Optimizing Nonlinear Control Allocation

Tor A. Johansen [1]

Abstract

Control allocation is commonly utilized in over-actuated mechanical systems in order to optimally generate a requested generalized force using a redundant set of actuators. Using a control-Lyapunov approach, we develop an optimizing control allocation algorithm in the form of a dynamic update law, for a general class of nonlinear systems. The asymptotically optimal control allocation in interaction with an exponentially stable trajectory-tracking controller guarantees uniform boundedness and uniform global exponential convergence.

1 Introduction

Consider the nonlinear system

$$\dot{x} = f(t, x, \tau) \quad (1)$$

$$\tau = h(t, x, u) \quad (2)$$

where $t \geq 0$ is time, $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^r$ is the control input vector, and $\tau \in \mathbb{R}^p$ is a vector of virtual controls, typically moments and forces in mechanical systems. During control design, the virtual control τ is treated as an available input, although it can only be manipulated indirectly via the input u through $\tau = h(t, x, u)$. Mapping the requested τ to an input u is the control allocation task. Assume given a virtual control τ_c, in terms of a state feedback law

$$\tau_c = k(t, x) \quad (3)$$

that uniformly exponentially stabilizes the origin of the system (1) with perfect control allocation, i.e. $\tau = \tau_c$. The state x typically represents the tracking error relative to a time-varying reference trajectory, possibly also including an exponentially stable observer error. The basic control allocation problem is then to solve the system of nonlinear algebraic equations (2) with respect to the control vector u subject to $\tau = \tau_c$. Since we consider fully- or over-actuated problems ($p \geq r$), this does not in general define a unique u and one usually introduces an instantaneous cost function, $J(t, x, u)$. The cost function may incorporate power consumption or cost of raw materials, for example, and we assume actuator limitations and operational constraints are embedded into J as penalty or barrier functions. The control allocation problem is then formulated in terms of solving the following nonlinear static minimization problem:

$$\min_u J(t, x, u) \quad \text{subject to} \quad \tau_c - h(t, x, u) = 0 \quad (4)$$

Optimizing solutions have been derived for certain classes of over-actuated systems, such as aircraft, marine vessels, and machines [1, 2, 3, 4, 5, 6, 7]. They all treat the control allocation problem as a static (or quasi-dynamic) problem that is solved independently of the dynamic control problem, generally considering linear models $\tau = Gu$, with $G \in \mathbb{R}^{p \times r}$. The main advantage of this is modularity through its hierarchical structure. In the present paper we take a Lyapunov-based design approach, and consider general nonlinear models. Essentially, we specify a control Lyapunov function and derive an exponentially convergent dynamic update law for u (similar to a gradient/Newton-like optimization or adaptive control law [8]) such that the control allocation problem is solved dynamically. The main contribution is a theoretical result that shows that it is not necessary to solve the optimization problem (4) exactly at each time instant (or sampling instant in a discrete-time implementation). It is shown that convergence and asymptotic optimality of the dynamic control allocation in combination with an exponentially stable trajectory-tracking nonlinear controller guarantees uniform boundedness and uniform global exponential convergence of the system. This is related to the concept of asymptotic optimality [9, 10] and sub-optimal control [11]. One advantage of this approach is computational efficiency, since the optimizing control allocation algorithm is implemented explicitly as a dynamic nonlinear controller with $r + p$ states to update. Solving (4) explicitly at each sampling instant requires a computationally more expensive numerical solution of a nonlinear program to guarantee optimality, [12], although the present results indicate that the computational complexity can be safety reduced by for example early termination of the iterative numerical optimization.

[1] Department of Engineering Cybernetics, Norwegian University of Science and Technology, N-7491 Trondheim, Norway. e-mail: tor.arne.johansen@itk.ntnu.no.

2 Lyapunov design

Assumption 1. The virtual controller (3) makes the origin uniformly globally exponentially stable, i.e. there exists a differentiable function $V_0 : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ and positive constants c_1, c_2, c_3 and c_4 such that

$$c_1 ||x||^2 \leq V_0(t, x) \leq c_2 ||x||^2 \quad (5)$$

$$\frac{\partial V_0}{\partial t}(t, x) + \frac{\partial V_0^T}{\partial x}(t, x) f(t, x, k(t, x)) \leq -c_3 ||x||^2 \quad (6)$$

$$\left|\left| \frac{\partial V_0}{\partial x}(t, x) \right|\right| \leq c_4 ||x|| \quad (7)$$

Assumption 2. There exists a constant $\varrho > 0$ such that

$$\frac{\partial h}{\partial u}(t, x, u) \frac{\partial h^T}{\partial u}(t, x, u) \geq \varrho I_p \quad (8)$$

Assumption 3. The function f is differentiable and satisfies $f(t, 0, 0) = 0$. Moreover, it is globally Lipschitz, uniformly in t, i.e. there exist constants L_x and L_τ such that $||f(t, x_1, \tau_1) - f(t, x_2, \tau_2)|| \leq L_x ||x_1 - x_2|| + L_\tau ||\tau_1 - \tau_2||$ for all $x_1, x_2 \in \mathbb{R}^n$, $\tau_1, \tau_2 \in \mathbb{R}^p$ and $t \geq 0$. The function h is twice differentiable and globally Lipschitz, uniformly in t, with $h(t, 0, 0) = 0$ and Lipschitz constant L_h in x and u. Finally, we require that k is differentiable and Lipschitz, uniformly in t, with $k(t, 0) = 0$.

Assumption 4. The cost function J is twice differentiable.

The optimization problem (4) is reformulated by introducing a vector of Lagrange multipliers $\lambda \in \mathbb{R}^p$ and the Lagrangian

$$\ell(u, \lambda, x, t) = J(t, x, u) + (\tau_c - h(t, x, u))^T \lambda \quad (9)$$

Local minima of (4) satisfy the first order optimality conditions for ℓ, and we define the limiting optimal set E^* accordingly:

$$E^* = \Big\{(x,u,\lambda) \in \mathbb{R}^{n+r+p} \;\Big|\; x = 0, \lim_{t\to\infty} \frac{\partial \ell}{\partial u}(u,\lambda,x,t) = 0, \lim_{t\to\infty} \frac{\partial \ell}{\partial \lambda}(u,\lambda,x,t) = 0\Big\}$$

For simplicity, it is assumed that the limit exists. The following control Lyapunov function is designed to attract the total state (x,u,λ) to E^* (notice that u and λ are yet unspecified, but will be states in the dynamic control allocation algorithm):

$$V(t,x,u,\lambda) = \sigma V_0(t,x) + \frac{1}{2}\left(\frac{\partial \ell^T}{\partial u}\frac{\partial \ell}{\partial u} + \frac{\partial \ell^T}{\partial \lambda}\frac{\partial \ell}{\partial \lambda}\right) \quad (10)$$

where $\sigma > 0$ is a constant that will be specified later. The arguments of ℓ are implicit in (10) to simplify the notation. The time-derivative of V along trajectories of (1) satisfies

$$\begin{aligned}\dot V &= \sigma \frac{\partial V_0}{\partial t}(t,x) \\ &+ \sigma \frac{\partial V_0^T}{\partial x}(t,x) f(t,x,k(t,x) + h(t,x,u) - k(t,x)) \\ &+ \left(\frac{\partial \ell^T}{\partial u}\frac{\partial^2 \ell}{\partial u^2} + \frac{\partial \ell^T}{\partial \lambda}\frac{\partial^2 \ell}{\partial u \partial \lambda}\right)\dot u \\ &+ \left(\frac{\partial \ell^T}{\partial u}\frac{\partial^2 \ell}{\partial \lambda \partial u}\right)\dot\lambda + \left(\frac{\partial \ell^T}{\partial u}\frac{\partial^2 \ell}{\partial x \partial u} + \frac{\partial \ell^T}{\partial \lambda}\frac{\partial^2 \ell}{\partial x \partial \lambda}\right) \\ &\cdot f(t,x,h(t,x,u)) + \frac{\partial \ell^T}{\partial \lambda}\frac{\partial^2 \ell}{\partial t \partial \lambda} + \frac{\partial \ell^T}{\partial u}\frac{\partial^2 \ell}{\partial t \partial u} \quad (11)\end{aligned}$$

where

$$\begin{aligned}\frac{\partial \ell}{\partial u}(u,\lambda,x,t) &= \frac{\partial J}{\partial u}(t,x,u) - \frac{\partial h^T}{\partial u}(t,x,u)\lambda \quad (12)\\ \frac{\partial \ell}{\partial \lambda}(u,\lambda,x,t) &= \tau_c - h(t,x,u) \quad (13)\end{aligned}$$

Define for notational convenience

$$\begin{aligned}\alpha &= \frac{\partial^2 \ell}{\partial u^2}\frac{\partial \ell}{\partial u} - \frac{\partial h^T}{\partial u}\frac{\partial \ell}{\partial \lambda} \\ \beta &= -\frac{\partial h}{\partial u}\frac{\partial \ell}{\partial u} \\ \delta &= \left(\frac{\partial \ell^T}{\partial u}\frac{\partial^2 \ell}{\partial x \partial u} - \frac{\partial \ell^T}{\partial \lambda}\frac{\partial h}{\partial x}\right) f(t,x,h(t,x,u)) \\ &\quad + \frac{\partial \ell^T}{\partial \lambda}\left(\dot\tau_c - \frac{\partial h}{\partial t}\right) + \frac{\partial \ell^T}{\partial u}\frac{\partial^2 \ell}{\partial t \partial u}\end{aligned}$$

and observe that

$$\frac{\partial^2 \ell}{\partial u \partial \lambda} = -\frac{\partial h^T}{\partial u}, \quad \frac{\partial^2 \ell}{\partial x \partial \lambda} = -\frac{\partial h}{\partial x} + \frac{\partial \tau_c}{\partial x}, \quad \frac{\partial^2 \ell}{\partial u^2} = \frac{\partial^2 J}{\partial u^2} - \sum_{i=1}^{p} \lambda_i \frac{\partial^2 h_i}{\partial u^2}$$

Hence, the definitions of α and β can be written

$$\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \mathbb{H}\begin{pmatrix}\frac{\partial \ell}{\partial u} \\ \frac{\partial \ell}{\partial \lambda}\end{pmatrix}, \text{ with } \mathbb{H} = \begin{pmatrix}\frac{\partial^2 \ell}{\partial u^2} & -\frac{\partial h^T}{\partial u} \\ -\frac{\partial h}{\partial u} & 0\end{pmatrix} \quad (14)$$

Assumption 5. There exist constants $\kappa_2 > \kappa_1 > 0$ such that $\kappa_1 I_r \le \frac{\partial^2 \ell}{\partial u^2} \le \kappa_2 I_r$.

Lemma 1 *Suppose assumptions 2 – 5 hold. Then $\alpha = 0$ and $\beta = 0$ is equivalent to $\frac{\partial \ell}{\partial u} = 0$ and $\frac{\partial \ell}{\partial \lambda} = 0$.*

Proof. Lemma 16.1 in [13] proves that $\mathbb{H}$ is bounded away from singularity due to Assumptions 2 and 5. □

Eq. (11) can be rewritten in the compact form

$$\begin{aligned}\dot V &= \sigma \frac{\partial V_0}{\partial t}(t,x) \\ &+ \sigma \frac{\partial V_0^T}{\partial x}(t,x) f(t,x,k(t,x) + h(t,x,u) - k(t,x)) \\ &+ \alpha^T \dot u + \beta^T \dot\lambda + \delta \quad (15)\end{aligned}$$

The $\alpha^T \dot u$ term in (15) is made negative definite by the first term of the dynamic update law

$$\dot u = -\Gamma \alpha + \zeta \quad (16)$$

With $\Gamma = \Gamma^T > 0$. Similarly, the $\beta^T \dot\lambda$ term in (15) is made negative by the first term of the dynamic update law

$$\dot\lambda = -W\beta + \phi \quad (17)$$

with $W = W^T > 0$. The last (indefinite) term in (15) is cancelled if the vector signals $\zeta(t) \in \mathbb{R}^r$ and $\phi(t) \in \mathbb{R}^p$ can be chosen such that the following scalar algebraic equation holds:

$$\alpha^T \zeta + \beta^T \phi + \delta = 0 \quad (18)$$

We will return to this issue shortly, and show that we can always find signals ζ and ϕ such that this equation holds for all $t \ge 0$. Using theorem 2.4.7 in [14], we get from (15), (16), and (17) with the algebraic constraint (18):

$$\begin{aligned}\dot V &= \sigma\left(\frac{\partial V_0}{\partial t}(t,x) + \frac{\partial V_0^T}{\partial x}(t,x) f(t,x,k(t,x))\right) \\ &- \alpha^T \Gamma \alpha - \beta^T W \beta \\ &+ \sigma \frac{\partial V_0^T}{\partial x}(t,x) R(t,x,u,\tau_c)(h(t,x,u) - \tau_c) \quad (19)\end{aligned}$$

with

$$R(t,x,u,\tau_c) = \int_0^1 \frac{\partial f}{\partial \tau}(t,x,s\tau_c + (1-s)h(t,x,u))ds \quad (20)$$

The following global convergence result shows that the last (indefinite) term in (19) is dominated by the other (negative) terms.

Proposition 1 *Consider the system (1), (2), (3), (16) and (17) with $\zeta(t)$ and $\phi(t)$ satisfying (18). If Assumptions 1 - 5 hold, then $||x(t)||$, $||\tau(t)||$ and $||\tau_c(t)||$ are uniformly bounded, and $(x(t), u(t), \lambda(t)) \to E^*$ as $t \to \infty$ with exponential convergence rate, for any initial conditions $x(0) \in \mathbb{R}^n$, $\lambda(0) \in \mathbb{R}^p$, and $u(0) \in \mathbb{R}^r$.*

Proof. First, we show that V is radially unbounded. Consider any $x_0 \in \mathbb{R}^n$, and some optimal $u_0 \in \mathbb{R}^r$ and $\lambda_0 \in \mathbb{R}^p$ such that $(x_0, u_0, \lambda_0) \in E^*$. Using theorem 2.4.7 in [14] it is straightforward to show that

$$\frac{\partial \ell}{\partial u} = \left.\frac{\partial^2 \ell}{\partial u^2}\right|_0 (u-u_0) - \left.\frac{\partial h}{\partial u}^T\right|_0 (\lambda - \lambda_0) \quad (21)$$

$$\frac{\partial \ell}{\partial \lambda} = -\left.\frac{\partial h}{\partial u}\right|_0 (u-u_0) \quad (22)$$

where $\left.\frac{\partial^2 \ell}{\partial u^2}\right|_0 = \int_0^1 \frac{\partial^2 \ell}{\partial u^2}(su_0 + (1-s)u, \lambda, x, t)ds$ and $\left.\frac{\partial h}{\partial u}\right|_0 = \int_0^1 \frac{\partial h}{\partial u}(su_0 + (1-s)u, \lambda, x, t)ds$. Hence,

$$\frac{\partial \ell^T}{\partial u}\frac{\partial \ell}{\partial u} + \frac{\partial \ell^T}{\partial \lambda}\frac{\partial \ell}{\partial \lambda} = \begin{pmatrix} u-u_0 \\ \lambda - \lambda_0 \end{pmatrix}^T \begin{pmatrix} \left.\frac{\partial^2 \ell}{\partial u^2}\right|_0 & -\left.\frac{\partial h^T}{\partial u}\right|_0 \\ -\left.\frac{\partial h}{\partial u}\right|_0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \left.\frac{\partial^2 \ell}{\partial u^2}\right|_0 & -\left.\frac{\partial h^T}{\partial u}\right|_0 \\ -\left.\frac{\partial h}{\partial u}\right|_0 & 0 \end{pmatrix} \begin{pmatrix} u-u_0 \\ \lambda - \lambda_0 \end{pmatrix}$$

Using Assumptions 2 and 5 it follows immediately that there exists a constant $k_0 > 0$ such that

$$\frac{\partial \ell^T}{\partial u}\frac{\partial \ell}{\partial u} + \frac{\partial \ell^T}{\partial \lambda}\frac{\partial \ell}{\partial \lambda} \geq k_0 \left(||u-u_0||^2 + ||\lambda - \lambda_0||^2 \right) \quad (23)$$

Since x_0 is arbitrary we conclude from Assumption 1 that V is radially unbounded in (u, λ, x). Next, from (19) and Assumption 1 it follows that

$$\begin{aligned} \dot{V} &\leq -c_3\sigma||x||^2 - \lambda_{min}(W)||\alpha||^2 - \lambda_{min}(\Gamma)||\beta||^2 \\ &\quad + 2\sigma L_\tau \left|\left|\frac{\partial \ell}{\partial \lambda}\right|\right| c_4 ||x|| \end{aligned} \quad (24)$$

From the definitions of α and β together with Assumptions 2, 3 and 5 one can derive

$$\left|\left|\frac{\partial \ell}{\partial \lambda}\right|\right| \leq \frac{L_h}{\varrho}||\alpha|| + \frac{\kappa_2 L_h^2}{\varrho^2}||\beta|| \quad (25)$$

Let $M = \max\left(\frac{L_\tau L_h c_4}{\varrho}, \frac{L_\tau L_h^2 c_4 \kappa_2}{\varrho^2}\right)$. Using Young's inequality $2ab \leq a^2/\mu + b^2\mu$ for $\mu > 0$, (24) and (25) lead to

$$\begin{aligned} \dot{V} &\leq -\sigma(c_3 - M\mu)||x||^2 - (\lambda_{min}(W) - \sigma M/\mu)||\alpha||^2 \\ &\quad -(\lambda_{min}(\Gamma) - \sigma M/\mu)||\beta||^2 \end{aligned} \quad (26)$$

Notice that $\mu > 0$ and $\sigma > 0$ are arbitrary constants. First, choose $\mu > 0$ such that $c_3 > M\mu$. Next, choose $\sigma > 0$ such that $\lambda_{min}(W) - \sigma M/\mu > 0$ and $\lambda_{min}(\Gamma) - \sigma M/\mu > 0$. Because $\mathbb{H}$ is bounded away from singularity (due to Assumptions 2 and 5), there exist constants $k_1, k_2, k_3, k_4 > 0$ such that

$$\dot{V} \leq -k_1||x||^2 - k_2\left|\left|\frac{\partial \ell}{\partial u}\right|\right|^2 - k_3\left|\left|\frac{\partial \ell}{\partial \lambda}\right|\right|^2 \leq -k_4 V \quad (27)$$

Uniform boundedness and exponential convergence follow directly. □

Consider the issue of solving (18) with respect to $\zeta \in \mathbb{R}^r$ and $\phi \in \mathbb{R}^p$. To achieve a well-defined unique solution to this time-varying scalar algebraic equation we solve a least-squares problem subject to (18). This leads to the Lagrangian

$$\mathbb{L}(\zeta, \phi, \nu) = \frac{1}{2}\left(\zeta^T\zeta + \phi^T\phi\right) + \nu\left(\alpha^T\zeta + \beta^T\phi + \delta\right) \quad (28)$$

where $\nu \in \mathbb{R}$ is a Lagrange multiplier. First order optimality conditions leads to ζ and ϕ being given by the solution to the following time-varying linear system of equations:

$$\begin{pmatrix} I_r & 0 & \alpha \\ 0 & I_p & \beta \\ \alpha^T & \beta^T & 0 \end{pmatrix} \begin{pmatrix} \zeta \\ \phi \\ \nu \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\delta \end{pmatrix} \quad (29)$$

Lemma 2 *Suppose Assumptions 2, 3 and 5 hold. Then (29) always has a unique solution for ζ and ϕ.*

Proof. Assuming $\alpha \neq 0$ or $\beta \neq 0$, the solution is indeed unique, see Lemma 16.1 in [13]. On the other hand, if $\alpha = 0$ and $\beta = 0$ it is evident from the definition of δ that $\delta = 0$, due to Lemma 1. The last equation in (29) then becomes trivial and $\phi = 0$, $\zeta = 0$ defines the solution (notice that ν is not uniquely defined in this case). □

Proposition 2 *Under the assumptions of Proposition 1, $\lambda(t)$ is uniformly bounded.*

Proof. Substituting into (17) gives

$$\dot{\lambda} = -W\frac{\partial h}{\partial u}\frac{\partial h^T}{\partial u}\lambda + \chi \quad (30)$$

with $\chi = W\frac{\partial h}{\partial u}\frac{\partial J}{\partial u} + \phi$. Consider the Lyapunov-like function $\mathbb{V}(\lambda) = \frac{1}{2}\lambda^T\lambda$. Its time-derivative is

$$\dot{\mathbb{V}} = -\lambda^T W\frac{\partial h}{\partial u}\frac{\partial h^T}{\partial u}\lambda + \lambda^T\chi \quad (31)$$

$$\leq -\varrho\lambda_{min}(W)||\lambda||^2 + ||\lambda|| \cdot ||\chi|| \quad (32)$$

Uniform boundedness of $\lambda(t)$ follows because $\varrho\lambda_{min}(W) > 0$ and Proposition 1 and Lemma 2 implies that $||\chi||$ is uniformly bounded. □

Remark 1. Notice that the matrices $\Gamma > 0$ and $W > 0$ may be chosen as time-varying, without changing any of the theoretical properties, provided they are bounded away from zero. Newton-like methods can therefore be implemented by taking

$$\begin{pmatrix} \dot{u} \\ \dot{\lambda} \end{pmatrix} = -\gamma\left(\mathbb{H}^T\mathbb{H} + \varepsilon I_{r+p}\right)^{-1}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} + \begin{pmatrix} \zeta \\ \phi \end{pmatrix} \quad (33)$$

where $\gamma > 0$ and $\varepsilon \geq 0$ are time-varying parameters. In a discrete-time implementation, γ may be chosen using a line search to guarantee descent between each sampling instant (in terms of a merit function), [13].

Remark 2. The terms involved in the algebraic constraint (18) arise because the optimal solution u to (4) is time-varying. The terms ζ and ϕ provides a feedforward-like compensation in the update laws for u and λ, seeking to maintain the time-varying optimum.

Remark 3. Although $\lambda(0)$ can be chosen arbitrarily, one can reduce transients by $\lambda(0) = \arg\min_\lambda V(0, x(0), u(0), \lambda)$.

Remark 4. If E^* contains a unique minimum (for example under some strict convexity assumption of J and additional assumptions on h), or the dynamics are time-invariant, one may extend the convergence result in a standard manner to global exponential stability. Robustness is then an inherent property.

3 Objectives and constraints

Usually, one wants to introduce a cost function $J(t,x,u)$ that captures multiple objectives such as minimizing power consumption, satisfying input constraints, and avoiding singularities. For example, power consumption can in some cases be approximated with the following term [3]

$$J_1(u) = \frac{1}{2}u^T Hu$$

Input constraints on the form $c(u) \leq 0$ can be added to the optimization problem through a barrier function of the form

$$J_2(u) = -w_2 \sum_i \log(-c_i(u)) \tag{34}$$

with $w_2 > 0$, [13]. J_2 will not be defined outside the admissible region, so Proposition 1 reduces to a local convergence result.

Since the control allocation algorithm explicitly computes $\dot{u}$, input rate constraints can be enforced by reducing the gain $\Gamma(t)$ sufficiently. Again, Proposition 1 may be reduced to a local convergence result because (ζ, ϕ) (which are not influenced by Γ) may require unacceptable high input rates, and in addition one can in general not find a strictly positive lower bound on $\inf_t \lambda_{min}(\Gamma(t)) \geq 0$.

Singular effector configurations must usually be avoided because they may lead to temporary loss of full controllability, [12]. This can be implemented by adding the following non-convex term to the criterion

$$J_3(t,x,u) = -w_3 \log\det\left(\frac{\partial h}{\partial u}(t,x,u)\frac{\partial h^T}{\partial u}(t,x,u) + \epsilon I_p\right)$$

with $w_3 > 0$, and $\epsilon > 0$. Notice that a finite value of $J_3(t,x,u)$ implies that Assumption 2 is always satisfied if $\epsilon = 0$. Hence, using this term in the criterion makes the control allocation algorithm avoid values of u where the effector configuration is singular, such that assumption 2 is effectively enforced.

4 Examples

4.1 Linear state feedback control with linear effector model

The example is intended to illustrate that the suggested approach is a natural nonlinear extension to the generalized inverse solution. We consider the over-actuated linear system

$$\dot{x} = Ax + B\tau, \qquad \tau = Gu \tag{35}$$

where the controller $\tau_c = -Kx$ is stabilizing such that $A - BK$ is Hurwitz. We assume G has full rank, such that the control allocation problem is non-singular and assumption 2 holds. If G does not have full rank, then the system (A, BG) might not be controllable even if (A, B) is controllable. Hence, this assumption is not restrictive. The cost function is the standard quadratic form $J(u) = \frac{1}{2}u^T Hu$, with $H = H^T > 0$, and it follows that $\frac{\partial \ell}{\partial u} = Hu - G^T\lambda$, $\frac{\partial \ell}{\partial \lambda} = \tau_c - Gu$. This leads to $\alpha = HHu - HG^T\lambda - G^T(\tau_c - Gu)$, $\beta = -G(Hu - G^T\lambda)$, and $\delta = -(\tau_c - Gu)^T Kx$. The control allocation algorithm is

$$\begin{aligned}\begin{pmatrix} \dot{u} \\ \dot{\lambda} \end{pmatrix} &= \begin{pmatrix} -\Gamma(HH + G^T G) & \Gamma HG^T \\ WGH & -WGG^T \end{pmatrix}\begin{pmatrix} u \\ \lambda \end{pmatrix} \\ &\quad + \begin{pmatrix} -\Gamma G^T Kx + \zeta \\ \phi \end{pmatrix}\end{aligned} \tag{36}$$

It is straightforward to verify that the system matrix in (36) is Hurwitz. Notice that $\zeta = 0$ and $\phi = 0$ must hold at the equilibrium point (origin). If x is considered as a constant input in (36), it is easily verified that

$$u = H^{-1}G^T(GH^{-1}G^T)^{-1}(-Kx) \tag{37}$$

$$\lambda = -(GH^{-1}G^T)^{-1}(-Kx) \tag{38}$$

defines the equilibrium point for (36). The solution (37) coincides with the conventional generalized inverse solution, e.g. [1], (or the Moore-Penrose pseudo-inverse $G^+ = G^T(GG^T)^{-1}$ if $H = I_r$). Using standard singular perturbation arguments (assuming the (u, λ)-dynamics are faster than the x-dynamics), it follows that the optimal solution (37) is asymptotically attained and that the suggested approach only differs from from the generalized inverse solution by a fast transient term due to the initial conditions $u(0)$ and $\lambda(0)$ not necessarily satisfying the optimality conditions.

4.2 Low-speed manoeuvering of over-actuated ship

This example is based on [15]. Consider a ship equipped with two rudder/propeller pairs at the stern, and one tunnel thruster at the bow. The nonlinear equations of motion in the horizontal plane are

$$\dot{\eta} = R(\psi)\nu \tag{39}$$

$$M\dot{\nu} + D\nu = \tau + d \tag{40}$$

where $\eta \in \mathbb{R}^3$ contains the (x, y)-position and heading ψ in an Earth-fixed coordinate frame, and $\nu \in \mathbb{R}^3$ the corresponding velocity components in a vessel-fixed coordinate frame. M is an inertia matrix, D is a linear damping matrix, $\tau \in \mathbb{R}^3$ contains surge and sway forces, as well as the yaw momentum, and $d \in \mathbb{R}^3$ is a vector of slowly time-varying disturbances. $R(\psi)$ is the rotation matrix from the vessel-fixed frame to the Earth-fixed frame. It is shown in [15] that the following controller is globally exponentially stabilizing under a reasonable assumption on ψ:

$$\tau_c = -K_I R^T(\psi)\xi - K_P R^T(\psi)(\eta - \eta^*) - K_D\nu \tag{41}$$

$$\dot{\xi} = \eta - \eta^* \tag{42}$$

The model and controller parameters in the simulation study are chosen in accordance with the scale model ship studied in [15]. At low speed, the surge force X (longitudinal) and sway force Y (lateral) produced by a propeller/rudder pair is given by [15]

$$T = \begin{cases} k_{Tp}\omega^2, & \omega \geq 0 \\ k_{Tn}|\omega|\omega, & \omega < 0 \end{cases} \tag{43}$$

$$L_r = \begin{cases} T(1 + k_{Ln}\omega)(k_{L\delta_1}\delta + k_{L\delta_2}|\delta|\delta), & \omega \geq 0 \\ 0, & \omega < 0 \end{cases} \tag{44}$$

$$D_r = \begin{cases} T(1 + k_{Dn}\omega)(k_{D\delta_1}|\delta| + k_{D\delta_2}\delta^2), & \omega \geq 0 \\ 0, & \omega < 0 \end{cases} \tag{45}$$

T is the nominal thrust, L_r is the rudder lift force, D_r is the rudder drag force, ω is the propeller angular velocity, and δ is the rudder angle. The surge and sway forces of each thruster are given as follows:

$$X = T - D_r, \qquad Y = L_r \tag{46}$$

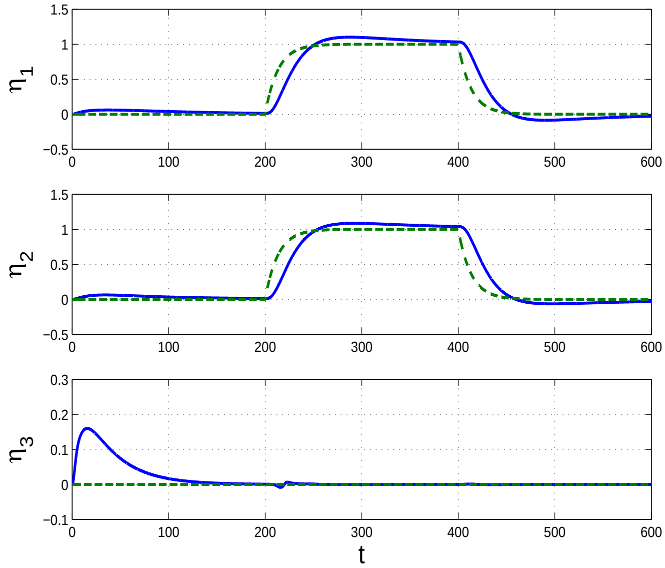

Figure 1: Simulation results - solid lines are positions while dashed lines are reference.

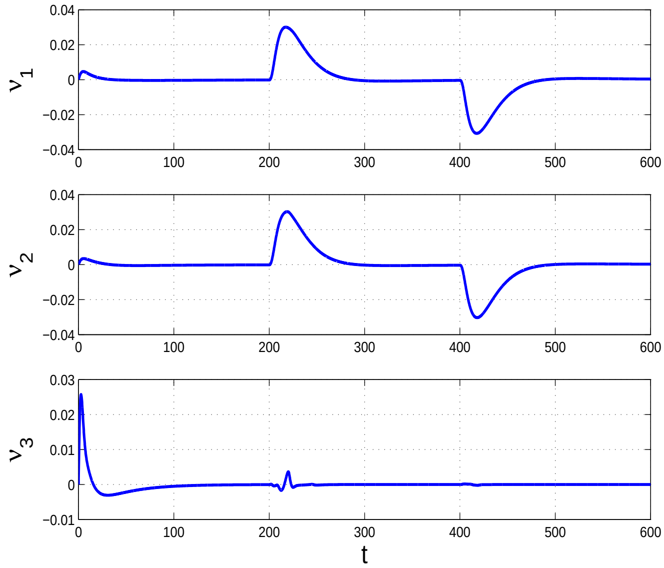

Figure 2: Simulation results - velocities.

For the bow tunnel thruster, the same model can be used with rudder angle $\delta \equiv 0$. Let the stern propeller/rudder pairs have index 1 and 2, and the bow tunnel thruster have index 3. Thus the virtual controls $\tau = (\tau_1, \tau_2, \tau_3)^T$ defined by

$$\tau_1 = X_1 + X_2 \quad (47)$$
$$\tau_2 = Y_1 + Y_2 + T_3 \quad (48)$$
$$\tau_3 = -\ell_{1,y}X_1 + \ell_{1,x}Y_1 - \ell_{2,y}X_2 + \ell_{2,x}Y_2 + \ell_{3,x}T_3 \quad (49)$$

are related to the control signals $u = (\omega_1, \omega_2, \omega_3, \delta_1, \delta_2)^T \in \mathbb{R}^5$ via the nonlinear model equations (43)- (46). The moment arms in (49) are defined by the location of the propulsion devices. Constraints $|\omega_i| \leq 18$ Hz, and $|\delta_i| \leq 0.61$ rad (35 deg) are implemented as a barrier function J_2 on the form (34) with $w_2 = 0.01$. In addition, the cost function measures relative power consumption and use of rudders

$$J(u) = \sum_{i=1}^{3} k_i |\omega_i| \omega_i^2 + \sum_{j=1}^{2} q_j \delta_j^2 + J_2(u) \quad (50)$$

with $k_1 = k_2 = 0.01$, $k_3 = 0.02$, and $q_1 = q_2 = 500$.

In the simulation example, we use (33) for optimization, with $\gamma = 1$, $\epsilon = 10^{-9}$. The simulation results are presented in Figures 1-5, with a constant disturbance d. Except for the initial transient (due to an arbitrary choice of $u(0)$ and $\lambda(0)$) and a short period around $t \approx 210$ where ω_3 saturates, we observe from Figure 3 that the attained generalized forces τ_c are very close to the commanded generalized forces τ. The control allocation problems is a non-convex optimization problem due to asymmetry in rudder lift with positive and negative surge thrust, cf. (44) and [15]. Hence, the control allocation algorithm only seeks a locally optimal solution. One benefit of the dynamic control allocation algorithm compared to a (quasi-)static approach is that one avoids chattering due to the fact that globally optimal control allocation is in fact discontinuous as a function of the requested generalized forces [15, 16].

5 Concluding remarks

Taking a control-Lyapunov design approach, an optimizing nonlinear control allocation algorithm is derived. The algorithm leads to asymptotic optimality, thus relaxing the computational complexity considerably compared to a direct nonlinear programming approach. At the same time, we guarantee global exponential convergence of the overall system comprising an uniformly globally exponentially stable trajectory-tracking nonlinear controller together with the control allocation algorithm. It is also interesting to observe that the method leads to feedforward-like terms that takes into account the fact that the optimum is time-varying.

Acknowledgements

The author is grateful to Thor I. Fossen and Roger Skjetne for insightful comments and suggestions for improvements.

References

[1] W. C. Durham, “Constrained control allocation,” *J. Guidance, Control and Dynamics*, vol. 16, pp. 717–725, 1993.

[2] D. Enns, “Control allocation approaches,” in *Proc. AIAA Guidance, Navigation and Control Conference and Exhibit, Boston MA*, 1998, pp. 98–108.

[3] O. J. Sørdalen, “Optimal thrust allocation for marine vessels,” *Control Engineering Practice*, vol. 5, pp. 1223–1231, 1997.

[4] S. P. Berge and T. I. Fossen, “Robust control allocation of overactuated ships; experiments with a model ship,” in *Preprints IFAC Conference on Maneuvering and Control of Marine Craft, Brijuni, Croatia*, 1997.

[5] W. Jackson, M. P. J. Fromherz, D. K. Biegelsen, J. Reisch, and D. Goldberg, “Constrained optimization based control of real time large-scale systems: Airjet object movement system,” in *Proc. IEEE Conf. Decision and Control, Orlando*, 2001.

[6] M. Bodson, “Evaluation of optimization methods for

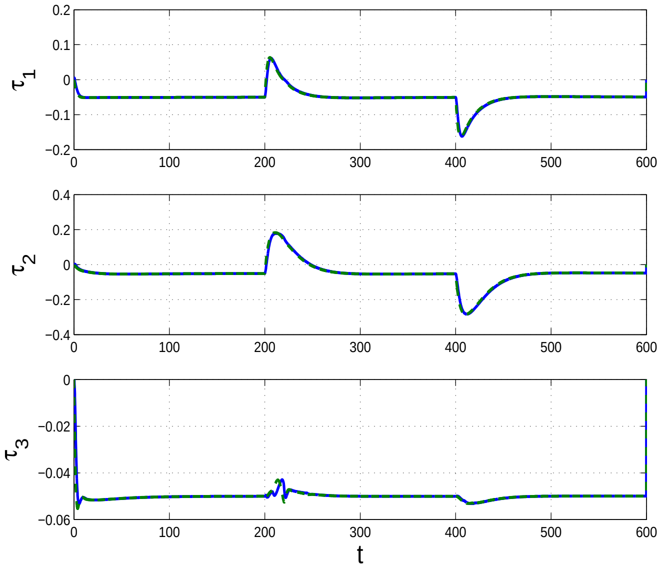

Figure 3: Simulation results - solid lines are attained generalized forces and dashed lines are generalized forces commanded by the controller.

control allocation," *J. Guidance, Control and Dynamics*, vol. 25, pp. 703–711, 2002.

[7] O. Härkegård, "Efficient active set algorithms for solving constrained least squares problems in aircraft control allocation," in *Proc. IEEE Conf. Decision and Control, Las Vegas NV*, 2002.

[8] M. Krstic, I. Kanellakopoulos, and P. Kokotovic, *Nonlinear and Adaptive Control Design*, Wiley and Sons, 1995.

[9] M. Cannon and B. Kouvaritakis, "Efficient constrained model predictive control with asymptotic optimality," *SIAM J. Optimization and Control*, vol. 41, pp. 60–82, 2002.

[10] T. A. Johansen and D. Sbárbaro, "Optimizing control of over-actuated linear systems with nonlinear output maps via control Lyapunov functions," in *European Control Conference, Cambridge, UK*, 2003.

[11] P. O. M. Scokaert, D. Q. Mayne, and J. B. Rawlings, "Suboptimal model predictive control (feasibility implies stability)," *IEEE Trans. Automatic Control*, vol. 44, pp. 648–654, 1999.

[12] T. A. Johansen, T. I. Fossen, and S. P. Berge, "Constrained nonlinear control allocation with singularity avoidance using sequential quadratic programming," *IEEE Trans. Control Systems Technology, accepted*, 2003.

[13] J. Nocedal and S. J. Wright, *Numerical Optimization*, Springer-Verlag, New York, 1999.

[14] R. Abrahamson, J. E. Marsden, and T. Ratiu, *Manifolds, Tensor Analysis and Applications (2nd Edition)*, Springer-Verlag, New York, 1988.

[15] K.-P. Lindegaard and T. I. Fossen, "Fuel efficient control allocation for surface vessels with active rudder usage: Experiments with a model ship," *IEEE Trans. Control Systems Technology*, vol. 11, pp. 850–862, 2003.

[16] T. A. Johansen, T. P. Fuglseth, P. Tøndel, and T. I. Fossen, "Optimal constrained control allocation in marine surface vessels with rudders," in *IFAC Conf. Manoeuvring and Control of Marine Craft, Girona*, 2003.

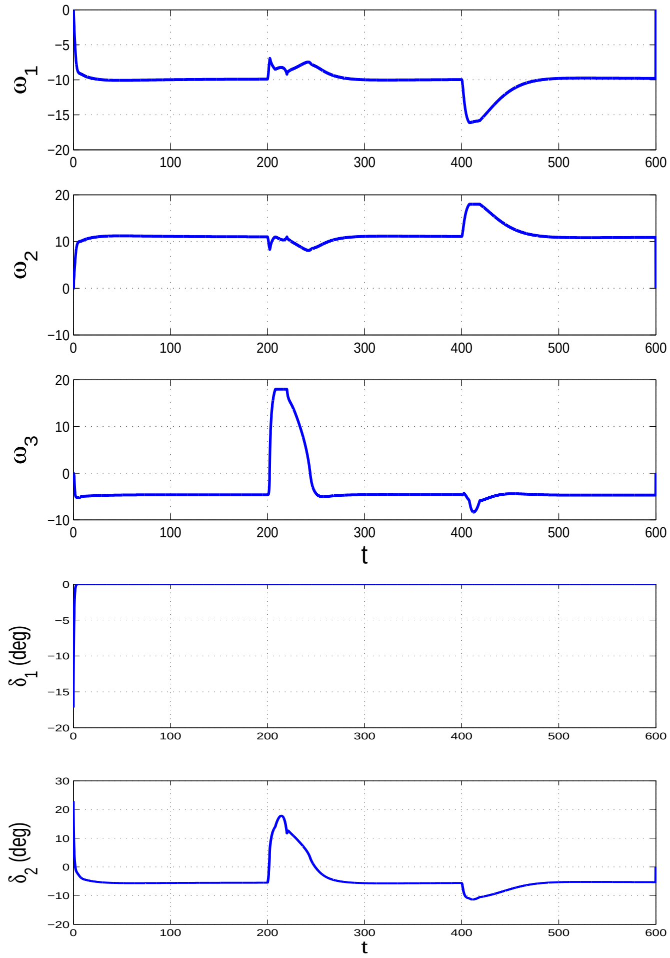

Figure 4: Simulation results - control signals computed by the control allocation algorithm.

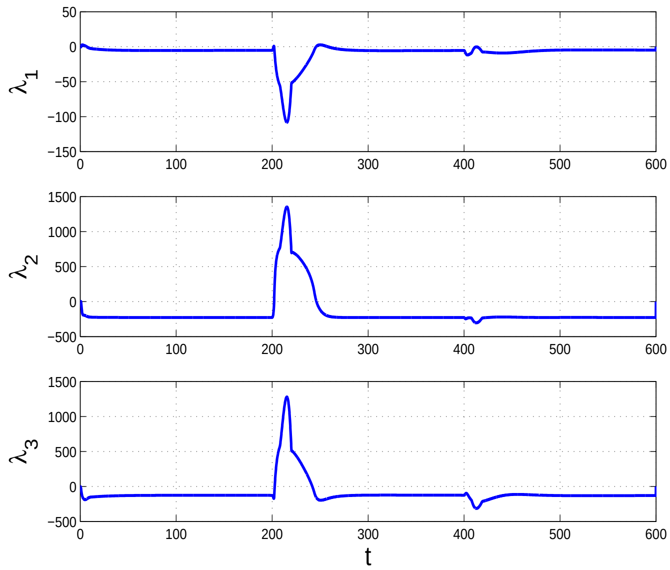

Figure 5: Simulation results - Lagrange multipliers.