APPENDICES

Appendix 1: the proof of Equation (6)

For $1 \le j \le m,\ \ 1 \le i \le n_l,$ and $1 \le l \le k,$ we have

$$E\left(\Delta Y_{ijl}\right) = \kappa'(\varpi(\mu(x_l)))\Delta t_{jl},$$

$$\frac{\partial \varpi(\mu(x_l))}{\partial a} = \frac{1}{V(\mu(x_l))}\frac{\partial \mu(x_l)}{\partial a},$$

and

$$\frac{\partial \varpi(\mu(x_l))}{\partial b} = \frac{1}{V(\mu(x_l))}\frac{\partial \mu(x_l)}{\partial b}.$$

Similarly, from Equation (3), note that

$$V\left(\mu(x_l)\right) = e^{d(a+bx_l)}, \quad \frac{\partial \mu(x_l)}{\partial a} = e^{a+bx_l}, \text{ and } \frac{\partial \mu(x_l)}{\partial b} = x_l e^{a+bx_l}.$$

By using the best asymptotic normality (BAN) property, we have $\hat{\mathbf{\Lambda}} \sim N\left(\mathbf{\Lambda}, I^{-1}(\mathbf{\Lambda})\right),$ where the elements I_{ij} in $\mathbf{I}(\mathbf{\Lambda})$ are as follows:

$$I_{11} = E\left(-\frac{\partial^2 \ln L(\Lambda)}{\partial a^2}\right) = N\lambda \sum_{l=1}^{k} e^{-(d-2)(a+bx_l)} t_l^* p_l,$$

$$I_{22} = E\left(-\frac{\partial^2 \ln L(\Lambda)}{\partial b^2}\right) = N\lambda \sum_{l=1}^{k} x_l^2 e^{-(d-2)(a+bx_l)} t_l^* p_l,$$

$$I_{12} = I_{21} = E\left(-\frac{\partial^2 \ln L(\Lambda)}{\partial a \partial b}\right) = N\lambda \sum_{l=1}^{k} x_l e^{-(d-2)(a+bx_l)} t_l^* p_l,$$

$$I_{13} = I_{31} = E\left(-\frac{\partial^2 \ln L(\Lambda)}{\partial a \partial \lambda}\right) = 0,$$

$$I_{23} = I_{32} = E\left(-\frac{\partial^2 \ln L(\Lambda)}{\partial b \partial \lambda}\right) = 0.$$

By the saddlepoint approximation (Jørgensen 1997), it can be shown that

$$c(\Delta y_{ijl} \mid \lambda, \Delta t_{jl}) \sim \sqrt{\frac{\lambda}{2\pi (\Delta t_{jl})^{1-d} (\Delta y_{ijl})^{d}}}.$$

Therefore,

$$I_{33}=E\left(-\frac{\partial^2 \ln L(\Lambda)}{\partial\lambda^2}\right)=E\left(-\sum_{l=1}^{k}\sum_{i=1}^{n_l}\sum_{j=1}^{m}\frac{\partial^2 \ln c(\Delta y_{ijl}\mid\lambda,\Delta t_{jl})}{\partial\lambda^2}\right)=\frac{Nm}{2\lambda^2}.$$

By the delta's method, we have

$$\mathrm{AVar}(\hat{\xi}_q|\mathbf{x},\mathbf{p})=\frac{1}{(f(\xi_q))^2}(\mathbf{h})^T\mathbf{I}^{-1}(\mathbf{\Lambda})\mathbf{h}\Big|_{\xi_q=\hat{\xi}_q}$$

$$=\frac{1}{(f(\hat{\xi}_q))^2 N}\left\{\frac{2\lambda^2h_2^2}{m}+\frac{h_1^2e^{a(d-2)}\sum_{l=1}^{k}e^{-b(d-2)x_l}x_l^2t_l^*p_l}{\lambda\sum_{u<v}^{k}e^{-b(d-2)(x_u+x_v)}(x_v-x_u)^2t_u^*t_v^*p_up_v}\right\}.$$

Appendix 2: the proof of Result 2

When $\{x_l\}_{l=1}^{k}$ are prefixed, the Lagrange function of the constrained-optimization problem stated in Equations (7-8) with k=3 can be expressed as follows:

$$M(p_1,p_2,p_3,\eta,\mu_1,\mu_2,\mu_3)=\frac{\sum_{l=1}^{3}p_lz_l^2x_l^2}{\sum_{u<v}^{3}z_u^2z_v^2(x_v-x_u)^2p_up_v}+\eta\left(\sum_{i=1}^{3}p_i-1\right)-\sum_{i=1}^{3}\varphi_ip_i, \quad \text{(A1)}$$

where $z_l=\sqrt{t_l^*}e^{-b(d-2)x_l/2}$, $l=1,2,3,$ and $(\eta,\ \varphi_1,\ \varphi_2,\ \varphi_3)$ are called KKT multipliers. Let $\Delta=\sum_{u<v}^{3}z_u^2z_v^2(x_v-x_u)^2p_up_v$ and $A_i=x_iz_i\sum_{l=1}^{3}x_lz_l^2p_l-z_i\sum_{l=1}^{3}x_l^2z_l^2p_l$.

Then, the KKT conditions are:

$$\frac{\partial M}{\partial p_i}=\frac{-A_i^2}{\Delta^2}+\eta-\varphi_i=0,\ i=1,2,3, \quad \text{(A2)}$$

$$\frac{\partial M}{\partial\eta}=\sum_{i=1}^{3}p_i-1=0, \quad \text{(A3)}$$

$$p_i\geq 0, i=1,2,3, \quad \text{(A4)}$$

$$\varphi_i\geq 0, i=1,2,3, \quad \text{(A5)}$$

and

$$\varphi_ip_i=0,\ i=1,2,3. \quad \text{(A6)}$$

Eight combinations satisfying Equation (A6) is also known as the complementary slackness conditions. These combinations can be further classified into four groups as shown in Table A1.

Table A1: The groups of conditions

Group	φ_1	φ_2	φ_3	p_1	p_2	p_3
Group 1	0	0	0	*	*	*
Group 2	0	0	>0	*	*	0
	0	>0	0	*	0	*
	>0	0	0	0	*	*
Group 3	0	>0	>0	*	0	0
	>0	0	>0	0	*	0
	>0	>0	0	0	0	*
Group 4	>0	>0	>0	0	0	0

(the symbol * denotes a non-zero value.)

Among these combinations, if all φ_i are non-zeros, then $p_i = 0,\ \forall i = 1,2,3,$ which contradicts to $\sum_{i=1}^{3} p_i = 1.$ Similarly, if only one of the values of φ_i is equal to 0, then $p_i = 1$ and $p_j = 0,\ \forall j \neq i.$ This situation reduces the problem to a single-level problem, which is not an ADT. Therefore, all the combinations for $\{p_i\}_{i=1}^{3}$ in Groups 3 and 4 are infeasible. Hence, only the combinations in Group 1 and Group 2 are needed to be considered.

In Group 1, note that if $\varphi_i = 0, i = 1,2,3,$ then from Equation (A2), we have

$$\eta = \frac{A_1^2}{\Delta^2} = \frac{A_2^2}{\Delta^2} = \frac{A_3^2}{\Delta^2}.$$

Hence, the solutions $\{p_l\}_{l=1}^{3}$ for the above simultaneous equations together with Equation (A3) are:

$$p_1 = \frac{z_2^2 z_3^2 x_2 x_3 (x_3 - x_2)}{z_2^2 z_3^2 x_2 x_3 (x_3 - x_2) + z_1^2 z_2^2 x_1 x_2 (x_2 - x_1) - z_1^2 z_3^2 x_1 x_3 (x_3 - x_1)},$$

$$p_2 = \frac{-z_1^2 z_3^2 x_1 x_3 (x_3 - x_1)}{z_2^2 z_3^2 x_2 x_3 (x_3 - x_2) + z_1^2 z_2^2 x_1 x_2 (x_2 - x_1) - z_1^2 z_3^2 x_1 x_3 (x_3 - x_1)},$$

and

$$p_3 = \frac{z_1^2 z_2^2 x_1 x_2 (x_2 - x_1)}{z_2^2 z_3^2 x_2 x_3 (x_3 - x_2) + z_1^2 z_2^2 x_1 x_2 (x_2 - x_1) - z_1^2 z_3^2 x_1 x_3 (x_3 - x_1)}.$$

Under the condition $x_1 < x_2 < x_3$, we have $p_1 p_2 < 0$ and $p_2 p_3 < 0$. Obviously, it will contradict to $p_i > 0,\ i = 1,2,3.$ Therefore, it is impossible to have a non-trivial optimum allocation under $k = 3.$

In Group 2, note that it implies that the test units only need to allocate to any two out of three stress levels. This is exactly the same as the optimum allocation for $k = 2.$ Therefore, the optimum allocation may belong to one of the following three possible allocations:

$$(p_1^*, p_2^*, 0),\ (p_1^*, 0, p_3^*),\ \text{ or }\ (0, p_2^*, p_3^*).$$

For the case of $(p_1^*, p_2^*, 0)$, substitute $(p_1^*, p_2^*) = ((1 + r_{12})^{-1},\ r_{12}(1 + r_{12})^{-1})$ and $(\varphi_1, \varphi_2) = (0,0)$ into the constraints stated in Equation (A5), and then solve φ_3 as follows:

$$\varphi_3 = \left(\frac{z_1 z_2 x_1 x_2}{z_1 x_1 + z_2 x_2} \right)^2 \left(-g_{12} + g_{13} + g_{23}\right) \times \left(-g_{12} - g_{13} - g_{23}\right).$$

Note that $-(g_{12} + g_{13} + g_{23}) < 0.$ Therefore, to ensure $\varphi_3 > 0,$ the condition that

$g_{13}+g_{32}<g_{12}$ is required. Similarly, $g_{21}+g_{13}>g_{23}$ and $g_{13}+g_{32}>g_{12}$ are the conditions for $(p_1^*, 0, p_3^*)$, while $g_{21}+g_{13}<g_{23}$ is the condition for $(0, p_2^*, p_3^*)$. Hence, we complete the proof of Result 2.

Appendix 3: the proof of Result 3

When $d \le 2$ and $t_1^* \le t_2^* \le t_3^*$, then $e^{-b(d-2)x_1/2} < e^{-b(d-2)x_2/2} < e^{-b(d-2)x_3/2}$. Hence,

$$\begin{aligned}
& g_{12}-g_{23}-g_{13} \\
&= \sqrt{t_1^* t_2^*}(x_2-x_1)e^{-b(d-2)(x_1+x_2)/2} - \sqrt{t_2^* t_3^*}(x_3-x_2)e^{-b(d-2)(x_2+x_3)/2} - \sqrt{t_1^* t_3^*}(x_3-x_1)e^{-b(d-2)(x_1+x_3)/2} \\
&\le \sqrt{t_1^* t_3^*}e^{-b(d-2)(x_1+x_3)/2}(x_2-x_1-x_3+x_1) - \sqrt{t_2^* t_3^*}(x_3-x_2)e^{-b(d-2)(x_2+x_3)/2} \\
&= -\sqrt{t_3^*}e^{-b(d-2)x_3/2}(x_3-x_2)(\sqrt{t_1^*}e^{-b(d-2)x_1/2} + \sqrt{t_2^*}e^{-b(d-2)x_2/2}) < 0.
\end{aligned}$$

This implies that $g_{12}-g_{23}$ is always less than g_{13}. From Result 2, it is impossible to have the setting of $(p_1^*, p_2^*, 0)$ for $d \le 2$. Similarly, when $d \ge 2$, it is impossible to have the setting of $(0, p_2^*, p_3^*)$.

Appendix 4: the proof of Result 4

For the case (i) of Result 4, since $x_H = 1$ and $p_H = 1 - p_L$, the optimum solutions of x_L^* and p_L^* can be obtained by the following equations:

$$\left.\frac{\partial G(x_L, p_L)}{\partial p_L}\right|_{(p_L, x_L)=(p_L^*, x_L^*)} = 0, \tag{A7}$$

and

$$\left.\frac{\partial G(x_L, p_L)}{\partial x_L}\right|_{(p_L, x_L)=(p_L^*, x_L^*)} = 0. \tag{A8}$$

From Equation (A7), we have $p_L^* = (1 + r_{LH})^{-1}$, where

$$r_{LH} = x_L^*(\sqrt{t_L^*/t_H^*})\exp\{b(d-2)(1-x_L^*)/2\}.$$

Setting $\alpha = b(d-2)/2$ and substituting p_L^* into Equation (A8), then we have

$$\frac{2\left[\mathrm{e}^{\alpha}\sqrt{t_L^*}+\mathrm{e}^{\alpha x_L^*}\sqrt{t_H^*}(1+\alpha(1-x_L^*))\right](\mathrm{e}^{\alpha x_L^*}\sqrt{t_H^*}+\mathrm{e}^{\alpha}\sqrt{t_L^*}x_L^*)}{t_L^* t_H^*(1-x_L^*)^3} = 0. \qquad \text{(A9)}$$

From Equation (A9), then we have either

$$e^{\alpha(1-x_L^*)} = -(1+\alpha(1-x_L^*))\sqrt{t_H^*/t_L^*}, \qquad \text{(A10)}$$

or

$$\mathrm{e}^{-\alpha(1-x_L^*)} = -\sqrt{t_L^*/t_H^*}\,x_L^*. \qquad \text{(A11)}$$

By setting $u = 1 - x_L^*$, $\gamma = -\sqrt{t_H^*/t_L^*}$, and $\beta = \alpha\gamma$ $(\neq 0)$, then Equation (A10) can be rewritten as

$$e^{\alpha u} = \gamma + \beta u. \qquad \text{(A12)}$$

By the Lambert W function, the solution for u in Equation (A12) can be expressed as follows:

$$u = -\frac{W\left(-\frac{\alpha}{\beta}e^{-\frac{\alpha\gamma}{\beta}}\right)}{\alpha} - \frac{\gamma}{\beta} = \frac{2}{b(2-d)}\left[1 + W\left(e^{-1}\sqrt{t_L^*/t_H^*}\right)\right].$$

Therefore, if $d < 2\left\{1 - \frac{1}{b}\left[1 + W\left(e^{-1}\sqrt{t_L^*/t_H^*}\right)\right]\right\}$, then

$$x_L^*(=1-u) = 1 - \frac{2}{b(2-d)}\left[1 + W\left(e^{-1}\sqrt{t_L^*/t_H^*}\right)\right] > 0. \qquad \text{(A13)}$$

Similarly, another solution for x_L^* can be obtained from Equation (A11):

$$x_L^* = \frac{2W\left(-\frac{b(2-d)}{2}\mathrm{e}^{\frac{1}{2}b(2-d)}\sqrt{t_H^*/t_L^*}\right)}{b(2-d)}. \qquad \text{(A14)}$$

Note that only x_L^* in Equations (A13) is the optimum solution due to the fact that when evaluating the Hessian matrix of $G(x_L, p_L)$ at $(x_L, p_L) = (x_L^*, p_L^*)$, the determinant of the Hessian matrix is

$$\frac{b^2(d-2)^2 \mathrm{e}^{2b(d-2)}\left[1+W\left(e^{-1}\sqrt{t_L^*/t_H^*}\right)\right]\left[b(d-2)+2W\left(e^{-1}\sqrt{t_L^*/t_H^*}\right)\right]^5}{16t_2^2\left[W\left(e^{-1}\sqrt{t_L^*/t_H^*}\right)\right]^5\left[2+b(d-2)+2W\left(e^{-1}\sqrt{t_L^*/t_H^*}\right)\right]} > 0,$$

if $d < 2\left\{1-\frac{1}{b}\left[1+W\left(e^{-1}\sqrt{t_L^*/t_H^*}\right)\right]\right\}$.

Note that if $d \geq 2\left\{1-\frac{1}{b}\left[1+W\left(e^{-1}\sqrt{t_L^*/t_H^*}\right)\right]\right\}$, we have $\left.\frac{\partial G(x_L, p_L)}{\partial x_L}\right|_{p_L=p_L^*} > 0.$

Therefore, the optimum lowest stress is $x_L^* = 0$.

The case (ii) of Result 4 can be proved similarly.