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Erratum

Optimizing Weakly Triangulated Graphs

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Due to an oversight on the part of the authors, the proof given in [1] for *The WT Two-Pair Theorem* is incomplete, and should be replaced with the following proof.

Recall that a *two-pair* is a pair of non-adjacent vertices in a graph, such that every chordless path between the two vertices has exactly two edges.

The WT Two-Pair Theorem *Let G be any weakly triangulated graph. Then either G is a clique or it contains a two-pair. Moreover, if C is any minimal cutset of G, then either C is a clique or C contains a two-pair of G.*

Proof. We first make two observations.

Observation 1. Let X be a set of vertices of G and let $\{y, z\}$ be a two-pair of $G - X$ such that every vertex of X is adjacent to both y and z. Then $\{y, z\}$ is a two-pair of G.

Observation 2. Let F be a clique of a graph G, and let B^* be the union of some connected components of $G - F$. Then any two-pair $\{x, y\}$ of $G - B^*$ is a two-pair of G.

We prove the Theorem by induction on the number of vertices of G. We may assume that G is not a clique. If G is disconnected, then we obtain a two-pair by taking two vertices lying in two distinct components of G. (A graph is disconnected if and only if the only minimal cutset is the empty set; we consider the empty set as a clique-cutset.) We may thus assume that G is connected. Let C be a minimal cutset of G, and let $B_1, \ldots, B_p$ be the components of $G - C$. Define $G[C]$ as the subgraph of G induced by C. We shall distinguish between two cases.

Case 1. C is a clique of G.

If there is a component B_j of $G - C$ such that $G - B_j$ is not a clique, then by the induction hypothesis the graph $G - B_j$ has a two-pair, which is also a two-pair of G by Observation 2 (where $F = C$ and $B^* = B_j$). Else, we must have that $p = 2$ and

$C \cup B_j$ induces a clique for $j = 1$ and 2. Then $\{x_1, x_2\}$ is a two-pair for any $x_1 \in B_1$ and $x_2 \in B_2$.

Case 2. $\bar{G}[C]$ is disconnected.

Let C^* be the set of vertices of some component of $\bar{G}[C]$ with at least two vertices (since C is not a clique, there must be such a set C^*). Note that every vertex of $C - C^*$ is a neighbor of every vertex of C^*, and that C^* is a minimal cutset, and not a clique, of $G - (C - C^*)$. Thus by inductive assumption, C^* contains a two-pair of $G - (C - C^*)$; this two-pair is also two-pair of G by Observation 1 (where $X = C - C^*$).

Case 3. $\bar{G}[C]$ is connected.

From Hayward's Theorem [2] it follows that in each component B_j of $G - C$, there is some vertex that is a neighbor of every vertex of C. If each B_j consists of a single vertex, then by the induction hypothesis the subgraph $G[C]$ contains a two-pair, which is also a two-pair of G by Observation 1 (where $X = B_1 \cup \cdots \cup B_p$). We may then assume without loss of generality that B_1 has at least two vertices. Let x be any vertex of B_1 that is a neighbor of all of C. We shall distinguish among three subcases.

Subcase 3.1. $G - x$ is disconnected.

Remarking that $C \cup B_2 \cup \cdots \cup B_p$ is contained in one single component of $G - x$, we define B^* to be the union of all the other components of $G - x$ (thus $B^* \neq \phi$ and $B^* \subset B_1 - x$), and $B_0 = B_1 - B^*$. Clearly $G[B_0]$ is connected, for otherwise any component of $G[B_0]$ not containing x would be a connected component of $G - C$, contradicting the definition of B_1. Note that C is a minimal cutset of the graph $G - B^*$, whose components are $B_0, B_2, \ldots, B_p$. By the inductive hypothesis C contains a two-pair of $G - B^*$, which is also a two-pair of G by Observation 2 (where B^* is as defined above and $F = \{x\}$).

Subcase 3.2. $G - x$ is connected and C is a minimal cutset of $G - x$.

By the induction hypothesis, C contains a two-pair of $G - x$, which is a two-pair of G by Observation 1 (where $X = \{x\}$).

Subcase 3.3. $G - x$ is connected and C is not a minimal cutset of $G - x$.

Let C' be a minimal cutset, contained in C, of $G - x$. Note that C' is not empty because $G - x$ is connected, and that $C'' = C - C'$ is not empty because C is not a minimal cutset of $G - x$. If C' is not a clique then, by the induction hypothesis, C' contain a two-pair of $G - x$, which is also a two-pair of G by Observation 1 (where $X = \{x\}$). Now we may assume that C' induces a clique.

Since C is a minimal cutset of G, each vertex of C'' must have at least one neighbor in each B_j. Therefore $B_2 \cup \cdots \cup B_p \cup C''$ is included in one single component of $(G - x) - C'$. Let B^* be another component of $(G - x) - C'$. Then we have $B^* \subset B_1 - x$. Furthermore, a vertex a of B^* cannot be adjacent to any vertex b of $C'' \cup (B_1 - x - B^*)$, because a and b are in different components of $(G - x) - C'$. It follows that $G[B_1 - B^*]$ is connected, for otherwise any component of $G[B_1 - B^*]$ not containing x would form a connected component of $G - C$, contradicting the definition of B_1. Thus C is a minimal cutset of $G - B^*$, the components of $(G - x) -$

C being $B_1 - B^*, B_2, \ldots, B_p$, since each vertex of C is a neighbor of the vertex x of $B_1 - B^*$. By the induction hypothesis, C contains a two-pair of $G - B^*$, which is also a two-pair of G by Observation 2 (where $F = C' \cup \{x\}$ and B^* is as defined above). This completes the proof. □

The above proof is essentially that of [3], with all instances of "even pair" replaced with "two-pair".

References

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