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Optimum Economic Growth in an Aggregative Model of Capital Accumulation: A Turnpike Theorem

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I. Introduction and Summary

Recent contributions to the theory of optimum economic growth, for example, in [1], [3], [8] or [10], like Ramsey's seminal article [6], have been primarily concerned with the implications of maximizing the social welfare generated by the entire stream of future consumption. As an alternative formulation, in this paper it is postulated that only the social welfare associated with future consumption over some finite period is of direct concern; generations beyond the horizon are accounted for only insofar as a lower bound on the terminal capital stock is prescribed. Then, within a closed, aggregative framework, the behavior of growth paths which are optimum with respect to this social welfare is investigated.

Our central result is to exhibit a general property of such optimum growth paths. Stated loosely, this property is that any economy pursuing optimum growth over a sufficiently long period would spend all except at most an

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[1] Work on this paper was begun while I was the recipient of a Haynes Foundation dissertation fellowship at Stanford University; it was also supported in part by the National Science Foundation under grants GS-420 to the University of Chicago and GS-88 to the Cowles Foundation for Research in Economics at Yale University. The paper itself has benefitted greatly from lively discussions at Stanford (in the Quantitative Analysis Workshop) and later at the University of Chicago (in an informal summer seminar on growth theory conducted by Professor H. Uzawa).

initial and final phase of the period performing nearly golden rule balanced growth (appropriately defined for a possibly nonzero social discount rate). Hence, we have generalized the turnpike property of essentially pure production models, originally elaborated by Dorfman, Samuelson and Solow [2], and later explored by many others, in an important direction: a similar result is shown to hold when the intrinsic value of consumption -- over and above its indirect contribution to the continuation of production -- is explicitly accounted for.

The plan of the paper is as follows: In the next section we introduce the now standard aggregative model of capital accumulation and a precise definition of the social welfare enjoyed within the economy described by this model. Section III characterizes and then discusses the general behavior of optimum growth paths, while in Section IV, the optimum growth turnpike property is presented. Finally, in the last section, the special case where social welfare is simply the discounted sum of per capita consumption is outlined; for this special case the optimum growth turnpike property is especially striking.

II. The Model and Definition of Social Welfare

We assume the closed, aggregative model of capital accumulation first closely analyzed by Solow [7]. Briefly, the behavior of this simple economy over time is described by three relations:

$$y = f(k) \,, \tag{1}$$

an aggregate production function, relating the output per capita of a single, homogeneous good to the capital-labor ratio;

$$c(t) + z(t) = y(t) \,, \quad c(t) \geq 0 \,, \quad z(t) \geq 0 \,, \tag{2}$$

the allocation of current output per capita between instantaneous consumption and gross investment per capita[2]; and

(3) $\dot{k}(t) = z(t) - \lambda k(t)$, with $k(0) = k^o$, given $\lambda = n + \mu > 0$, $k^o > 0$,

the growth of capital per head from the historically given initial capital-labor ratio, when the labor force (and population) is expanding exogenously at the positive rate n , and the capital stock is depreciating, independently of use, at the positive rate μ .

The production function is assumed to exhibit, in addition to constant returns to scale, positive marginal productivity of either factor as well as a diminishing marginal rate of substitution between factors, expressed by the conditions

(4) $f(k) > 0$, $f'(k) > 0$, $f''(k) < 0$ for $k > 0$.

Moreover, both the importance and limitation of roundaboutness in production are assumed to be further represented by the conditions

(5)
$$\lim_{k \to 0} f(k) = 0 , \quad \lim_{k \to \infty} f(k) = \infty ,$$
$$\lim_{k \to 0} f'(k) = \infty , \quad \lim_{k \to \infty} f'(k) = 0 .$$

Finally, as should be clear from (1), there is no technical change in this economy.

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2 It is necessary in what follows that the allocation process be mildly well-behaved. Therefore, it is convenient at this point to simply assume the stronger condition that $c(t)$ and $z(t)$ are piecewise continuous. This can be interpreted as an approximation of the fact that the planning and execution of abrupt changes in any given allocation scheme require time.

It is worthwhile mentioning that the aggregative "neoclassical" technology outlined in the preceding paragraphs is a special case of the two-sector "neoclassical" technology of Uzawa [9]; most of the results to be presented carry over directly into that more general formulation, in which the available techniques for producing capital goods differ from those for producing consumption goods. Because it simplifies the exposition, and also because no further insight is gained otherwise, we choose to carry out the analysis for the special case.

Social welfare over any finite period $[0, T]$ is presumed to be adequately represented by the functional

$$W = \int_0^T U[c(t)]e^{-\delta t}dt \ , \quad \text{given} \quad -\infty < \delta < \infty \ , \tag{6}$$

that is, by the discounted sum over the period of some index, $U(\cdot)$, of the rate of per capita consumption. For short, we refer to this index as the instantaneous utility function.

In the major part of the paper, the latter is assumed to increase at a decreasing rate,

$$U'(c) > 0 \ , \quad U''(c) < 0 \quad \text{for} \quad c > 0 \ , \tag{7}$$

and to increase very rapidly for small but very slowly for large rates of per capita consumption,

$$\lim_{c \to 0} U'(c) = \infty \ , \quad \lim_{c \to \infty} U'(c) = 0 \ . \tag{8}$$

(7) can be interpreted, by reference to the discrete analogue of (6), to represent a diminishing marginal rate of substitution between rates of per

capita consumption at any two points of time. And the limit conditions (8) are essentially first, one possible continuous generalization of the imposition of a minimum subsistence level, and second, the condition of non-saturation.

For the last section of the paper, however, the instantaneous utility function becomes

$$U(c) = c \,, \tag{7'}$$

simple per capita consumption -- the latter assumed to be explicitly constrained by

$$c \geq \underline{c} > 0 \,, \tag{8'}$$

a minimum subsistence level -- which implies that social welfare is represented by

$$W = \int_0^T c(t)e^{-\delta t}dt \,, \tag{6'}$$

the discounted sum of per capita consumption.

Finally, as mentioned in the introduction, we assume that the growth paths which this economy is free to follow are constrained by the additional non-technological requirement

$$k(T) \geq k^T \,, \quad \text{given} \quad k^T \geq 0 \,, \tag{9}$$

that the terminal capital-labor ratio be at least as large as some prescribed minimum.

III. Optimum Growth Paths Within a Finite Horizon

The problem confronting, say, the technical staff of the central planning board is to determine the growth path $\left\{c(t),\ z(t),\ k(t)\ :\ 0 \leq t \leq T\right\}$ maximizing the welfare criterion (6) subject to the feasibility constraints (1)-(3) and the terminal condition (9). For easy reference, this problem can be stated concisely as:

$$
(10)\quad \left\{\begin{array}{lll}
\text{specify the growth path} & & (c,\ z,\ k) \\
\text{which maximizes social welfare} & & \int_0^T U(c)e^{-\delta t}dt \\
\text{subject to} & c + z = f(k)\ ,\quad c \geq 0\ ,\quad z \geq 0\ , & \\
 & \dot{k} = z - \lambda k\ ,\quad \text{with}\quad k(0) = k^o\ ,\quad \text{and}\quad k(T) \geq k^T & \\
\text{and given} & 0 < T < \infty\ ,\quad -\infty < \delta < \infty\ ,\quad \lambda > 0\ ,\quad k^o > 0\ ,\quad \text{and} & \\
 & k^T \geq 0\ . &
\end{array}\right.
$$

In (10) and hereafter, variables such as c, z and k are understood to be functions of t when not explicitly so denoted.

In order that (10) be a meaningful problem, it is necessary that k^T and T be chosen so that the minimum terminal capital-labor ratio is attainable within the existing technology and feasible within the prescribed period. Formally, these restrictions can be expressed by the constraint

$$(k^T,\ T) \in A$$

where A, the set of attainable and feasible terminal parameters, is defined by

$$A = \left\{(k^T,\ T):\ \ 0 \leq k^T < \bar{\bar{k}}\ ,\quad g(k^T,\ T) \geq 0\right\}\ ,$$

with

$$\bar{\bar{k}} = \hat{k}\ , \quad \text{for} \quad k^o \leq \hat{k} \tag{11}$$

$$= k^o\ , \quad \text{for} \quad \hat{k} < k^o\ ,$$

where

$$f(\hat{k}) = \lambda\hat{k}\ ,$$

and

$$g(k^T, T) = T - \int_{k^o}^{k^T} \frac{dk}{f(k) - \lambda k}\ , \quad \text{for} \quad k^o < k^T \tag{12}$$

$$= \int_{k^o}^{\max\,(k^T,\ \hat{k})} \frac{dk}{f(k) - \lambda k} - T\ , \quad \text{for} \quad k^o > k^T\ .$$

(11) and (12) follow directly from the structure of the technology postulated in the last section: the former defines the maximum attainable capital-labor ratio starting from any $k^o > 0$, while the latter merely displays the difference between the prescribed period and the minimum time required to go from k^o to k^T , when $k^o < k^T$ (or between the maximum time permitted in going from k^o to k^T and the prescribed time, when $k^o > k^T$, the less interesting case).

Assuming that the pair (k^T, T) is in the constraint set A , we have still left unanswered the natural questions of why the terminal capital-labor condition $k(T) \geq k^T$ is appropriate, and by what criterion actual values for k^T and T would be chosen.

As to the first question, it should be obvious that any path providing both more social welfare <u>and</u> a larger terminal capital stock would always be preferred. Thus, it is reasonable to expect that targets would be expressed as minimum objectives in the practical planning situation which our formulation idealizes. Moreover, it is shown in the sequal that, with the terminal capital-labor condition expressed as a specific desired target, the optimum growth path may exhibit a conspicuous inconsistency between maximizing social welfare and reaching the desired terminal structure.

With regard to the second question, presumably k^T and T could be set in accordance with any goals -- including some traditionally outside the purview of economics, for example, national power and influence as represented by industrial potential, or rapid industrialization for its own sake. On the other hand, we need not reject the possibility that they could be set in accordance with the goals usually presupposed in welfare economics. For example, k^T and T could be any particular values which would also maximize long run social welfare.

Whatever the basis for the choice of k^T and T, a certain minimum impingement of welfare considerations would probably be widely accepted: namely, the requirement that the nation be no worse off at the end than at the beginning of the planning period -- in terms of national wealth as measured by real capital per head -- that is, that

$$k^T \geq k^o .$$

As it does not especially facilitate our argument, we do not assume the latter, except to the extent that it helps provide some consistency to our diagrammatics.

As a last comment before presenting the solution to (10), let us emphasize that the (effective) social discount rate δ need not be positive. Indeed, for any finite social discount rate and any feasible growth path generating consumption per capita c, from (2), (4)-(8), and (11) it follows that

$$\int_0^T U(c)e^{-\delta t}dt \leq \frac{U[f(\bar{\bar{k}})]}{\delta}(1-e^{-\delta T}) < \infty ,$$

social welfare is bounded from above.[3] Thus, one obvious and permissible interpretation of the welfare functional (6) is purely classical: it represents total individual welfare over the period under consideration. On our other assumptions -- in particular that the current labor force $L(t)$ and the current total population $P(t)$ grow at the same positive rate n, so that the former is a fixed proportion γ of the latter -- such an interpretation can be expressed explicitly by

$$W = \int_0^T P(t)\, u[c(t)]dt = \int_0^T U[c(t)]e^{-\delta t}dt ,$$

where

$$U[c(t)] = \frac{L(o)}{\gamma} u\,[c(t)] \quad \text{and} \quad \delta = -n < 0 .$$

Now the index $U(\cdot)$ is simply a constant multiple of the instantaneous utility function of the representative individual in our egalitarian economy $u(\cdot)$, providing a motivation for the reference introduced in the previous section.

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[3] Of course, if δ is positive, or if δ is zero and a suitable level of saturation is admitted, then our formulation of the problem (10) is also meaningful for the limiting case $T\to\infty$. However, as Koopmans [3, pp. 27-29, 58-64] has shown, even if a modification of Ramsey's bliss device is used, if δ is negative, then the limiting case $T\to\infty$ of the corresponding problem has no solution. This result is relevant to our later interpretation of the optimum growth turnpike.

In order to characterize the optimum growth paths, we appeal to Pontryagin's Maximum Principle (theorem 7, p. 69 in [4], which applies to the problem (10) if the variables are redefined in such a way that the gross investment ratio s ,

$$0 \leq s = \frac{z}{f(k)} \leq 1 ,$$

can be treated as an explicit control parameter). By introducing the imputed price of a unit of gross investment per head,

$$q = q(t) , \tag{13}$$

and the imputed value of gross national product per capita,

$$\psi = U(c) + qz , \tag{14}$$

and then applying the theorem to the Hamiltonian expression representing the present imputed value of net national product per capita,

$$(\psi - q\lambda k)e^{-\delta t}$$

the following theorem is obtained:

<u>The necessary conditions for an optimum growth path are that there exists a continuous imputed price (13) such that</u>

$$\dot{q} = \frac{d\left(\int_t^{\infty} \frac{\partial\psi}{\partial k} e^{-(\delta+\lambda)(\tau-t)} dt\right)}{dt} = (\delta+\lambda)\, q - U'(c)\, f'(k) , \tag{15}$$

and

$$q(T) \left\{k(T) - k^T\right\} = 0 , \tag{I}$$

the imputed price changes as if, say, the central planning board exercises perfect foresight with respect to the marginal, imputed value product of capital, while the terminal imputed price is zero if $k(T) > k^T$ or non-negative if $k(T) = k^T$;

(16) $$c + z = f(k) \ , \quad c \geq 0 \ , \quad z \geq 0 \ ,$$

and

(17) $$\left(\frac{\partial\psi}{\partial z}\right)_{c=f(k)-z} = U'(c) + q \leq 0 \ , \text{ with strict equality for } z > 0 \ ,$$

the allocation between current consumption and current gross investment per capita is feasible and maximizes the imputed value of gross (and net) national product at each point of time;

(18) $$\dot{k} = z - \lambda k \ , \quad \text{with} \quad k(0) = k^o \ ,$$

and

(II) $$k(T) \geq k^T \ ,$$

the growth of the capital-labor ratio is feasible, while the terminal capital-labor ratio is at least as large as the prescribed minimum.

It should be clear that all the above valuations are in terms of the instantaneous utility of the rate of per capita consumption.

Conditions (15)-(18), (I) and (II) are also sufficient. In demonstrating this fact it is further easily shown that if an optimum growth path exists, then it is unique. Hence, we can state as a second theorem:

Suppose we have found a feasible path (c^1, z^1, k^1) and an imputed price q^1 which satisfy (15)-(18), (I) and (II). Then consider any other distinct feasible path (c^2, z^2, k^2) satisfying (16), (18) and (II). By "distinct" is meant

$$k^1(\tau) \neq k^2(\tau) \quad \text{for some} \quad 0 < \tau < T ,$$

which, because of the continuity of k on any feasible path, implies

(19) $$k^1(t) \neq k^2(t) \quad \text{for} \quad t \in I(\tau) ,$$

where $I(\tau)$ is some open interval around τ. It follows that

(20) $$\int_0^T \left\{U(c^1) - U(c^2)\right\} e^{-\delta t} \, dt > 0 ,$$

the first path is strictly better than the second.

To prove this result we perform various manipulations on the integral in (20). Thus,

$$\int_0^T \left\{U(c^1) - U(c^2)\right\} e^{-\delta t} \, dt = ,$$

adding and substracting $U'(c^1)\,(c^1-c^2)$ and the zero expressions $q^1(z^j - \lambda k^j - \dot{k}^j)$, $j = 1, 2,$ derived from (18), within the braces under the integral,

$$\int_0^T \Big\{U(c^1) - U(c^2) - U'(c^1)(c^1-c^2) + U'(c^1)(c^1-c^2)$$
$$+ q^1(z^1-\lambda k^1-\dot{k}^1) - q^1(z^2-\lambda k^2-\dot{k}^2)\Big\} e^{-\delta t} \, dt = ,$$

adding and subtracting $U'(c^1)f'(k^1)(k^1-k^2)$, substituting for c^j , $j = 1, 2$, from (16), and rearranging terms, all again within the braces under the integral,

$$\int_0^T \Big\{U(c^1) - U(c^2) - U'(c^1)(c^1-c^2) + \Big(q^1 - U'(c^1)\Big)(z^1-z^2)$$
$$+ \Big(U'(c^1)\, f'(k^1) - \lambda q^1\Big)(k^1-k^2)$$
$$+ U'(c^1)\Big(f(k^1) - f(k^2) - f'(k^1)(k^1-k^2)\Big) - q^1(\dot{k}^1-\dot{k}^2\Big\}\, e^{-t}\, dt > ,$$

utilizing the strict concavity of $f(\cdot)$ in conjunction with (19), and the concavity of $U(\cdot)$, and integrating by parts the expression

$$\int_0^T q^1(\dot{k}^1-\dot{k}^2)\, e^{-\delta t}\, dt ,$$

$$\int_0^T \Big\{\Big(q^1-U'(c^1)\Big)(z^1-z^2) + \Big(\dot{q}^1 + U'(c^1)f'(k^1) - (\delta+\lambda)q^1\Big)(k^1-k^2)\Big\}\, e^{-\delta t}dt -$$
$$\Big[q^1\, (k^1-k^2)\, e^{-\delta t} \Big|_0^T \geq ,$$

applying the optimality conditions (15) and (17) ,

$$- \Big[q^1\, (k^1-k^2)\, e^{-\delta t} \Big|_0^T \geq 0 ,$$

substituting from the initial condition in (18) and the terminal conditions (I) and (II).

An analysis of the solutions to the pair of autonomous differential equations (15) and (18), given the relations (16) and (17), is presented in detail in [1], and these results are exhibited here in Figure I. The general behavior of the unique optimum growth path for any given initial

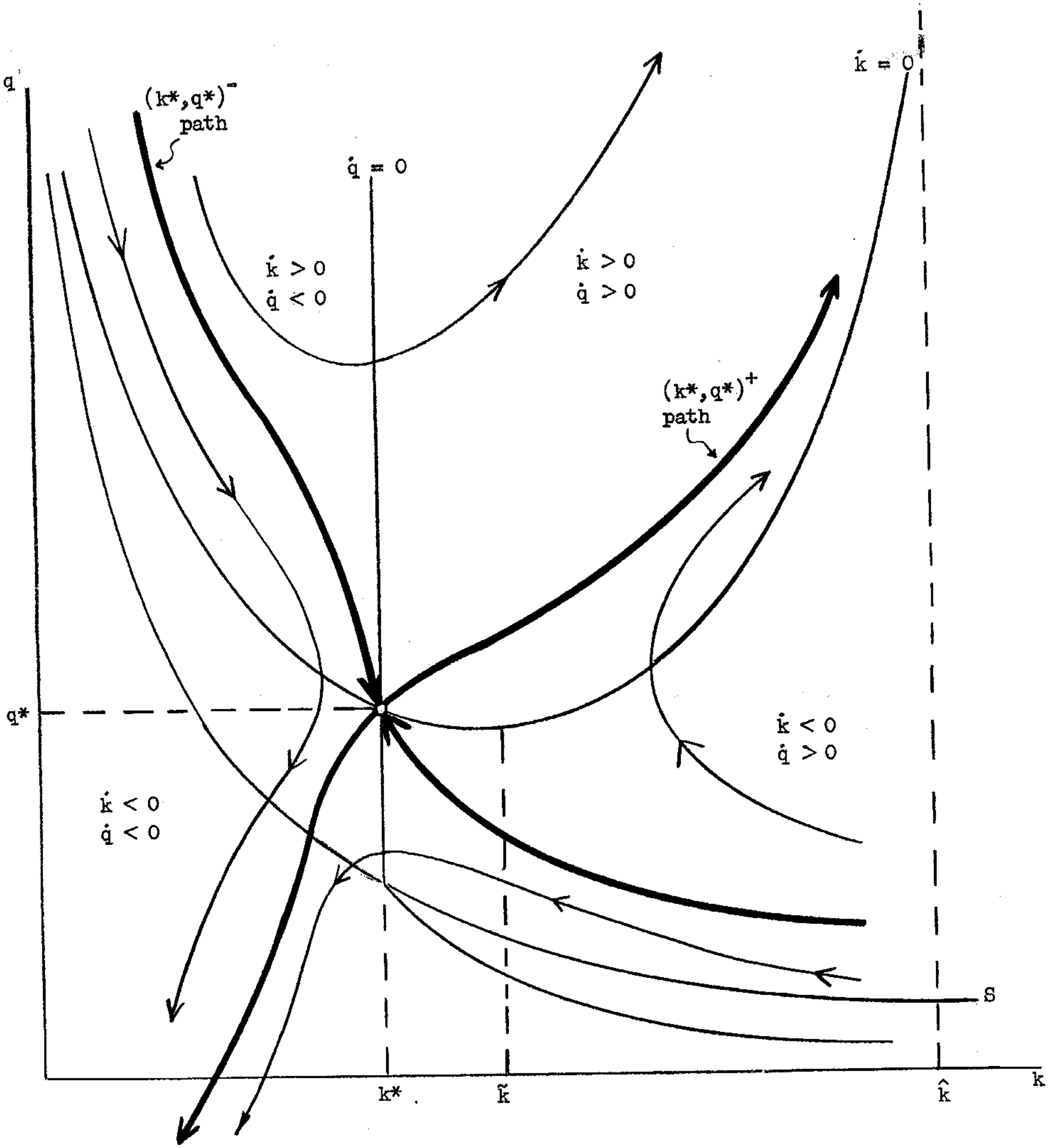

FIGURE I

The Nature of the Solutions to the Optimality Conditions capital-labor ratio, $k^o > 0$, and any prescribed minimum terminal capital-labor ratio and time horizon which are attainable and feasible, $(k^T, T) \in A$ -- referred to hereafter as the unique optimum growth path specified by (k^o, k^T, T) -- can thus be derived from a close examination of this phase diagram. Therefore, before we present the central results of this paper, we detour and mention the several features depicted in Figure I.

$\dot{k} = 0$, $\dot{q} = 0$ The singular curves for each differential equation considered separately. They divide the half-plane $k \geq 0$ into four regions of behavior for the system of differential equations, as labelled and illustrated.

S The curve for which $z = 0$ and (17) is satisfied with equality. Thus it can best be represented as the boundary of specialization in consumption: for all points on or below (above) S gross investment is zero (strictly positive). Notice that our assumption (8), that marginal utility approaches infinity as consumption per capita approaches zero, in conjunction with the optimality conditions specifying imputation and allocation, imply that the contrary specialization in investment is never optimum over a finite interval of time.

$\hat{k}$ Previously defined by (11), $\hat{k}$ can be further interpreted as the maximum long run attainable capital-labor ratio: for any $k^o > 0$, specialization in investment would result in an asymptotic approach to $\hat{k}$. Although the cases for which $k^o \geq \hat{k}$ are probably not particularly interesting -- a point which would be debatable in an open model allowing foreign aid or virtually unlimited borrowing in the international capital market -- as they present no essential complication, we will only implicitly restrict our attention to the cases for which $k^o < \hat{k}$.

$\tilde{k}$ This is the golden rule capital-labor ratio, defined by

$$f'(\tilde{k}) = \lambda .$$

Thus, balanced growth at $\tilde{k}$ yields the golden rule growth path, denoted by $(\tilde{c}, \tilde{z}, \tilde{k})$,

$$\tilde{c} = f(\tilde{k}) - \lambda\tilde{k} \ , \ \tilde{z} = \lambda\tilde{k} \ .$$

It can be interpreted as the one balanced growth path which would be forever voluntarily maintained as optimum, given the ethical judgment that there is to be no discrimination among generations, i.e., a zero social discount rate, either because of size or timing.

(k^*, q^*) The point representing the unique balanced growth path and imputed price defined by the singular solution of the system (15)-(18),

$$f'(k^*) = \delta + \lambda \ ,$$

$$z^* = \lambda k^* \ ,$$

$$c^* = f(k^*) - z^* \ ,$$

and

$$q^* = U'(c^*) \ .$$

For $\delta \neq 0$, (c^*, z^*, k^*) differs from the golden rule growth path. However, it is proved in [1] that if there is social impatience, i.e., a positive social discount rate, then it is again the one balanced growth path which would be forever voluntarily maintained as optimum. Though (c^*, z^*, k^*) is not such a "best" balanced growth path when the social discount rate is negative, we take the liberty of referring to any balanced growth path at a capital-labor ratio k^* as a <u>(modified) golden rule growth path</u>.

It should be mentioned explicitly that the (modified) golden rule growth path is attainable if and only if $k^* < \hat{k}$, which condition provides a lower bound on δ . That is, if $k^* < \hat{k}$, then

$$\delta + \lambda = f'(k^*) > f'(\hat{k})$$

or

$$\delta > f'(\hat{k}) - \lambda = \rho ,$$

where ρ is some finite negative number, as $\lambda = f'(\tilde{k}) > f'(\hat{k})$ by the strict concavity of $f(\cdot)$. The implication for our analysis is that, in any society in which the material well-being of the average individual of tomorrow's generation is vastly more important than that of today's -- perhaps because of rapid population growth -- although the optimum growth path specified by (k^o, k^T, T) always exists, for large T the turnpike becomes the maximum attainable balanced growth path $(0, \hat{z}, \hat{k})$,

$$\hat{z} = f(\hat{k}) ,$$

and not the (modified) golden rule growth path (c^*, z^*, k^*) . We neglect this possibility and hereafter simply assume $\delta > \rho$.

$(k^*, q^*)^-$ and $(k^*, q^*)^+$ paths The singular point (k^*, q^*) is a saddle point, with its stable branches the $(k^*, q^*)^-$ path, and its unstable branches the $(k^*, q^*)^+$ path. It is also shown in [1] that for $\delta > 0$, some portion of the $(k^*, q^*)^-$ path is the unique optimum growth path specified by $(k^o, 0, \infty)$.

Before continuing the analysis, two comments about emphasis and notation will be helpful. First, although the system (15)-(18) along with the terminal conditions (I) and (II) characterize any unique optimum growth path in terms of the real variables c , z , and k and the imputed price q , most of the discussion in the remainder of this and the following section will emphasize only the real stock variable k and the imputed price q . This is possible because we can reformulate (16) and (17) as

(17') $$c = \min \left\{ h(q) , f(k) \right\} = c(k, q)$$

and

(16') $$z = f(k) - c(k, q) = z(k, q) ,$$

where

(21) $$h(q) = U'^{-1}(q) \geq 0 , \quad \text{for} \quad q \geq 0$$

$$= +\infty , \quad \text{for} \quad q < 0 ,$$

with

(22) $$h'(q) = \frac{1}{U''[h(q)]} < 0$$

and

(23) $$\lim_{q \to 0} h(q) = \infty , \quad \lim_{q \to \infty} h(q) = 0 ,$$

utilizing the postulated properties of the instantaneous utility function (7) and (8). Hence, the system (15)-(18) can be reduced to a pair of autonomous differential equations in the variables k and q ,

(18') $$\dot{k} = z(k, q) - \lambda k, \quad \text{with} \quad k(0) = k^0$$

and

(15') $$\dot{q} = (\delta+\lambda)q - U'[c(k, q)]\, f'(k) \, .$$

Such simplification is primarily a matter of convenience; the same relations (16') and (17') which enable us to ignore the real flow variables c and z also readily allow us to convert results in terms of k and q , where relevant, into terms of c or z .

Second, in accordance with this emphasis, we adopt the procedure of referring to a particular solution to the system (15)-(18) as the (k^j, q^j) path, where the point (k^j, q^j) is some easily distinguished feature of the particular solution. Also, the backward and forward (in time) segments of the (k^j, q^j) path from the point (k^j, q^j) are expressly referred to as the $(k^j, q^j)^-$ and $(k^j, q^j)^+$ paths, respectively. And, when the need arises, any segment of a (k^j, q^j) path is explicitly denoted by $\left\{\left(k^j(t), q^j(t)\right) : 0 \leq t \leq T\right\}$ with initial or terminal values $\left(k^j(0) , q^j(0)\right)$ or $\left(k^j(T), q^j(T)\right)$ -- though occasionally for the last we drop superscripts if the possibility of resulting confusion is small.

For example, a distinguishing feature of the solution discussed in the third preceding paragraph is its asymptotic properties with respect to the point (k^*, q^*) . Hence, it is referred to as the (k^*, q^*) path, and its branches as the $(k^*, q^*)^-$ and $(k^*, q^*)^+$ paths, while later on, some segment of the solution will be denoted by $\left\{\left(k^*(t),\ q^*(t)\right):\ 0 \le t \le T_1^q\right\}$.

Now, in order both to prepare the way for the next section as well as to gain further insight into the present formulation and solution of the optimum growth problem, it is worthwhile utilizing the information implicit in Figure I to scrutinize more closely some particular optimum growth paths. Specifically, for this discussion we assume that k^o and k^T are given such that $0 < k^o < k^T < k^*$, but allow T to take on all values permitted within A. One possible interpretation is that we are restricting attention to a relatively underdeveloped economy with moderate growth ambitions (provided T is sufficiently large and δ is close to zero or negative). Similar observations can be made concerning the analogous set of optimum growth paths given any configuration of initial and minimum terminal capital-labor ratios.

Within the set of particular optimum growth paths specified by (k^o, k^T, T), given $0 < k^o < k^T < k^*$ with $(k^T, T) \epsilon A$, we can distinguish two types. On the first type of path, k increases steadily from $k(0) = k^o$ to $k(T) = k^T$, while q decreases steadily to $q(T) = q^T \ge q^\alpha$, where q^α is defined by

$$\dot{k} = 0 \Big|_{k = k^T} \quad . \tag{24}$$

Let us denote these as (k^T, q^T) paths. On the second type of path, k first increases from $k(0) = k^o$ to $k^{T^*} \epsilon (k^T, k^*)$, and then decreases to $k(T) \ge k^T$, while again q decreases steadily to $0 \le q(T) = q^T < q^\alpha$. Let us denote these as (k^{T^*}, q^{T^*}) paths, where q^{T^*} is defined by

(25) $$\dot{k} = 0 \Big|_{k = k^{T^*}} \, .$$

Observe that each type of path, exemplified in Figure II, is labelled according to the point at which k approaches closest to k^* .

By the foregoing definitions, $\dot{k} > 0$ on any (k^T, q^T) or $(k^{T^*}, q^{T^*})^-$ path, while $\dot{k} < 0$ on any $(k^{T^*}, q^{T^*})^+$ path. On the other hand, with respect to the behavior over time of per capita consumption, as $\dot{q} < 0$ on either type of path, from (17') and (22)

(26) $$\dot{c} = h'(q)\dot{q} > 0$$

when the economy is not specializing in consumption, but

(27) $$\dot{c} = f'(k)\dot{k} = \lambda f'(k)\dot{k} < 0$$

when the economy is specializing in consumption. Thus, as a glance at Figure II will verify, $\dot{c} > 0$ on any (k^T, q^T) or $(k^{T^*}, q^{T^*})^-$ path, while $\dot{c} > 0$ initially but possibly $\dot{c} < 0$ finally on any $(k^{T^*}, q^{T^*})^+$ path.

Such behavior can, of course, be related to the length of the period of direct concern [0, T], for short, the planning period. More precisely, a falling capital-labor ratio would be observed if and only if our relatively underdeveloped economy pursues optimum growth for a planning period of length $T > \tau_1$, and decreasing per capita consumption for a planning period of length $T > \tau_2 > \tau_1$.

The general method by which τ_1 and τ_2 are determined also underlies our later results. Hence, it is useful to detail it here.

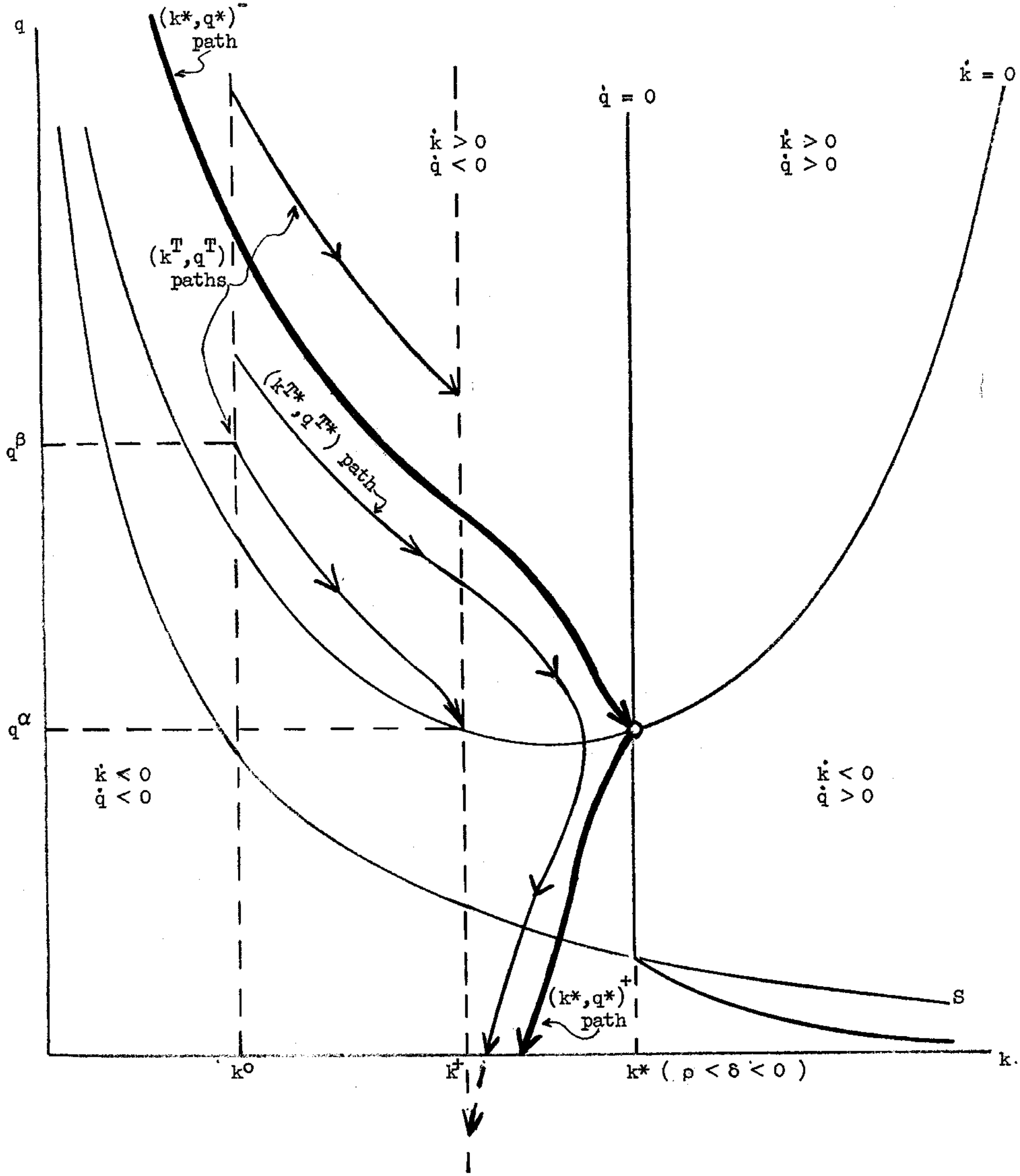

FIGURE II

The Optimum Growth Paths Specified by
(k^o, k^T, T), given $0 < k^o < k^T < k^*$ with $(k^T,T) \in A$

Given the set of optimum growth paths from $k(0) = k^o$ to $k(T) \geq k^T$, suppose we want to compare the lengths of the planning periods associated with portions (including the whole) of any two. Then, if one path traverses the interval $[k^i, k^j]$ $\left(\text{or } [q^i, q^j]\right)$ while the other traverses at least the same interval, both monotonically, by comparing the rates of change of k (or q), the relative lengths of the planning periods associated with the common interval can be easily ascertained. Note that such an unequivocal comparison is only possible here by virtue of the facts a) that the solution to the pair (15') and (18') through any point (k, q) for which $k \geq 0$ is unique,[4] and b) that from (15')-(18') and (22)

$$\text{(28)} \qquad \frac{\partial \dot{k}}{\partial q} = -h'(q) > 0 \,, \quad \text{for } (k, q) \text{ above } S$$

$$= 0 \,, \quad \text{for } (k, q) \text{ below } S \,,$$

and

$$\text{(29)} \qquad \frac{\partial \dot{q}}{\partial k} = -f''(k)q > 0 \,, \quad \text{for } (k, q) \text{ above } S \,.$$

$$= -U''[f(k)]f'(k)^2 + U'[f(k)]\, f''(k) > 0 \,, \quad \text{for } (k, q) \text{ below } S \,.$$

Thus, for example, by comparing $\dot{k}$ over $[k^o, k^T]$ on each (k^T, q^T) path and $\dot{q}$ over $[q^\alpha, q^\beta]$ on each (k^{T*}, q^{T*}) path with the same quantities

---

[4] It is straightforward to show that the RHS of (15') and (18') satisfies the Lipschitz condition on any closed, bounded and convex region of the half plane $k \geq 0$. Hence, this pair satisfies the conditions of the basic existence and uniqueness theorem for systems of differential equations. See, for example, pp. 20-22 and 159-167 in [5], which contains an especially clear statement and proof of this theorem, assuming that the RHS of the system has continuous partial derivatives (though actually requiring only that the RHS satisfies the Lipschitz condition).

on the particular (k^T, q^T) path for which $q(T) = q^\alpha$ (see Figure II), we demonstrate that τ_1 is defined by the planning period associated with this particular (k^T, q^T) path. Similarly, a comparison of each (k^{T*}, q^{T*}) path with the particular (k^{T*}, q^{T*}) path for which $q(T) = U'[f(k^T)]$ (or $z(T)$ just becomes zero) yields the planning period associated with the latter to define $\tau_2 > \tau_1$.

Such comparisons[5] also enable us to assert the uniform approach of first, the (k^T, q^T) path to the minimum time feasible path -- which would appear in our diagram as the horizontal line at $q = \infty$ from k^o to k^T -- as T decreases from τ_1, and second, the (k^{T*}, q^{T*}) path to the portions of the (k^*, q^*) path sketched in Figure II as T increases from τ_1. As a preface to the next section, we stress that the last implies that the optimum growth path spends a relatively and absolutely longer middle length of the planning period close to the (modified) golden rule growth path.

As a final comment, consider the fact that for any (k^{T*}, q^{T*}) path which terminates beyond the minimum terminal target, that is, with $q^{T*}(T) = 0$ and $k^{T*}(T) > k^T$, its continuation which returns to $q(\tau) < 0$ and $k(\tau) = k^T$ (see Figure II) also nominally satisfies the optimality conditions for some $\tau > T$. But our earlier exposition of (I) ruled out this continued (k^{T*}, q^{T*}) path as a possible optimum growth path -- the

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5 In conjunction with the theorem concerning the uniform continuity with respect to initial conditions of the solutions to systems of differential equations whose RHS satisfy the Lipschitz condition. Again see [5], pp. 192-199.

reason being that it is easily shown that the (k^{T*}, q^{T*}) path with a comparable planning period $[0, \tau]$ but which terminates beyond the minimum terminal target is strictly better. The interpretation of this result is straightforward and, in addition, helps to clarify the relation between prescribing an arbitrary terminal target while simultaneously attempting to maximize an interim welfare criterion: Suppose that, instead of the terminal condition $k(T) \geq k^T$, we had postulated the exact terminal target $k(T) = k^T$. Then, any continued (k^{T*}, q^{T*}) path would be optimum for some T. However, the resultant negative imputed prices on the final portion of such a path mean that to use the existing capital stock to provide positive gross investment would be definitely detrimental, in particular, in reaching the desired target. That is, if negative gross investment were permitted within the technology available, say, in the extreme case by some finite maximum rate of deliberate destruction, then it would be optimum over the final stage of that growth path. On the other hand, noting that $U'(c)$, the marginal value of per capita consumption, is always positive, it would also be true that over that same final stage the use of the existing capital stock to provide positive consumption is always definitely beneficial. Hence, there would be some inconsistency between both reaching the arbitrary terminal target and maximizing welfare while doing so. The upshot of this discussion is therefore that by introducing the relaxed terminal condition, we preclude this particular sort of inconsistency, and thereby end up with possibly both more welfare and more capital -- though it should be mentioned that there remains in our formulation a more fundamental inconsistency arising from the somewhat arbitrary imposition of any terminal constraints, that is, the implicit discrimination against generations beyond the horizon.

IV. The Optimum Growth Turnpike Property

We are now in a position to present our central result, stated precisely in the following theorem:

Given any positive number $\epsilon > 0$, *define the closed, rectangular* ϵ-*neighborhood* $N(\epsilon)$ *of the (modified) golden rule growth path by*

$$N(\epsilon) = \left\{(k,\ q):\ |k-k^*| \leq \epsilon\ ,\ |q-q^*| \leq \epsilon\right\}\ . \tag{30}$$

Then, for the unique optimum growth path $\left\{\left(k(t),\ q(t)\right):\ 0 \leq t \leq T\right\}$ *specified by the initial and terminal parameters* $(k^o,\ k^T,\ T)$, *there exist two finite times* $0 \leq T_1 < \infty$ *and* $0 \leq T_2 < \infty$,

$$T_1 = T_1(\epsilon,\ k^o)\ ,\quad T_2 = T_2(\epsilon,\ k^T)\ , \tag{31}$$

such that $\left(k(t)\ ,\ q(t)\right) \in N(\epsilon)$ *whenever* $T_1 \leq t \leq T-T_2$.

Thus, defining the "sufficiently long period" mentioned in the introduction by

$$\mathcal{T} > T_1 + T_2\ , \tag{32}$$

the theorem asserts a strong turnpike property for optimum growth over any planning period $[0,\ \mathcal{T}]$, in the sense that it states that such growth occurs within an arbitrarily small neighborhood of the "best" balanced growth path except possibly over some initial or terminal phase. We also emphasize that this theorem is stated in terms of both the real stock variable k and the imputed price variable q -- the reason being that the concept of optimum growth advanced in this paper is intimately related to the flow variable c , and the turnpike property thereby encompasses it, along with the remaining flow variable of the model z, by virtue of the relations (16') and (17').

That the theorem is true has already been suggested by the delineation of the limiting (k^T, q^T) and (k^{T^*}, q^{T^*}) paths; further heuristic proof is developed by reference to the constructions presented in Figures III and IV. Without loss of generality we can assume that $|k^o - k^*| > \epsilon$ and $|k^T - k^*| > \epsilon$. We also only consider the case for which $k^o < k^*$, as the alternative case is essentially similar, and, for the first part of the discussion, the subcase analyzed in the last section for which $0 < k^o < k^T < k^*$.

Thus, for given k^o and k^T, $0 < k^o < k^T < k^*$, we initially examine a set of (k^T, q^T) and (k^{T^*}, q^{T^*}) paths. In order to make relevant comparisons in this set, we distinguish the lengths of planning period associated with first, the particular $(k^{T^*}, q^{T^*})^-$ and $(k^{T^*}, q^{T^*})^+$ paths on which $k^{T^*} = k^* - \epsilon$,[6] and second, the portions of the $(k^*, q^*)^-$ and $(k^*, q^*)^+$ paths which just intersect the lines $k = k^o$ and $q = q^* + \epsilon$ or the lines $q = q^* - \epsilon$ and the prior of $k = k^T$ or $q = 0$, respectively. Denote the former by T_1^k and T_2^k, and the latter by T_1^q and T_2^q, as in Figure III.

Employing the method outlined earlier, it is easily established through direct comparison that $T_1^k + T_2^k$ is longer than the planning period associated

6 It is implicitly assumed in the ensuing argument that the curve $\dot{k} = 0$ going away from the point (k^*, q^*) intersects the line $k = k^* \mp \epsilon$ before it intersects either of the lines $q = q^* \pm \epsilon$ (see Figure III). Then this particular (k^{T^*}, q^{T^*}) path is the optimum path which just enters $N(\epsilon)$; for $k^{T^*} > k^* - \epsilon$ $(k^{T^*} < k^* - \epsilon)$ the (k^{T^*}, q^{T^*}) path enters (does not enter) $N(\epsilon)$. An argument analogous to that in the text can be presented to cover each of the other possibilities, but for our purpose it would entail needless added complication.

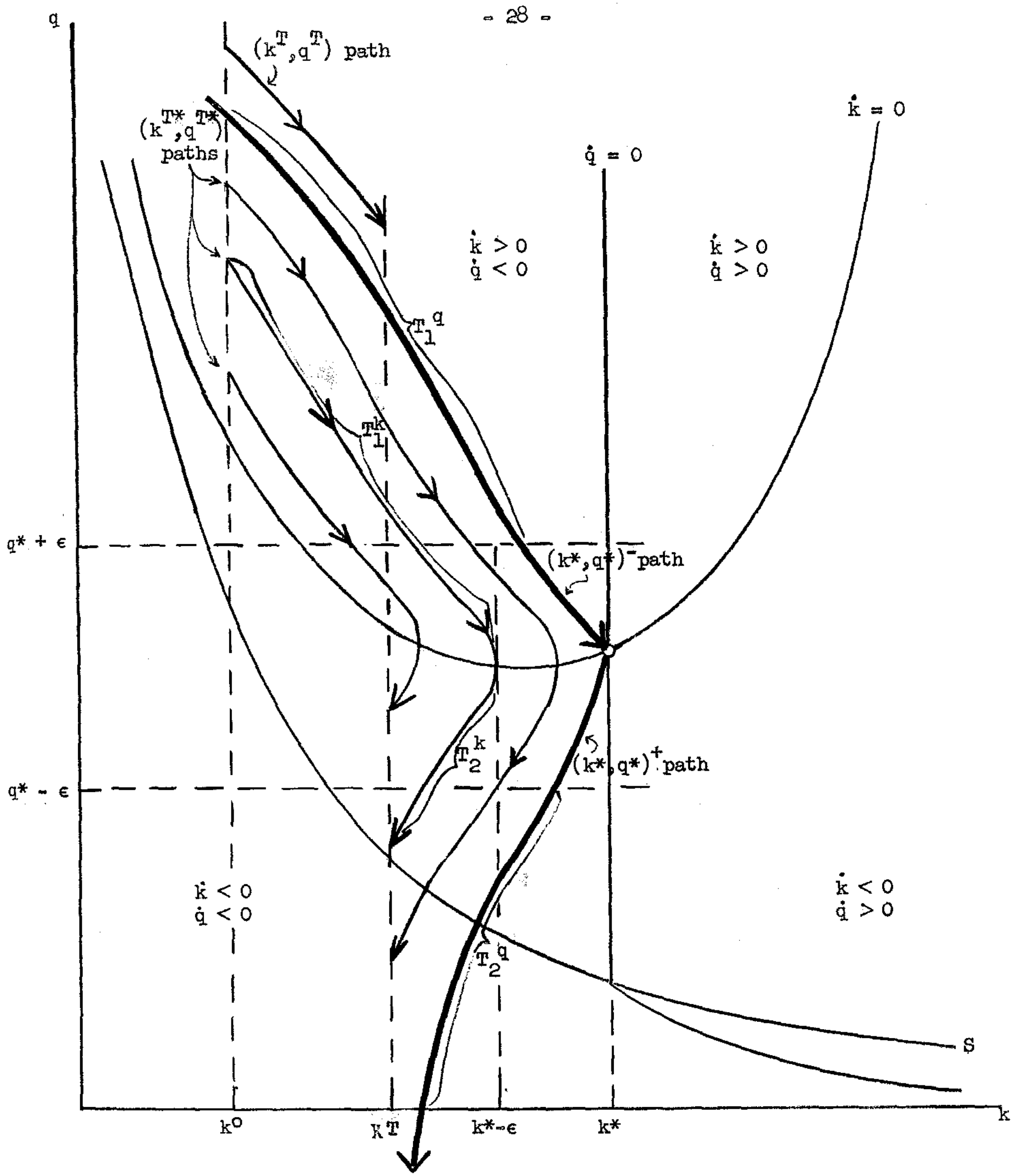

FIGURE III

The Optimum Growth Turnpike Property: $k^o < k^T < k^*$

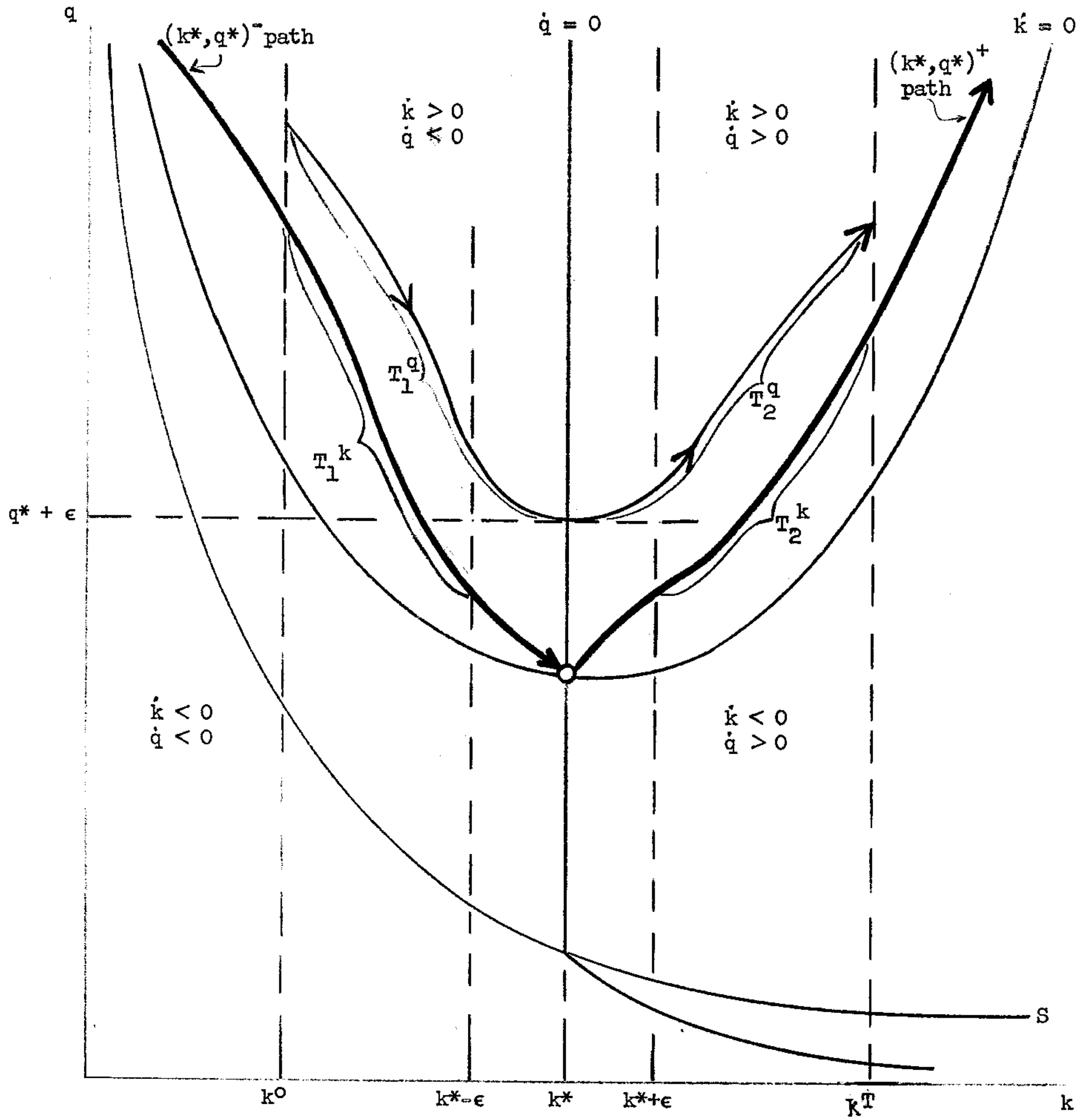

FIGURE IV

The Optimum Growth Property: $k^o < k^* < k^T$

with any (k^{T^*}, q^{T^*}) path which does not enter $N(\epsilon)$, and therefore is also longer than the planning period associated with any (k^T, q^T) path. Of equal consequence, it is likewise easily demonstrated that either T_1^k or T_1^q (either T_2^k or T_2^q) is longer than the length of planning period associated with the entrance of any (k^{T^*}, q^{T^*}) path into $N(\epsilon)$ $\Big($the departure of any (k^{T^*}, q^{T^*}) path out of $N(\epsilon)\Big)$.

Hence, we have justified the use of the lengths of planning period T_1^k or T_1^q and T_2^k or T_2^q to define the times whose existence is asserted T_1 and T_2 by

$$(28) \qquad T_1 = \max\,(T_1^k\,,\, T_1^q) \quad \text{and} \quad T_2 = \max\,(T_2^k\,,\, T_2^q)\,,$$

in the case of any (k^T, q^T) path or any (k^{T^*}, q^{T^*}) path which does not enter $N(\epsilon)$, because the segment of planning period $[T_1, T\text{-}T_2]$ is empty, while in the case of any (k^{T^*}, q^{T^*}) path which does enter $N(\epsilon)$, because the path enters or departs $N(\epsilon)$ in at most a length of time T_1 or T_2, respectively.

The arguments establishing the times T_1 and T_2 for the other subcases are also quite straightforward:

For $k^T < k^o < k^*$, we can distinguish (k^o, q^o) paths, on which both k and q decrease steadily from $k(0) = k^o$ and $q(0) = q^o \leq q^\alpha$, where q^α is defined by

$$(21') \qquad \dot{k} = 0\Big|_{k = k^o}\,,$$

from (k^{o^*}, q^{o^*}) paths, on which k first increases from $k(0) = k^o$ to $k^{o^*} \in (k^o, k^*)$, and then decreases to $k(T) \geq k^T$, while q decreases steadily, where q^{o*} is defined by

(22') $$\dot{k} = 0 \Big|_{k = k^{o*}} .$$

Therefore, this subcase is essentially the same as that discussed in the preceding paragraphs, but with the superscript T replaced by the superscript o.

For $k^o < k^* < k^T$, on all optimum growth paths k increases steadily from $k(0) = k^o$ to $k(T) = k^T$, while q first decreases to $q^{\infty*} \epsilon (q^*, \infty)$, and then increases. Denote these paths, in analogy with our other notation, as $(k^{\infty*}, q^{\infty*})$ paths, where $k^{\infty*}$ is defined by

(29) $$\dot{q} = 0 \Big|_{q = q^{\infty*}} \quad \text{or} \quad k^{\infty*} = k^* .$$

Then, the lengths of the planning period associated with first, the particular $(k^{\infty*}, q^{\infty*})^-$ and $(k^{\infty*}, q^{\infty*})^+$ paths on which $q^{\infty*} = q^* + \epsilon$, and second, the portions of the $(k^*, q^*)^-$ and $(k^*, q^*)^+$ paths which just intersect the lines $k = k^o$ and $k = k^* - \epsilon$ or the lines $k = k^* + \epsilon$ and $k = k^T$, respectively, can be utilized to define T_1 and T_2 as in (28). This subcase is illustrated in Figure IV.

It only remains to show that T_1 depends primarily on the parameters k^o and ϵ, and T_2 primarily on k^T and ϵ. (Of course, these times also depend on the parameters and functions defining the underlying model.) This is most directly accomplished thusly: For notational convenience we again concentrate on the subcase for which $0 < k^o < k^T < k^*$. Then, on either the particular $(k^{T*}, q^{T*})^-$ and $(k^{T*}, q^{T*})^+$ paths on which $k^{T*} = k^* - \epsilon$ (note that the choice of these particular paths depends on ϵ) or the $(k^*, q^*)^-$ and $(k^*, q^*)^+$ paths, both k and q are monotonic functions of time. It follows that on any of these paths k is a unique function of q, and conversely. Hence, by separating variables and integrating, (15') and (18') yield

$$T_1^k = -\int_{k^*-\epsilon}^{k^o} \frac{dk^{T*}}{z[k^{T*}, q^{T*}(k^{T*})] - \lambda k^{T*}} = T_1^k(k^o, \epsilon),$$

$$T_1^q = -\int_{q^*+\epsilon}^{q^*(k^o)} \frac{dq^*}{(\delta+\lambda)q^* - U'\left(c[k^*(q^*), q^*]\right) f'[k^*(q^*)]} = T_1^q(k^o, \epsilon),$$

$$T_2^k = \int_{k^*-\epsilon}^{\max\left(k^{T*}(0),\, k^T\right)} \frac{dk^{T*}}{z[k^{T*}, q^{T*}(k^{T*})] - \lambda k^{T*}} = T_2^k(k^T, \epsilon),$$

and

$$T_2^q = \int_{q^*-\epsilon}^{\max\left(0,\, q^*(k^T)\right)} \frac{dq^*}{(\delta+\lambda)q^* - U'\left(c[k^*(q^*), q^*]\right) f'[k^*(q^*)]} = T_2^q(k^T, \epsilon),$$

or from (28),

(30) $$T_1 = \max\left\{T_1^k(k^o, \epsilon),\ T_1^q(k^o, \epsilon)\right\} = T_1(k^o, \epsilon)$$

and

$$T_2 = \max\left\{T_2^k(k^T, \epsilon)\ ,\ T_2^q(k^T, \epsilon)\right\} = T_2(k^T, \epsilon)\ . \tag{31}$$

V. The Optimum Growth Turnpike Property: A Special Case

In this section we discuss optimum growth over the planning period $[0, T]$ where the criterion of social welfare is given by (6') while per capita consumption is constrained by (8'). To simplify the discussion, several relatively uninteresting possibilities are excluded. In particular, we assume that the minimum subsistence level is low enough so that

$$\underline{k} < k^o\ , \tag{32}$$

provision of subsistence consumption would not entail steady deterioration of the capital-labor ratio, and

$$\underline{k} < k^* < \overline{k}\ , \tag{33}$$

the (modified) golden rule growth path provides more than subsistence consumption, where $\underline{k} < \overline{k}$ are thus assumed to be the two solutions to

$$f(k) = k + \underline{c}\ ,$$

the equation defining balanced growth which just provides subsistence consumption. Obviously, (33) is equivalent to imposing upper and lower bounds on the permissible range of the social discount rate,

$$f'(\overline{k}) - \lambda < \delta < f'(\underline{k}) - \lambda\ .$$

Furthermore, consistency in setting the minimum terminal capital-labor ratio now requires the assumption that

$$(34) \qquad f(k^T) \geq \underline{c} \, ,$$

subsistence consumption be possible at that capital-labor ratio. (34) and (8') imply that the set of attainable and feasible terminal parameters is now of the form

$$A = \left\{ (k^T, T) \; : \; \underline{k} \leq k^T < \bar{\bar{k}} \, , \; g(k^T, T) \geq 0 \right\} ,$$

with

$$(11') \qquad \bar{\bar{k}} = \bar{k} \, , \quad \text{for} \quad k^o \leq \bar{k}$$

$$= k^o , \quad \text{for} \quad \bar{k} < k^o < \infty \, ,$$

and

$$(12') \qquad g(k^T, T) = T - \int_{k^o}^{k^T} \frac{dk}{f(k) - \lambda k - \underline{c}} \, , \quad \text{for} \quad k^o \leq k^T$$

$$= \int_{k^o}^{\max(k^T, \bar{k})} \frac{dk}{f(k) - \lambda k - \underline{c}} - T \, , \quad \text{for} \quad k^o > k^T .$$

The foregoing assumptions are depicted satisfied in Figure V. (Note that we would also avoid the complications assumed away by (32)-(34) if the necessity of some minimum subsistence level were simply ignored.)

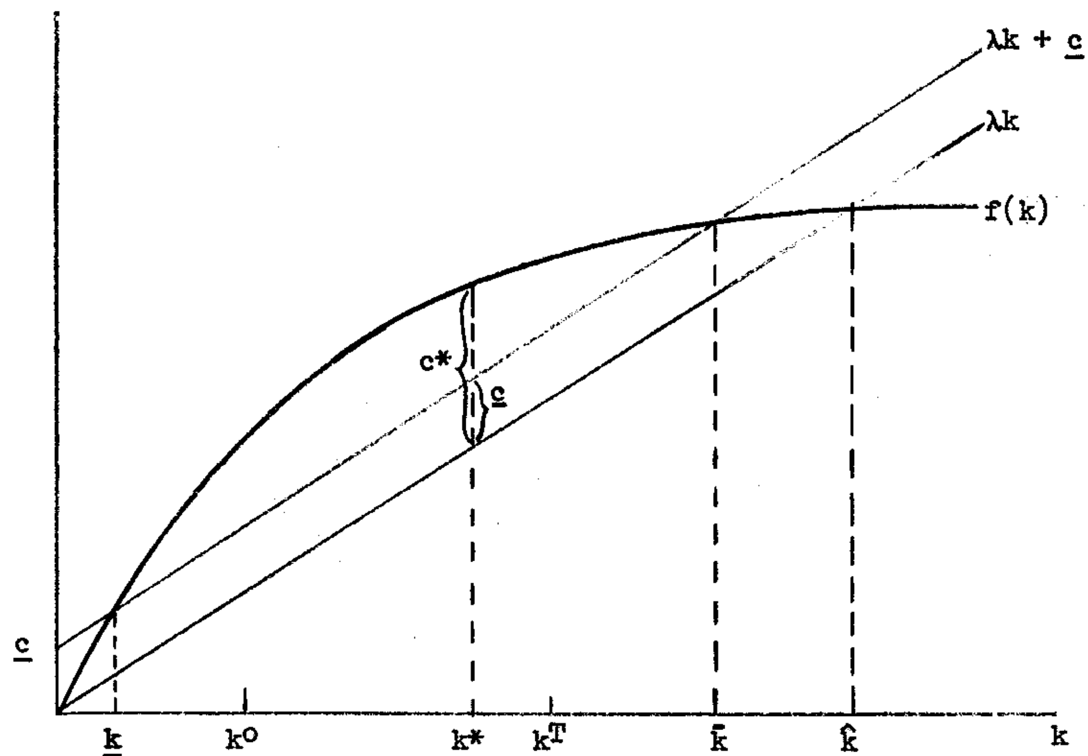

FIGURE V

For this special case, the imputed value of gross national product per capita becomes

$$(14'') \qquad \psi = c + qz + (p-1)(c-\underline{c}) = pc + qz - (p-1)\underline{c} \ ,$$

where now p as well as q is an imputed price, the marginal value or imputed price of a unit of consumption per capita. Then, by applying an extension of Pontryagin's Maximum Principle (presented in Chapter 6 of [4]) to the Hamiltonian representing the present imputed value of net national product per capita,

$$(\psi-q\lambda k)e^{-\delta t} \ ,$$

and pursuing an argument like that on pp. 11-13, it follows that the necessary and sufficient conditions for an optimum growth path have exactly the same interpretation as before, but that (15), (16) and (17) become

$$(15'') \qquad \dot{q} = (\delta+\lambda)q - pf'(k) \ ,$$

$$(16'') \qquad c + z = f(k) \ , \quad c \geq \underline{c} \ , \quad z \geq 0 \ ,$$

and

$$(17'') \qquad p \geq 1 \ , \ \text{with strict equality for} \ \ c > \underline{c}$$

$$q \leq p \ , \ \text{with strict equality for} \ \ z > 0 \ ,$$

while (18), (I) and (II) remain unchanged. Likewise, the same argument which establishes sufficiency also establishes uniqueness.

A straightforward reformulation of the system (15'')-(17'') and (18) enables us to distinguish three possible phases of optimum growth:

Phase I
$$\begin{cases} q = p \geq 1 \\ c = \underline{c},\ z = f(k)-\underline{c} \\ \dot{q} = \left(\delta+\lambda-f'(k)\right) q \\ \dot{k} = f(k)-\lambda k-\underline{c}\ , \end{cases}$$

specialization in investment above provision of subsistence consumption;

Phase II
$$\begin{cases} q^* = p^* = 1 \\ c = c^*,\ z = z^* \\ \dot{q}^* = \delta+\lambda-f'(k^*) = 0 \\ \dot{k}^* = f(k^*) - \lambda k^*-c^* = 0\ , \end{cases}$$

non-specialization on the (modified) golden rule growth path; and

Phase III
$$\begin{cases} q \leq p = 1 \\ c = f(k),\ z = 0 \\ \dot{q} = (\delta+\lambda)q-f'(k) \\ \dot{k} = -\lambda k\ , \end{cases}$$

specialization in consumption. These three phases, along with the behavior of some particular optimum growth paths, are illustrated in Figure VI (assumed to be self explanatory in the context of the rest of the paper).

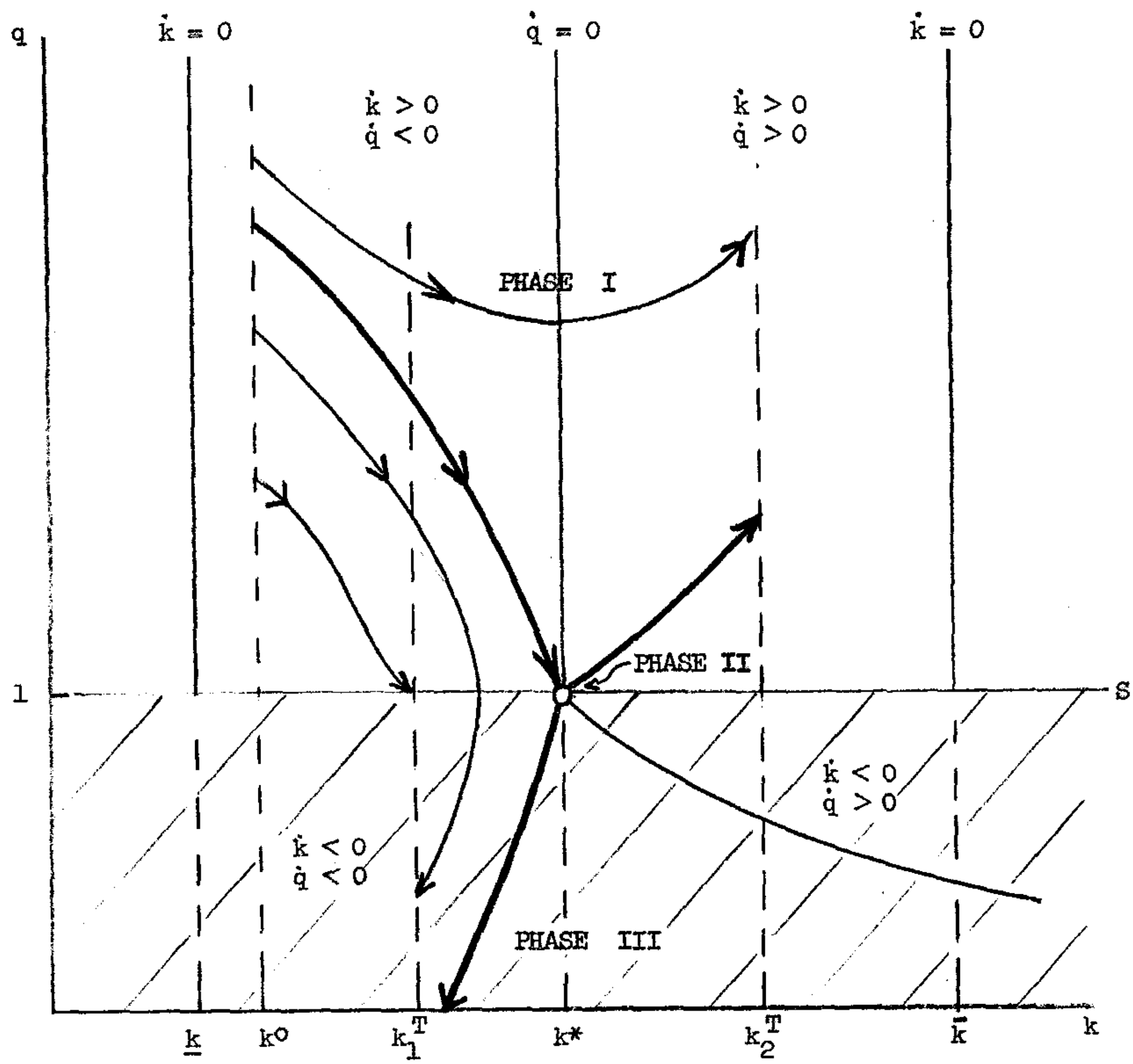

FIGURE VI

The Optimum Growth Turnpike Property: A Special Case

The phases which are actually achieved on the unique optimum growth path specified by (k^o, k^T, T) depend on these latter parameters. However, because the economy is either specializing or performing balanced growth, it is possible to strengthen the turnpike property for this special case to:

For the unique optimum growth path $\left\{\left(c(t), z(t), k(t), q(t), p(t)\right): 0 \leq t \leq T\right\}$ specified by the initial and terminal parameters (k^o, k^T, T) there exist two finite times $0 \leq T_1 < \infty$, and $0 \leq T_2 < \infty$,

(35) $$T_1 = T_1(k^o), \quad T_2 = T_2(k^T),$$

such that if $T > T_1 + T_2$, then $\left(c(t), z(t), k(t), q(t), p(t)\right)$ is in Phase I $(k^o < k^*)$ or Phase III $(k^o > k^*)$ when $0 \leq t \leq T_1$, switches into Phase II when $T_1 \leq t \leq T-T_2$, and returns to Phase I $(k^T > k^*)$ or Phase III $(k^T < k^*)$ when $T-T_2 \leq t \leq T$.

Thus, for any sufficiently long planning period, the course of optimum growth is first, specialization to achieve the (modified) golden rule growth path, second, balanced growth actually on the (modified) golden rule growth path,[7] and third, again specialization, but to achieve a terminal state in which the transversality conditions (I) and (II) are satisfied.

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[7] This result is specific to the aggregative model of capital accumulation. With differing technology in the consumption goods and capital goods industries, in general the non-specialized phase entails only an asymptotic approach to balanced growth. For a detailed treatment of two-sector optimum growth under the criterion (6') (with $T = \infty$, $\delta > 0$) and constraint (8') see [10].

For proof we merely exhibit T_1 and T_2 derived directly from the capital-labor ratio growth equation for Phase I or III:

(36) $$T_1(k^o) = \int_{k^o}^{k^*} \frac{dk}{f(k)-\lambda k-\underline{c}}\ , \quad \text{for} \quad k^o \leq k^*$$

$$= \int_{k^o}^{k^*} \frac{dk}{\lambda k}\ , \quad \text{for} \quad k^o > k^*\ ,$$

and

(37) $$T_2(k^T) = \min\left(\int_{k^*}^{k^T} \frac{dk}{\lambda k}\ , \tau\right)\ , \quad \text{for} \quad k^T < k^*$$

$$= \int_{k^*}^{k^T} \frac{dk}{f(k)-\lambda k-\underline{c}}\ , \quad \text{for} \quad k^T > k^*\ ,$$

where τ is defined implicitly by solving for $q(\tau) = 0$ in Phase III, given $\Big(k(0),\ q(0)\Big) = (k^*, 1)$,

$$1 = \int_0^{\tau} f'[k^* e^{-\lambda s}] e^{-(\delta+\lambda)s} ds\ .$$

Hence, because the initial values of p and q can always be adjusted to correspond to Phase I $(k^o < k^*)$ or Phase II $(k^o > k^*)$ and yet yield $p(T_1) = q(T_1) = 1$, while allocation can always be adjusted to switch into

Phase I $(k^T > k^*)$ or Phase III $(k^T < k^*)$ at time $T-T_2$ and thereby result in $q(T)\left\{k(T)-k^T\right\} = 0$, it follows that the growth path described in the theorem must be the unique optimum growth path for $T > T_1 + T_2$.

As a concluding remark to this section we observe that with Harrod neutral or labor efficiency augmenting technical progress at the constant rate θ, the preceding analysis also applies, provided a) the real variables c, z and k are reinterpreted as being measured in terms of efficient labor units, b) the (effective) social discount rate δ and growth parameter λ are understood to include a contribution from θ,

$$\delta = \delta' - \theta \quad \text{and} \quad \lambda = n + \mu + \theta$$

and c) further assumptions are made concerning the specification of k^T and $\underline{c}$. For example, $\underline{c}$ could be prescribed in either efficient labor or per capita units, or some combination in between, in the face of foreseeable progress. For the former the analysis remains unchanged in the sense that again the simplifying assumptions (32) and (33) are appropriate. On the other hand, for either of the latter, with sufficiently long planning periods the constraint (8') would eventually become relatively unimportant, and a complete analysis would be modified accordingly. We forego detailed discussion of the ramifications stemming from different, possible specifications of k^T and $\underline{c}$.

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