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THE OPTIMUM AXIAL FLOW TAPER IN A COUNTERCURRENT GAS CENTRIFUGE

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ABSTRACT

The effect of an axially varying countercurrent circulation rate in a gas centrifuge on the efficiency factors, e_I, the ideality efficiency, and e_C, the circulation efficiency, is investigated and compared with the case in which the countercurrent circulation rate is constant throughout the centrifuge. The optimum variation of the centrifuge parameter m, which is a measure of the countercurrent circulation rate, as a function of axial position in the centrifuge is determined. It is shown that when the countercurrent circulation rate has its optimum value at every axial position in the centrifuge, the product of the efficiency factors, $e_I \times e_C$, can exceed 81 per cent, the nominal upper limit of the value of the product of the efficiency factors for a constant countercurrent circulation rate, and can be quite close to unity. This is illustrated by numerical examples based on a centrifuge previously described in the literature.

INTRODUCTION

It has previously been shown that the separative work produced per unit time by a countercurrent gas centrifuge used for isotope separation, denoted by δU, can be expressed by an equation of the form

$$\delta U = e_I \times e_C \times e_F \times \delta U_{max} \quad ,$$

where e_I is the ideality efficiency, e_C is the circulation efficiency, e_F is the flow profile efficiency, and δU_{max} is the maxiumum theoretical separative capacity of the gas centrifuge. The basis for this formulation of the separative work equation, along with the definition of δU_{max}, can be found in the appendices to this report. It has also previously been shown that when the countercurrent flow profile and the countercurrent circulation rate are axially invariant, that is, the same at all axial positions throughout the centrifuge (the case which has been called pure axial flow by some and non-decaying axial flow by others), the circulation efficiency is given by

$$e_C = \frac{m^2}{m^2 + 1} \quad ,$$

and the ideality efficiency, e_I, has a maximum value of about 81 per cent. The definition of the centrifuge parameter m, which is a measure of the countercurrent circulation rate, can be found in Appendix I which contains a review of the Onsager-Cohen formulation of the gradient equation for the countercurrent gas centrifuge.

In this work the effect of an axially varying countercurrent circulation rate in the gas centrifuge, that is, of an axially varying m-value, on the efficiency factors e_I and e_C is investigated and the optimum relationship for m as a function of axial position in the centrifuge is determined. It is shown that when the countercurrent circulation rate has its optimum value at every axial position in the centrifuge, the product of the efficiency factors, $e_I \times e_C$, can exceed 81 per cent and can be quite close to unity. This is illustrated by numerical examples based on centrifuges whose physical characteristics are taken from the literature.

For completeness, the performance of centrifuges in which the magnitude of the countercurrent circulation is axially invariant is reviewed and it is shown how the value of m which maximizes the separative work produced by the centrifuge per unit time can be determined. The details are worked out for two cases: in the first case the quantity $x(1-x)$ which appears in the gradient equation is treated as if it were constant, in the second case, called the dilute approximation, the quantity $x(1-x)$ is replaced in the gradient equation by x alone. In the first case the analysis is simplified appreciably and some useful insight regarding centrifuge performance gained; the second case is of more practical interest in uranium isotope separation problems.

THEORY DEVELOPMENT

When the product withdrawal rate, P, and the waste withdrawal rate, W, of a countercurrent gas centrifuge are both small with respect to the magnitude of the countercurrent circulation rate, L, the separative performance of the gas centrifuge is satisfactorily described by the standard Onsager-Cohen formulation of the gradient equation. The gradient equation for the enriching section of a gas centrifuge (that is, for that part of the centrifuge which lies between the feed point and the product withdrawal point) can therefore be written in the form

$$\frac{m^2+1}{2m} S_0 \frac{dx}{dz} = \psi x(1-x) - \frac{P(y_P - x)}{mL_0} . \qquad (1)$$

The derivation of this equation is presented in Appendix I along with an explanation of the notation used. Furthermore, when the product withdrawal rate is small with respect to the magnitude of the countercurrent circulation rate, the separative work produced per unit time by a cylindrical volume element of the centrifuge of length dz and cross sectional area equal to that of the centrifuge, located in the enriching section of the centrifuge, is given by

$$d(\delta U) = P(y_P - x)\frac{dx}{dz} v''(x)dz , \qquad (2)$$

where $v''(x)$ denotes the second derivative of the value function of the concentration x.

Combining the two preceding equations by substituting the expression for dx/dz obtained from equation (1) into equation (2), one obtains the following expression for the separative work produced per unit time per unit length of centrifuge in the enriching section

$$\frac{d(\delta U)}{dz} = 2P(y_P - x)\frac{m\psi x(1-x) - \frac{P(y_P - x)}{L_0}}{(m^2+1)S_0} v''(x) . \qquad (3)$$

The value of m which maximizes the separative work produced per unit time per unit length of centrifuge may now be found by setting the derivative of $d(\delta U)/dz$ with respect to m equal to zero. This procedure leads to the following expression for the optimum value of m in the enriching section as a function of the local concentration x and hence as an implicit function of axial position in the centrifuge

$$M^2 - \frac{2P(y_P - x)}{L_0\psi x(1-x)} M - 1 = 0 , \qquad (4)$$

where M denotes the local optimum value of the centrifuge parameter m. For convenience, let $\phi_E(x)$, a function of the local concentration in the enriching section of the centrifuge and hence a function of axial position, be defined by

$$\phi_E(x) \equiv \frac{P(y_P - x)}{L_0 \psi x(1-x)} \quad . \tag{5}$$

It should be evident that $\phi_E(x)$ is equal to zero at the top of the enriching section where x is equal to y_P and, assuming that the centrifuge parameters ψ and L_0 are constant, increases monotonically as the feed point, that is, as the bottom of the enriching section is approached. Equation (4) can now be rewritten in the form

$$M^2 - 2\phi_E M - 1 = 0 \ , \quad \text{or} \quad \phi_E = \frac{M^2 - 1}{2M} \ , \tag{6}$$

from which, since by definition M must be greater than zero, it follows that

$$M = \phi_E + \sqrt{\phi_E^2 + 1} \quad . \tag{7}$$

With the definition of the function ϕ_E given by equation (5), the gradient equation, equation (1), can be rewritten in the form

$$\frac{dx}{dz} = \frac{2\psi x(1-x)}{S_0} \frac{m - \phi_E}{m^2 + 1} \tag{8}$$

which, when m has its optimum value at every axial position in the enriching section, reduces to

$$\frac{dx}{dz} = \frac{\psi x(1-x)}{S_0 M} \quad . \tag{9}$$

Thus, from the combination of equations (7) and (9) and the definition of $\phi_E(x)$, one can obtain expressions for x and ϕ_E, and therefore for M, as a function of the axial position in the enriching section of the centrifuge. Similar equations valid for the stripping section of the centrifuge (that is, for that part of the centrifuge which lies between the feed point and the waste withdrawal point) can be obtained by replacing the quantities P and Py_P where they occur in the preceding equations by $-W$ and $-Wx_W$, respectively. Thus the gradient equation for the stripping section of a gas centrifuge, the counterpart of equation (1), is written

$$\frac{m^2 + 1}{2m} S_0 \frac{dx}{dz} = \psi x(1-x) - \frac{W(x - x_W)}{mL_0} \quad . \tag{10}$$

If one defines $\phi_S(x)$, a function of the local concentration in the stripping section of the centrifuge and hence a function of axial position, by

$$\phi_S(x) \equiv \frac{W(x - x_W)}{L_0 \psi x(1-x)} \ , \tag{11}$$

it follows that equations (6) through (9), with ϕ_E replaced by ϕ_S, are also valid for the stripping section of the centrifuge.

When m has its optimum value at every axial position in the centrifuge, the equation for the separative work produced per unit time per unit length of centrifuge, equation (3), reduces to

$$\frac{d(\delta U)}{dz} = \frac{M^2 - 1}{M^2} \frac{L_0 \psi^2}{2S_0} \cdot \{x^2 (1-x)^2 v''(x)\} \ , \tag{12}$$

or, more simply, to

$$\frac{d(\delta U)}{dz} = \frac{M^2 - 1}{M^2} \frac{L_0 \psi^2}{2S_0} \ , \tag{13}$$

since the quantity in brackets in equation (12) is by virtue of the definition of the value function equal to unity. Equation (13) can be compared with

$$\frac{d(\delta U)}{dz} = \frac{4\phi(m - \phi)}{m^2 + 1} \frac{L_0 \psi^2}{2S_0} \ , \tag{14}$$

where ϕ is equal to ϕ_E in the enriching section and ϕ is equal to ϕ_S in the stripping section, which gives the separative work produced per unit time per unit length of centrifuge for the case in which m has any arbitrary value whatever. Since the quantity, $L_0\psi^2/(2S_0)$, is the maximum separative work, adjusted for the countercurrent flow profile efficiency, that can be produced per unit time per unit length of centrifuge (which is shown to be true in Appendix II), the coefficiencts of this quantity in equations (13) and (14) can be regarded as the local values of the product of the ideality efficiency and the circulation efficiency, $e_I \times e_C$. From the integration of these equations over the length of the centrifuge one can obtain the overall effective or average values of the efficiency factors for the centrifuge.

ILLUSTRATIVE APPLICATIONS

A. THE APPROXIMATION: $x(1-x) = q$

We consider here the case of a centrifuge in which the axial end-to-end enrichment is sufficiently small that the quantity, $x(1-x)$, can be treated as a constant, and we denote this constant value of $x(1-x)$ by q. This approximation simplifies the analysis considerably and provides some insight regarding the function M, its variation with axial position, and its effect upon the efficiency factors of the centrifuge. In the next section, Section B, we consider the more realistic dilute approximation.

Under the assumption that the quantity, $x(1-x)$, is equal to a constant, q, the definition of the function $\phi_E(x)$, given by equation (5), can be rewritten in the form

$$\phi_E(x) = \frac{P}{L_0 \psi q} (y_P - x) = \frac{p}{\psi q} (y_P - x) \tag{A.1}$$

where $p \equiv \dfrac{P}{L_0}$.

Differentiating $\phi_E(x)$ with respect to z, the axial coordinate, assuming that the centrifuge parameters L_0 and ψ are independent of axial position, one obtains

$$\frac{d\phi_E}{dz} = -\frac{p}{\psi q}\frac{dx}{dz} \quad . \tag{A.2}$$

The gradient equation for the case in which *m* has its optimum value at every axial position, equation (9), can now also be rewritten in the form

$$\frac{dx}{dz} = \frac{\psi q}{S_0 M} \quad . \tag{A.3}$$

The combination of equations (A.2) and (A.3), accomplished by substituting the above expression for dx/dz into equation (A.2), yields

$$\frac{d\phi_E}{dz} = -\frac{p}{S_0 M} \quad . \tag{A.4}$$

Changing the dependent variable in the above equation from ϕ_E to M by means of equation (6) and changing the independent variable from z to s, where s, which is defined in the enriching section by $(Z-z)/S_0$ and in the stripping section by z/S_0, is the axial distance from the end of the centrifuge to the point under consideration measured in multiples of the minimum stage length S_0, one obtains

$$\frac{dM}{ds} = 2p\,\frac{M}{M^2+1} \quad . \tag{A.5}$$

Integration of equation (A.5), assuming that the centrifuge parameters ψ, L_0, and S_0, are constant throughout the centrifuge, yields

$$M^2 + 2\,\ell n\,M = 4ps + C \ , \tag{A.6}$$

and evaluating the constant of integration, C, from the boundary condition at the end of the centrifuge, M is equal to unity when s is equal to zero (which follows from equation (7), since $\phi_E(x)$ is equal to zero at the end of the centrifuge), one obtains the following expression for the optimum value of *m* as a function of axial position

$$M^2 + 2\,\ell n\,M = 4ps + 1 \quad . \tag{A.7}$$

As derived and written, with p equal to P/L_0 and s the distance from the top of the enriching section to the point under consideration measured in minimum stage lengths, equation (A.7) applies to the enriching section of the centrifuge; however, with minor modification it can also be applied to the stripping section of the centrifuge. It is necessary only to replace p by w, where w is defined as the waste withdrawal rate W divided by L_0, and to interpret s as the distance from the bottom of the stripping section to the point under consideration measured in minimum stage lengths. The optimum value of *m* as a function of axial position in the stripping section is therefore given by

$$M^2 + 2\,\ell n\, M = 4ws + 1 \ , \qquad \text{(A.8)}$$

where w is equal to W/L_0.

The optimum value of m is shown as a function of axial position in either the enriching section or the stripping section in Figure A.1; it increases monotonically from its minimum value of unity at either end of the centrifuge as the distance from the end of the centrifuge increases until the feed point of the centrifuge is reached. It follows from equation (A.7) for the enriching section and its counterpart, equation (A.8), for the stripping section, that for M to be continuous at the feed point of the centrifuge, pS_E must equal wS_S, where S_E is the length of the enriching section and S_S is the length of the stripping section, both expressed as multiples of the minimum stage length, S_0. It can also be shown from the foregoing equations that when M is continuous at the feed point of the centrifuge, the feed concentration, x_F, matches the concentration in the centrifuge at the feed point and no mixing losses are incurred.

The rate of separative work production per unit length of centrifuge when m has its optimum value at every axial position, given by equation (13), can be written with s as the independent variable in the following form for the enriching section

$$\frac{d(\delta U)}{ds} = - \frac{M^2 - 1}{M^2} \frac{L_0 \psi^2}{2} \ . \qquad \text{(A.9)}$$

Changing the independent variable from s to M by means of equation (A.5), one obtains

$$\frac{d(\delta U)}{dM} = - \frac{(M^2 - 1)(M^2 + 1)}{M^3} \frac{L_0 \psi^2}{4p} \ . \qquad \text{(A.10)}$$

Integration from the bottom of the enriching section, that is, from the feed point, where the optimum value of m will be denoted by M_F, to the top of the enriching section where the optimum value of m is equal to unity, under the assumption that the centrifuge parameters, ψ, L_0, and S_0, are axially invariant (that is, assuming that the countercurrent flow profile is independent of axial position), yields

$$\delta U(\text{enricher}) = \frac{(M_F^2 - 1)^2}{2M_F^2} \frac{L_0 \psi^2}{4p} \ . \qquad \text{(A.11)}$$

Dividing this result by the maximum rate of separative work production of the enriching section adjusted for the countercurrent flow profile efficiency, that is, dividing by the quantity $L_0\psi^2 S_E/2$, one obtains the following expression for the product of the ideality efficiency and the circulation for the enriching section of the centrifuge

$$e_I \times e_C = \frac{1}{4pS_E} \frac{(M_F^2 - 1)^2}{M_F^2} \ . \qquad \text{(A.12)}$$

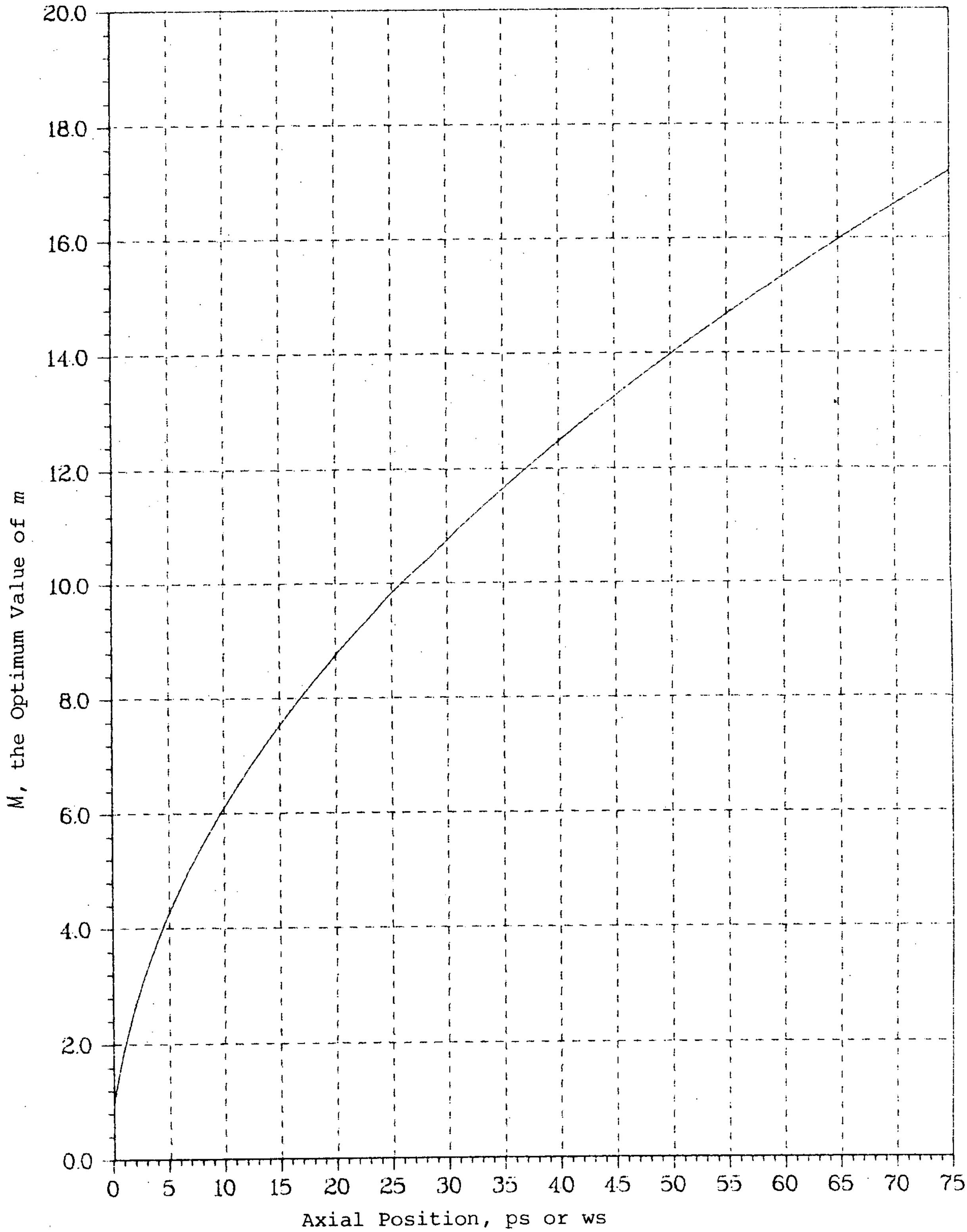

FIGURE A.1. THE VARIATION OF THE OPTIMUM VALUE OF *m* WITH AXIAL POSITION

Assumption: $x(1-x) = q$

Combining this result and equation (A.7), the above expression for the product of the efficiency factors can be written as a function solely of the value of M at the centrifuge feed point as follows

$$e_I \times e_C = \frac{(M_F^2 - 1)^2}{M_F^2(M_F^2 + 2\,\ell n\, M_F - 1)} \quad . \tag{A.13}$$

When M is continuous at the feed point of the centrifuge, this equation is also applicable to the stripping section of the centrifuge. The value of the product of the ideality efficiency and the circulation efficiency, $e_I \times e_C$, for both the enriching section and the stripping section is shown as a function of the length of the enriching section, pS_E, or of the stripping section, wS_S, in Figure A.2 for the case in which m has its optimum value everywhere in the centrifuge. It can be seen that for values of pS_E or wS_S greater than about 5.5 the value of the product of the efficiency factors, $e_I \times e_C$, exceeds 0.815, the upper limit of the value of this product for the case in which m is constant throughout the centrifuge, and for large values of pS_E or wS_S the value of the product of the efficiency factors approaches unity.

In general, small departures from the optimum circulation rate should not affect the performance of the centrifuge appreciably. Let us test the validity of this statement by considering the case of a centrifuge in which the axial variation of m is given by

$$m(s) = 2\sqrt{ps} \tag{A.14}$$

in the enriching section and by

$$m(s) = 2\sqrt{ws} \tag{A.15}$$

in the stripping section. Equations (A.14) and (A.15) are reasonably good approximations of equations (A.7) and (A.8), respectively, over the whole range of the arguments, ps and ws, and are somewhat simpler in form. A comparison of the value of m as a function of axial position given by equation (A.14) or (A.15) with the value of M given by equation (A.7) or (A.8) is presented in Figure A.3.

The value of the product of the efficiency factors, $e_I \times e_C$, for a centrifuge in which the variation of m with axial position is given by equations (A.14) and (A.15) can be evaluated as follows. The gradient equation for the enriching section for any arbitrary value of m, given by equation (8), can now be written in the form

$$\frac{dx}{dz} = \frac{2\psi q}{S_0} \frac{m - \phi_E}{m^2 + 1} \quad . \tag{A.16}$$

Replacing dx/dz in the above equation by its equivalent given by equation (A.2), and changing the independent variable from z to s, one obtains

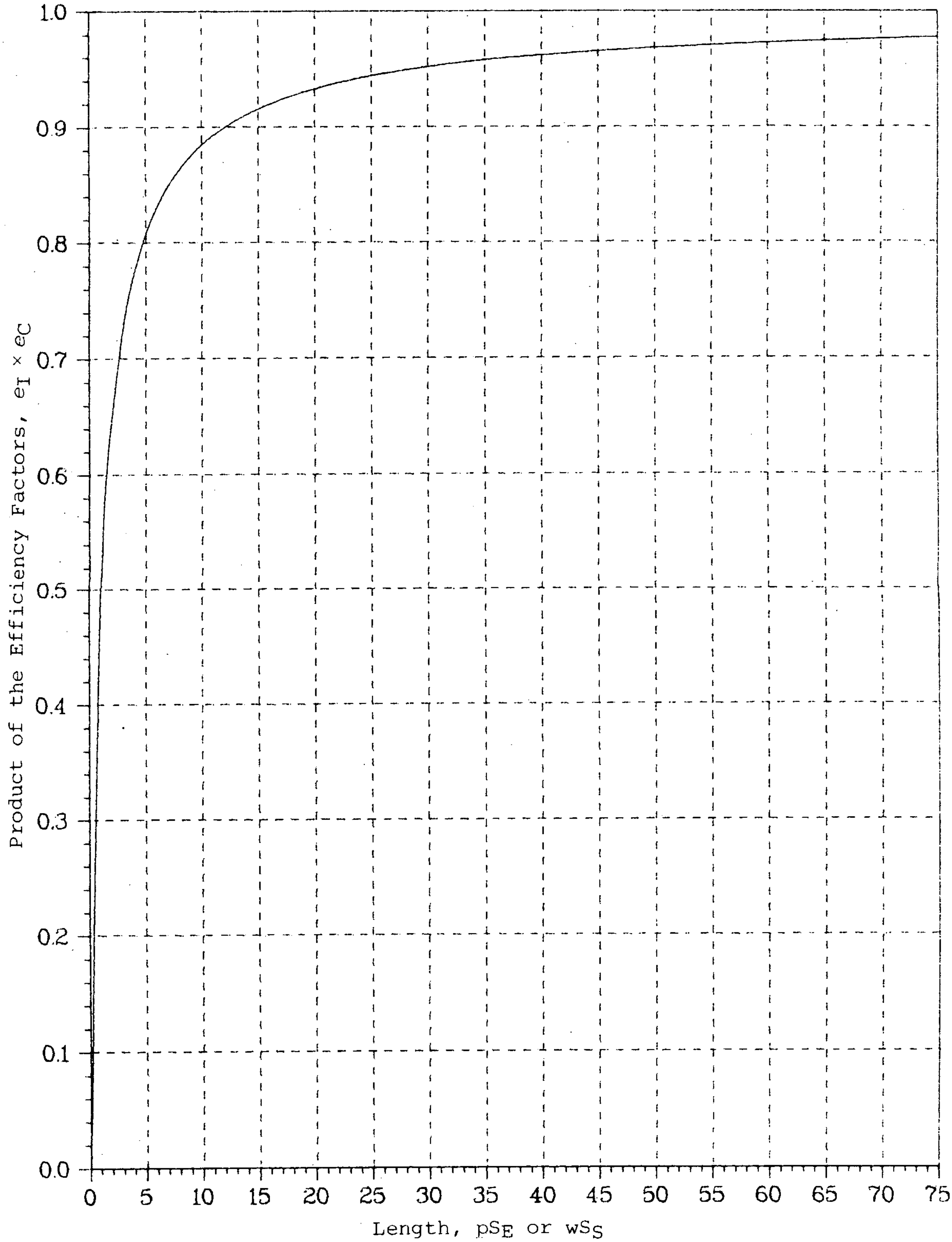

FIGURE A.2. THE VARIATION OF THE PRODUCT OF THE EFFICIENCY FACTORS, $e_I \times e_C$, WITH ENRICHER OR STRIPPER LENGTH WHEN m HAS ITS OPTIMUM VALUE EVERYWHERE

Assumption: $x(1-x) = q$

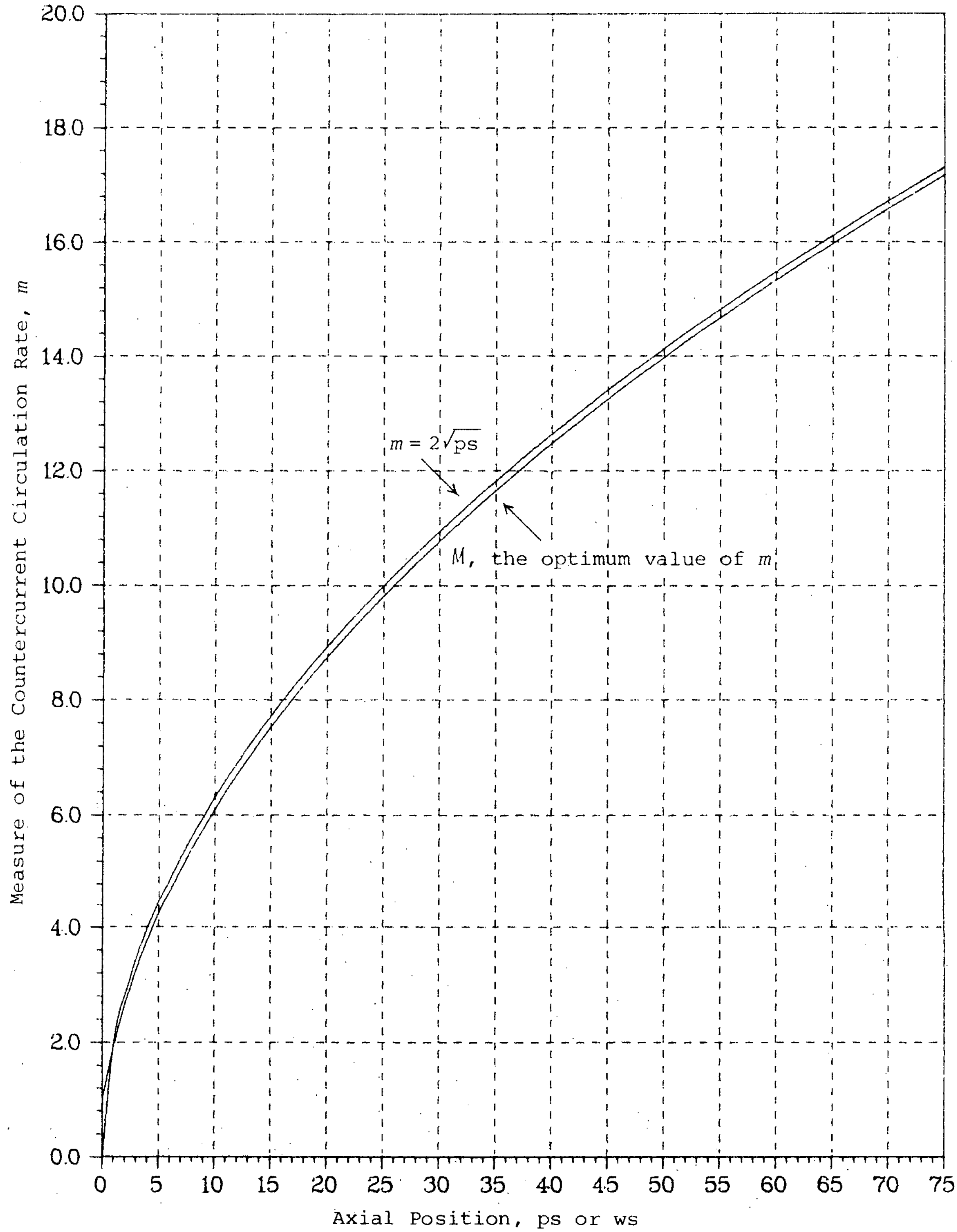

FIGURE A.3. COMPARISON OF THE VALUE OF m GIVEN BY EQUATION (A.14) OR (A.15) WITH THE OPTIMUM VALUE OF m GIVEN BY EQUATION (A.7) OR (A.8)

Assumption: $x(1-x) = q$

$$\frac{d\phi_E}{dz} = - \frac{2p}{S_0} \frac{m - \phi_E}{m^2 + 1} \quad , \tag{A.17}$$

and

$$\frac{d\phi_E}{ds} = 2p \frac{m - \phi_E}{m^2 + 1} \quad . \tag{A.18}$$

The independent variable can now be changed from s to m by means of equation (A.14) with the result

$$\frac{d\phi_E}{dm} + \frac{m}{m^2 + 1} \phi_E = \frac{m^2}{m^2 + 1} \quad . \tag{A.19}$$

The integrating factor for this first order linear differential equation is $\sqrt{m^2 + 1}$; its solution can therefore be written

$$(\phi_E)_F = \frac{m_F}{2} - \frac{1}{2\sqrt{m_F^2 + 1}} \ln \{m_F + \sqrt{m_F^2 + 1}\} , \tag{A.20}$$

where the subscript F denotes the value of the variable at the bottom of the enriching section, that is, at the feed point.

From the combination of equation (A.17) and equation (14), which gives the separative work produced per unit time per unit length by a centrifuge in which m has any arbitrary value, one obtains

$$\frac{d(\delta U)}{d\phi_E} = - \frac{L_0 \psi^2}{p} \phi_E \tag{A.21}$$

and integration over the length of the enriching section, assuming that the centrifuge parameters are constant, yields

$$\delta U(\text{enricher}) = \frac{(\phi_E)_F^2}{pS_E} \{\frac{L_0 \psi^2 S_E}{2}\} \quad . \tag{A.22}$$

Since the term in brackets in the above equation is just the maximum separative work adjusted for the countercurrent flow profile efficiency produced per unit time by the enriching section, it follows that the product of the efficiency factors for the enriching section is given by

$$e_I \times e_C = \frac{(\phi_E)_F^2}{pS_E} = \frac{1}{pS_E} \{\frac{m_F}{2} - \frac{1}{2\sqrt{m_F^2 + 1}} \ln (m_F + \sqrt{m_F^2 + 1})\}^2 \tag{A.23}$$

which can be written in the equivalent form

$$e_I \times e_C = \{1 - \frac{1}{m_F\sqrt{m_F^2 + 1}} \ln (m_F + \sqrt{m_F^2 + 1})\} \quad . \tag{A.24}$$

When m is continuous at the feed point of the centrifuge, equation (A.24) is valid also for the stripping section of the centrifuge. The product of the efficiency factors, $e_I \times e_C$, calculated from the above relationship is shown as a function of the length of the enriching or stripping sections, that is, as a function of pS_E or wS_S, in Figure A.4, where it is compared with the maximum value of the product of these efficiency factors obtained for the case in which m has its optimum value at every axial position in the centrifuge. It can be seen that, over the range of values of pS_E or wS_S of interest, the value of the product of the efficiency factors obtained when m(s) is prescribed by equation (A.14) or (A.15) is lower than the maximum value of the product of the efficiency factors obtained when m(s) is prescribed by equation (A.7) or (A.8) by only a few tenths of one per cent. Thus it can be concluded that a very good approximation of the optimum flow taper in a countercurrent gas centrifuge is one in which the magnitude of the countercurrent circulation rate is directly proportional to the square root of the distance from the point under consideration to the end of the centrifuge and in which the constant of proportionality is $2\sqrt{PL_0/S_0}$ for the enriching section and $2\sqrt{WL_0/S_0}$ for the stripping section.

For completeness and for purposes of comparison, it is appropriate to review here the calculation of the values of the efficiency factors of the centrifuge for the case of a centrifuge in which the parameter m is constant. When m is constant, equation (A.18) can be integrated directly over the length of the enriching section, from the top of the enriching section where s and ϕ_E are both equal to zero to the bottom of the enriching section where s is equal to S_E and ϕ_E is equal to $(\phi_E)_F$, with the result

$$(\phi_E)_F = m\left(1 - e^{-\frac{2pS_E}{m^2+1}}\right) . \qquad \text{(A.25)}$$

Substituting this expression for $(\phi_E)_F$ into equation (A.22), one finds that the separative work produced per unit time by an enriching section operating with a constant (that is, axially invariant) m is given by

$$\delta U(\text{enricher}) = \frac{m^2}{pS_E}\left(1 - e^{-\frac{2pS_E}{m^2+1}}\right)^2 \left\{\frac{L_0\psi^2 S_E}{2}\right\} . \qquad \text{(A.26)}$$

Clearly, the rate of separative work production by the enriching section will be at its maximum value for specified values of the quantities p and S_E when $(\phi_E)_F$ has its maximum value. It follows from equation (A.25) that $(\phi_E)_F$ will be at its maximum value when m satisfies the equation (obtained by setting the derivative of $(\phi_E)_F$ with respect to m equal to zero)

$$e^Q - 2Q + \frac{Q^2}{pS_E} = 1 , \qquad \text{(A.27)}$$

where $Q = \dfrac{2pS_E}{m^2+1}$.

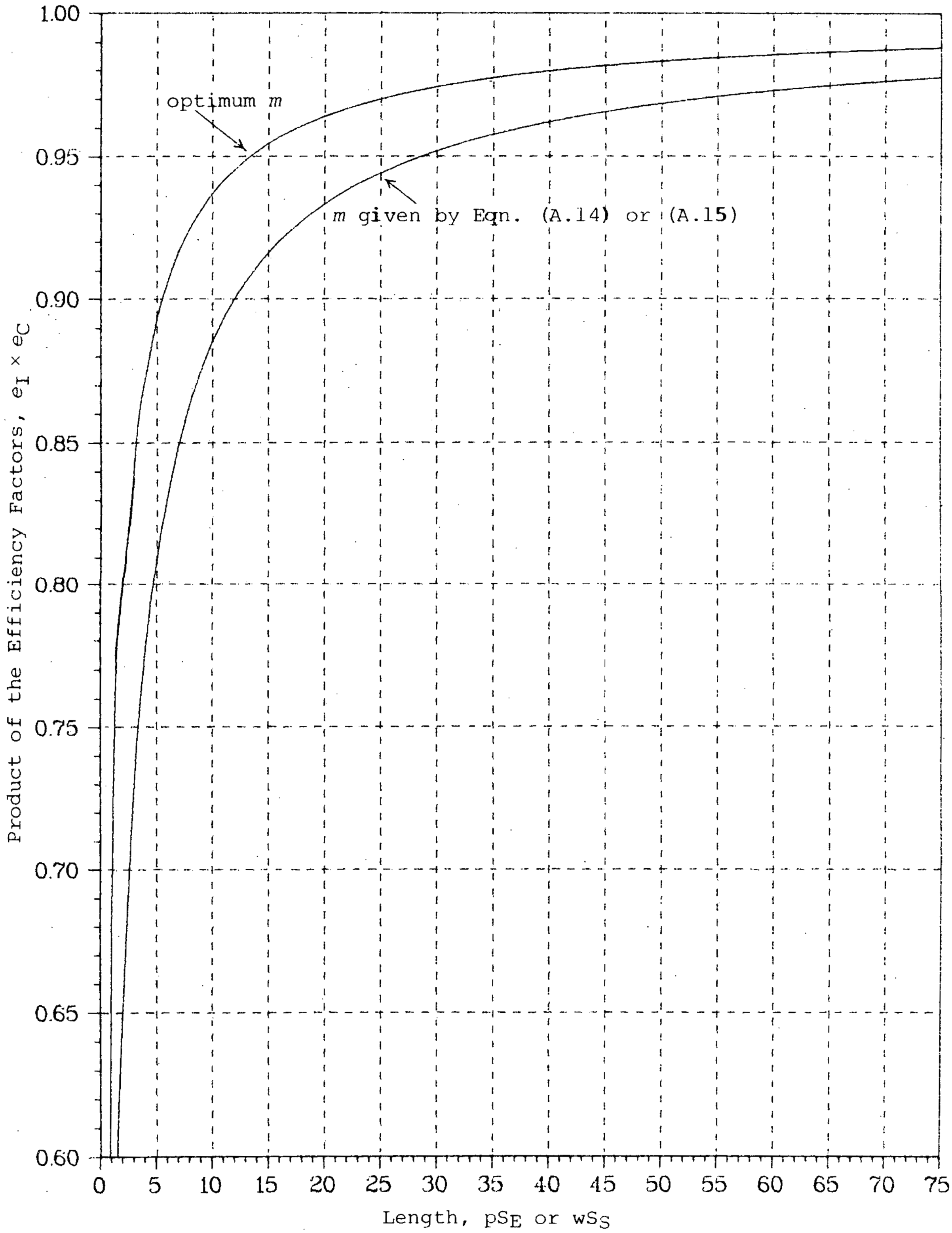

FIGURE A.4. COMPARISON OF THE PRODUCT OF THE CENTRIFUGE EFFICIENCY FACTORS, $e_I \times e_C$, FOR THE CASE WHEN m IS GIVEN BY EQUATIONS (A.14) AND (A.15) WITH THE CASE IN WHICH m HAS ITS OPTIMUM VALUE

Assumption: $x(1-x) = q$

Similar equations with p replaced by w and S_E replaced by S_S apply for the stripping section of the centrifuge. The optimum value of m for the case in which m is constant, calculated by means of the preceding equation, is shown as a function of the length of the enriching or stripping section, that is, as a function of the quantity pS_E or wS_S in Figure A.5. It may be noted that, as the value of pS_E becomes large, the value of Q in equation (A.27) approaches a limiting value of 1.2564; therefore, for large values of pS_E, the optimum value of the parameter m is given approximately by $1.262\sqrt{pS_E}$.

The value of the product of the efficiency factors, $e_I \times e_C$, for this case is obtained by dividing the expression for the separative work produced per unit time, given for the enriching section by equation (A.26), by the maximum rate of separative work production adjusted for the countercurrent flow profile efficiency and, for the enriching section, is given by

$$e_I \times e_C = \frac{m^2}{pS_E}\left(1 - e^{-\frac{2pS_E}{m^2+1}}\right)^2 . \qquad \text{(A.28)}$$

An equation valid for the stripping section is obtained by replacing the quantity pS_E by wS_S. The value of this product of the efficiency factors is shown as a function of the length of the enriching or stripping section, that is, as a function of the quantity pS_E or wS_S, in Figure A.6 for the case in which the parameter m is constant and has its optimum value. It can be seen from a comparison of this curve with the curve of Figure A.2 which gives the product of the efficiency factors for the case in which m has its optimum value at every axial position in the centrifuge that, in general, an appreciable improvement in the efficiency of a centrifuge can be obtained, with respect to the case in which m is constant, by properly adjusting the local values of the magnitude of the countercurrent circulation rate.

For the case in which m is constant it is a simple matter to resolve the product of the centrifuge efficiency factors, $e_I \times e_C$, into its components: the circulation efficiency, e_C, is given by the standard expression

$$e_C = \frac{m^2}{m^2 + 1}$$

and the value of e_I can be obtained by dividing the product of the efficiency factors by e_C. These components of the efficiency for the case in which m is constant are also shown in Figure A.6. It can be observed that the ideality efficiency, e_I, is already quite close to its maximum value of 0.8145 when pS_E or wS_S is equal to 10 and increases only very slightly as pS_E or wS_S is increased.

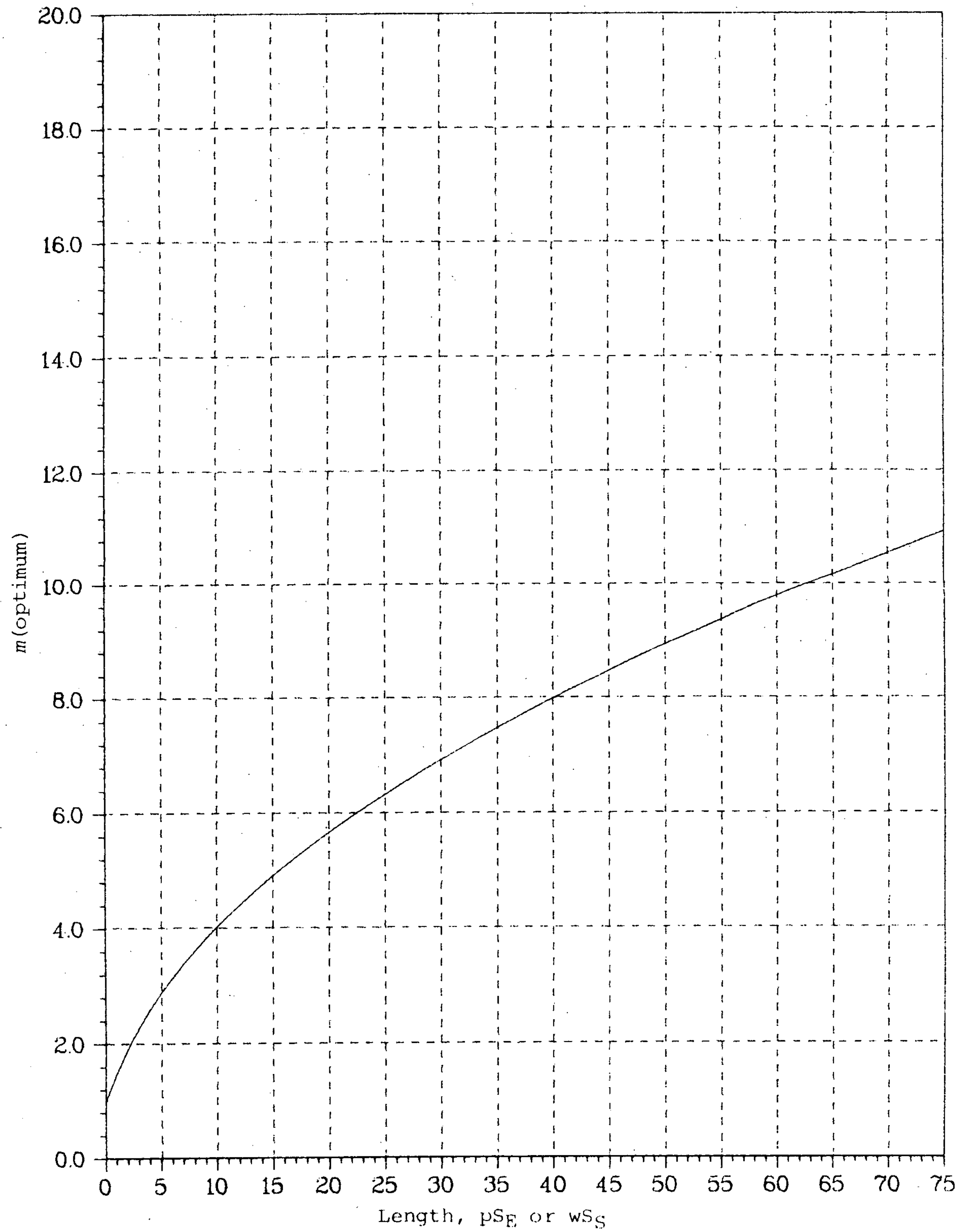

FIGURE A.5. THE VARIATION OF THE OPTIMUM VALUE OF *m*, FOR THE CASE IN WHICH *m* IS CONSTANT, WITH THE LENGTH OF THE ENRICHING OR STRIPPING SECTION

Assumption: $x(1-x) = q$

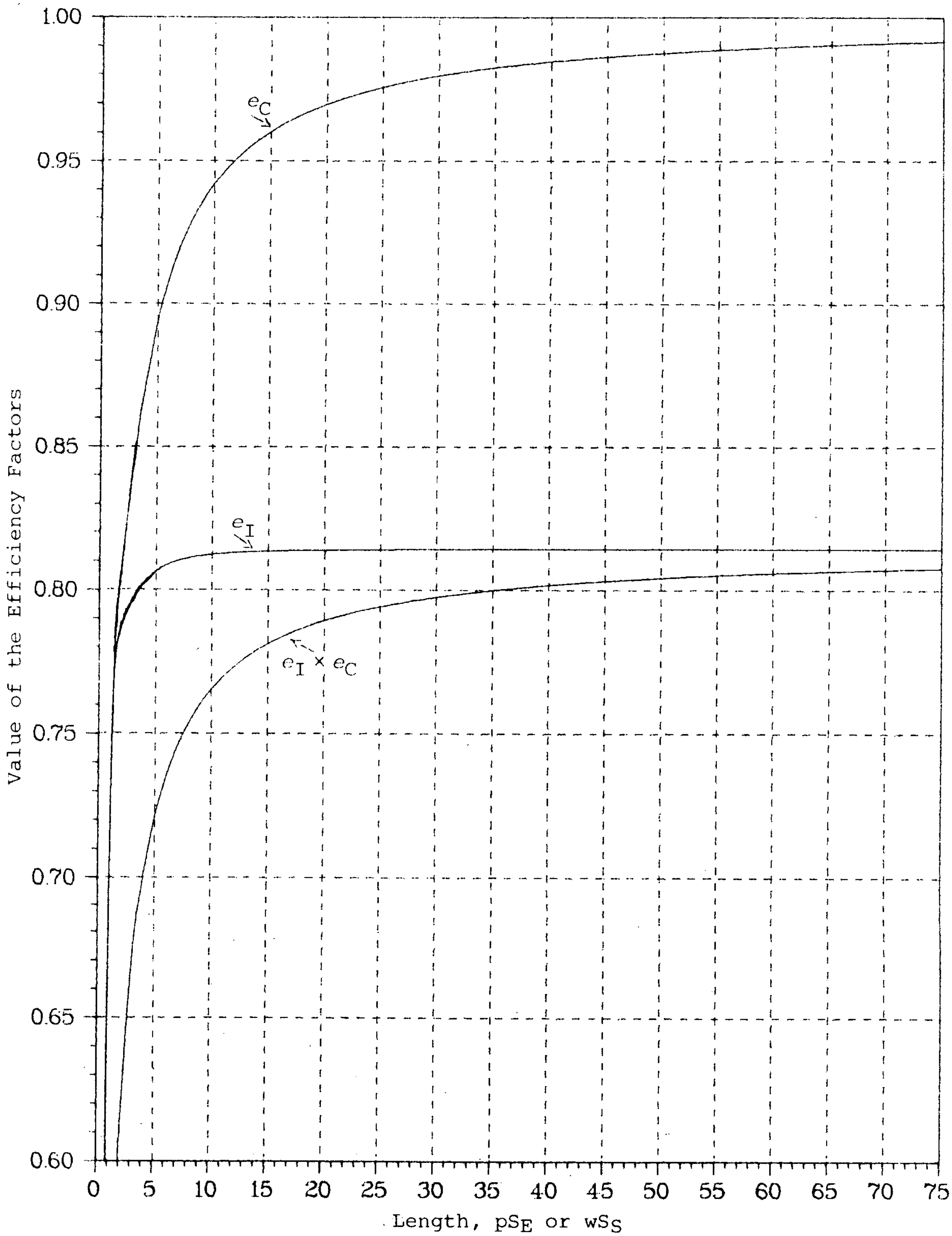

FIGURE A.6. THE VARIATION OF THE CENTRIFUGE EFFICIENCY FACTORS FOR THE CASE IN WHICH m IS CONSTANT AND HAS ITS OPTIMUM VALUE WITH ENRICHER OR STRIPPER LENGTH

Assumption: $x(1-x) = q$

Numerical Examples

The centrifuge dimensions and operating characteristics used by May[1] in his paper, *Separation Parameters of Gas Centrifuges*, are used here to illustrate the application of the preceding equations. The values of the centrifuge parameters, ψ, L_0, and S_0, presented in Table A.1 are calculated from the model of the internal flow in a gas centrifuge developed by Parker[2] and Lotz[3], and summarized in a recent paper by the author. They agree quite closely with the values of the centrifuge parameters presented by May calculated with a somewhat different model for the internal flow.

TABLE A.1

CHARACTERISTICS OF THE CENTRIFUGE MODEL

Centrifuge Length (cm)	335.3		
Centrifuge Radius (cm)	9.144		
Operating Temperature (K)	300.0		
ρD, UF_6 (g/cm/sec)	2.257×10^{-4}		
Peripheral Speed (m/s)	400.0	500.0	700.0
$A^2 = MV^2/(2RT)$	11.29	17.64	34.57
$\delta U(\max) = \frac{\pi Z \rho D}{2}\left(\frac{\Delta M V^2}{2RT}\right)^2$ (SWU/yr)	23.47	57.30	220.11
ψ	0.02091	0.02093	0.02093
L_0 (g UF_6/s)	0.03107	0.04007	0.05752
S_0 (cm)	3.818	2.960	2.062
$L_0\psi^2 Z/(2S_0)$ (SWU/yr)	12.72	21.19	43.67
e_F, Flow Profile Efficiency	0.542	0.370	0.198

Taking the feed rate to the centrifuge to be the 1000 kg UF_6/yr used by May, and assuming that the centrifuge cut, P/F, is equal to 0.5 so that the centrifuge product and waste withdrawal rates are equal, and assuming that the feed is introduced at the mid-point of the centrifuge so that one-half of the centrifuge serves as the enriching section and one-half as the stripping section, one can evaluate the performance of the centrifuge based on the preceding equations with the results presented in Table A.2.

TABLE A.2

CALCULATED PERFORMANCE OF THE CENTRIFUGE MODEL

Assumption: $x(1-x) = q$

Product Rate ($g\,UF_6/s$)	0.01585		
Waste Rate ($g\,UF_6/s$)	0.01585		
Enricher Length (cm)	167.65		
Stripper Length (cm)	167.65		
Peripheral Speed (m/s)	400.0	500.0	700.0
$p = w$	0.5103	0.3957	0.2756
$S_E = S_S$	43.92	56.64	81.30
$pS_E = wS_S$	22.41	22.41	22.41
Constant *m*			
m(optimum)	6.001	6.001	6.001
L(optimum) ($g\,UF_6/s$)	0.1865	0.2405	0.3452
$e_I \times e_C$	0.7921	0.7921	0.7921
δU (SWU/yr)	10.08	16.79	34.59
Variable *m*			
M_F	9.284	9.284	9.284
L_F ($g\,UF_6/s$)	0.2884	0.3720	0.5340
$e_I \times e_C$	0.9393	0.9393	0.9393
δU (SWU/yr)	11.95	19.91	41.02

It is evident from the results of these calculations that both the optimum value of the centrifuge parameter *m* and the associated maximum value of the product of the centrifuge efficiency factors, $e_I \times e_C$, are independent of the peripheral speed of the centrifuge. It can also be seen that the performance of the centrifuge, that is, the rate of production of separative work, is greater by about 18.5 per cent for this centrifuge when operated with the optimum countercurrent circulation rate at every axial position than when operated with the best constant countercurrent circulation rate.

B. THE DILUTE APPROXIMATION: $x(1-x) = x$

Consider now the case of a centrifuge processing a gas mixture in which the concentration of the desired component is sufficiently small that the quantity $(1-x)$ is essentially equal to unity. Since the mole fraction of the U-235 isotope in naturally occurring uranium fed to an isotope separation cascade is only 0.0072, the dilute approximation is a case of real practical interest. The results of the analysis for this case, however, cannot be expressed in quite so simple a form as was possible for the approximation used in Part A since in this case, as will be seen, they depend on two groups of centrifuge parameters instead of on only the single quantity, pS_E for the enriching section, or wS_S for the stripping section.

Under the assumption that the quantity, $x(1-x)$, can be replaced, wherever it occurs in the equations, by x without introducing any measureable error, the definition of the function $\phi_E(x)$, given by equation (5), can be rewritten in the form

$$\phi_E(x) = \frac{P}{L_0\psi}\left(\frac{y_P}{x} - 1\right) = \frac{p}{\psi}\left(\frac{y_P}{x} - 1\right) , \qquad (B.1)$$

where $p \equiv \frac{P}{L_0}$.

Differentiating $\phi_E(x)$ with respect to z, the axial coordinate, assuming that the centrifuge parameters, L_0 and ψ, are independent of axial position, one obtains

$$\frac{d\phi_E}{dz} = - \frac{p}{\psi}\,\frac{y_P}{x^2}\,\frac{dx}{dz} . \qquad (B.2)$$

The gradient equation for the case in which m has its optimum value at every axial position in the centrifuge, equation (9), can now also be rewritten in the following form

$$\frac{dx}{dz} = \frac{\psi x}{S_0 M} . \qquad (B.3)$$

The combination of equations (B.2) and (B.3), accomplished by substituting the above expression for dx/dz into equation (B.2), yields

$$\frac{d\phi_E}{dz} = - \frac{p}{S_0 M}\,\frac{y_P}{x} . \qquad (B.4)$$

Expressing the quantity y_P/x in terms of ϕ_E by means of the relationship of equation (B.1) and changing the independent variable from z to s, where s, which is defined as in the preceding section by $(Z-z)/S_0$ in the enriching section and by z/S_0 in the stripping section, is the distance along the z-axis from the end of the centrifuge to the point under consideration measured in multiples of the minimum stage length S_0, assuming that the centrifuge parameter S_0 is also independent of axial position, the preceding equation can be written

$$\frac{d\phi_E}{ds} = \frac{p}{M}\left(1 + \frac{\psi\phi_E}{p}\right) = \frac{p}{M}(1 + r\phi_E) \ , \tag{B.5}$$

where $r \equiv \dfrac{\psi}{p}$.

Changing the dependent variable in the above equation from ϕ_E to M by means of equation (6), one obtains the simple first order differential equation

$$\frac{dM}{ds} = p\frac{rM^2 + 2M - r}{M^2 + 1} \tag{B.6}$$

which can be solved by quadrature with the following result

$$\frac{M-1}{r} - \frac{1}{r^2}\ln\frac{rM^2 + 2M - r}{2} + \frac{\sqrt{1 + r^2}}{r^2}\ln\left\{\frac{1 + rM - \sqrt{1 + r^2}}{1 + r - \sqrt{1 + r^2}}\,\frac{1 + r + \sqrt{1 + r^2}}{1 + rM + \sqrt{1 + r^2}}\right\} = ps \tag{B.7}$$

where use has been made of the boundary condition at the top of the enriching section that M is equal to unity when s is equal to zero. As derived and written, with p equal to P/L_0 and s the distance from the top of the enriching section to the point under consideration, equation (B.7) applies to the enriching section of a centrifuge. The optimum value of m, calculated as a function of axial position in the enriching section, that is, as a function of the quantity ps, for various values of the parameter r, by means of equation (B.7), is shown in Figure B.1. It may be noted that when r is equal to zero, equation (B.7) reduces to equation (A.7); thus the curve in Figure B.1 for r equal to zero is identical to the curve presented in Figure A.1.

In the dilute approximation the equations for the stripping section differ somewhat in form from those for the enriching section. The definition of the function $\phi_S(x)$ for the stripping section is given by

$$\phi_S(x) = \frac{w}{L_0\psi}\left(1 - \frac{x_W}{x}\right) = \frac{w}{\psi} \ , \tag{B.8}$$

where $w \equiv \dfrac{W}{L_0}$.

Differentiating $\phi_S(x)$ with respect to z, the axial coordinate, one obtains

$$\frac{d\phi_S}{dz} = \frac{w}{\psi}\,\frac{x_W}{x^2}\,\frac{dx}{dz} \ , \tag{B.9}$$

which, when combined with the gradient equation, equation (B.3), yields

$$\frac{d\phi_S}{dz} = \frac{w}{S_0 M}\,\frac{x_W}{x} \ . \tag{B.10}$$

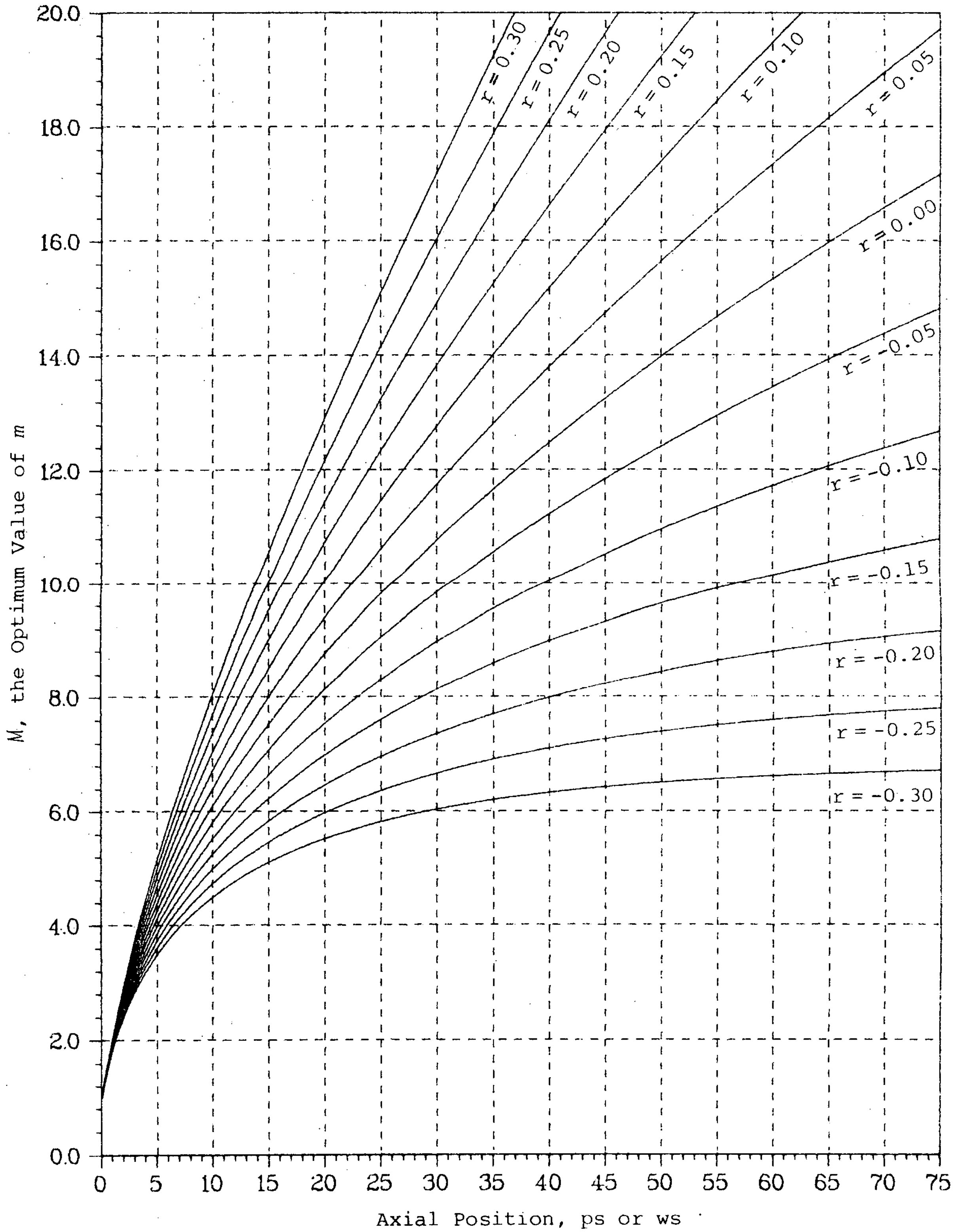

FIGURE B.1. THE VARIATION OF THE OPTIMUM VALUE OF *m* WITH AXIAL POSITION:
r = ψ/p for the enriching section
r = -ψ/p for the stripping section

Assumption: dilute approximation

Expressing the quantity x_W/x in terms of ϕ_S by means of the relationship of equation (B.8) and changing the independent variable from z to s, the preceding equation can be written

$$\frac{d\phi_S}{ds} = \frac{w}{M}\left(1 - \frac{\psi\phi_S}{w}\right) = \frac{w}{M}(1 + r\phi_S) \; , \tag{B.11}$$

where $r \equiv \frac{\psi}{w}$.

Thus, by defining r in the stripping section as the negative of the ratio of ψ to w, one can obtain equations for the stripping section which are similar in appearance to those for the enriching section. Changing the dependent variable in the above equation from ϕ_S to M by means of equation (6), one obtains

$$\frac{dM}{ds} = w \, \frac{rM^2 + 2M - r}{M^2 + 1} \; , \tag{B.12}$$

which can be solved as before, making use of the boundary condition at the bottom of the stripping section that M is equal to unity when s is equal to zero, with the result that M in the stripping section is given as a function of axial position by an equation of the same form as equation (B.7), but with the quantity ps on the right hand side of the equation replaced by ws. The optimum value of m as a function of axial position in the stripping section, that is, as a function of the quantity ws, for various values of the parameter r is also shown in Figure B.1. Positive values of the parameter r correspond to enriching sections, negative values of the parameter r correspond to stripping sections. It follows from the results of the calculations presented in Figure B.1 that in order for M to be continuous at the feed point of the centrifuge, the quantity wS_S must be greater than pS_E.

The rate at which separative work is produced per unit length of centrifuge when m has its optimum value at every axial position, given by equation (13), can be written with s as the independent variable in the following form for the enriching section

$$\frac{d(\delta U)}{ds} = - \frac{M^2 - 1}{M^2} \, \frac{L_0 \psi^2}{2} \; . \tag{B.13}$$

Changing the independent variable from s to M by means of equation (B.6), one obtains

$$\frac{d(\delta U)}{dM} = - \frac{(M^2 - 1)(M^2 + 1)}{M^2 (rM^2 + 2M - r)} \, \frac{L_0 \psi^2}{2p} \; . \tag{B.14}$$

Integration from the bottom of the enriching section, that is, from the feed point, where the optimum value of m is denoted by M_F, to the top of the enriching section, where M is equal to unity, yields the following expression for the separative work produced per unit time by the enriching section

$$\delta U(\text{enricher}) = \frac{1}{r}\left\{M_F - \frac{1}{M_F} - \frac{2}{r}\ln\frac{rM_F^2 + 2M_F - r}{2M_F}\right\}\frac{L_0\psi^2}{2p} \quad . \tag{B.15}$$

Dividing this result by the maximum rate of separative work production of the enriching section adjusted for the countercurrent flow profile efficiency, that is, dividing by the quantity $L_0\psi^2 S_E/2$, one obtains the following expression for the product of the ideality and circulation efficiency factors for the enriching section of the centrifuge

$$e_I \times e_C = \frac{1}{pS_E}\frac{1}{r}\left\{M_F - \frac{1}{M_F} - \frac{2}{r}\ln\frac{rM_F^2 + 2M_F - r}{2M_F}\right\} \quad . \tag{B.16}$$

The product of the ideality and circulation efficiency factors for the stripping section of the centrifuge is given by an equation of the same form as equation (B.16), but with the quantity pS_E replaced by wS_S and in which the permissible values of r are equal to or less than zero. The values of the product of the ideality efficiency and the circulation efficiency, $e_I \times e_C$, for both the enriching section and the stripping section are shown as a function of the length of the enriching section, pS_E, or of the stripping section, wS_S, for various values of the parameter r in Figure B.2. It may be noted that when r is equal to zero, equation (B.16) reduces to equation (A.12); thus the curve in Figure B.2 for r equal to zero is identical to the curve presented in Figure A.2. One would expect, based on the results of the calculations presented in Figure A.2, that the centrifuge efficiency calculated assuming the dilute approximation would not differ appreciably from the efficiency calculated assuming that the quantity x(1-x) is constant.

For purposes of comparison we will now consider the calculation of the values of the efficiency factors of the centrifuge for the case of a centrifuge in which the parameter m is constant. The gradient equation for the enriching section for any arbitrary value of m, given by equation (8), can be written in the form

$$\frac{dx}{dz} = \frac{2\psi x}{S_0}\frac{m - \phi_E}{m^2 + 1} \tag{B.17}$$

when the dilute approximation is applicable. Replacing dx/dz in the above equation by its equivalent given by equation (B.2), one obtains

$$\frac{d\phi_E}{dz} = -\frac{2p}{S_0}\frac{y_P}{x}\frac{m - \phi_E}{m^2 + 1} \quad . \tag{B.18}$$

Expressing the quantity y_P/x in terms of ϕ_E by means of the relationship of equation (B.1), and changing the independent variable from z to s, the preceding equation can be written

$$\frac{d\phi_E}{ds} = 2(p + \psi\phi_E)\frac{m - \phi_E}{m^2 + 1} \quad , \tag{B.19}$$

which, when m is constant, can be integrated directly over the length of

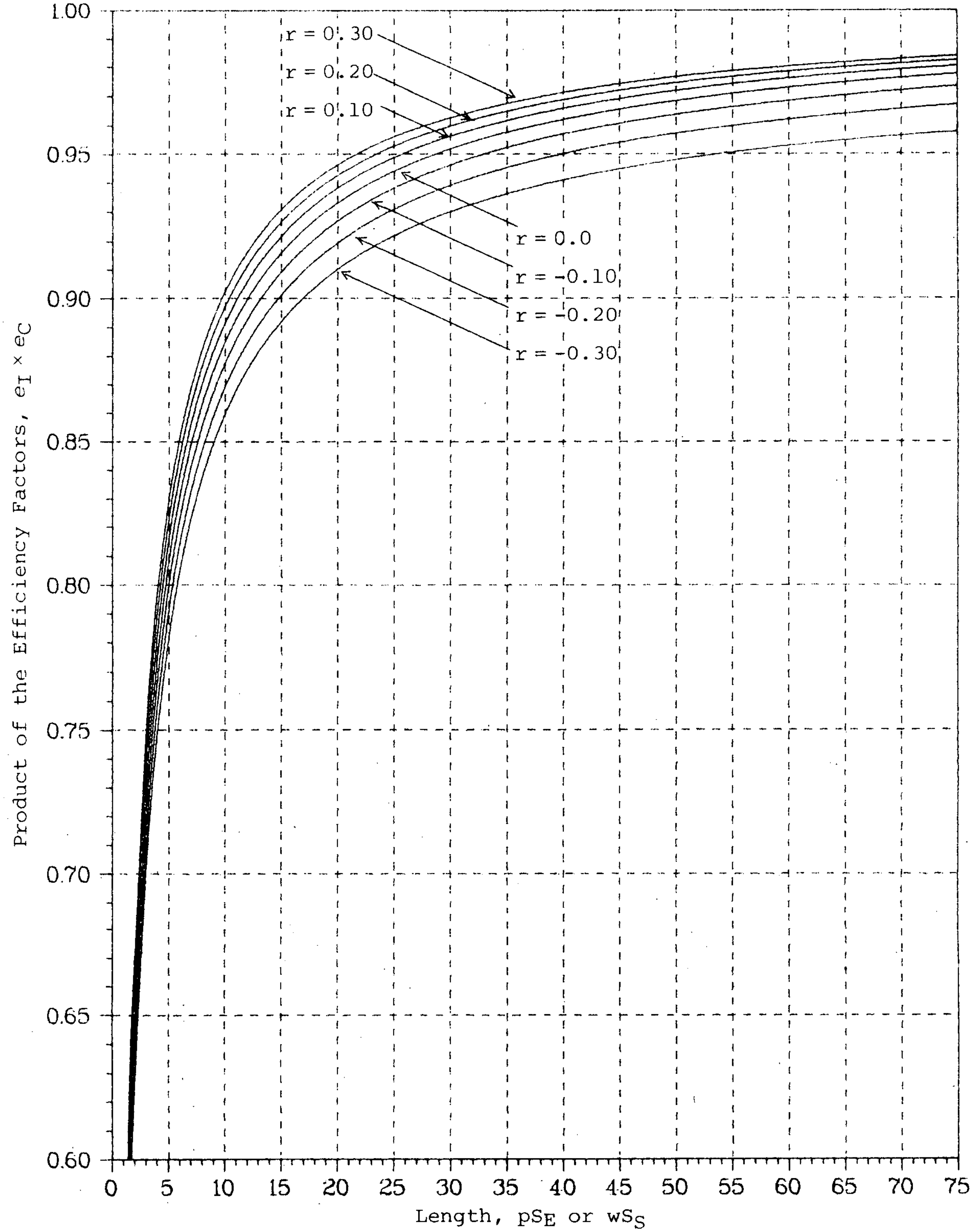

FIGURE B.2. THE VARIATION OF THE PRODUCT OF THE EFFICIENCY FACTORS, $e_I \times e_C$, WITH ENRICHER OR STRIPPER LENGTH WHEN m HAS ITS OPTIMUM VALUE EVERYWHERE: $r = \psi/p$ for the enriching section
$r = -\psi/w$ for the stripping section

Assumption: dilute approximation

the enriching section, from the top of the enriching section where s and ϕ_E are both equal to zero to the bottom of the enriching section where s is equal to S_E and ϕ_E is equal to $(\phi_E)_F$, with the result

$$(\phi_E)_F = m \frac{1 - e^{-(1+rm)Q}}{1 + rme^{-(1+rm)Q}} \quad , \qquad (B.20)$$

where $Q = \frac{2pS_E}{m^2+1}$ and r is equal to ψ/p.

From the combination of equation (B.18) and equation (14), which gives the separative work produced per unit time per unit length by a centrifuge in which m has any arbitrary value, one obtains

$$\frac{d(\delta U)}{d\phi_E} = -\frac{2}{p}\frac{L_0\psi^2}{2}\frac{\phi_E}{1 + r\phi_E} \quad , \qquad (B.21)$$

and integration over the length of the enriching section yields

$$\delta U(\text{enricher}) = \frac{2}{pS_E}\left\{\frac{(\phi_E)_F}{r} - \frac{1}{r^2}\ell n\,[1 + r(\phi_E)_F]\right\}\left\{\frac{L_0\psi^2 S_E}{2}\right\} \quad . \qquad (B.22)$$

Since the final term in brackets in the above equation is just the maximum separative work adjusted for the countercurrent flow profile efficiency produced per unit time by the enriching section, the product of the ideality and circulation efficiency factors for the enriching section for the case in which m is constant is given by

$$e_I \times e_C = \frac{2}{pS_E}\left\{\frac{(\phi_E)_F}{r} - \frac{1}{r^2}\ell n\,[1 + r(\phi_E)_F]\right\} \quad . \qquad (B.23)$$

It follows that the rate of production of separative work by the enriching section, and also the efficiency of the enriching section, will be at a maximum for a given value of the parameter r and of the quantity pS_E when the quantity $(\phi_E)_F$ has its maximum value. And it follows from equation (B.20) that $(\phi_E)_F$ will be at its maximum value when m satisfies the equation (obtained by setting the derivative of $(\phi_E)_F$ with respect to m equal to zero)

$$\frac{e^{(1+rm)Q} - 1 + m(1+rm)Q\left\{r - \frac{m(1+rm)Q}{pS_E}\right\}}{\left\{e^{(1+rm)Q} + rm\right\}^2} = 0 \quad . \qquad (B.24)$$

Similar equations with p replaced by w, S_E replaced by S_S, and in which r is equal to the negative of the ratio of ψ to w are valid for the stripping section of the centrifuge. The optimum values of m for the case in which m is constant, calculated by means of the preceding equation, are shown as a function of the length of the enriching or stripping section, that is, as a function of pS_E or wS_S, for various values of the parameter r in Figure B.3. The values of the product of the ideality efficiency and the circulation

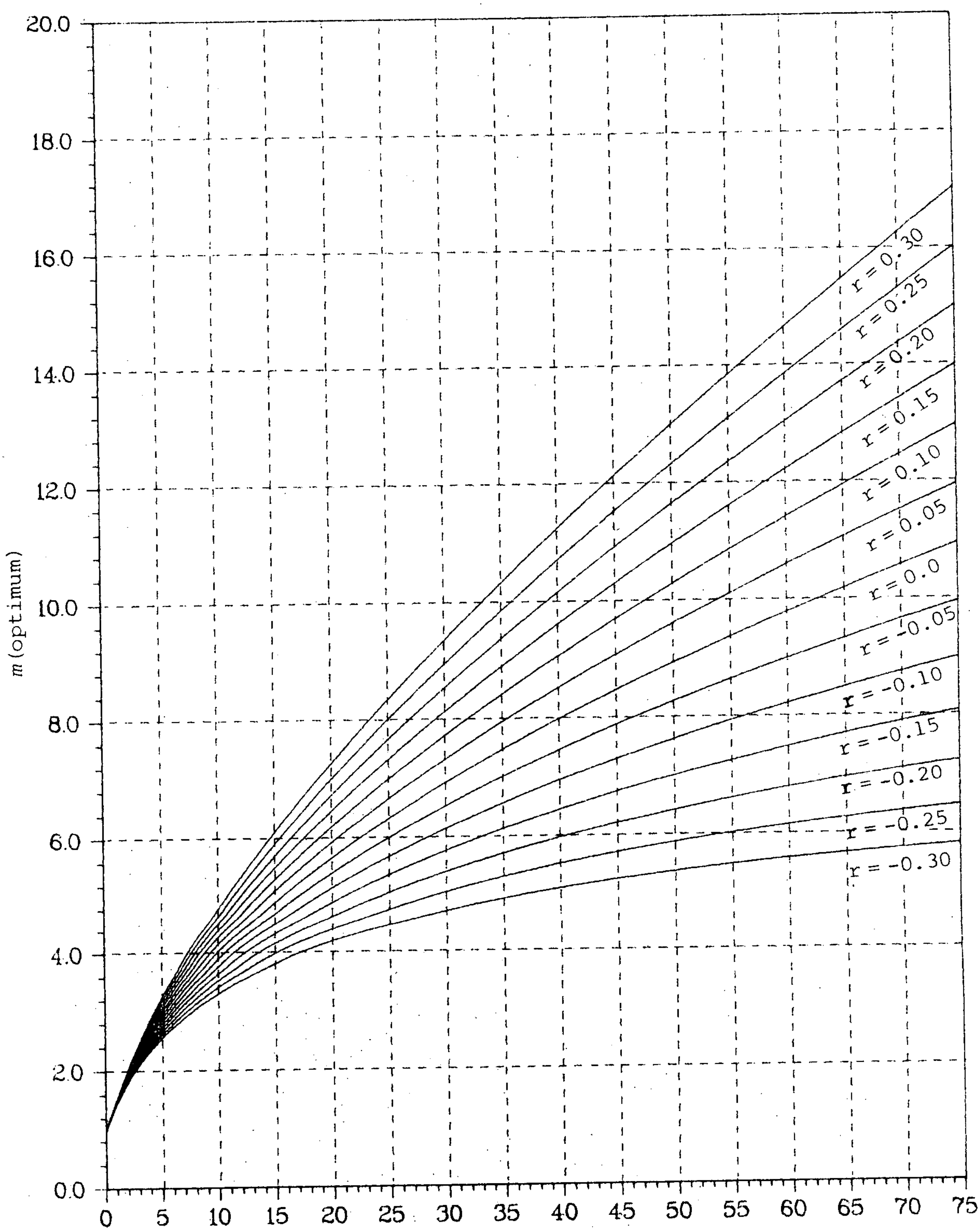

FIGURE B.3. THE VARIATION OF THE OPTIMUM VALUE OF *m*, FOR THE CASE IN WHICH *m* IS CONSTANT, WITH THE LENGTH OF THE ENRICHER OR STRIPPER: $r = \psi/p$ for the enriching section
$r = -\psi/w$ for the stripping section

Assumption: dilute approximation

efficiency, $e_I \times e_C$, for both the enriching section and the stripping section, for the case in which m is constant and has its optimum value, are shown as a function of the length of the enriching or stripping section, for various values of the parameter r, in Figure B.4. As in the preceding case, positive values of the parameter r correspond to enriching sections, negative values of the parameter r correspond to stripping sections. Again, when r is equal to zero, the curves are identical to those in the corresponding case of Part A.

Numerical Examples

The same examples which were considered in Part A of this paper will now be re-examined under the assumption that the dilute approximation is valid. The centrifuge dimensions, operating characteristics, and the values of the centrifuge parameters, ψ, L_0, and S_0, are those given in Table A.1.

Under the simplifying assumption applied in Part A, optimum centrifuge performance coincided with symmetrical centrifuge operation, that is, optimum centrifuge performance occurred when the enrichment provided by the enriching section and the stripping section were equal; when the dilute approximation applies, this situation does not occur. Since symmetrical centrifuge operation is desirable for centrifuges which are to be incorporated into separation cascades, this requirement, which can be expressed by the constraint that the ratio y_P/x_F equal the ratio x_F/x_W, is imposed in the examples. It is also required that there be no mixing of unequal concentrations at the feed point of the centrifuge; satisfying this condition requires that $(\phi_E)_F$ be equal to $(\phi_S)_F$. It is also required that the centrifuge parameter m be continuous at the feed point of the centrifuge. For the case of a centrifuge operating with a constant m-value, these constraints make it impossible for both the enriching section and the stripping section to operate with their optimum values of the parameter m given in Figure B.3: the value of the parameter m which is optimum for the cascade can be regarded as the result of a compromise between the optimum value for the enriching section and for the stripping section. The results of the calculations of centrifuge performance under the above mentioned constraints are summarized in Table B.1.

The most obvious fact arising from the comparison of the results presented in Table A.2 and Table B.1 is that, although the ratios p/w and S_E/S_S are appreciably different depending upon whether the assumption $x(1-x) = q$ or the dilute approximation is used, the values of the parameter m, the centrifuge efficiencies, and the separative work output of the centrifuge are essentially unchanged. Again it can be seen that, for the centrifuge model under consideration, the performance of the centrifuge is about 18.5 per cent higher when operated with the optimum countercurrent circulation rate at every axial position in the centrifuge than when operated with the best constant countercurrent circulation rate.

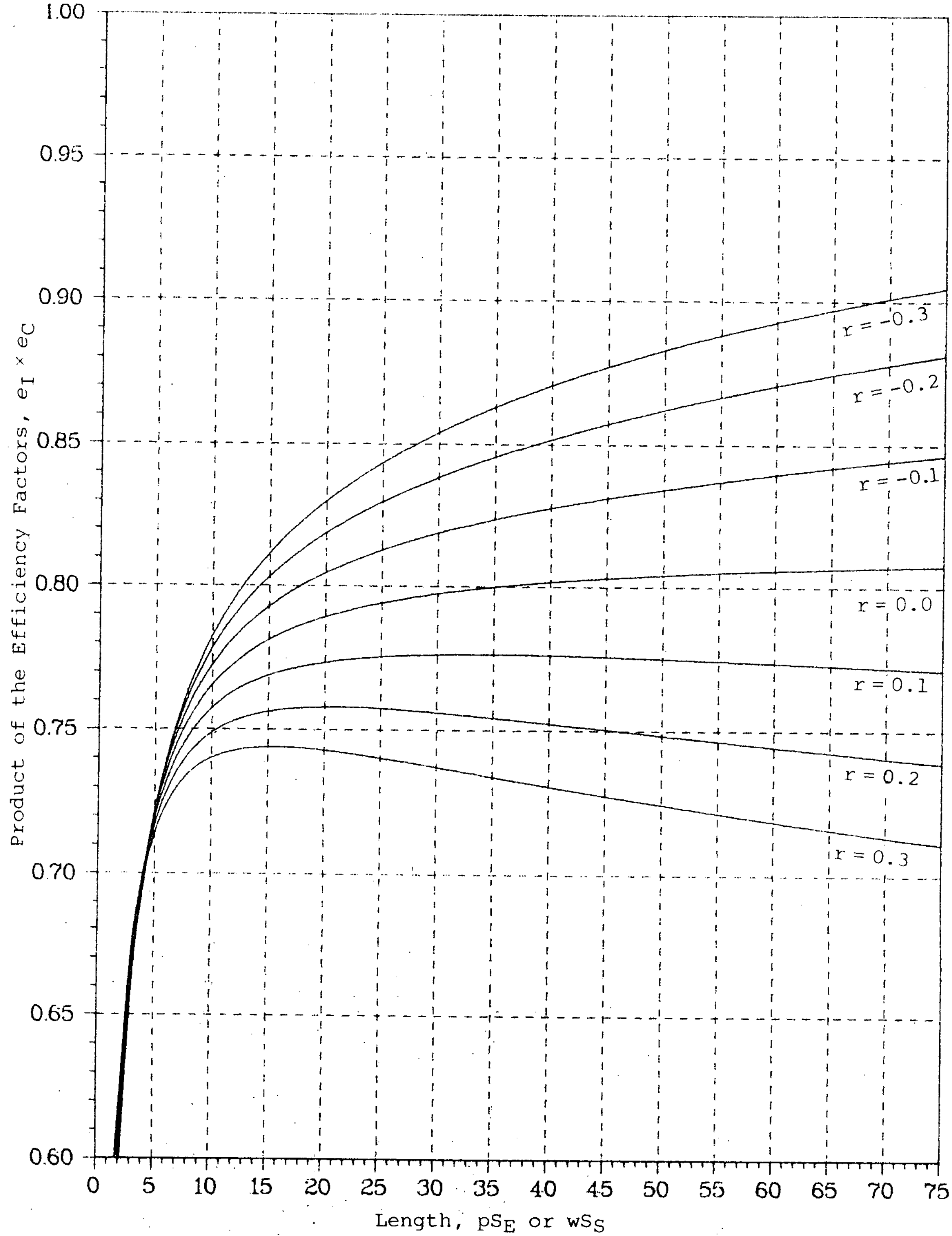

FIGURE B.4. THE VARIATION OF THE PRODUCT OF THE EFFICIENCY FACTORS, $e_I \times e_C$, WITH ENRICHER OR STRIPPER LENGTH FOR THE CASE IN WHICH m IS CONSTANT AND HAS ITS OPTIMUM VALUE:
$r = \psi/p$ for the enriching section
$r = -\psi/w$ for the stripping section

Assumption: dilute approximation

TABLE B.1

CALCULATED PERFORMANCE OF THE CENTRIFUGE MODEL

Assumption: Dilute Approximation

Feed Rate ($g\ UF_6/s$)	0.03171		
Centrifuge Length (cm)	335.3		
Peripheral Speed (m/s)	400.0	500.0	700.0
$p + w$	1.0206	0.7913	0.5513
$S_E + S_S$	87.830	113.278	162.608
Constant *m*			
p	0.4663	0.3517	0.2318
w	0.5543	0.4397	0.3195
S_E	43.168	55.396	78.743
S_S	44.662	57.882	83.865
pS_E	20.130	19.482	18.250
wS_S	24.757	25.448	26.796
m(optimum)	5.995	5.991	5.980
L(optimum) ($g\ UF_6/s$)	0.1863	0.2401	0.3440
$y_P/x_F = x_F/x_W$	1.1887	1.2501	1.3786
$e_I \times e_C$	0.7919	0.7917	0.7913
δU (SWU/yr)	10.07	16.78	34.56
Variable *m*			
p	0.4624	0.3478	0.2279
w	0.5582	0.4435	0.3234
S_E	42.713	54.638	77.178
S_S	45.117	58.640	85.430
pS_E	19.751	19.003	17.588
sS_S	25.185	26.010	27.627
M_F	9.269	9.260	9.235
L_F ($g\ UF_6/s$)	0.2880	0.3710	0.5312
$y_P/x_F = x_F/x_W$	1.2072	1.2753	1.4190
$e_I \times e_C$	0.9391	0.9389	0.9386
δU (SWU/yr)	11.95	19.90	40.99

APPENDIX I

THE ONSAGER-COHEN FORMULATION OF THE GRADIENT EQUATION

The equation of continuity for the process gas in a countercurrent gas centrifuge at steady state, taking both the axial and the radial convective transport into account, can be written

$$\frac{1}{r}\frac{\partial}{\partial r}(rcu) + \frac{\partial}{\partial z}(cw) = 0 . \qquad (I.1)$$

The equation of continuity for the desired component of the binary isotopic gas mixture which is the process gas in the countercurrent gas centrifuge at steady state, taking both the axial and the radial convective transport and the axial and the radial diffusive transport into account, can be written

$$\frac{1}{r}\frac{\partial}{\partial r}(rcu\xi + rJ_r) + \frac{\partial}{\partial z}(cw\xi + J_z) = 0 . \qquad (I.2)$$

In these equations

r is the radial coordinate of length in the centrifuge,
z is the axial coordinate of length in the centrifuge,
c is the molar density of the process gas,
u is the gas velocity in the radial direction,
w is the gas velocity in the axial direction,
ξ is the mole fraction of the desired component in the process gas,
J_r is the diffusive flux of the desired component in the radial direction, and
J_z is the diffusive flux of the desired component in the axial direction.

The diffusive flux terms are given by the expressions

$$J_r = -cD\{\frac{M_2 - M_1}{RT}\,\omega^2 r\xi(1-\xi) + \frac{\partial\xi}{\partial r}\} \text{ , and} \qquad (I.3)$$

$$J_z = -cD\frac{\partial\xi}{\partial z} , \qquad (I.4)$$

in which D is the mutual diffusion coefficient for the two components of the process gas mixture,

M_1 is the molecular weight of the desired component,
M_2 is the molecular weight of the other component,
R is the gas constant (8.3143×10^7 erg/°K/mole),
T is the absolute temperature of the process gas, and
ω is the angular velocity of the process gas which will be assumed to be constant and equal to Ω, the angular velocity of the centrifuge.

Since the radial transport of both the process gas and the desired component must vanish at the rotor wall, that is where r is equal to a where a is the radius of the centrifuge, integration of equation (I.1) with respect to r yields

$$rcu = \int_r^a r' \frac{\partial}{\partial z}(cw)dr' \qquad (I.5)$$

and the integration of equation (I.2) with respect to r yields

$$rcu\xi + rJ_r = \int_r^a r' \frac{\partial}{\partial z}(cw\xi + J_z)dr' \ . \qquad (I.6)$$

Replacing the quantity rcu in the above equation by its equivalent given by equation (I.5), replacing J_r and J_z by means of the expressions given by equations (I.3) and (I.4), and solving the resulting equation for the partial derivative of ξ with respect to r, one obtains

$$\frac{\partial\xi}{\partial r} = -\frac{1}{rcD}\int_r^a r' \frac{\partial}{\partial z}\left(cw\xi - cD\frac{\partial\xi}{\partial z}\right)dr' + \frac{\xi}{rcD}\int_r^a r' \frac{\partial}{\partial z}(cw)dr' - \frac{\Delta M\omega^2}{RT} r\xi(1-\xi), \qquad (I.7)$$

where ΔM is equal to $M_2 - M_1$. This form of the equation of continuity is particularly convenient for use in the formulation of the gradient equation.

Consider now the net axial transport of the desired component in a countercurrent gas centrifuge. The net axial transport of the desired component, $\mathcal{T}$, can be obtained by integrating the expression for the molar flux of the desired component in the axial direction over the cross sectional area of the centrifuge and can be written

$$\mathcal{T} = \int_0^a 2\pi rcw\xi dr - \int_0^a 2\pi rcD\frac{\partial\xi}{\partial z}dr \ , \qquad (I.8)$$

where a is the radius of the centrifuge rotor. The first integral on the right hand side of this equation represents the net axial transport of the desired component due to axial convection; the second integral represents the net axial transport of the desired component due to axial diffusion. At steady state, $\mathcal{T}$ anywhere in the enriching section of a centrifuge is equal to Px_p where P is the product withdrawal rate and x_p is the mole fraction of the desired component in the product stream, and $\mathcal{T}$ anywhere in the stripping section of the centrifuge is equal to $-Wx_w$ where W is the waste or tails withdrawal rate and x_w is the mole fraction of the desired component in the waste or tails stream.

It is now convenient to define a function G(r,z), sometimes called the stream function, which is equal to the next axial transport of process gas in the centrifuge between a cylinder of radius r and the rotor wall. This function is given by

$$G(r,z) = \int_r^a 2\pi r' cwdr' \ . \qquad (I.9)$$

It follows from this definition of the G-function that

$$G(0,z) = T, \quad G(a,z) = 0, \quad \text{and} \quad \frac{\partial G}{\partial r} = -2\pi rcw ,$$

where T is the net axial transport of process gas in the centrifuge. At steady state, T anywhere in the enriching section of a centrifuge is equal to P, the product withdrawal rate, and T anywhere in the stripping section of the centrifuge is equal to -W, where W is the waste or tails withdrawal rate. By virtue of the above definition of the G-function, equation (I.8) for the net axial transport of the desired component can be rewritten in the form

$$T = -\int_0^a \frac{\partial G}{\partial r}\xi\, dr - \int_0^a 2\pi rcD\frac{\partial \xi}{\partial z}dr \tag{I.10}$$

and, provided that both $G(r,z)$ and $\xi(r,z)$ are continuous functions of r, the first integral on the right hand side of the equation can be integrated by parts with the result

$$T = -T\xi(0,z) + \int_0^a G\frac{\partial \xi}{\partial r}dr - \int_0^a 2\pi rcD\frac{\partial \xi}{\partial z}dr . \tag{I.11}$$

Substitution of the expression for the partial derivative of ξ with respect to r given by equation (I.7) into the above equation for the net axial transport of the desired component yields the gradient equation for the countercurrent gas centrifuge in the following form

$$T = T\xi(0,z) - \int_0^a G\frac{\Delta M\omega^2}{RT}r\xi(1-\xi)dr + \int_0^a \frac{G\xi}{rcD}dr \int_r^a r'\frac{\partial}{\partial z}(cw)dr'$$

$$- \int_0^a \frac{G}{rcD}dr \int_r^a r'\frac{\partial}{\partial z}(cw\xi)dr' + \int_0^a \frac{G}{rcD}dr \int_r^a r'\frac{\partial}{\partial z}\left(cD\frac{\partial \xi}{\partial z}\right)dr'$$

$$- \int_0^a 2\pi rcD\frac{\partial \xi}{\partial z}dr . \tag{I.12}$$

Assuming that the cD product which is independent of the pressure can be treated as being independent of the radial variable r, that is, assuming that the gas in the centrifuge is essentially isothermal, the preceding equation can be rearranged and written in the following form:

$$\frac{1}{2\pi cD}\int_0^a \frac{G}{r}dr \int_r^a 2\pi r' cw\frac{\partial\xi}{\partial z}dr' + 2\pi cD\int_0^a r\frac{\partial\xi}{\partial z}dr - \int_0^a \frac{G}{R}dr \int_r^a r'\frac{\partial^2\xi}{\partial z^2}dr'$$

$$+ \frac{1}{2\pi cD}\int_0^a \frac{G}{r}dr \{ \int_r^a 2\pi r'\xi\frac{\partial(cw)}{\partial z}dr' - \xi\int 2\pi r'\frac{\partial(cw)}{\partial z}dr' \}$$

$$= -\frac{\Delta M\omega^2}{RT}\int_0^a Gr\xi(1-\xi)dr - \{T - T\xi(0,z)\} . \qquad (I.13)$$

The standard Onsager-Cohen form of the gradient equation is obtained directly from equation (I.13) by the application of the following three simplifying assumptions. These assumptions, because of their importance in the Onsager-Cohen formulation of the gradient equation, could be called the Onsager-Cohen assumptions.

1. It is assumed that the radial concentration gradient of the desired component is sufficiently small that the concentration of the desired component, ξ, and its derivatives with respect to z can be treated as being essentially independent of the radial variable r. This assumption permits the removal of the concentration terms from inside the integrals of equation (I.13) and reduces the partial differential equation to an ordinary differential. Letting x(z) represent the radially averaged value of $\xi(r,z)$, the mole fraction of the desired component, the application of this assumption permits one to rewrite equation (I.13) in the simpler form

$$\frac{1}{2\pi cD}\int_0^a \frac{G^2}{r}dr \cdot \frac{dx}{dz} + \pi a^2 cD\frac{dx}{dz} - \frac{1}{2}\int_0^a G\frac{a^2 - r^2}{r}dr \cdot \frac{d^2x}{dz^2}$$

$$= -\frac{\Delta M\omega^2}{RT}\int_0^a Grdr \cdot x(1-x) - (T-Tx) . \qquad (I.14)$$

2. It is assumed that the term in equation (I.13) or (I.14) containing the second derivative of the concentration of the desired component with respect to z is sufficiently small that it may be neglected. The application of this assumption reduces the gradient equation from a second order differential equation to one of first order. The gradient equation can now be written in the form

$$\{\frac{1}{2\pi cD}\int_0^a \frac{G^2}{r}dr + \pi a^2 cD\}\frac{dx}{dz} = \{-\frac{\Delta MV^2}{a^2RT}\int_0^a Grdr\}x(1-x) - (T-Tx) , \qquad (I.15)$$

where V, equal to ωa, is the peripheral speed of the centrifuge rotor.

3. It is assumed that the net axial transport of process gas, T, in the centrifuge is sufficiently small compared with the countercurrent circulation rate, that is, compared with either the total process gas upflow

or total process gas downflow rate in the centrifuge, that terms of the order of T/L, where

$$L = \frac{1}{2}\int_0^a 2\pi r|cw|dr \ , \qquad \text{(I.16)}$$

can be neglected with respect to unity. This assumption allows both the quantity L and the G-function to be treated as being independent of the net axial transport of process gas in the centrifuge.

It is convenient to define a set of centrifuge separation parameters based on the standard Onsager-Cohen form of the gradient equation. Let each term of equation (I.15) be divided by the quantity, L, with the result

$$\left\{\frac{1}{2\pi cD}\int_0^a \frac{G^2}{L}\frac{dr}{r} + \frac{\pi a^2 cD}{L}\right\}\frac{dx}{dz} = \left\{-\frac{\Delta MV^2}{a^2RT}\int_0^a \frac{G}{L} r dr\right\} - \frac{T - Tx}{L} \ , \qquad \text{(I.17)}$$

which can be considered to be the standard form of the Onsager-Cohen gradient equation. By adopting the following definitions for the centrifuge separation parameters

LS_c (the convective contribution to the stage length)

$$= \frac{1}{2\pi cD}\int_0^a \frac{G^2 dr}{Lr} \ , \qquad \text{(I.18)}$$

Sd/L (the diffusive contribution to the stage length)

$$= \frac{\pi a^2 cD}{L} \ , \qquad \text{(I.19)}$$

and ψ (the stage separation constant of a theoretical stage)

$$= -\frac{\Delta MV^2}{a^2RT}\int_0^a \frac{G}{L} r dr \ , \qquad \text{(I.20)}$$

the Onsager-Cohen gradient equation can be written in the form

$$\left\{LS_c + \frac{Sd}{L}\right\}\frac{dx}{dz} = \psi x(1-x) - \frac{T - Tx}{L} \ , \qquad \text{(I.21)}$$

where S_c, S_d, and ψ are independent of the magnitude of the countercurrent circulation, that is, independent of L. The term in brackets in equation (I.21) can be regarded as the length of a theoretical stage in the centrifuge. By adopting the following definitions for three additional centrifuge separation parameters

L_0 (the value of L which minimizes the stage length) $= \sqrt{S_d/S_c}$, (I.22)

S_0 (the minimum value of the stage length) $= 2\sqrt{S_c S_d}$, (I.23) and m (the ratio of the actual circulation rate, L, to L_0) = L/L_0 , (I.24)

the Onsager-Cohen gradient equation can be written in the alternative form

$$\left\{\frac{m^2+1}{2m} S_0\right\} \frac{dx}{dz} = \psi x(1-x) - \frac{T - Tx}{mL_0} \quad . \tag{I.25}$$

It may be noted that the value of L which minimizes the stage length also has the effect of maximizing the axial enrichment at total reflux, that is, when T is equal to zero.

APPENDIX II

THE SEPARATIVE WORK PRODUCED BY A CENTRIFUGE

The separative work produced per unit time by an incremental length, dz, of a countercurrent isotope separation element, such as a countercurrent gas centrifuge, is obtained from

$$\frac{d(\Delta U)}{dz} = (T - Tx)\frac{dx}{dz}\, v''(x) \quad , \qquad \text{(II.1)}$$

where v"(x) is the second derivative of the value function and is equal to $[x^2(1-x)^2]^{-1}$. Substituting for (T - Tx) in the above equation using the Onsager-Cohen formulation of the gradient equation, equation (I.25), one obtains

$$\frac{d(\Delta U)}{dz} = \left\{ mL_0\psi(1-x) - \frac{m^2+1}{2} L_0 S_0 \frac{dx}{dz} \right\} \frac{dx}{dz} v''(x) \quad . \qquad \text{(II.2)}$$

It can be seen from this equation that the separative work produced by the incremental length, dz, of the centrifuge will be a maximum when

$$\frac{dx}{dz} = \frac{m\psi x(1-x)}{(m^2+1)S_0} \quad . \qquad \text{(II.3)}$$

The maximum separative work output of the incremental length, dz, of the centrifuge is therefore given by

$$\frac{d(\Delta U)}{dz} = \frac{m^2}{m^2+1} \frac{L_0\psi^2}{2S_0} \quad . \qquad \text{(II.4)}$$

A lower separative work output than that given by the above equation, attributable to the fact that dx/dz differs from its optimal value given by equation (II.3), is said to result from a departure from ideality and is taken into account by an ideality efficiency factor, e_I. The term $m^2/(1+m^2)$ which depends on the countercurrent circulation rate and approaches unity as the countercurrent circulation becomes large is called the circulation efficiency, e_C. The quantity $L_0\psi^2/2S_0$ which depends on the shape of the countercurrent axial velocity profile can be regarded as the maximum separative work, taking into account the gas flow pattern in the centrifuge, which could be produced per unit length of centrifuge. Thus the ratio of this term to the maximum theoretical separative capacity of the gas centrifuge is called the flow profile efficiency, e_F. The maximum theoretical separative capacity of a countercurrent gas centrifuge is given by the well-known relationship

$$\Delta U_{(\text{theoretical max})} = \frac{\pi ZcD}{2} \left(\frac{\Delta M V^2}{2RT}\right)^2 \quad . \qquad \text{(II.5)}$$

When the shape of the axial velocity profile is invariant, that is, when L_0, ψ, and S_0 are constant over the length of the centrifuge, equation (II.4) can be integrated with respect to z and the result expressed in the form

$$\Delta U = e_I \times e_C \times e_F \times \Delta U_{(\text{theoretical max})} \quad . \qquad \text{(II.6)}$$

Here e_I and e_C represent the values of the ideality efficiency and the circulation efficiency, respectively, averaged over the length of the centrifuge.

REFERENCES

1. Parker, H., University of Virginia, *Private Communication*, 1966.

2. Lotz, M., "Die rien axiale Strömung in einer Gegenstrom-Gasultranzentrifuge", *Atomkernergie* (22), pp 41-45 (1973).

3. Von Halle, E., *The Countercurrent Gas Centrifuge for the Enrichment of U-235*, Union Carbide Corporation, Nuclear Division, Oak Ridge Gaseous Diffusion Plant, Oak Ridge, Tennessee, November 1977 (K/OA-4058).

4. May, W. G., "Separation Parameters of Gas Centrifuges", *Developments in Uranium Enrichment* (Edited by Manson Benedict), AIChE Symposium Series 169, Volume 73, pp 30-38 (1977).