruin probability

In the simplest case of exponential jumps of rate  $\mu$  and  $\sigma = 0$ , the formula for the ruin probability is

$$\Psi(x) = P_x[\exists t \ge 0 : X_t < 0] = \frac{1}{1+\theta} \exp\left(-\frac{x\theta\mu}{1+\theta}\right) = \frac{1}{1+\theta} \exp\left(-\frac{x\theta m_1^{-1}}{1+\theta}\right),\tag{30}$$

where  $\theta = \frac{c - \lambda m_1}{\lambda m_1}$  is the loading coefficient. By plugging the correct mean of the claims in the second formula yields the **simplest approximation** for processes with finite mean claims.

More sophisticated is the Renyi exponential approximation

$$\Psi_R(x) = \frac{1}{1+\theta} \exp\left(-\frac{x\theta \widehat{m}_2^{-1}}{1+\theta}\right), \widehat{m}_2 = \frac{m_2}{2m_1};$$
 (31)

This formula can be obtained as a two point Padé approximation of the Laplace transform, which conserves also the value  $\Psi(0) = (1+\theta)^{-1}$  [AP14]. It may be also derived heuristically from the first formula in (30), via replacing  $\mu$  by the correct "excess mean" of the **excess/severity density** 

$$f_e(x) = \frac{\overline{F}(x)}{m_1} = \frac{1 - F(x)}{m_1},$$

which is known to be  $\widehat{m}_2$ . Heuristically, it makes more sense to approximate  $f_e(x)$  instead of the original density f(x), since  $f_e(x)$  is a monotone function, and also an important component of the Pollaczek-Khinchine formula for the Laplace transform  $\widehat{\Psi}(s) = \int_0^\infty e^{-sx} F(dx)$  – see [Ram92, AP14].

More moments are put to work in the de Vylder approximation

$$\Psi_{DV}(x) = \frac{1}{1+\widetilde{\theta}} \exp\left(-\frac{x\widetilde{\theta}\widehat{m}_3^{-1}}{1+\widetilde{\theta}}\right), \ \widehat{m}_3 := \frac{m_3}{3m_2}, \ \widetilde{\lambda} = \frac{9m_2^3}{2m_3^2}\lambda, \ \widetilde{c} = c - \lambda m_1 + \widetilde{\lambda}\widehat{m}_3, \ \widetilde{\theta} = \frac{2m_1m_3}{3m_2^2}\theta = \frac{\widehat{m}_3}{\widehat{m}_2}\theta.$$

$$(32)$$

Interestingly, the result may be expressed in terms of the so-called "normalized moments"

$$\widehat{m}_i = \frac{m_i}{i \ m_{i-1}} \tag{33}$$

introduced in [BHT05].

The de Vylder approximation parameters above may be obtained either from

- 1. equating the first three cumulants of our process to those of a process with exponentially distributed claim sizes of mean  $\widehat{m}_3$ , and modified  $\lambda, c$  [DV78] (however  $p = c \lambda m_1 = E_0[X_1]$  must be conserved, since this is the first cumulant), or
- 2. a Padé approximation of the Laplace transform of the ruin probabilities [ACFH11].

The second derivation via Padé shows that higher order approximations may be easily obtained as well. They might not be admissible, due to negative values, but packages for "repairing" the non-admissibility are available – see for example [DcSA16].

The first derivation of the de Vylder approximation is a **process approximation** (i.e., independent of the problem considered); as such, it may be applied to other functionals of interest besides ruin probabilities  $(W_q(x),$  dividend barriers, etc.), simply by plugging the modified parameters in the exact formula for the ruin probability of the simpler process.

## 4.2 Three two point Padé approximations of the Laplace transform $\widehat{W}_q$ of scale function

The simplest approximations for the scale function  $W_q(x)$  will now be derived heuristically from the following example.

**EXAMPLE 2.** The Cramér-Lundberg model with exponential jumps Consider the Cramér-Lundberg model with exponential jump sizes with mean  $1/\mu$ , jump rate  $\lambda$ , premium rate c>0, and Laplace exponent  $\kappa(s)=s\left(c-\frac{\lambda}{\mu+s}\right)$ . Solving  $\kappa(s)-q=0 \Leftrightarrow cs^2+s(c\mu-\lambda-q)-q\mu=0$  for s yields two distinct solutions  $\gamma_2\leq 0\leq \gamma_1=\Phi_q$  given by

$$\gamma_1(\mu, \lambda, c) = \gamma_1 = \frac{1}{2c} \left( -(\mu c - \lambda - q) + \sqrt{(\mu c - \lambda - q)^2 + 4\mu qc} \right),$$
  
$$\gamma_2(\mu, \lambda, c) = \gamma_2 = \frac{1}{2c} \left( -(\mu c - \lambda - q) - \sqrt{(\mu c - \lambda - q)^2 + 4\mu qc} \right).$$

The W scale function is:

$$W_q(x) = \frac{A_1 e^{\gamma_1 x} - A_2 e^{\gamma_2 x}}{c(\gamma_1 - \gamma_2)} \Leftrightarrow \widehat{W}_q(s) = \frac{s + \mu}{cs^2 + s(c\mu - \lambda - q) - q\mu},\tag{34}$$

where  $A_1 = \mu + \gamma_1, A_2 = \mu + \gamma_2$ .

Furthermore, it is well-known and easy to check that the function  $W'_q(x)$  is in this case unimodal with global minimum at

$$b_{DeF} = \frac{1}{\gamma_1 - \gamma_2} \begin{cases} \log \frac{(\gamma_2)^2 A_2}{(\gamma_1)^2 A_1} = \log \frac{(\gamma_2)^2 (\mu + \gamma_2)}{(\gamma_1)^2 (\mu + \gamma_1)} & \text{if } W_q''(0) < 0 \Leftrightarrow (q + \lambda)^2 - c\lambda \mu < 0 \\ 0 & \text{if } W_q''(0) \ge 0 \Leftrightarrow (q + \lambda)^2 - c\lambda \mu \ge 0 \end{cases},$$
(35)

since  $W_q''(0) = \frac{(\gamma_1)^2(\mu + \gamma_1) - (\gamma_2)^2(\mu + \gamma_2)}{c(\gamma_1 - \gamma_2)} = \frac{(q + \lambda)^2 - c\lambda\mu}{c^3}$  and that the optimal strategy for the de Finetti problem is the barrier strategy at level  $b_{DeF}$  (see for example [APP07], [AGVA19, Sec. 3]).

Plugging now the respective parameters of the de Vylder type approximations in the exact formula (34) for the Cramèr-Lundberg process with exponential claims, we obtain three approximations for  $\widehat{W}_q$ :

- 1. "Naive exponential" approximation obtained by plugging  $\mu^{-1} \to m_1$  in (34) (as was done, for a different purpose) in (30)
- 2. Renyi  $\S$ , obtained by plugging  $\mu^{-1} \to \widehat{m}_2, \lambda_R \to \lambda \frac{m_1}{\widehat{m}_2}$  (since c is unchanged, the latter equation is equivalent to the conservation of  $\rho = \frac{\lambda m_1}{c}$ , and to the conservation of  $\theta$ , so this coincides with the Renyi ruin approximation used in (31).)
- 3. De Vylder, obtained by plugging  $\mu^{-1} \to \widehat{m}_3, \widetilde{\lambda} \to \lambda \frac{9m_3^2}{2m_3^2}, \widetilde{c} = c \lambda m_1 + \widetilde{\lambda} \widehat{m}_3$ .

**REMARK 10.** In the case of exponential claims, these three approximations are exact, by definition (or check that for exponential claims all the normalized moments are equal to  $\mu^{-1}$ ).

**REMARK 11.** The conditions for the non-negativity of the barrier is  $W_q''(0_+) < 0 \Leftrightarrow (\frac{\lambda+q}{c})^2 < \frac{\lambda}{c}f(0)$ . Here, this condition is satisfied for the exact when  $\theta > \frac{(\lambda+q)^2(1-\rho)}{\lambda^2f(0)m_1}$ .

 $<sup>\</sup>S$ This is called DeVylder B) method in [GSS08, (5.6-5.7)], since it is the result of fitting the first two cumulants of the risk process.

It is shown in [AHPS19, Prop. 1] that the three de Vylder type approximations are two-point Padé approximations of the Laplace transform (hence higher order generalizations are immediately available).

We recall that two-point Padé approximations incorporate into the Padé approximation two initial values of the function (which can be derived easily via the initial value theorem, from the Pollaczek-Khinchine Laplace transform):

$$W_q(0_+) = \lim_{s \to \infty} \widehat{sW_q}(s) = \frac{1}{c},\tag{36}$$

$$W_q'(0_+) = \lim_{s \to \infty} s \left( \frac{s}{\kappa(s) - q} - W_q(0_+) \right) = \frac{q + \lambda}{c^2}.$$
 (37)

In our case, incorporating both  $W_q(0_+), W'_q(0_+)$  leads to the natural exponential approximation which is therefore the best near x = 0. Incorporating none of them yields the de Vylder approximation, which is the best asymptotically. Incorporating only  $W_q(0_+)$  leads to Renyi, which is expected to be the best in an intermediate regime.

Note that when the jump distribution has a density f, it holds that :

$$W_q''(0_+) = \lim_{s \to \infty} s \left( s \left( \frac{s}{\kappa(s) - q} - W_q(0_+) \right) - W_q'(0_+) \right) = \frac{1}{c} \left( (\frac{\lambda + q}{c})^2 - \frac{\lambda}{c} f(0) \right). \tag{38}$$

Thus,  $W_q''(0)$  already requires knowing  $f_C(0)$  (which is a rather delicate task starting from real data); therefore we will not incorporate into the Padé approximation more than two initial values of the function.

We recall below in Proposition 5 three types of two-point Padé approximations [AHPS19, Prop. 1], and particularize them to the case when the denominator degree is n = 2 (which are further illustrated below).

### PROPOSITION 5. Three matrix exponential approximations for the scale function.

1. To secure both the values of  $W_q(0)$  and  $W_q'(0)$ , take into account (36) and (37), i.e. use the Padé approximation

$$\widehat{W}_q(s) \sim \frac{\sum_{i=0}^{n-1} a_i s^i}{c s^n + \sum_{i=0}^{n-1} b_i s^i}, a_{n-1} = 1, b_{n-1} = c a_{n-2} - \lambda - q.$$

For n=2 we recover the "natural exponential" approximation of plugging  $\mu \to \frac{1}{m_1}$  in (34):

$$\widehat{W}_q(s) \sim \frac{\frac{1}{m_1} + s}{cs^2 + s\left(\frac{c}{m_1} - \lambda - q\right) - \frac{q}{m_1}},\tag{39}$$

used also (for a different purpose) in (30).

2. To ensure only  $W_q(0) = \frac{1}{c}$ , we must use the Padé approximation

$$\widehat{W}_q(s) \sim \frac{\sum_{i=0}^{n-1} a_i s^i}{c s^n + \sum_{i=0}^{n-1} b_i s^i}, a_{n-1} = 1.$$

For n = 2, we find

$$\widehat{W}_{q}(s) \sim \frac{\frac{2m_{1}}{m_{2}} + s}{cs^{2} + \frac{s(2cm_{1} - 2\lambda m_{1}^{2} - m_{2}q)}{m_{2}} - \frac{2m_{1}q}{m_{2}}} = \frac{\frac{1}{\widehat{m}_{2}} + s}{cs^{2} + s\left(\frac{c}{\widehat{m}_{2}} - \lambda \frac{m_{1}}{\widehat{m}_{2}} - q\right) - \frac{q}{\widehat{m}_{2}}},\tag{40}$$

where  $\widehat{m}_2 = \widehat{m}_2 = \frac{m_2}{2m_1}$  is the first moment of the excess density  $f_e(x)$ . Note that it equals the scale function of a process with exponential claims of rate  $\widehat{m}_2^{-1}$  and with  $\lambda$  modified to  $\lambda_R = \lambda \frac{m_1}{\widehat{m}_2}$ . Since c is unchanged, the latter equation is equivalent to the conservation of  $\rho = \frac{\lambda m_1}{c}$ , and to the conservation of  $\theta$ , so this coincides with the Renyi approximation  $\S$  used in (31).

This equation is important in establishing the nonnegativity of the optimal dividends barrier.

<sup>§</sup>This is called DeVylder B) method in [GSS08, (5.6-5.7)], since it is the result of fitting the first two cumulants of the risk process.

3. The pure Padé approximation yields for n = 2

$$\begin{split} \widehat{W}_{q}(s) &\sim \frac{s + \frac{3m_{2}}{m_{3}}}{s^{2} \left(c - \lambda m_{1} + \lambda \frac{3m_{2}^{2}}{2m_{3}}\right) + s\left(c\frac{3m_{2}}{m_{3}} - \frac{3m_{1}m_{2}}{m_{3}}\lambda - q\right) - \frac{3m_{2}}{m_{3}}q} \\ &= \frac{s + \frac{1}{\widehat{m}_{3}}}{\widetilde{c}s^{2} + s\left(\widetilde{c}\frac{1}{\widehat{m}_{3}} - \widetilde{\lambda} - q\right) - \frac{1}{\widehat{m}_{3}}q}, \ \widetilde{c} = c - \lambda m_{1} + \widetilde{\lambda}\widehat{m}_{3}, \widetilde{\lambda} = \lambda \frac{9m_{2}^{3}}{2m_{3}^{2}}. \end{split}$$

Note that both the coefficient of  $s^2$  in the denominator coincides with the coefficient  $\widetilde{c}$  in the classic de Vylder approximation, since  $\widetilde{\lambda}\widehat{m}_3 = \lambda \frac{9m_2^3}{2m_3^2} \frac{m_3}{3m_2} = \lambda \frac{3m_2^2}{2m_3}$ , and so does the coefficient of s, since

$$c\frac{3m_2}{m_3}-\frac{3m_1m_2}{m_3}\lambda=\widetilde{c}\frac{1}{\widehat{m}_3}-\widetilde{\lambda}=\left(c-\lambda m_1+\widetilde{\lambda}\widehat{m}_3\right)\frac{1}{\widehat{m}_3}-\widetilde{\lambda}.$$

# 5 Examples of computations involving scale function and dividend value approximations

Our goal in this section is to investigate whether exponential approximations are precise enough to yield reasonable estimates for quantities important in control like

- 1. the dominant exponent  $\Phi_q$  of  $W_q(x)$
- 2. the last local minimum of  $W'_q(x)$ ,  $b_{DeF}$ , which yields, when being the global minimum, the optimal De Finetti barrier
- 3.  $W_a''(0)$ , which determines if  $b_{DeF} = 0$
- 4. the functional  $J_0$  yielding the maximum dividends with capital injections.

All the examples considered involve a Cramèr-Lundberg model with rational Laplace transform  $\widehat{W}_q(s)$  (since in this case, the computation of  $W_q, Z_q$  is fast and in principle arbitrarily large precision may be achieved with symbolic algebra systems).

- 1. For the first three problems, we will use de Vylder type approximations. Graphs of  $W'_q$ ,  $W''_q$  and some tables summarizing the simulation results will be presented. We note that in most of the cases that we observed, the de Vylder approximation of  $\Phi_q$  deviates from the exact value the least see for example Table 2. For the De Finetti barrier, the "winner" depends on the size of  $b_{DeF}$ . Unsurprisingly, when near 0, the natural exponential approximation wins, and as  $b_{DeF}$  increases, Renyi and subsequently the de Vylder approximation take the upper hand see for example Table 3.
- 2. For the computation of  $J_0$ , we provide, besides the exact value, also two approximations:
  - (a) For a given density of claims f one computes an exponential density approximation  $f_e(x) = \frac{1}{m_1} \exp(-\frac{x}{m_1})$  where  $m_1$  is the first moment of f. Subsequently, W, Z,  $J_0$  and a, b are obtained using the exponential approximation  $f_e$ . Quantities obtained by this method would be referred to with an affix 'expo pure'.
  - (b) For a given density of claims f, the value function is computed via the formula which assumes exponential claims in equation 3, but the "ingredients" W, Z,  $\overline{F}$  and the mean function m are the correct ingredients corresponding to our original density f. Quantities obtained by this method would be referred to with an affix 'expo CI'.

It turns out that the pure expo approximation works better for large  $\theta$ , and the correct ingredients approximation works better for small  $\theta$ .

Note that we only included tables illustrating approximating  $J_0$  for the first two examples, to keep the length of the paper under control, but similar results were obtained for the other examples.

### 5.1 A Cramér-Lundberg process with hyperexponential claims of order 2

We take a look at a Cramér-Lundberg process with density function  $f(x) = \frac{2}{3}e^{-x} + \frac{2}{3}e^{-2x}$  with  $\lambda = 1$ ,  $\theta = 1$  and  $q = \frac{1}{10}$ .

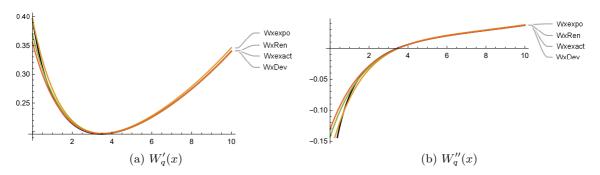


Figure 7: Exact and approximate plots of  $W_q'(x)$  and  $W_q''(x)$  for  $f(x)=\frac{2}{3}e^{-x}+\frac{2}{3}e^{-2x},\ \theta=1,\ q=\frac{1}{10}$ .

	Dominant exponent	Percent relative error	Optimal barrier	Percent relative error
	$\Phi_q$	$(\Phi_q)$	$b_{DeF}$	$(b_{DeF})$
Exact	0.110113	0	3.45398	0
Expo	0.110657	0.494313	3.51173	1.67191
Dev	0.110115	0.00195933	3.48756	0.972251
Renyi	0.110078	0.0321413	3.5323	2.26744

Table 1: Exact and approximate values of  $\Phi_q$  and  $b_{DeF}$  for  $f(x)=\frac{2}{3}e^{-x}+\frac{2}{3}e^{-2x}$ , theta = 1,  $q=\frac{1}{10}$ , as well as percent relative errors, computed as the absolute value of the difference between the approximation and the exact, divided by the exact, times 100. Relative errors for  $\Phi_q$  are less than 0.5%, with the pure exponential approximation proving to be the worst and the DeVylder the best approximations, respectively. The optimal barrier  $b_{DeF}$  is also best approximated by DeVylder, with Renyi being the worst at 2.26%.

$\theta$	Closest approximation	$\Phi_q$ exact	$\Phi_q$ approximation	$\%$ error $\Phi_q$
1	Dev	0.110113	0.110115	0.00195933
0.9	Dev	0.120328	0.120331	0.00269878
0.8	Dev	0.132452	0.132457	0.00380056
0.7	Dev	0.147017	0.147025	0.00548411
0.6	Dev	0.16475	0.164763	0.00812643
0.5	Dev	0.186652	0.186675	0.0123901
0.4	Dev	0.214122	0.214163	0.0194631
0.3	Dev	0.249118	0.249196	0.0315039
0.2	Dev	0.294396	0.294551	0.0524528
0.1	Dev	0.353829	0.354145	0.0894466

Table 2: Exact and the winning DeVylder approximate values of  $\Phi_q$  for  $f(x) = \frac{2}{3}e^{-x} + \frac{2}{3}e^{-2x}$  when  $\theta$  varies.

$\theta$	Closest approximation	Barrier exact	Barrier approx	% error Barrier
1	Dev	3.45398	3.48756	0.972251
0.9	Dev	3.20191	3.23103	0.909487
0.8	Dev	2.90951	2.93074	0.729628
0.7	Dev	2.57043	2.57742	0.272088
0.6	Dev	2.1804	2.16054	0.910666
0.5	Ren	1.74216	1.75266	0.60278
0.4	Expo	1.2735	1.29456	1.65378
0.3	Expo	0.81068	0.652264	19.5412

Table 3: Exact and approximate values of  $b_{DeF}$  for for  $f(x) = \frac{2}{3}e^{-x} + \frac{2}{3}e^{-2x}$  when  $\theta$  varies. As  $b_{DeF}$  approaches 0, errors of all the approximations increase dramatically, with the pure exponential approximation performing better than the rest. Meanwhile, as  $b_{DeF}$  increases, Renyi and subsequently the de Vylder approximation take the upper hand. For  $\theta = 0.2$  and  $\theta = 0.1$ , all the approximations yield a 0 barrier approximate for exact barrier values of 0.392105 and 0.0354538 respectively, hence failing to predict the non-zero barrier.

	J0					
$\theta$	J0 exact	J0 expo pure	J0 expo pure error	J0 expo CI	J0 expo CI error	
1	5.95034	5.99151	0.691856	6.26009	5.20551	
0.9	5.15579	5.17573	0.386663	5.45269	5.7584	
0.8	4.39383	4.38494	0.202205	4.67042	6.29489	
0.7	3.68299	3.63933	1.18555	3.92937	6.68958	
0.6	3.04577	2.96728	2.57704	3.25112	6.7423	
0.5	2.50331	2.39942	4.15022	2.65901	6.21974	
0.4	2.06833	1.9585	5.31006	2.17044	4.93725	
0.3	1.74095	1.65616	4.86984	1.78984	2.80878	
0.2	1.50439	1.44242	4.11969	1.50871	0.286672	
0.1	1.30271	1.25324	3.79748	1.30271	0	

Table 4: Values of  $J_0$  compared with approximations using all exponential inputs ( $J_0$  expo pure) and actual inputs but computed using the exponential formula ( $J_0$  expo CI). The pure exponential approximation does a good job of approximating  $J_0$  for higher values of  $\theta$  considered, while the exponential CI approximation seemed to fair better for lower  $\theta$  values

	a					
$\theta$	a exact	a expo pure	a expo pure error	a expo CI	a expo CI error	
1	3.9669	3.99434	0.691861	4.17339	5.20551	
0.9	3.4372	3.45049	0.386665	3.63512	5.7584	
0.8	2.92922	2.9233	0.202204	3.11361	6.29489	
0.7	2.45533	2.42622	1.18555	2.61958	6.68958	
0.6	2.03051	1.97818	2.57704	2.16741	6.7423	
0.5	1.66888	1.59961	4.15022	1.77268	6.21974	
0.4	1.37888	1.30566	5.31006	1.44696	4.93725	
0.3	1.16063	1.10411	4.86983	1.19323	2.80878	
0.2	1.00293	0.961612	4.11969	1.0058	0.286672	
0.1	0.868476	0.835496	3.79748	0.868476	0	

Table 5: Values of a compared with approximations using all exponential inputs (a expo pure) and actual inputs but computed using the exponential formula (a expo CI)

	b					
$\theta$	b exact	b expo pure	b expo pure error	b expo CI	b expo CI error	
1	1.41036	1.46188	3.65293	1.25374	11.1045	
0.9	1.37645	1.44439	4.93621	1.23362	10.3761	
0.8	1.31492	1.40417	6.78809	1.19529	9.09781	
0.7	1.21057	1.32258	9.25207	1.12775	6.84178	
0.6	1.04634	1.17215	12.0245	1.01753	2.7529	
0.5	0.810767	0.920406	13.5229	0.853397	5.25805	
0.4	0.510085	0.538725	5.61475	0.634716	24.4335	
0.3	0.17425	0.0105496	93.9457	0.376872	116.282	
0.2	0	0	0	0.105322	100	
0.1	0	0	0	0	0	

Table 6: Values of b compared with approximations using all exponential inputs (b expo pure) and actual inputs but computed using the exponential formula (b expo CI)

#### 5.2 A Cramér-Lundberg process with hyperexponential claims of order 3

Consider a Cramér-Lundberg process with density function  $f(x)=\frac{12}{83}e^{-x}+\frac{42}{83}e^{-2x}+\frac{150}{83}e^{-3x}$ , and c=1,  $\lambda=\frac{83}{48},\ \theta=\frac{263}{235},\ p=\frac{263}{498},\ q=\frac{5}{48}.$  The Laplace exponent of this process is  $\kappa(s)=s-\frac{12s}{83(s+1)}-\frac{21s}{83(s+2)}-\frac{50s}{83(s+3)}$  and from this one can

invert  $\frac{1}{\kappa(s)-q}=\widehat{W}_q(s)$  to obtain the scale function  $^1$ 

$$W_q(x) = -0.0813294e^{(-2.60997x)} - 0.179472e^{(-1.68854x)} - 0.373887e^{(-0.779311x)} + 1.63469e^{(0.18198x)}.$$

From this, we see that the dominant exponent is  $\Phi_q = 0.18198$ .

Figure 8 shows the exact and approximate plots of the first two derivatives of  $W_q$ . The exact plots are labelled Wxexact, and coloured as the darkest. The plots of  $W'_q$  exhibit noticeable unique minima around x=2, with the exact one being at  $b_{DeF}=1.89732$ , which is the optimal barrier that maximizes dividends here. Note that the approximations are practically indistinguishable from the exact around this point (which is our main object of interest here).

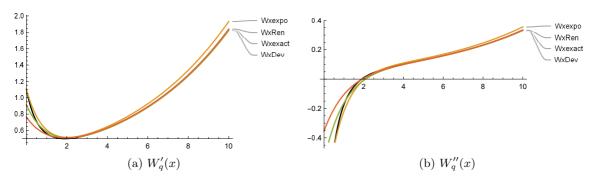


Figure 8: Exact and approximate plots of  $W_q'(x)$  and  $W_q''(x)$  for  $f(x) = \frac{12}{83}e^{-x} + \frac{42}{83}e^{-2x} + \frac{150}{83}e^{-3x}$ , c = 1,

<sup>&</sup>lt;sup>1</sup>Laplace inversion done via Mathematica; coefficients and exponents are decimal approximations of the real values.

	Dominant exponent	Percent relative error	Optimal barrier	Percent relative error
	$\Phi_q$	$(\Phi_q)$	$b_{DeF}$	$(b_{DeF})$
Exact	0.18198	0	1.89732	0
Expo	0.184095	1.162215628	2.04608	7.840532962
Renyi	0.181708	0.149466974	2.08136	9.699997892
Dev	0.182011	0.017034839	1.91233	0.79111589

Table 7: Exact and approximate values of  $\Phi_q$  and  $b_{DeF}$  for  $f(x) = \frac{12}{83}e^{-x} + \frac{42}{83}e^{-2x} + \frac{150}{83}e^{-3x}$ , c = 1,  $q = \frac{5}{48}$ , as well as percent relative errors, computed as the absolute value of the difference between the approximation and the exact, divided by the exact, times 100. Relative errors for the  $\Phi_q$  value are less than 1.25%, even for the worse natural exponential approximation, and the DeVylder approximation is the winner. The optimal barrier  $b_{DeF}$  is well approximated only by DeVylder.

$\theta$	Closest approximation	$\Phi_q$ exact	$\Phi_q$ approximation	$\%$ error $\Phi_q$
263/235	Dev	0.18198	0.182011	0.0168217
243/235	Dev	0.194712	0.194754	0.0213671
223/235	Dev	0.209221	0.209279	0.0274827
203/235	Dev	0.225876	0.225957	0.0358309
183/235	Dev	0.245146	0.245262	0.0474032
163/235	Dev	0.267635	0.267806	0.0637063
143/235	Dev	0.294126	0.294382	0.0870647
123/235	Dev	0.325643	0.326038	0.121115
103/235	Dev	0.363539	0.364163	0.171618
83/235	Dev	0.40961	0.410625	0.247788
63/235	Dev	0.466261	0.46796	0.364457
43/235	Dev	0.536719	0.539647	0.545532
23/235	Dev	0.62533	0.630516	0.829419
3/235	Dev	0.737962	0.747389	1.27736

Table 8: Exact and the winning DeVylder approximate values of  $\Phi_q$ , for  $f(x) = \frac{12}{83}e^{-x} + \frac{42}{83}e^{-2x} + \frac{150}{83}e^{-3x}$ ,  $\theta$  varies.

$\theta$	Closest approximation	Barrier exact	Barrier approx	% error Barrier
263/235	Dev	1.89732	1.91233	0.791183
243/235	Dev	1.79954	1.78002	1.08482
183/235	Ren	1.45224	1.52484	4.9989
163/235	Ren	1.31579	1.33691	1.60463
143/235	Ren	1.16804	1.12368	3.79796
123/235	Expo	1.00898	1.04123	3.19653
103/235	Expo	0.839228	0.794964	5.27444
83/235	Expo	0.660338	0.513179	22.2854
63/235	Expo	0.474896	0.196234	58.6785
43/235	Expo	0.286563	0	100
23/235	Expo	0.0998863	0	100
3/235	All	0	0	0

Table 9: Exact and approximate values of  $b_{DeF}$  for  $f(x) = \frac{12}{83}e^{-x} + \frac{42}{83}e^{-2x} + \frac{150}{83}e^{-3x}$ , when  $\theta$  varies. Unsurprisingly, when  $b_{DeF}$  is near 0, the natural exponential approximation wins, but has poor performance, and as  $b_{DeF}$  increases, Renyi and subsequently the de Vylder approximation take the upper hand. For the smallest three values of  $\theta$ , all of the approximations yielded  $\tilde{b}_{DeF} = 0$  as the optimal barrier, with this being true only for  $\theta = 3/235$ .

We move now to the dividend problem with capital injections with cost  $k \ge 1$  as in Theorem 1. One can compute the value function  $J_0$  at x = 0 in terms of W, Z,C, S, and G – see equation 5.

To provide a more concrete example, fixing  $q = \frac{5}{48}$ , P = 0, k = 3/2 as input parameters we compute for values of  $J_0$  as a function of  $\theta$ , with results summarized in the tables below. The tables provide comparisons of the computed optimal quantities  $J_0$ , a, and b to an approximation using all exponential inputs (referred to as  $J_0$ , a, and b expo pure) and to an approximation which uses actual inputs but computed using the exponential formula as described in equation 3 (referred to as  $J_0$ , a, and b expo CI).

		J0				
$\theta$	$J_0$	$J_0$ expo pure	$J_0$ expo pure error	$J_0 \exp \mathrm{CI}$	$J_0$ expo CI error	
263/235	3.7747	3.76883	0.155556	4.11784	9.09041	
243/235	3.41491	3.38603	0.845606	3.74156	9.5654	
223/235	3.0636	3.00802	1.81444	3.36985	9.99637	
203/235	2.72335	2.63828	3.1238	3.00466	10.3296	
183/235	2.39737	2.28225	4.80185	2.64879	10.4871	
163/235	2.08958	1.94765	6.79226	2.3062	10.3665	
143/235	1.80446	1.64396	8.8946	1.9823	9.85516	
123/235	1.54668	1.38072	10.73	1.68379	8.86472	
103/235	1.32041	1.16526	11.7499	1.4178	7.37587	
83/235	1.12864	1.00194	11.2258	1.19022	5.45555	
63/235	0.972835	0.88785	8.73585	1.00404	3.20798	
43/235	0.852739	0.789923	7.36635	0.859039	0.738837	
23/235	0.751597	0.701299	6.69218	0.751597	0	
3/235	0.660372	0.620567	6.02761	0.660372	0	

Table 10: Values of  $J_0$  compared with approximations using all exponential inputs ( $J_0$  expo pure) and actual inputs but computed using the exponential formula ( $J_0$  expo CI). The pure exponential approximation does a good job of approximating  $J_0$  for higher values of  $\theta$  considered, while the exponential CI approximation seemed to fair better for lower  $\theta$  values

	a				
$\theta$	a	a expo pure	a expo pure error	$a \exp CI$	$a \exp$ CI error
263/235	2.51647	2.74523	0.155536	2.51255	9.09042
243/235	2.27661	2.49437	0.845596	2.25735	9.56541
223/235	2.0424	2.24657	1.81443	2.00535	9.99638
203/235	1.81557	2.00311	3.1238	1.75885	10.3296
183/235	1.59825	1.76586	4.80185	1.5215	10.4871
163/235	1.39306	1.53747	6.79226	1.29844	10.3665
143/235	1.20298	1.32153	8.8946	1.09598	9.85516
123/235	1.03112	1.12252	10.73	0.920479	8.86472
103/235	0.880271	0.945198	11.7499	0.77684	7.37587
83/235	0.752428	0.793477	11.2258	0.667962	5.45555
63/235	0.648557	0.669362	8.73585	0.5919	3.20798
43/235	0.568493	0.572693	7.36635	0.526616	0.738838
23/235	0.501065	0.501065	6.69218	0.467532	0
3/235	0.440248	0.440248	6.02761	0.413711	0

Table 11: Values of a compared with approximations using all exponential inputs (a expo pure) and actual inputs but computed using the exponential formula (a expo CI)

	b				
$\theta$	b	b expo pure	b expo pure error	$b \exp CI$	b expo CI error
263/235	0.709355	0.805116	13.4997	0.677918	4.43179
243/235	0.695874	0.801936	15.2416	0.671779	3.46259
223/235	0.677601	0.794377	17.2337	0.662801	2.18425
203/235	0.653005	0.779265	19.3352	0.649805	0.490147
183/235	0.620126	0.751601	21.2012	0.631097	1.76912
163/235	0.576553	0.704104	22.123	0.604293	4.81128
143/235	0.519526	0.627369	20.7579	0.566198	8.98366
123/235	0.446259	0.511076	14.5246	0.512961	14.947
103/235	0.354524	0.346046	2.39143	0.440755	24.3231
83/235	0.243362	0.126054	48.2032	0.347059	42.6099
63/235	0.113593	0	100	0.231975	104.216
43/235	0	0	0	0.0987484	0
23/235	0	0	0	0	0
3/235	0	0	0	0	0

Table 12: Values of b compared with approximations using all exponential inputs (b expo pure) and actual inputs but computed using the exponential formula (b expo CI)

To provide a point of comparison, we fix  $q = \frac{5}{48}$ , and compute the de Finetti barrier to be  $b_{DeF} = 1.89732$  and the corresponding dividend value function when starting at x = 0 to be  $J_{DeF} = 1.99847$ .

k	$J_0$ % deviation	$J_0 - J_{DeF}$	$J_0$	b% deviation	b
1	154.123	3.0801	5.07857	100	0
2	65.714	1.31327	3.31174	43.0208	1.08108
3	42.863	0.856604	2.85507	25.4792	1.4139
4	31.4465	0.628448	2.62692	17.6853	1.56178
5	24.6995	0.493611	2.49208	13.4234	1.64264
6	20.2855	0.4054	2.40387	10.7759	1.69287
7	17.1884	0.343504	2.34197	8.98441	1.72686
8	14.9014	0.2978	2.29627	7.69619	1.7513
9	13.1463	0.262724	2.26119	6.72732	1.76968
10	11.758	0.23498	2.23345	5.97306	1.78399
100	1.11095	0.0222019	2.02067	0.533448	1.8872
1000	0.110409	0.00220648	2.00067	0.0527257	1.89632
10000	0.0109536	0.000218903	1.99869	0.00533526	1.89722

Table 13: Values of  $J_0$  and b in presence of capital injections compared to the case where capital injections are non-existent,  $J_{DeF} = 1.99847$  and  $b_{DeF} = 1.89732$ . As k is increased one can see that  $J_0$  and b approaches  $J_{DeF}$  and  $b_{DeF}$ . This is expected since higher costs of injecting capital makes it less viable, hence it is treated like the concept does not exist.

### 5.3 A Cramér-Lundberg process with oscillating density and scale function

In the following example, we study a Cramèr-Lundberg model with density of claims given by

$$f(x) = ue^{-ax} 2\cos^2\left(\frac{\omega x + \phi}{2}\right) = ue^{-ax} (1 + \cos(\omega x + \phi)) =$$
$$= e^{-ax} (u + u\cos(\phi)\cos(\omega x) - u\sin(\phi)\sin(\omega x))$$

where

$$u = \frac{a\left(a^2 + \omega^2\right)}{a^2 + \omega^2 + a^2\cos(\phi) - a\omega\sin(\phi)}.$$

Assuming further that  $a=1, \phi=2, \omega=20$ , and that  $\theta=1, q=1/10$ , the Laplace exponent for this

process is 
$$\kappa(s) = \frac{s\left(2.09898s^3 + 5.29695s^2 + 843.502s + 420.846\right)}{(s+1.)(s^2 + 2.s + 401.)}$$
 and the scale function is 
$$W_q(x) = 0.824723e^{0.0881484x} - 0.348141e^{-0.540677x} \\ + e^{-1.0117x}\cos(19.9957x) \left( -(0.000285494 + 0.0000804151i)\sin(39.9914x) - (0.0000804151 + 0.000285494i) + (-0.0000804151 + 0.000285494i)\cos(39.9914x) \right) \\ + e^{-1.0117x}\sin(19.9957x) \left( -(0.0000804151 - 0.000285494i)\sin(39.9914x) + (0.000285494 + 0.0000804151i)\cos(39.9914x) - (0.000285494 - 0.0000804151i) \right).$$

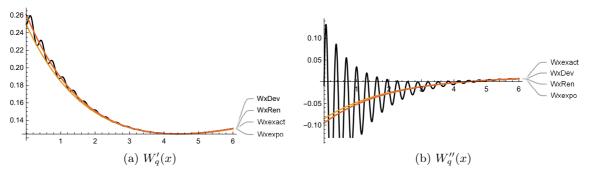


Figure 9: Plots of  $W_q'(x)$ , and  $W_q''(x)$  of the exact solution and the approximations for  $f(x) = ue^{-ax}2\cos^2\left(\frac{\omega x + \phi}{2}\right)$ ,  $\theta = 1$ ,  $q = \frac{1}{10}$ .

	Dominant exponent	Percent relative error	Optimal barrier	Percent relative error
	$\Phi_q$	$(\Phi_q)$	$b_{DeF}$	$(b_{DeF})$
Exact	0.0881484	0	4.38201	0
Expo	0.0878658	0.32053	4.42263	0.927122
Renyi	0.0881481	0.000314617	4.39788	0.362284
Dev	0.0881484	6.11743*10^-6	4.39745	0.352331

Table 14: Exact and approximate values of  $\Phi_q$  and  $b_{DeF}$  for  $f(x) = ue^{-ax}2\cos^2\left(\frac{\omega x + \phi}{2}\right)$ ,  $\theta = 1$ ,  $q = \frac{1}{10}$ . The DeVylder approximation wins on both fronts.

Clearly, our completely monotone approximation cannot fully reproduce functions like  $W'_q(x), W''_q(x)$  in examples like this where oscillations occur (note however that the de Finetti optimal barrier is well approximated here). If a more exact reproduction is necessary, higher order approximations should be used.

# 6 The maximal error of exponential approximations $J_0$ along one parameter families of Cramér-Lundberg processes

In this section, we provide the two approximations for the dividend value with capital injections  $J_0$ , and the dividend barrier b, for two one parameter families of Cramér-Lundberg processes, with densities given respectively by:

$$f(x) = k_{\epsilon} \left[ e^{-x} + \epsilon e^{-2x} \right] \tag{41}$$

$$f(x) = k_{\epsilon} \left[ \frac{12}{83} e^{-x} + \epsilon \left( \frac{42}{83} e^{-2x} + \frac{150}{83} e^{-3x} \right) \right], \tag{42}$$

where  $k_{\epsilon}$  is the normalization constant, and compute the maximal error of approximation when  $\epsilon \in (0, \infty)$  and  $\theta \approx 1$ . For this choice, the pure exponential approximation works considerably better,  $\forall \epsilon$ .

	J0					
	J0 exact	J0 expo pure	J0 expo pure error	J0 expo CI	J0 expo CI error	
0.001	7.1879	7.18802	0.0016603	7.18849	0.00827967	
0.01	7.17075	7.17193	0.0164663	7.17666	0.0824782	
0.1	7.008	7.01863	0.151653	7.06358	0.793041	
1	5.95034	5.99151	0.691856	6.26009	5.20551	
10	4.20175	4.19406	0.183122	4.40089	4.73941	
100	3.66909	3.6654	0.100631	3.69555	0.721228	
1000	3.6025	3.60208	0.0117065	3.60523	0.075585	

Table 15:  $\lambda = 1$ ,  $\theta = 1$ ,  $q = \frac{1}{10}$  k = 3/2 and P = 0. As expected, the errors decrease both as  $\epsilon$  goes to zero and infinity since the densities approach an exponential density.

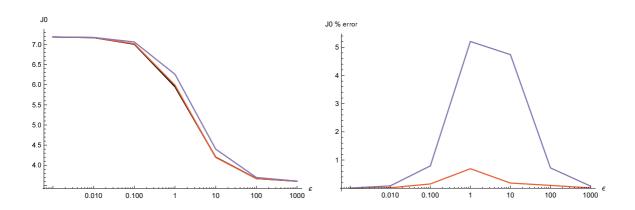


Figure 10:  $J_0$  values and errors plotted against  $\epsilon$ . Errors peak at  $\epsilon=1$ .

We do the same thing for the family of densities given by  $f(x) = k_{\epsilon} \left[ \frac{12}{83} e^{-x} + \epsilon \left( \frac{42}{83} e^{-2x} + \frac{150}{83} e^{-3x} \right) \right].$ 

	J0					
$\epsilon$	J0 exact	J0 expo pure	J0 expo pure error	J0 expo CI	J0 expo CI error	
0.001	7.95508	7.95771	0.0330111	7.96565	0.132796	
0.01	7.68765	7.71127	0.307292	7.78772	1.30166	
0.1	6.06176	6.15381	1.5186	6.62641	9.31507	
1	3.7747	3.76883	0.155556	4.11784	9.09041	
10	3.1382	3.1379	0.00959692	3.23354	3.03813	
100	3.05894	3.06427	0.174306	3.12284	2.08921	
1000	3.0508	3.05678	0.196219	3.11149	1.98956	

Table 16:  $\lambda = 1$ , c = 1,  $q = \frac{5}{48}$  k = 3/2 and P = 0. As  $\epsilon$  goes to zero, the density becomes exponential hence the decrease in errors. As  $\epsilon$  goes to infinity, the density approaches a hyper exponential density of order 2, but still both methods of approximating  $J_0$  yield reasonable results.

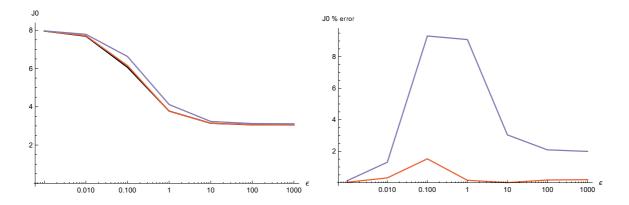


Figure 11:  $J_0$  values and errors plotted against  $\epsilon$ . Errors peak at  $\epsilon = 0.1$ .

### 7 The profit function when the claims are distributed according to a matrix exponential jumps density

Consider now the more general case when the claims are distributed according to a matrix exponential density generated by a row vector  $\vec{\beta}$  and by an invertible matrix B of order n, which are such that the vector  $\vec{\beta}e^{xB}$  is decreasing componentwise to 0, and  $\vec{\beta}.\mathbf{1} \neq 0$ , with  $\mathbf{1}$  a column vector. As customary, we restrict w.l.o.g. to the case when  $\vec{\beta}$  is a probability vector, and  $\vec{\beta}.\mathbf{1} = 1$ , so that

$$\overline{F}(x) = \vec{\beta}e^{xB}\mathbf{1}$$

is a valid survival function.

The matrix versions of our functions are:

$$\begin{cases}
C_{a}(x) = \lambda \int_{0}^{x} W_{q}(x-y) \ \overline{F}(y+a) \ dy = \lambda \vec{\beta} \int_{0}^{x} W_{q}(x-y) \ e^{yB} \ dy \ e^{aB} \mathbf{1} = \vec{C}(x) e^{aB} \mathbf{1} \\
m_{a}(y) = \int_{0}^{a} z f(y+z) dz = \vec{\beta} \ e^{yB} \int_{0}^{a} z \ e^{zB}(-B) \ dz \ \mathbf{1} = \vec{\beta} \ e^{yB} M(a) \mathbf{1}
\end{cases} ,$$

$$G_{a}(x) = \lambda \int_{0}^{x} W_{q}(x-y) \ m_{a}(y) \ dy = \vec{C}(x) M(a) \mathbf{1}$$
(43)

where

$$\begin{cases} C(x) = \lambda \int_0^x W_q(x-y) e^{yB} dy \\ \vec{C}(x) = \lambda \vec{\beta} \int_0^x W_q(x-y) e^{yB} dy \end{cases}$$

$$(44)$$

The product formulas (43) may also be established directly in the phase-type case, using the conditional independence of the ruin probability of the overshoot size.

We derive first these extensions from scratch for  $(\vec{\beta}, B)$  phase-type densities, in order to highlight their probabilistic interpretation. Later, we will show that the matrix exponential jumps case follows as a particular case of [GK17].

Recall first [AA10] that  $\Psi_q(x) = \vec{\Psi}_q(x)\mathbf{1}$ , where  $\vec{\Psi}_q(x)$  is a vector whose components represent the probability that ruin occurs during a certain phase, and that the conditional independence of ruin and overshoots translates into the product formula

$$\Psi_q(x,y) := P_x[T_{0,-} < \infty, X_{T_{0,-}} < -y] = \vec{\Psi}_q(x)e^{yB}\mathbf{1}. \tag{45}$$

To take advantage of this, it is convenient to replace from the beginning  $Z_q(x)$  by  $\Psi_q(x)$ , taking advantage of the formula [AKP04, Kyp14]

$$Z_q(x) = \Psi_q(x) + W_q(x) \frac{q}{\Phi_q} \Longrightarrow C(x) = \left(c - \frac{q}{\Phi_q}\right) W_q(x) - \Psi_q(x). \tag{46}$$

Alternatively, one may introduce a vector function

$$\vec{Z}_q(x) := \vec{\Psi}_q(x) + W_q(x) \frac{q}{\Phi_q} \mathbf{1}. \tag{47}$$

On the other hand, the mean function may be written as

$$m_a = \int_0^a y \ F(dy) = -a\overline{F}(a) + \int_0^a \overline{F}(x) dx = \vec{\beta} M(a) \mathbf{1}, M(a) = -B^{-1} - e^{aB} \left( aI_n - B^{-1} \right).$$

The following result follows in the phase-type case just as in the exponential case [AGLW20]; in the matrix exponential jumps case, it may be obtained from [GK17]:

**PROPOSITION 6.** For a Cramèr-Lundberg process (compound Poisson) with matrix exponential jumps of type  $(\vec{\beta}, B)$ , it holds that

1.

$$J_{x} = \begin{cases} kG_{a}(x) + J_{0}S_{a}(x) = kG_{a}(x) + \frac{1 - kG'(b)}{S'(b)}S_{a}(x), & x \in [0, b] \\ kx + J_{0} & x \in [-a, 0], \\ 0 & x \leq -a \end{cases}$$

$$(48)$$

where

$$\begin{cases}
C(x) = \lambda \int_{0}^{x} W_{q}(x - y) e^{yB} dy \\
\vec{C}(x) = \lambda \vec{\beta} \int_{0}^{x} W_{q}(x - y) e^{yB} dy = cW_{q}(x)\vec{1} - \vec{Z}_{q}(x) = (c - \frac{q}{\Phi_{q}})W_{q}(x)\vec{1} - \vec{\Psi}_{q}(x) \\
G_{a}(x) = \vec{C}_{q}(x)M(a)\mathbf{1} \\
R_{a}(x) = S_{a}(x) - Z_{q}(x) = \vec{C}_{q}(x)e^{aB}\mathbf{1}
\end{cases} , (49)$$

and

$$J_0 = \frac{1 - k\vec{C}_q'(b) \ M(a)\mathbf{1}}{qW_q(b) + \vec{C}_q'(b)e^{aB}\mathbf{1}}.$$
 (50)

2. For fixed a, the optimality equation  $\frac{\partial}{\partial b}J_0^{a,b}=0$  simplifies to

$$J_0 = \frac{k\vec{C}_q''(b) \ M(a)\mathbf{1}}{qW_a'(b) + \vec{C}_a''(b)e^{aB}\mathbf{1}}.$$
 (51)

**REMARK 12.** The additive separation of a, b which was the basis of proving optimality in the exponential case does not seem possible anymore, but (50) allows the numeric computation of the optimum.

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## Optimizing dividends and capital injections limited by bankruptcy, and practical approximations for the Cramér-Lundberg process

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### Abstract

The recent papers Gajek-Kucinsky(2017) and Avram-Goreac-Li-Wu(2020) investigated the control problem of optimizing dividends when limiting capital injections stopped upon bankruptcy. The first paper works under the spectrally negative Lévy model; the second works under the Cramér-Lundberg model with exponential jumps, where the results are considerably more explicit. The current paper has three purposes. First, it illustrates the fact that quite reasonable approximations of the general problem may be obtained using the particular exponential case studied in Avram-Goreac-Li-Wu(2020). Secondly, it extends the results to the case when a final penalty P is taken into consideration as well besides a proportional cost k>1 for capital injections. This requires amending the "scale and Gerber-Shiu functions" already introduced in Gajek-Kucinsky(2017). Thirdly, in the exponential case, the results will be made even more explicit by employing the Lambert-W function. This tool has particular importance in computational aspects and can be employed in theoretical aspects such as asymptotics.

**Keywords:** dividend problem, capital injections, penalty at default, scale functions, Lambert-W function, De Vylder-type approximations, rational Laplace transform

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