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OPTIMUM MIXING OF INERTIAL NAVIGATOR DATA AND RADAR DATA

by

Michel Claude Brayard

June 1969

Degree of Master of Science

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Michel Claude Brayard

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Submitted to the Department of Aeronautics and Astronautics on June 24, 1969, in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

The filtering problem is studied for a system composed of an inertial navigation system giving continuous indication of position and velocity, and a radar or some other external device giving continuous or discrete positional information. After a brief review of the results of the filtering theory using the state variables representation, an error analysis yields three different mathematical models for the I.N.S., represented by a set of linear differential equations that can be written with the state variables method. From there, the equations for both the continuous and the discrete filtering schemes are derived, assuming the only error sources are a white noise at the acceleration level in the I.N.S. and a white or Gaussian (in the discrete measurement mode) noise in the radar. Functional relations are obtained between the position and velocity root mean square errors and the following characteristic quantities: noises in both the I.N.S. and the radar and operating time (time between the measurements) for the discrete filter scheme.

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The publication of this thesis does not constitute approval by the National Aeronautics and Space Administration or the Measurement Systems Laboratory of the findings or the conclusions contained therein. It is published only for the exchange and stimulation of ideas.

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NOTATIONS

Underlined lower or upper case letters are column vectors. For instance $\underline{x}$ denotes a vector with components: x_1, x_2, ...,x_n; $\underline{x}'$ is the transpose of $\underline{x}$ and is a row vector. Capital letters denote matrices, except for the noises N and R which can be either scalar or matrices. The transpose of the matrix M is denoted M'.

Scalars

t	time
t_m^-	time just before the measurement at t_m
t_m^+	time just after the measurement at t_m
R_e	earth radius
ω_{ie}	earth rate
C_x, C_y, C_z	misalignment angles of the platform with respect to the navigation frame
L	latitude
l	longitude
p_{ij}	elements of the matrix P_x ; i^{th} row, j^{th} column
m_{ij}	continuous filtering compensation terms
k_{ij}	elements of the gain matrix K; i^{th} column, j^{th} row
a, b, c	parameters for models 1 and 2
RMX - RMY	r.m.s. north and east position errors
RVX - RVY	r.m.s. north and east velocity errors
OPT or Δt	operating time

Vectors

$\underline{x}$	state vector
$\underline{u}$	I.N.S. noise vector
$\underline{v}$	radar noise vector

$\underline{y}$ output of the linear system

$\underline{m}$ measurement vector

$\underline{x}$ optimum estimate of $\underline{x}$

$\underline{f}$ specific force

$\underline{g}$ gravity vector

$\underline{C}$ misalignment angles vector

$\underline{W}^{c}_{ab}$ angular rate of frame b with respect to frame a coordinatized in frame c

Matrices

F system parameters matrix

G input distribution matrix

H output distribution matrix

$\Phi(t,s)$ state transition matrix from time s to time t

I unit matrix

Q or N I.N.S. noise power spectral densities matrix

R radar noise power spectral densities matrix or covariance matrix

[PF] performance function

[P] velocity computation matrix (page 23)

K gain matrix

Others

p Laplace operator

$\dot{x}$ time derivative of the quantity x

c subscript denoting a variable computed by the system

n subscript or superscript denoting navigation frame

cm " " " platform frame

i " " " inertial frame

e " " " earth frame

sk superscript denoting a skew-symmetric matrix

CHAPTER I

INTRODUCTION

The design of an optimal filter to yield actual position and velocity, when given data from an Inertial Navigation System and radar, depends upon the model chosen to represent the I.N.S.

The degree of sophistication of the filter must be matched with the accuracy of the available instruments and measurements, and since a short operating time for the reset of the I.N.S. lowers the accuracy requirements, this fact must also be considered in designing the filter.

We can consider 3 models for the I.N.S.:

model I : platform misalignment and cross-coupling between the axes are neglected; so, it is possible to study the two channels separately.

model 2 : platform misalignment angles are introduced but the cross-coupling is still neglected.

model 3 : finally, both misalignment angles and cross coupling are taken into account; this is theoretically the most accurate filter, but also the most complex one.

It will be shown later that the steady state r.m.s. errors do not depend on the model chosen to represent the I.N.S.

Although the filter cannot work in a continuous way because of the time required for the computations of the varying gains, it may appear useful to study this continuous filtering which is the optimum and may be used as a reference for choosing the parameters of the discrete filter.

As a first approach to this problem, only two error sources will be considered: I.N.S. noise which appears as a white noise at the acceleration level and measurement noise (radar noise) which is also considered as a white noise. These two noises are to be characterized in the following way: the I.N.S.noise by its power spectral density N or Q in all the cases; the measurement noise by R, which is a power spectral density in the continuous process and a covariance matrix in the discrete one. This difference will be necessary in part 5-3, in order to relate any discrete process to one particular continuous one.

From both the theory and the results of the computation, functional relations are derived between the noises, the operating time and the 'steady state' r.m.s. position and velocity errors. The final charts I7 through I9 enable one to choose the instruments in order to match the accuracy requirements for a given mission.

It is to be understood that,whenever the position radar appears in the text, any other external position information can be used instead, including radio-navigation and observation, provided that the noise in this information is a Gaussian white noise.

Furthermore, only the errors in the optimum estimate are to be studied. This means that we shall not study the way this estimate must be generated from both informations. Although some equations as well as some signal flow diagrams for this estimator are given in this paper, only the variance equation, yielding the covariance matrix and the errors, is solved.

CHAPTER 2

FILTERING THEORY : PRINCIPAL RESULTS

None of the equations of the filtering theory are derived. Only the principal results that will be used throughout this paper are summarized. Further information about these equations and their derivation can be found in references [1] and [2] .

2.1 State transition method

Any linear dynamical system described by a set of ordinary differential equations can always be represented by the single equation:

$$\dot{\underline{x}}(t) = F(t)\ \underline{x}(t) + G(t)\ \underline{u}(t) \qquad (2\text{-}1)$$

$\underline{x}(t)$ is the <u>state vector</u> of dimension n , its coordinates x_i are the state variables.
$F(t)$ is the <u>system parameters matrix</u>.
$G(t)$ is the (n.l) <u>input distribution matrix</u>.
$\underline{u}(t)$ is a vector of dimension l called <u>control vector</u>.

It is assumed that both matrices F(t) and G(t) are continuous functions of time.

The system is completely described when the vector output $\underline{y}(t)$, which can be of any dimension m less than n, is written as a linear combination of the state variables:

$$\underline{y}(t) = H(t)\ \underline{x}(t) \tag{2-2}$$

where H(t) is the (m.n) output distribution matrix.

The block diagram of this system is given on figure 1.

F(t) represents the dynamics of the system.
G(t) " the constraints due to inputs.
H(t) " the constraints on observing the system from outputs.

The general solution of this system (see [3] and [4]) is:

$$\underline{x}(t) = \Phi(t,t_o)\ \underline{x}(t_o) + \int_{t_o}^{t} \Phi(t,s)\ G(s)\ \underline{u}(s)\ ds \tag{2-3}$$

where $\Phi(t,s)$ is the state transition matrix, which is always nonsingular and obeys the differential equation:

$$\frac{\partial \Phi(t,s)}{\partial t} = F(t)\ \Phi(t,s) \tag{2-4}$$

with the "initial condition" $\Phi(t,t) = I$ (unit matrix).

2.2 Stochastic linear differential equation

When a linear dynamical system is driven by some vector noise $\underline{v}(t)$, its state obeys the equation (2-1), namely:

$$\dot{\underline{x}}(t) = F(t)\,\underline{x}(t) + G(t)\,\underline{u}(t) \tag{2-5}$$

If $\underline{u}(t)$ is a white noise - i.e. a noise the power spectral density of which is a constant over some bandwidth- this equation becomes a <u>stochastic linear differential equation</u> and its solution is again

$$\underline{x}(t) = \Phi(t,t_o)\,\underline{x}(t_o) + \int_{t_o}^{t} \Phi(t,s)\,G(s)\,\underline{u}(s)\,ds$$

In order to study the statistics of $\underline{x}(t)$, we define

$$\begin{cases} \text{mean of } \underline{x}(t) = \underline{m}_x(t) = E\left[\underline{x}(t)\right] \\ \text{covariance matrix of } \underline{x}(t) = P_x(t) = E\left[\underline{x}(t).\underline{x}'(t)\right] \end{cases}$$

Then, it can be easily found that:

$$\begin{cases} \underline{m}_x(t) = \Phi(t,t_o)\,\underline{m}_x(t_o) \\ P_x(t) = \Phi(t,t_o)\,P_x(t_o)\,\Phi'(t,t_o) \\ \qquad + \int_{t_o}^{t} \Phi(t,s)\,G(s)\,Q(s)\,G'(s)\,\Phi'(t,s)\,ds \end{cases} \tag{2-6}$$

where Q(s) will be called the strength of the white noise and is defined by the relation

$$E\left[\underline{u}(t).\underline{u}'(t+s)\right] = Q(t)\,\delta(s) \tag{2-7}$$

$\delta(s)$ is a <u>l-dimension vector delta function</u> such that

$$\begin{cases} \delta(s) = 0 \quad \text{if } s \text{ is not } 0 \\ \int_{-\infty}^{+\infty} ds_1 \int_{-\infty}^{+\infty} ds_2 \ldots\ldots \int_{-\infty}^{+\infty} ds_L \; \delta(s) = 1 \end{cases}$$

2.3 The general filtering theory

The message is a random process $\underline{x}(t)$ generated by a model obeying a stochastic linear differential equation:

$$\dot{\underline{x}}(t) = F(t)\,\underline{x}(t) + G(t)\,\underline{u}(t) \qquad (2\text{-}8)$$

The observed signal, or measurement is

$$\underline{m}(t) = H(t)\,\underline{x}(t) + \underline{v}(t) \qquad (2\text{-}9)$$

where $\underline{u}(t)$ and $\underline{v}(t)$ are white noises with zero means and covariance matrices

$$\left|\begin{array}{l} P_u(t) = E\left[\underline{u}(t)\ \underline{u}'(s)\right] = Q(t)\,\delta(t-s) \\ P_v(t) = E\left[\underline{v}(t)\ \underline{v}'(s)\right] = R(t)\,\delta(t-s) \\ \text{and}\quad E\left[\underline{u}(t)\ \underline{v}(t)'\right] = 0 \end{array}\right.$$

We assume the matrices $Q(t)$ and $R(t)$ are non-negative definite matrices continuously differentiable.

Figure 2 represents the block diagram of this system.

The filtering problem is then to determine from the measurement $\underline{m}(t)$ the best estimate $\hat{\underline{x}}(t)$, best in the sense that it maximizes the conditional probability density function $f_{x/M}(a/b)$ of the state vector $\underline{x}$ conditioned on the values (a priori past values as well as actual ones) of the measurements.

We only give the results of this theory: see [1] for free system and continuous filtering, and [5] for discrete filtering.

2.3.1 Free system

The system simply obeys the equation (2-8). No measurement is taken so that the best estimate $\hat{\underline{x}}(t)$ is $\underline{m}_x(t)$ with variance $P_x(t)$ given by equation (2-6).

It is easier to write these equations as differential equations instead of integral ones. Differentiating (2-6) and using (2-4) yield:

$$\begin{cases} \dot{\hat{\underline{x}}}(t) = F(t)\ \hat{\underline{x}}(t) \\ \dot{P}_x(t) = F(t)\ P_x(t) + P_x(t)\ F'(t) + G(t)\ Q(t)\ G'(t) \end{cases} \tag{2-10}$$

2.3.2 Continuous filtering

Measurement $\underline{m}(t)$ is now taken in a continuous way. It can be shown that, in this case, the above equations become:

$$\begin{cases} \dot{\hat{\underline{x}}}(t) = F(t)\hat{\underline{x}}(t) + P_x(t)H'(t)R^{-1}(t)\left[\underline{m}(t)-H(t)\hat{\underline{x}}(t)\right] \\ \dot{P}_x = FP + PF' + GQG' - PH'R^{-1}HP \end{cases} \tag{2-11}$$

In the last equation the variable t has been dropped and this will stand throughout this paper whenever there is no ambiguity.

2.3.3 Discrete filtering

The measurement can only be taken at discrete times. The time between two measurements is assumed constant and called the <u>operating time</u>. The scheme is the following one:

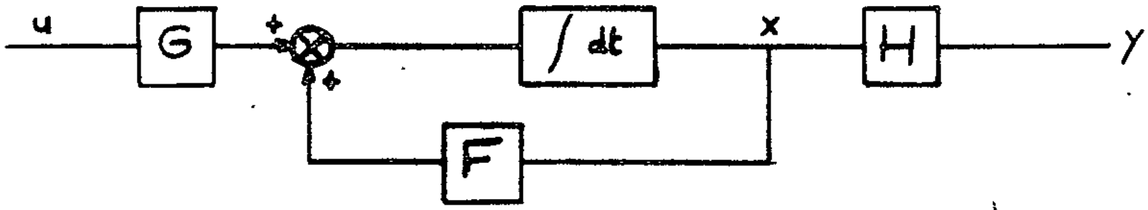


figure 1 : diagram of the dynamical system

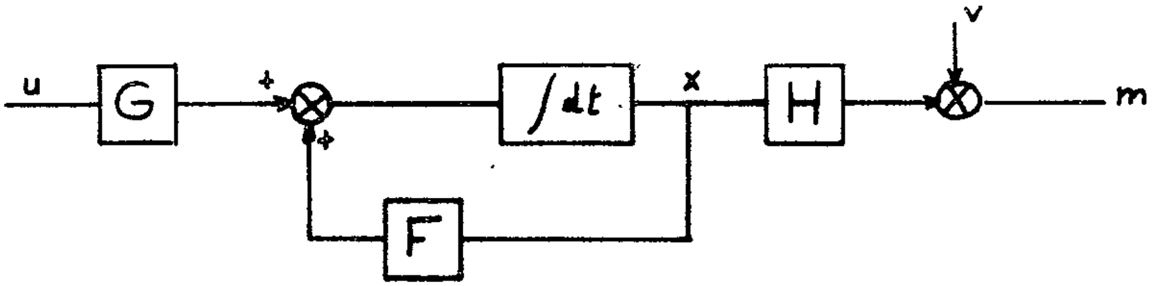


figure 2 : diagram of the measurement process

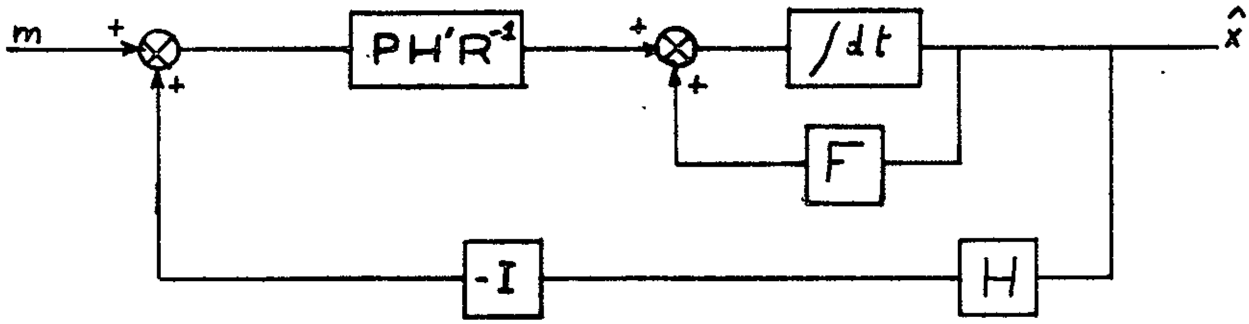


figure 3 : continuous filter diagram

Between the measurements, the system is free and obeys (2-10)
At each measurement, we compute

$$K(t_m) = P'(t_m)H'(t_m)\left[H(t_m)P'(t_m)H'(t_m)+R(t_m)\right]^{-1}$$

Then the best estimate is given by

$$\begin{cases} \hat{\underline{x}}(t_m) = K(t_m)\underline{m}(t_m) \quad \text{with covariance matrix:} \\ P_x(t_m^+) = \left[I-K(t_m)H(t_m)\right] P_x(t_m^-) \end{cases} \qquad (2\text{-}12)$$

In this last equation,'-' means just before the measurement and '+' just after the measurement.

The diagram of the continuous filter is given in figure 3.

2.4 The filtering problem in this paper

The general filtering theory assumes that one can take measurements of some coordinates of the state vector $\underline{x}$ which is itself generated by the system.

The problem is not quite the same here and is to be understood in a different way:

1. the state vector $\underline{x}(t)$, which can include position and velocity informations obeys the homogeneous equation:

$$\dot{\underline{x}}(t) = F(t)\,\underline{x}(t)$$

where $F(t)$ depends primarily upon the trajectory.

2. some indications about position, velocity and other co-

ordinates of the state vector are available as outputs of an inertial navigation system. But due to measurements and instruments inaccuracies, some noise is added so that the output obeys equation (2-8); we limit ourselves in this paper to the case where the only noise source is the gyro drift. Furthermore, following [6] and [7], we assume that this drift can be well enough approximated by a white noise at the acceleration level.

3. on another hand, some components of the state vector are available as outputs to an external device - external to the inertial system - This includes altimeter, Doppler and position radar, etc... Here again, the inaccuracy yields a noise term in equation (2-9) which represents the external information.

Thus, we have here two different measurements of the same vector and we want to get from them the best estimate of this vector.

From now on, the external device will be a position radar. The measurement will be the difference between the position given by the inertial navigator and the position given by the radar. This measurement is considered as the error in the indication of the position. Filtering this position error yields the best estimate of the error on the state vector. The best estimate of the state vector itself is then obtained by substracting the estimated error from the output of the inertial system.

The general block diagram of these operations is given in figure 4.

This approach allows us to use an error analysis for the inertial navigator.

The following chapter investigates the different possible models for this navigator.

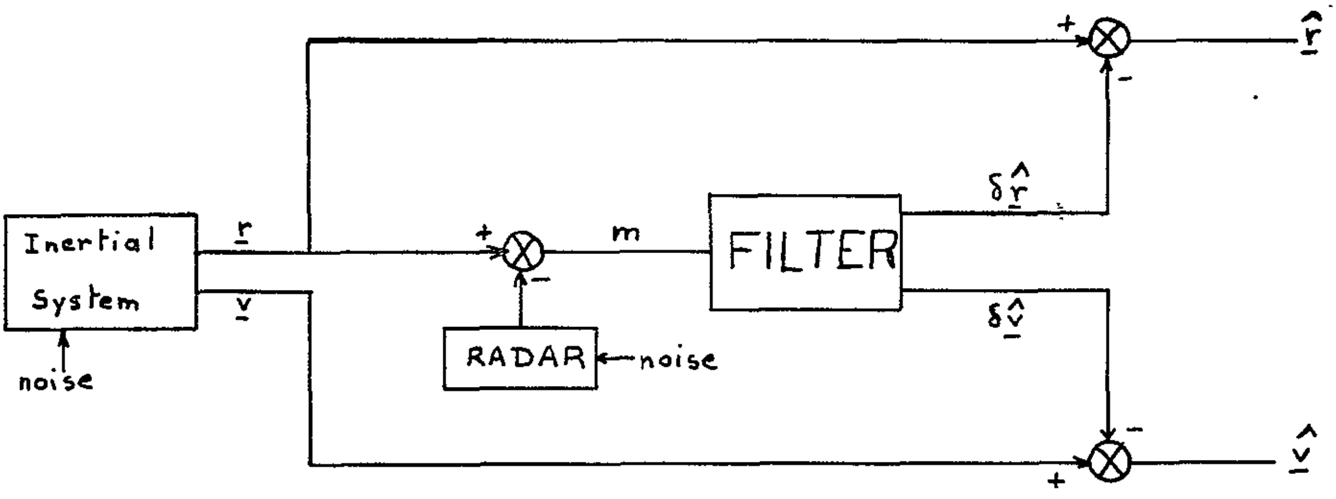


figure 4 : block - diagram of the system

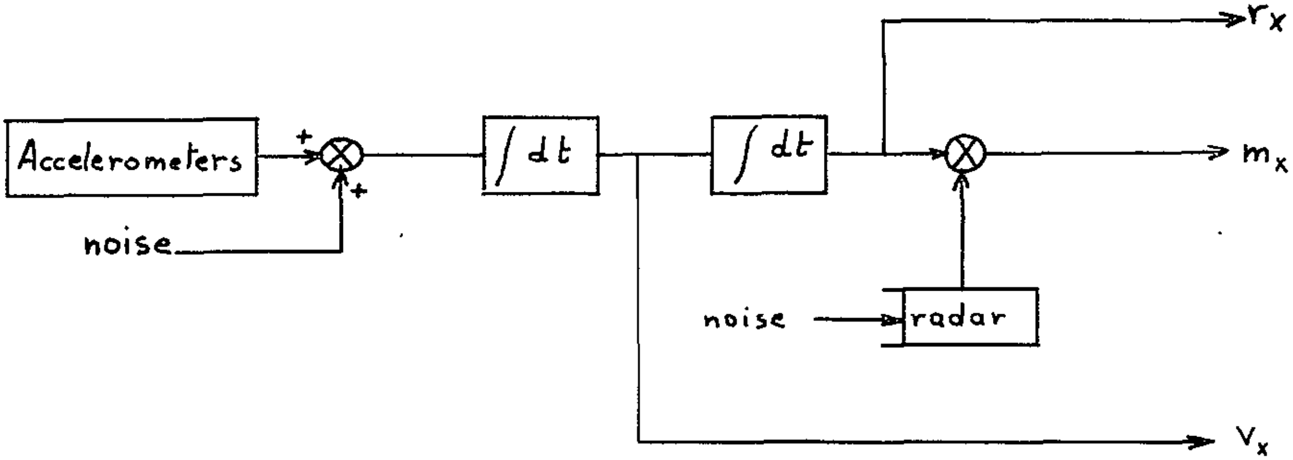


figure 5 : model 1 - single axis operation

CHAPTER 3

THE THREE POSSIBLE MODELS

3.1 Introduction

The complexity of the filter increases with the complexity of the model representing the inertial navigator. Indeed the filter gain $K(t)$ depends on $F(t)$ and the dimensions of the matrix F increase with the number of variables in the system.

On another hand, the accuracy obtained after filtering depends both on the complexity of the model and on the "power" of the noise. Thus it could be useless to filter with a complex model if the noise is important.

In this paper, 3 models are analysed:

1. the first one neglects both platform misalignment and cross-coupling between the axes.
2. the second one takes into account the platform misalignment angles..
3. the last one is a complete 3-axes model but assumes a steady state - i.e. "en route" conditions-.

In this chapter, we determine for all of these models the various matrices involved in the filtering theory: F, G, H, Q, R.

3.2 Model 1 : no cross-coupling; no misalignment

It is simply considered that the velocity vector is the time derivative of the position vector. A simple diagram of this I.N.S. is given in figure 5.

Consider the state vector $\underline{x}$ composed of the position and velocity errors as:

$$\underline{x} = \begin{bmatrix} dr_x \\ dr_y \\ dv_x \\ dv_y \end{bmatrix}$$

Then the equation (2-8) can be written as:

$$\begin{bmatrix} d\dot{r}_x \\ d\dot{r}_y \\ d\dot{v}_x \\ d\dot{v}_y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dr_x \\ dr_y \\ dv_x \\ dv_y \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \qquad (3\text{-}1\text{-}1)$$

$$\dot{\underline{x}} = F \quad \underline{x} + G \quad \underline{u}$$

In this equation u_x and u_y are the two white noises on both the x and the y channels.

The measurement equation (2-9) is:

$$\begin{bmatrix} dr_x \\ dr_y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} dr_x \\ dr_y \end{bmatrix} + \begin{bmatrix} v_x \\ v_y \end{bmatrix} \qquad (3\text{-}1\text{-}2)$$

or $$\underline{m} = H \quad \underline{x} + \underline{v}$$

Finally, the matrices Q and R are here:

$$Q = \begin{bmatrix} N & O \\ O & N \end{bmatrix} \qquad R = \begin{bmatrix} R & O \\ o & R \end{bmatrix} \qquad (3\text{-}1\text{-}3)$$

where N is the power spectral density of the I.N.S. noise
and R " " " radar noise.

3.3 Model 2 : no cross-coupling

The functional block diagram of a single axis inertial navigator instrumenting the navigation frame is given in figure 8. The platform misalignment is now accounted for: this means that a component of the gravity vector is falsely sensed by at least one of the accelerometers.

A misalignment angle about the z-axis (down) does not introduce a large error because the acceleration of the vehicle is usually small in comparison to g. On the contrary, a misalignment angle C_x (C_y) about the x-axis (y-axis) causes the accelerometer along the y-axis (x-axis) to sense a component of gravity $\pm g C_x$ ($\pm g C_y$) in the small angles approximation.

Thus we neglect the angle C_z about the down-axis; then there is no coupling between the x(north)-axis and the y(east)-axis. The z(vertical) channel follows exactly the equations of model 1 and the other two channels can be studied separately.

As shown in figure 9, the misalignment angles C_x and C_y are defined as correction angles -i.e. they are the angles the navigation frame should rotate about its X and Y axes to align itself with the platform axes –.

Throughout this paper the accelerometer outputs are supposed to be the components of the specific force $\underline{f}$ along the sensitive axes of any of these accelerometers. The specific force here is gravity minus acceleration ; so:

$$\underline{f} = \underline{g} - p_i^2 R_{EP}$$

The effects of C_x and C_y are the following: when C_x is positive, the instrumented east axis is below the horizon; therefore the y-accelerometer senses $+ g C_x$. When C_y is positive, the instrumented north axis is above the horizon; therefore the x-accelerometer senses $- g C_y$. Then, when the vehicle is moving, the accelerometers sense

$$\begin{cases} -R_e\, p^2 L - g\, C_y & \text{along the x-axis} \\ -R_e \cos L\, p^2 l + g\, C_x & \text{along the y-axis} \end{cases}$$

R_e is the earth radius; L the latitude and l the longitude.

Therefore, the errors are $\begin{cases} + \dfrac{g}{R_e} C_y \text{ on } p^2 L \\[2ex] - \dfrac{g}{R_e} C_x \text{ on } \cos L\, p^2 l \end{cases}$

The error model is given in figure 10.

We can write:

$p\,(dr_x) = dv_x$	$p\,(dr_y) = dv_y$
$p\,(dv_x) = + g\, C_y$	$p\,(dv_y) = - g\, C_x$
$p\,(C_y) = - \frac{1}{R_e} dv_x$	$p\,(C_x) = \frac{1}{R_e}\, dv_y$

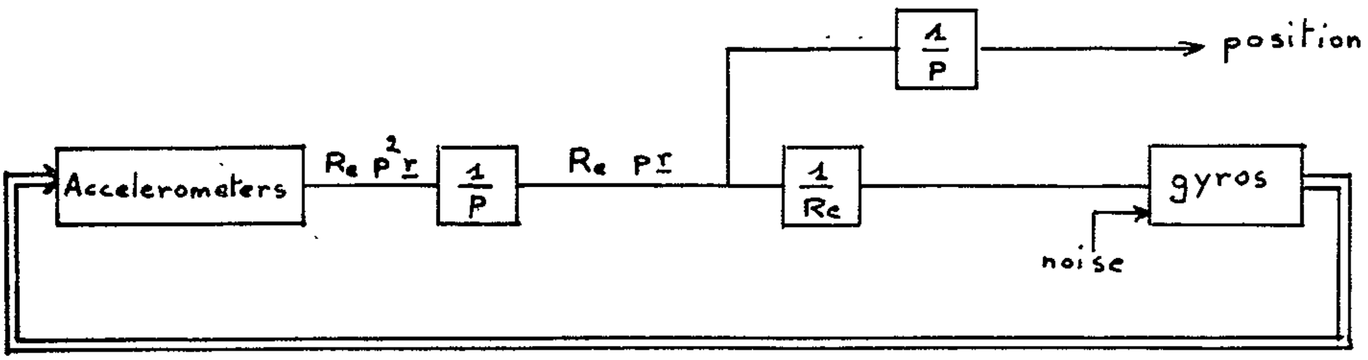


figure 8 : model 2 - single axis operation

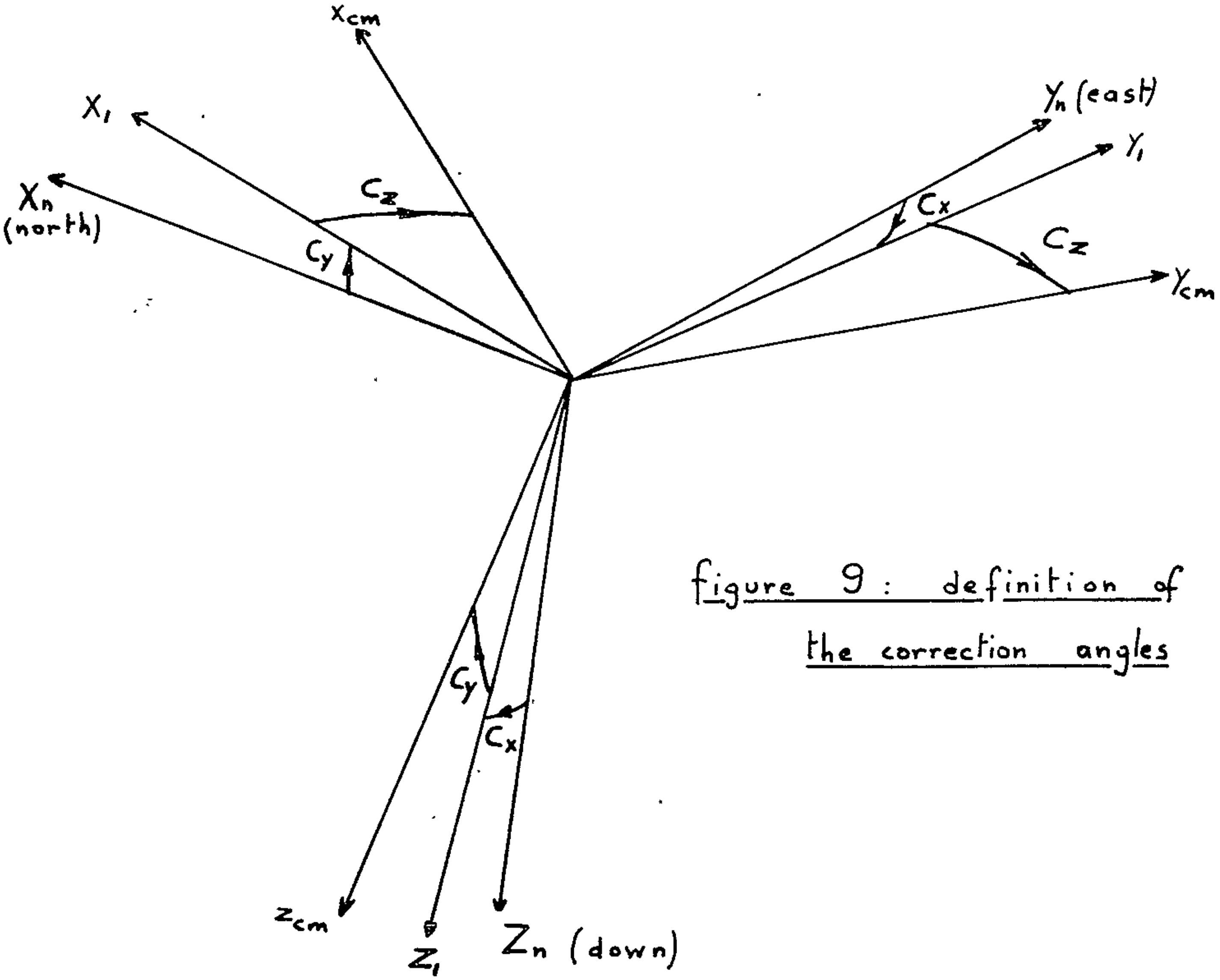


figure 9 : definition of the correction angles

Defining the state vector as

$$\underline{x} = \begin{bmatrix} dr_x \\ dr_y \\ dv_x \\ dv_y \\ C_x \\ C_y \end{bmatrix}$$

the equations (2-8) and (2-9) can be written:

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g \\ 0 & 0 & 0 & 0 & -g & 0 \\ 0 & 0 & 0 & +1/R_e & 0 & 0 \\ 0 & 0 & -1/R_e & 0 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \qquad (3\text{-}2\text{-}1)$$

and for the measurement:

$$\begin{bmatrix} dr_x \\ dr_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} v_x \\ v_y \end{bmatrix} \qquad (3\text{-}2\text{-}2)$$

Finally the matrices Q amd R are :

$$Q = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} \qquad R = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \qquad (3\text{-}2\text{-}3)$$

Let us remark that here the first four rows and columns of the matrix F are exactly the same as in model 1. This means that model 1 can be studied as a special case of model 2. This will be useful later on.

3.4 Model 3

The functional block diagram of the navigator is almost the same as in the previous model, but the misalign-

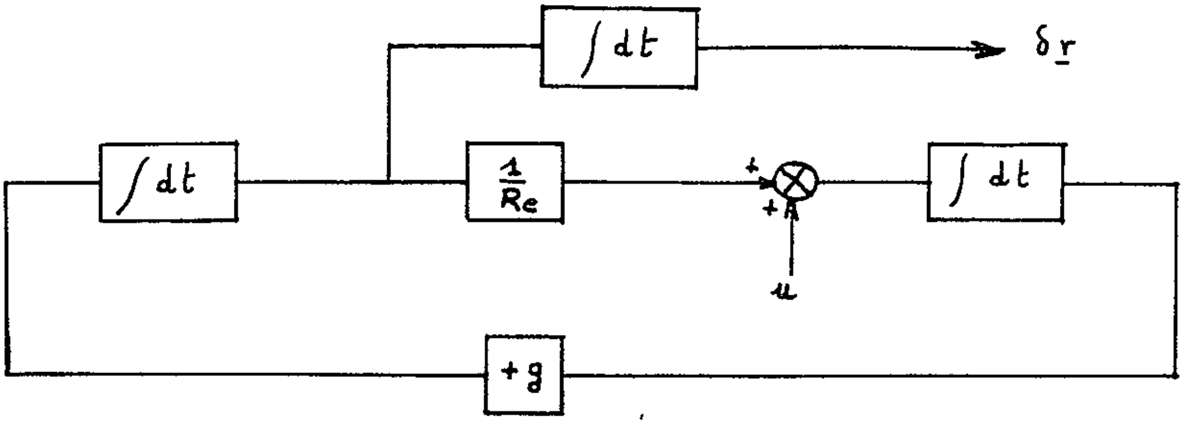


figure 10 : model 2 - error model

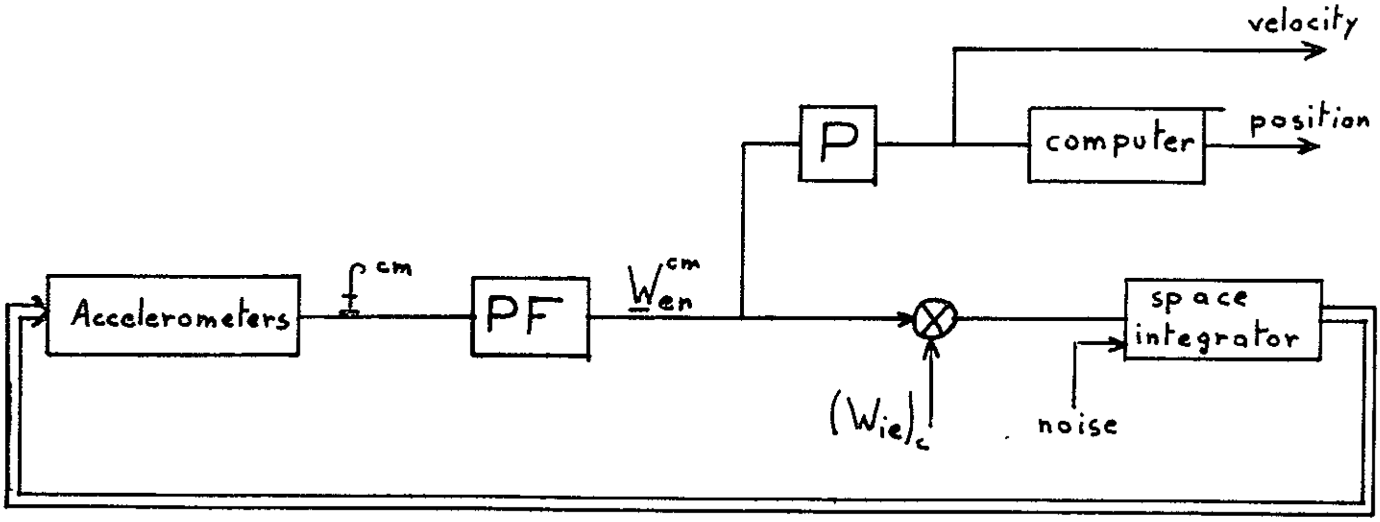


figure 11 : functional block diagram of the I.N.S.

ment about the z-axis is no longer neglected. Furthermore there is now an earth rate compensation which will introduce some other errors due to computed latitude feedback. See [8] and [9].

Let us call "n" the navigation frame and "cm" the controlled member frame - this is the platform axes -. With the same sign convention for the misalignment angles as before (the angles are positive when the n-frame rotates about its positive axes to get aligned with the cm-frame) the <u>direction cosines matrix</u> C_n^{cm} can be written:

$$C_n^{cm} = \begin{bmatrix} 1 & C_z & -C_y \\ -C_z & 1 & C_x \\ C_y & -C_x & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -C_z & C_y \\ C_z & 0 & -C_x \\ -C_y & C_x & 0 \end{bmatrix}$$

$$\text{or} \qquad C_n^{cm} = I - C \qquad (3\text{-}3)$$

where C is the <u>skew-symmetric matrix associated</u> with the rotation vector $\underline{C}$ (components C_x, C_y, C_z). This matrix is defined for any vector $\underline{V}$ by the relation :

$$\underline{V} \times \underline{W} = [V]\ \underline{W}$$

Let us assume for the moment that position and velocity are given in terms of latitude and longitude (angles instead of distances); thus:

$$\begin{cases} dv_x = p(L_c - L) \\ dv_y = p(l_c - l) \end{cases} \quad \text{and} \quad \begin{cases} dr_x = L_c - L = (dv_x)/p \\ dr_y = l_c - l = (dv_y)/p \end{cases}$$

In these relations, the subscript "c" stands for "computed" because L_c is one of the outputs of the I.N.S. and may differ from the true latitude L.

A-1: the first assumption is to neglect Coriolis effects and vertical acceleration.

Then the specific force in the n-frame is:

$$\underline{f}^n = \begin{bmatrix} -R_e\ p^2 L \\ -R_e\ \cos L\ p^2 l \\ g \end{bmatrix} \qquad (3\text{-}4)$$

and the accelerometers output is the specific force coordinatized in the cm-frame which can be written

$$\underline{f}^{cm} = C_n^{cm}\ \underline{f}^n$$

From $\underline{f}^n$, $\underline{W}_{en}^n$ (angular velocity of the n-frame with respect to the earth frame, coordinatized in the nav.-frame) is given by the relation:

$$(W_{en}^n)_c = \begin{bmatrix} \cos L\ pl \\ -pL \\ -\sin L\ pl \end{bmatrix} = \begin{bmatrix} 0 & -1/R_e p & 0 \\ 1/R_e p & 0 & 0 \\ 0 & \tan L_c/R_e p & 0 \end{bmatrix} \begin{bmatrix} -R_e\ p^2 L \\ -R_e\ \cos L\ p^2 l \\ g \end{bmatrix}$$

or $$(W_{en}^n)_c = \left[PF\right]_c\ \underline{f}^n \qquad (3\text{-}5)$$

In the expression of the <u>performance function</u> $\left[PF\right]_c$, the computed latitude comes only into $\tan L_c$.

But

$$\tan L_c = \tan(L+dL) = \tan L + dL/\cos^2 L + \text{high order terms}$$

We can, without introducing a great error, assume

$\tan L_c = \tan L$ under the following assumption:

A-2: the system is working far from the pole.

At a latitude of 45 degrees, the resulting error is only 2 dL (with dL in radians) and the approximation is quite good.

From $\underline{W}^n_{en}$, the velocity indication (in terms of longitude and latitude rates) is given by:

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} pL \\ pl \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1/\cos L_c & 0 & 0 \end{bmatrix} \underline{W}^n_{en}$$

or

$$(\underline{V}^n)_c = [P]\, \underline{W}^n_{en} \qquad (3\text{-}6)$$

The command angular velocity the space integrator must receive in order to operate properly is:

$$\underline{W}^n_{cmd} = (\underline{W}^n_{en})_c + (\underline{W}^n_{ie})_c \qquad (3\text{-}7)$$

Thus $(\underline{W}^n_{ie})_c = \begin{bmatrix} \omega_{ie} \cos L_c \\ 0 \\ -\omega_{ie} \sin L_c \end{bmatrix}$ must be added to $\underline{W}^n_{en}$

The equation for the angular motion of the controlled member is now:

$$\underline{W}^{cm}_{icm} = \underline{W}^{cm}_{in} + \underline{W}^{cm}_{ncm} \qquad (3\text{-}8)$$

where $\underline{W}^{cm}_{icm}$ is the commanded angular velocity given by equation (3-7) and $\underline{W}_{ncm}^{cm}$, which is the rate at which the controlled member goes away from the n-frame, is related to the correction angles by:

$$\underline{W}_{ncm}^{cm} = \begin{bmatrix} pC_x \\ pC_y \\ pC_z \end{bmatrix} = p\,\underline{C} \qquad (3\text{-}9)$$

Noting that $\underline{W}_{in}^{cm} = C_n^{cm}\,\underline{W}_{in}^{n}$ and using equations (3-5), (3-7) and (3-9), (3-7) can be written in the form:

$$C_n^{cm}\,(\underline{W}_{ie}^{n} + \underline{W}_{en}^{n}) + p\,\underline{C} = [PF]\,C_n^{cm}\,\underline{f}^{n} + (\underline{W}_{ie}^{n})_c \qquad (3\text{-}10)$$

Expanding $\sin L_c$ and $\cos L_c$ in terms of $dL = L_c - L$ up to the first order, we get:

$$(\underline{W}_{ie}^{n})_c = \underline{W}_{ie}^{n} - \begin{bmatrix} \omega_{ie}\sin L \\ 0 \\ \omega_{ie}\cos L \end{bmatrix} dL = \underline{W}_{ie}^{n} + d\underline{W}_{ie}^{n}$$

Using (3-6), (3-10) becomes:

$$[I - C]\,(\underline{W}_{ie}^{n} + \underline{W}_{en}^{n}) + p\,\underline{C} = [PF]\,[I - C]\,\underline{f}^{n} + \underline{W}_{ie}^{n} + d\underline{W}_{ie}^{n} \qquad (3\text{-}11)$$

Some simplifications are possible. Indeed, from the principle of operation of the I.N.S.

$$[PF]\,\underline{f}^{n} = \underline{W}_{en}^{n}$$

Further use of the relation $[A]\,\underline{V} = \underline{A}\times\underline{V} = -\,\underline{V}\times\underline{A} = -[V]\,\underline{A}$

yields finally:

$$p\,\underline{C} = d\underline{W}^n_{ie} + [PF]\left[\underline{f}^n\right]^{sk} \underline{C} - \left[\underline{W}^n_{ie}\right]^{sk} \underline{C} \tag{3-12}$$

To be able to get a relation of the form (2-8), we need a first order differential equation between the several errors. Therefore we need to relate the terms $[PF]\left[\underline{f}^n\right]^{sk} \underline{C}$ to some other error terms. It turns out to be easily relatable to $d\underline{v}$. Indeed, from (3-5) and (3-6)

$$\underline{v} + d\underline{v} = [P]\,[PF]\,[I - C]\,\underline{f}^n$$

Hence $$d\underline{v} = -\,[P][PF]\;C\,\underline{f}^n$$

or $$d\underline{v} = [P][PF]\left[\underline{f}^n\right]^{sk} \underline{C}$$

We can develop this matrix product. Assumption A-2 allows us to write $-1/\cos L_c = -1/\cos L$ (the error involved is of the same order as the one involved when writing $\tan L_c = \tan L$).

Furthermore, with the last assumption :

<u>A-3</u>: the vehicle is slowly moving on the surface of the earth

the terms in pL and pl in the product $[PF]\left[\underline{f}^n\right]^{sk}$ can be neglected. Thus

$$\begin{bmatrix} dv_x \\ dv_y \end{bmatrix} = \begin{bmatrix} 0 & g/R_e p & 0 \\ -\,g/R_e \cos L\; p & 0 & 0 \end{bmatrix} \begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} \tag{3-14}$$

Now we wish to relate $[PF]\left[\underline{f}^n\right]^{sk}\underline{C}$ to $d\underline{v}$. It is easy to see that

$$[PF]\left[\underline{f}^n\right]^{sk}\underline{C} = \begin{bmatrix} 0 & \cos L \\ -1 & 0 \\ 0 & -\sin L \end{bmatrix} \begin{bmatrix} dv_x \\ dv_y \end{bmatrix} \tag{3-14}$$

Substituting (3-14) into (3-12) yields finally:

$$\begin{bmatrix} pC_x \\ pC_y \\ pC_z \end{bmatrix} = -\begin{bmatrix} \omega_{ie}\sin L \\ 0 \\ \omega_{ie}\cos L \end{bmatrix} dL + \begin{bmatrix} 0 & \cos L \\ -1 & 0 \\ 0 & -\sin L \end{bmatrix} \begin{bmatrix} dv_x \\ dv_y \end{bmatrix} \cdots$$

$$\cdots - \begin{bmatrix} 0 & \omega_{ie}\sin L & 0 \\ -\omega_{ie}\sin L & 0 & -\omega_{ie}\cos L \\ 0 & \omega_{ie}\cos L & 0 \end{bmatrix} \begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} \tag{3-15}$$

The last equation of this error analysis is

$$\begin{bmatrix} p\,(dr_x) \\ p\,(dr_y) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} dv_x \\ dv_y \end{bmatrix} \tag{3-16}$$

From equations (3-13), (3-15) and (3-16), defining

$$\underline{x} = \begin{bmatrix} dr_x \\ dr_y \\ dv_x \\ dv_y \\ C_x \\ C_y \\ C_z \end{bmatrix}$$

where all the quantities involved are angles and angular rates we can write $\dot{\underline{x}} = F\,\underline{x} + G\,\underline{u}$ with:

$$F = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & +\frac{g}{R_e} & 0 \\ 0 & 0 & 0 & 0 & -\frac{g}{R_e \cos L} & 0 & 0 \\ -\omega_{ie}\sin L & 0 & 0 & \cos L & 0 & -\omega_{ie}\sin L & 0 \\ 0 & 0 & -1 & 0 & +\omega_{ie}\sin L & 0 & +\omega_{ie}\cos L \\ \omega_{ie}\cos L & 0 & 0 & -\sin L & 0 & -\omega_{ie}\cos L & 0 \end{bmatrix} \qquad (3\text{-}17)$$

and:

$$G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \underline{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

To get an equation of the same form as in the previous models, $d\underline{r}$ and $d\underline{v}$ must be expressed as functions of distance. Remarking that

$$\begin{bmatrix} r_x \\ r_y \\ v_x \\ v_y \end{bmatrix}_{\text{n.miles}} = \begin{bmatrix} R_e & 0 & 0 & 0 \\ 0 & R_e\cos L & 0 & 0 \\ 0 & 0 & R_e & 0 \\ 0 & 0 & 0 & R_e\cos L \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ v_x \\ v_y \end{bmatrix}_{\text{minutes}}$$

it follows that we can write :

$\dot{\underline{x}} = F\,\underline{x} + G\,\underline{u}$ with now

$$F = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & 0 & -g & 0 & 0 \\ -\dfrac{\omega_{ie}\sin L}{R_e} & 0 & 0 & +\dfrac{1}{R_e} & 0 & -\omega_{ie}\sin L & 0 \\ 0 & 0 & -\dfrac{1}{R_e} & 0 & \omega_{ie}\sin L & 0 & \omega_{ie}\cos L \\ -\dfrac{\omega_{ie}\cos L}{R_e} & 0 & 0 & -\dfrac{\tan L}{R_e} & 0 & -\omega_{ie}\cos L & 0 \end{bmatrix} \tag{3-18-1}$$

Then position is in nautical miles and velocity in nautical miles per unit time.

The measurement is $\underline{m} = H\,\underline{x} + \underline{v}$ with

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{3-18-2}$$

Finally, the matrices Q and R are the same as before:

$$Q = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} \qquad R = \begin{bmatrix} R & 0 \\ 0 & R \end{bmatrix} \tag{3-18-3}$$

N and R are power spectral densities of white noises and thus are to be constant over some frequency interval. Therefore, the units must be:

$$\begin{cases} (\text{n. miles})^2 . \text{sec} & \text{for } R \\ (\text{n. miles})^2 / (\text{sec})^3 & \text{for } N \end{cases}$$

Let us recall the 3 assumptions made to derive this equation:

A-1: neglect Coriolis effects and vertical acceleration

A-2: operation far from the pole

A-3: vehicle slowly moving.

CHAPTER 4

FILTER EQUATIONS

4.1 Introduction

As it was pointed out in chapter 1, we are not interested in finding the output of the filter for given I.N.S. and radar outputs, but rather in measuring how accurate this indication is. Therefore, in this chapter, only the variance equation (the differential equation for $P_x(t)$) will be studied and solved.

It follows from equations (3-1), (3-2) and (3-18) that the F matrices are very similar in all the cases. The 12 equations governing the $P_x(t)$ matrix in model 2 are a particular case of the 28 equations of model 3; in the same manner, the 6 equations of model 1 are a particular case of the 12 equations of model 2.

Therefore, after an analytic solution for model 1, the only model that can be handled in a simple way, the equations for model 3 will be derived using 2 parameters "a" and "b" to allow a single study of the 3 cases.

4.2 Analytic solution of model 1

The equations of this model are simple enough to be analytically solved. This allows us to find a closed form answer to the continuous filtering problem.

Let us go back to equation (3-1-1). Since one of the main assumptions of this model is that there be no cross-coupling between the channels, the rank of all the matrices can be reduced by writing

$$\begin{bmatrix} d\dot{r}_x \\ d\dot{v}_x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} dr_x \\ dv_x \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_x \qquad (4\text{-}1\text{-}1)$$

$$\underline{\dot{x}} \quad = \quad F \quad \underline{x} \quad + \quad G \quad \underline{u}$$

and

$$dr_x = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} dr_x \\ dr_y \end{bmatrix} + v_x \qquad (4\text{-}1\text{-}2)$$

$$\underline{m} \quad = \quad H \quad \underline{x} \quad + \quad \underline{v}$$

Thus the matrices Q and R become <u>scalars</u> N and R . If we let

$$P_x(t) = \begin{bmatrix} p_{11}(t) & p_{13}(t) \\ p_{13}(t) & p_{33}(t) \end{bmatrix}$$

the variance equation can be written (dropping the variable t):

$$\begin{bmatrix} \dot{p}_{11} & \dot{p}_{13} \\ \dot{p}_{13} & \dot{p}_{33} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{13} \\ p_{13} & p_{33} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{13} \\ p_{13} & p_{33} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ 1 \end{bmatrix} N \begin{bmatrix} 0 & 1 \end{bmatrix} - \begin{bmatrix} p_{11} & p_{13} \\ p_{13} & p_{33} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{R} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p_{11} & p_{13} \\ p_{13} & p_{33} \end{bmatrix} \qquad (4\text{-}2)$$

4-2-1 Steady state

The steady state solution is the solution of the right hand side of this equation when the left hand side is 0.

The answer is straightforward

$$\begin{cases} (p_{11})_{s.s.} = \sqrt{2}\; R^{3/4}\; N^{1/4} \\ (p_{13})_{s.s.} = \sqrt{NR} \\ (p_{33})_{s.s.} = \sqrt{2}\; R^{1/4}\; N^{3/4} \end{cases}$$

Thus, the steady state r.m.s. errors are:

$$\begin{cases} \text{in position} \quad RMX = 2^{1/4}\, R^{3/8}\, N^{1/8} \\ \text{in velocity} \quad RVX = 2^{1/4}\, R^{1/8}\, N^{3/8} \end{cases} \tag{4-3}$$

Now the optimum estimate is given by equation (2-11) with $K(t) = P(t)\; H'(t)\; R^{-1}$

Let us call $\omega_n = \left[\dfrac{N}{R}\right]^{1/4}$; then $K_{s.s.} = \begin{bmatrix} \sqrt{2}\,\omega_n \\ \omega_n^2 \end{bmatrix}$

And:

$$\begin{bmatrix} d\dot{\hat{r}}_x \\ d\dot{\hat{v}}_x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d\hat{r} \\ d\hat{v} \end{bmatrix} + \begin{bmatrix} \sqrt{2}\,\omega_n \\ \omega_n^2 \end{bmatrix} \left(m - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} d\hat{r} \\ d\hat{v} \end{bmatrix} \right) \tag{4-4}$$

The optimum filter block diagram is given in figure 6.

4-2-2 Response to I.N.S. and radar noise

From the diagram, it is easy to get the response of the filter to:

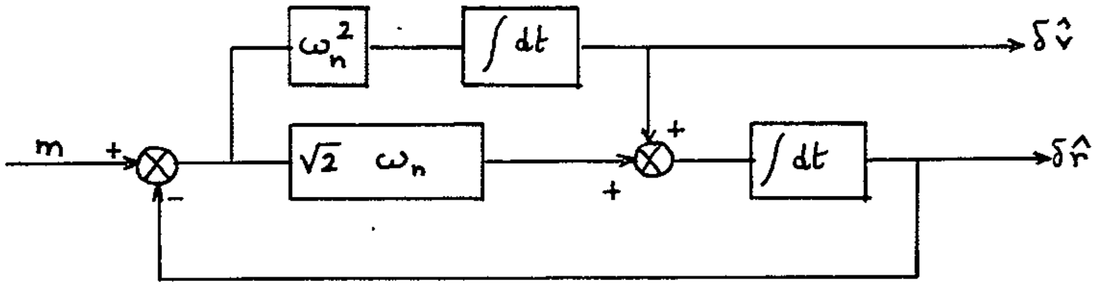


figure 6 : model 1 ; optimum filter block diagram

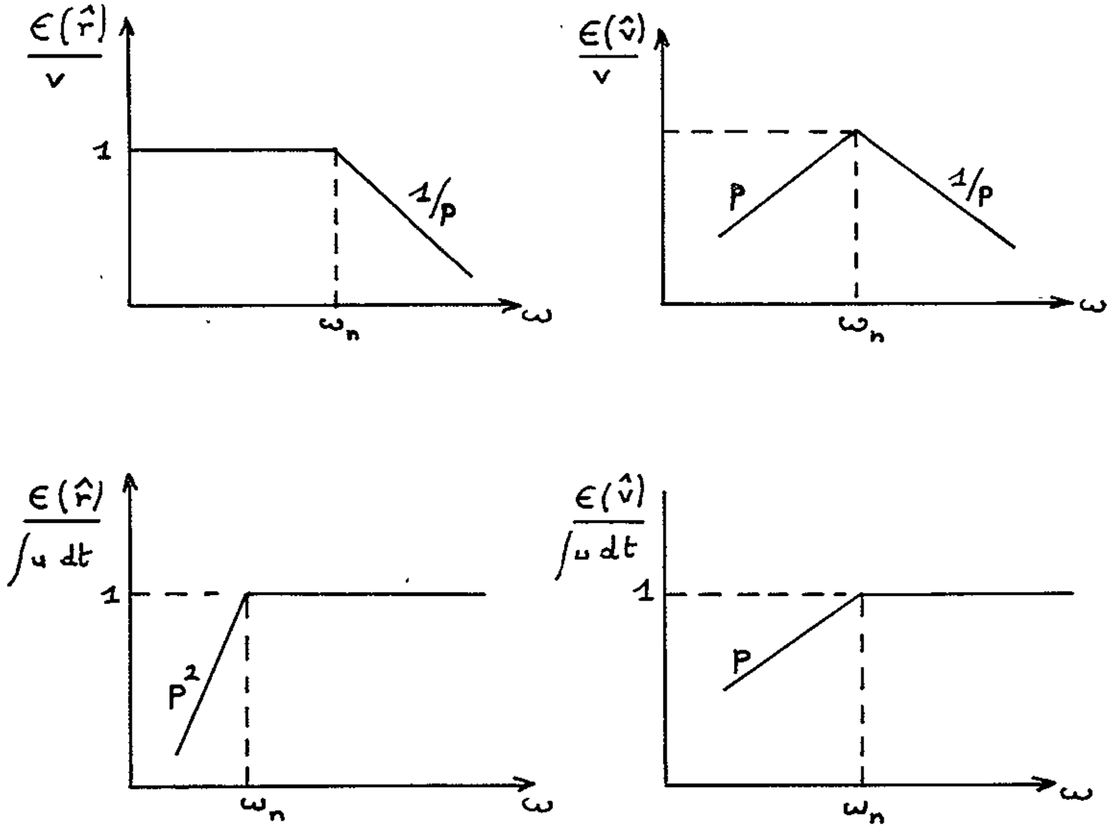


figure 7 : model 1; responses to I.N.S and radar noises

#1 radar noise :

on position $$\frac{e(\hat{r})}{R} = \frac{\sqrt{2}\,\omega_n\, p + \omega_n^2}{p^2 + \sqrt{2}\omega_n\, p + \omega_n^2}$$

on velocity $$\frac{e(\hat{v})}{R} = \frac{\omega_n^2\, p}{p^2 + \sqrt{2}\omega_n\, p + \omega_n^2}$$

#2 I.N.S. noise

on position $$\frac{e(\hat{r})}{u/p} = \frac{p^2}{p^2 + \sqrt{2}\,\omega_n\, p + \omega_n^2}$$

on velocity $$\frac{e(\hat{v})}{u/p} = \frac{p^2 + \sqrt{2}\,\omega_n\, p}{p^2 + \sqrt{2}\,\omega_n\, p + \omega_n^2}$$

These response functions are plotted on figure 7.

4-2-3 Closed form solution for the free system

The initial conditions can be taken as

$$P_x(t=0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The variance equation for the free system is:

$$\begin{bmatrix} \dot{p}_{11} & \dot{p}_{13} \\ \dot{p}_{13} & \dot{p}_{33} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} p_{11} & p_{13} \\ p_{13} & p_{33} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{13} \\ p_{13} & p_{33} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}$$

or: $$\begin{cases} \dot{p}_{11} = 2\, p_{13} \\ \dot{p}_{13} = p_{33} \\ \dot{p}_{33} = N \end{cases}$$

With the initial condition 0, the solution to this equation is:

$$P_x(t) = N \begin{bmatrix} t^3/3 & t^2/2 \\ t^2/2 & t \end{bmatrix}$$

Then, the r.m.s. errors are:

$$\begin{cases} \text{on position} & RMX = (N/3)^{1/2}\, t^{3/2} \\ \text{on velocity} & RVX = N^{1/2}\, t^{1/2} \end{cases}$$

These results are plotted on graphs 1 and 3 (upper curves)

The steady state covariance matrix in continuous filtering and the free system r.m.s. errors are about the only things we can analytically study even in this simple model. As soon as we go into the transient solution for the continuous case, we get the following set of non-linear differential equations:

$$\begin{cases} \dot{p}_{11} = 2\, p_{13} - p_{11}^2/R \\ \dot{p}_{13} = p_{33} - p_{11}\, p_{13}/R \\ \dot{p}_{33} = N - p_{13}^2/R \end{cases}$$

These equations are not easy to solve (see [10]) and it is better to solve them on a computer as a particular case of the most complicated model, as shown below.

4.3 The variance equations

From now on, the covariance matrix will be taken in the form:

$$P_x(t) = \left[p_{ij}\right] = \begin{bmatrix} p_{11} & p_{12} & p_{13} & \cdots & p_{17} \\ \vdots & & & & \vdots \\ p_{17} & \cdots & \cdots & \cdots & p_{77} \end{bmatrix}$$

Since $P_x(t) = E\left[\underline{x}(t)\ \underline{x}'(t)\right]$ with the usual definition for the state vector $\underline{x}(t)$ (chapter 3), the off diagonal terms are the correlations between the different state variables.

For instance $p_{36} = E\left[dv_x\ C_y\right]$ because all the state variables are zero-meaned quatities (error terms). The diagonal terms are the variances (squares of the corresponding r.m.s. errors) for the same reason.

For instance $p_{44} = E\left[dv_y^2\right] = (\text{y-velocity rms error})^2$

Let us come back to the system parameters matrix F of the 3 models, as they appear in equations (3-1-1), (3-2-1) and (3-18-1), and define three additional parameters a, b and c such that:

$$\begin{cases} a = 0 \text{ for model 1 and 1 otherwise} \\ b = 0 \text{ for model 2 and 1 otherwise} \\ c = ab \end{cases}$$

Then it is possible to represent the 3 models by the following matrix F:

$$F = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a\,g & 0 \\ 0 & 0 & 0 & 0 & -a\,g & 0 & 0 \\ -c\,\dfrac{\omega_{ie}\sin L}{R_e} & 0 & 0 & +\dfrac{a}{R_e} & 0 & -c\,\omega_{ie}\sin L & 0 \\ 0 & 0 & -\dfrac{a}{R_e} & 0 & c\,\omega_{ie}\sin L & 0 & c\,\omega_{ie}\cos L \\ -c\,\dfrac{\omega_{ie}\cos L}{R_e} & 0 & 0 & -\dfrac{a\tan L}{R_e} & 0 & -c\,\omega_{ie}\cos L & 0 \end{bmatrix} \quad (4\text{-}5)$$

The differential equations for the covariance matrix will be derived with this expression for F. As for the number of equations necessary to study each of the cases, it appears to be :

in model 1- 6 equations corresponding to the 6 non zero quatities p_{11}, p_{13}, p_{33}, p_{22}, p_{24}, p_{44} for both the x- and y-channels.

in model 2- 6 more non zero terms: p_{16}, p_{36}, p_{66} for the x-channel and p_{25}, p_{45}, p_{55} for the y-channel. Therefore, 12 equations are necessary.

in model 3- 28 equations because no quantity is a-priori zero.

4.3.1 Free system equations

Equation (2-10) is:

$$\dot{P} = F\,P + P\,F' + G\,Q\,G'$$

All the quantities have been defined and the result is the following set of 28 equations:

1) $\dot{p}_{11} = 2\,p_{13}$

2) $\dot{p}_{33} = 2ag\,p_{36} + N$

3) $\dot{p}_{13} = p_{33} + ag\,p_{16}$

4) $\dot{p}_{22} = 2p_{24}$

5) $\dot{p}_{44} = -2ag\,p_{45} + N$

6) $\dot{p}_{24} = p_{44} - ag\,p_{25}$

- -

7) $\dot{p}_{25} = p_{45} - b\,\dfrac{\omega_{ie}\sin L}{R_e}\,p_{12} + \dfrac{a}{R_e} - b\omega_{ie}\sin L\,p_{26}$

8) $\dot{p}_{45} = -g - b\,\frac{\omega_{ie}\sin L}{R_e}\,p_{12} + \frac{1}{R_e}\,p_{24} - b\,\omega_{ie}\sin L\,p_{26}$

9) $\dot{p}_{55} = -2b\,\frac{\omega_{ie}\sin L}{R_e}\,p_{15} + \frac{2}{R_e}\,p_{45} - 2b\,\omega_{ie}\sin L\,p_{56}$

10) $\dot{p}_{16} = p_{36} - \frac{1}{R_e}\,p_{13} + b\,\omega_{ie}\sin L\,p_{15} + b\,\omega_{ie}\cos L\,p_{17}$

11) $\dot{p}_{36} = +g\,p_{66} - \frac{1}{R_e}\,p_{33} + b\,\omega_{ie}\sin L\,p_{35} + b\,\omega_{ie}\cos L\,p_{37}$

12) $\dot{p}_{66} = -\frac{2}{R_e}p_{36} + 2b\,\omega_{ie}\sin L\,p_{56} + 2b\,\omega_{ie}\cos L\,p_{67}$

---

13) $\dot{p}_{12} = p_{23} + p_{14}$

14) $\dot{p}_{23} = p_{34} + g\,p_{26}$

15) $\dot{p}_{14} = p_{34} - g\,p_{15}$

16) $\dot{p}_{34} = g\,p_{46} - g\,p_{35}$

17) $\dot{p}_{15} = p_{35} - \frac{\omega_{ie}\sin L}{R_e}p_{11} + \frac{1}{R_e}p_{14} - \omega_{ie}\sin L\,p_{16}$

18) $\dot{p}_{35} = gp_{56} - \frac{\omega_{ie}\sin L}{R_e}p_{13} + \frac{1}{R_e}p_{34} - \omega_{ie}\sin L\,p_{36}$

19) $\dot{p}_{26} = +p_{46} - \frac{1}{R_e}p_{23} + \omega_{ie}\sin L\,p_{25} + \omega_{ie}\cos L\,p_{27}$

20) $\dot{p}_{46} = -gp_{56} - \frac{1}{R_e}p_{34} + \omega_{ie}\sin L\,p_{45} + \omega_{ie}\cos L\,p_{47}$

21) $\dot{p}_{56} = -\frac{\omega_{ie}\sin L}{R_e}p_{16} + \frac{1}{R_e}p_{46} - \omega_{ie}\sin L\,p_{66} - \frac{1}{R_e}p_{35} + \omega_{ie}\sin L\,p_{55}$

$+\,\omega_{ie}\cos L\,p_{57}$

22) $\dot{p}_{17} = p_{37} - \frac{\omega_{ie}\cos L}{R_e}p_{11} - \frac{\tan L}{R_e}p_{14} - \omega_{ie}\cos L\,p_{16}$

23) $\dot{p}_{27} = p_{47} - \frac{\omega_{ie}\cos L}{R_e}p_{12} - \frac{\tan L}{R_e}p_{24} - \omega_{ie}\cos L\,p_{26}$

24) $\dot{p}_{37} = gp_{67} - \frac{\omega_{ie}\cos L}{R_e}p_{13} - \frac{\tan L}{R_e}p_{34} - \omega_{ie}\cos L\,p_{36}$

25) $\dot{p}_{47} = -gp_{57} - \frac{\omega_{ie}\cos L}{R_e}p_{14} - \frac{\tan L}{R_e}p_{44} - \omega_{ie}\cos L\,p_{46}$

$$26)\ \dot{p}_{57} = -\frac{\omega_{ie}\sin L}{R_e}p_{17} + \frac{1}{R_e}p_{47} - \omega_{ie}\sin L\ p_{67} - \frac{\omega_{ie}\cos L}{R_e}p_{15} - \frac{\tan L}{R_e}p_{45} - \omega_{ie}\cos L\ p_{56}$$

$$27)\ \dot{p}_{67} = -\frac{1}{R_e}p_{37} + \omega_{ie}\sin L\ p_{57} + \omega_{ie}\cos L\ p_{77} - \frac{\omega_{ie}\cos L}{R_e}p_{16} - \frac{\tan L}{R_e}p_{46} - \omega_{ie}\cos L\ p_{66}$$

$$28)\ \dot{p}_{77} = -2\,\frac{\omega_{ie}\cos L}{R_e}p_{17} - 2\frac{\tan L}{R_e}p_{47} - 2\,\omega_{ie}\cos L\ p_{67}$$

(4-6)

In the second part of these equations (#7 through #12) the coefficient a has been dropped; in the last part (#13 through #28) both coefficient a and b have been dropped.

This system of linear first order differential equations may be solved given some initial conditions and yields the r.m.s. errors in the state variable estimators.

4.3.2 Continuous filtering compensation terms

To get the equation (2-11) the quatity

$$M(t) = P_x(t)\ H'\ R^{-1}\ H\ P_x(t)$$

must be substracted from the foregoing equations.

Of course $M'(t) = M(t)$ so that only 28 terms must be computed as functions of the p_{ij}. Assuming that $M(t)$ is written as $M(t) = \left[m_{ij} \right]$ the following set of equations is obtained:

$$1)\ m_{11} = (p_{11}^2 + p_{12}^2)/R \qquad 2)\ m_{33} = (p_{13}^2 + p_{23}^2)/R$$

3) $m_{13} = (p_{11}p_{13}+ p_{12}p_{23})/R$	4) $m_{22}= (p_{12}^2+ p_{22}^2)/R$
5) $m_{44} = (p_{14}^2+ p_{24}^2)/R$	6) $m_{24}= (p_{12}p_{14}+ p_{22}p_{24})/R$
7) $m_{25} = (p_{12}p_{15}+ p_{22}p_{25})/R$	8) $m_{45}= (p_{14}p_{15}+ p_{24}p_{25})/R$
9) $m_{55} = (p_{15}^2+ p_{25}^2)/R$	10) $m_{16}= (p_{11}p_{16}+ p_{12}p_{26})/R$
11) $m_{36} = (p_{13}p_{16}+ p_{23}p_{26})/R$	12) $m_{66}= (p_{16}^2+ p_{26}^2)/R$
13) $m_{12} = (p_{11}p_{12}+ p_{12}p_{22})/R$	14) $m_{23}= (p_{12}p_{13}+ p_{22}p_{23})/R$
15) $m_{14} = (p_{11}p_{14}+ p_{12}p_{24})/R$	16) $m_{34}= (p_{13}p_{14}+ p_{23}p_{24})/R$
17) $m_{15} = (p_{11}p_{15}+ p_{12}p_{25})/R$	18) $m_{35}= (p_{13}p_{15}+ p_{23}p_{25})/R$
19) $m_{26} = (p_{12}p_{16}+ p_{22}p_{26})/R$	20) $m_{46}= (p_{14}p_{16}+ p_{24}p_{26})/R$
21) $m_{56} = (p_{15}p_{16}+ p_{25}p_{26})/R$	22) $m_{17}= (p_{11}p_{17}+ p_{12}p_{27})/R$
23) $m_{27} = (p_{12}p_{17}+ p_{22}p_{27})/R$	24) $m_{37}= (p_{13}p_{17}+ p_{23}p_{27})/R$
25) $m_{47} = (p_{14}p_{17}+ p_{24}p_{27})/R$	26) $m_{57}= (p_{15}p_{17}+ p_{25}p_{27})/R$
27) $m_{67} = (p_{16}p_{17}+ p_{26}p_{27})/R$	28) $m_{77}= (p_{17}^2 + p_{27}^2)/R$

(4-7)

These are the continuous filtering compensation terms written as functions of the p_{ij}, some among them can be 0, depending on the model (the different modelsare delimited by the two dotted lines).

To get the continuous filter variance equation, m_{ij} is to be substracted from the right hand side of equation (4-6) :

$$\dot{p}_{ij} = (\dot{p}_{ij})_{\text{free system}} - m_{ij} \tag{4-8}$$

When the equation (4-8) are solved, the r.m.s. errors are given by:

$$\begin{cases} \sqrt{p_{11}} = \text{r.m.s. error in x-position} = RMX \\ \sqrt{p_{22}} = \text{" \quad " \quad y-position} = RMY \\ \sqrt{p_{33}} = \text{" \quad " \quad x-velocity} = RVX \\ \sqrt{p_{44}} = \text{" \quad " \quad y-velocity} = RVY \end{cases}$$

p_{55}, p_{66} and p_{77} are the r.m.s. misalignment angles.

Let us find, in this case of continuous filtering, the form of the optimal filter. The optimum gains matrix is:

$K(t) = P(t)\, H'\, R^{-1}$ and is equal to:

$$K(t) = \frac{1}{R}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \\ p_{13} & p_{23} \\ p_{14} & p_{24} \\ p_{15} & p_{25} \\ p_{16} & p_{26} \\ p_{17} & p_{27} \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \\ k_{13} & k_{23} \\ k_{14} & k_{24} \\ k_{15} & k_{25} \\ k_{16} & k_{26} \\ k_{17} & k_{27} \end{bmatrix} \tag{4-9}$$

The optimum estimate obeys equation (2-11):

$$\dot{\hat{\underline{x}}} = F\,\hat{\underline{x}} + K\,(\underline{m} - H\,\hat{\underline{x}})$$

If $\left| \begin{array}{l} m_x \text{ is the measurement along the x-axis} \\ m_y \text{ " \quad " \quad " \quad y-axis} \end{array} \right.$

and if $\left| \begin{array}{l} \tilde{m}_x = m_x - d\hat{r}_x \\ \tilde{m}_y = m_y - d\hat{r}_y \end{array} \right.$

equation (2-11) can be written:

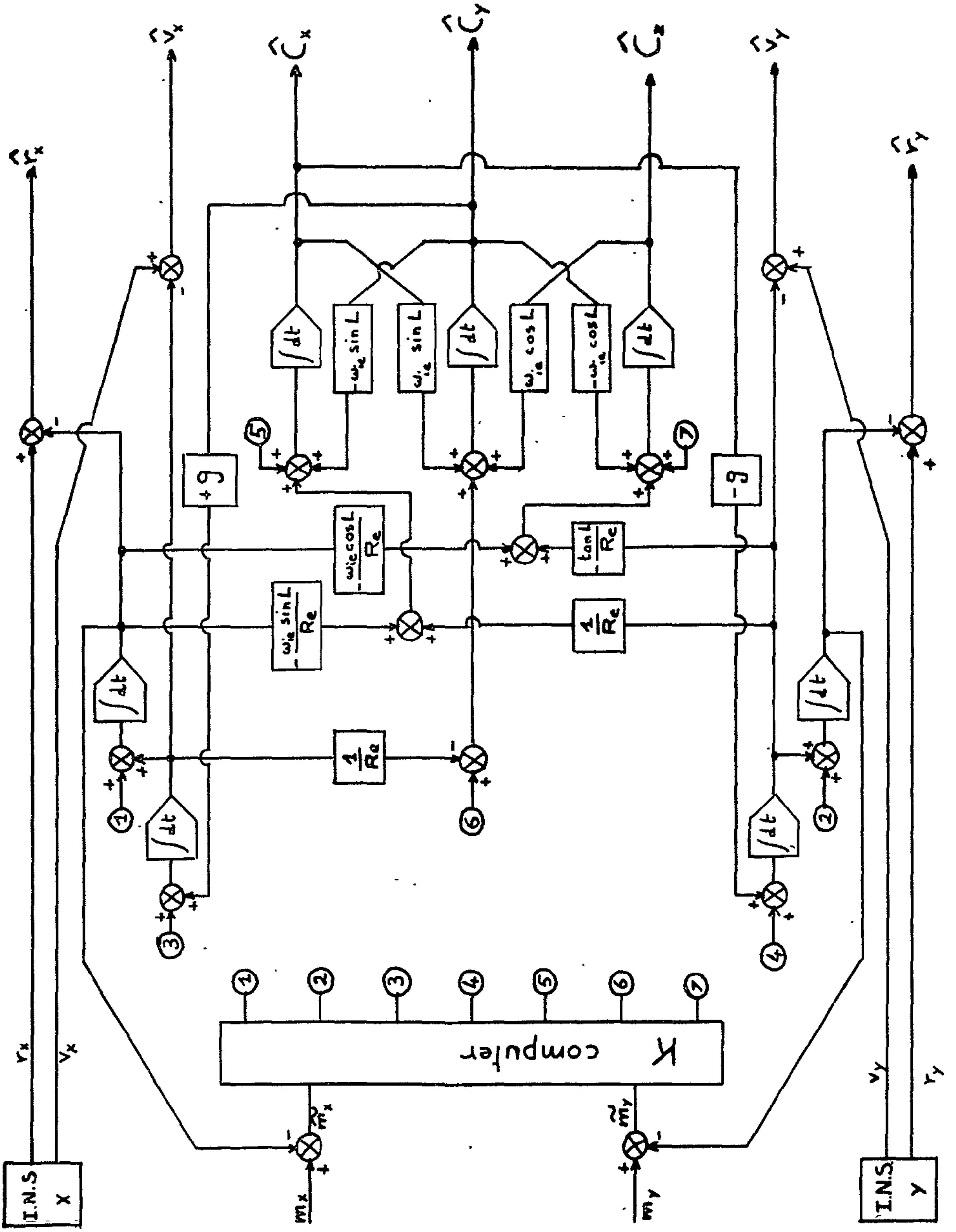


figure 12 : model 3 ; optimum filter signal flow diagram

$$\left|\begin{aligned}
p(d\hat{r}_x) &= d\hat{v}_x + k_{11}\tilde{m}_x + k_{12}\tilde{m}_y \\
p(d\hat{r}_y) &= d\hat{v}_y + k_{12}\tilde{m}_x + k_{22}\tilde{m}_y \\
p(d\hat{v}_x) &= ag\hat{C}_y + k_{13}\tilde{m}_x + k_{23}\tilde{m}_y \\
p(d\hat{v}_y) &= -ag\,\hat{C}_x + k_{14}\tilde{m}_x + k_{24}\tilde{m}_y \\
p(\hat{C}_x) &= -c\,\frac{\omega_{ie}\sin L}{R_e} d\hat{r}_x - a\,\frac{d\hat{v}_y}{R_e} - c\,\omega_{ie}\sin L\,\hat{C}_y + k_{15}\tilde{m}_x + k_{25}\tilde{m}_y \\
p(\hat{C}_y) &= -a\,\frac{d\hat{v}_x}{R_e} + c\,\omega_{ie}\sin L\,\hat{C}_x + c\,\omega_{ie}\cos L\,\hat{C}_z + k_{16}\tilde{m}_x + k_{26}\tilde{m}_y \\
p(\hat{C}_z) &= -c\,\frac{\omega_{ie}\cos L}{R_e}\,d\hat{r}_x - c\,\frac{\tan L}{R_e}\,d\hat{v}_y - c\,\omega_{ie}\cos L\,\hat{C}_y + k_{17}\tilde{m}_x + k_{27}\tilde{m}_y
\end{aligned}\right.$$

The signal flow diagram of the optimum filter for model 3 is given in <u>figure 12</u>. For purpose of simplicity any quantity $k_{1j}\tilde{m}_x + k_{2j}\tilde{m}_y$ (with $k_{21} = k_{12}$) has been replaced by the number "j".

These quantities are computed from the 14 gains of equations (4-9) and the measured quantities $\tilde{m}_x$ and $\tilde{m}_y$. These computations are not shown on this figure.

4.3.3 Discrete filtering

Between the measurements, during the operating time OPT, the covariance matrix obeys equations (4-6). At each measurement it is updated using equation (2-12); for this it is necessary to compute

$$K = P'\,H'\,(\,H\,P'\,H' + R\,)^{-1}$$

It is easy to see that

$$H\,P\,H' + R = \begin{bmatrix} p_{11} + R & p_{12} \\ p_{12} & p_{22} + R \end{bmatrix}$$

Since the general filtering theory assumes that R is a positive definite matrix and since the covariance matrix is non-negative definite, it is always possible to invert the matrix $H P H' + R$, even when the initial condition is $P(t=0) = 0$, provided that we do not update P at the initial time. This is straightforward and can easily be shown using the Schwartz inequality and the fact that P becomes more and more positive during the operating time (from equation (2-10)).

Denoting

$$D = \det(H P H' + R) = p_{11}p_{22} - p_{12}^2 + R(p_{11}+p_{22}) + R^2$$

we can get :

$$K(t) = \frac{1}{D}\begin{bmatrix} p_{11}(p_{22}+R) - p_{12}^2 & p_{12}(p_{11}+R) - p_{11}p_{12} \\ p_{12}(p_{22}+R) - p_{12}p_{22} & p_{22}(p_{11}+R) - p_{12}^2 \\ p_{13}(p_{22}+R) - p_{12}p_{23} & p_{23}(p_{11}+R) - p_{12}p_{13} \\ p_{14}(p_{22}+R) - p_{12}p_{24} & p_{24}(p_{11}+R) - p_{12}p_{14} \\ p_{15}(p_{22}+R) - p_{12}p_{25} & p_{25}(p_{11}+R) - p_{12}p_{15} \\ p_{16}(p_{22}+R) - p_{12}p_{26} & p_{26}(p_{11}+R) - p_{12}p_{16} \\ p_{17}(p_{22}+R) - p_{12}p_{27} & p_{27}(p_{11}+R) - p_{12}p_{17} \end{bmatrix} \tag{4-10}$$

Once this is computed, the optimum estimate of $\underline{x}$ is:

$$\hat{\underline{x}} = K \begin{bmatrix} m_x \\ m_y \end{bmatrix}$$

The covariance matrix is updated by $(I - K H) P$; in other words, p_{ij} is to be changed into :

$$p_{ij} - k_{i1}p_{1j} - k_{i2}p_{2j} \tag{4-11}$$

The new covariance matrix $P(t_m^+)$ becomes the new initial condition for the equations (4-6), until the next measurement is taken, when a new $P(t_m^+)$ becomes available.

4.4 Computer program

As it was already pointed out in part 4.2.3, the analytic solution of these equations is not easy to obtain, even in the simplest model. Therefore, this work must be done on a computer.

The computer program used for this paper is presented in Appendix B.

The first program (pages B-1 through B-10) solves for the covariance matrix in the 3 modes : free system, continuous, and discrete filtering; writesthe r.m.s. errors, and punches some of these results for later use on a plotter. The main inputs are : ALAT1 (latitude in degrees); TF (final time); AQ (inertial navigator noise in ft /sec) AR (radar noise in ft .sec), and three different operating times OPT1, OPT2, OPT3 in sec.

CHAPTER 5

RESULTS

The computer program used to get the results of this chapter is given in Appendix B. It was worked out on an I.B.M. 360 in the M.I.T. Computation Center.

For the discrete filtering, 3 operating times were studied: 30 seconds, 3 minutes and 18 minutes.

For any of the 3 modes, several cases for both the I.N.S. and the radar noise have been worked out; N, the I.N.S. noise, ranging from 10^{-2} to 10^{+2} $(\text{feet})^2/(\text{sec})^3$ (power spectral density of white noise over some frequency range) and R, the radar noise, ranging from 10 to 10^{+6} $(\text{feet})^2.\text{sec}$. These values seem to cover all the practical cases.

Before exposing the results, it is better to give some explanations about the curves of Appendix A.

The first 10 graphs are actual outputs from the computer and represent the variation of RMX (r.m.s. error in x-position indication) and RVX (r.m.s. error in x-velocity indication) as functions of time in minutes. The units are nautical miles for RMX and feet/sec for RVX.

Graphs 1 and 2 (3 and 4) are RMX (RVX) for the free system and for the 3 models.

Graphs 5 through 12 give RMX and RVX for models 1 and 2 only. There are 4 curves on each graph: 1 for the continuous filtering and 3 for each operating time in the discrete filtering.

Graphs 13 through 19 have been established from the printed outputs and the units are : feet for RMX and ft/sec for RVX. Graphs 13 through 17 are concerned with continuous filtering only and chart 17 yields the expected RMX and RVX for given I.N.S. noise N and radar noise R, both in usual units.

The last 2 graphs are concerned with discrete filtering. From them we can find the influence of the operating time on both RMX and RVX given some values for the r.m.s. errors in the continuous filter (see below).

5.1 Free system (graphs 1, 2, 3 and 4)

According to the results of part 4.2.3, RMX and RVX increase as $t^{3/2}$ for RMX and $t^{1/2}$ for RVX for a given inertial noise in model 1. This is observable on graphs 1 and 2 for RMX, 3 and 4 for RVX, where the upper curve represents model 1.

The 2 other curves on these graphs represent the free system r.m.s. errors and are almost the same. The improvement in the error by comparison with the model 1 is due to the Sculer oscillation (indeed the time between 2 consecutive points of both curves where the slope is minimum appears to be 42 minutes, half of the Sculer period). This improvement with respect to model 1 reaches 80 percent for RMX and 30 percent for RVX after 84 minutes.

The tiny difference between the curves representing models 2 and 3 was also observed for other values of N and clearly shows that model 3 is not better than model 2 when there is no filtering.

Furthermore, up to 14 minutes for RMX and 8 minutes for RVX, the 3 curves are almost coincident. This means that the 3 models yield the same r.m.s. errors in free mode up to 14 minutes for position error and 8 minutes for velocity error.

Of course, since no radar is used here, the graphs do not depend on R. Furthermore, looking at the upper curve on graphs 1 and 2 as well as 3 and 4, the influence of N can be found to match equation of part 4.2.3. In graphs 1 and 2 for instance, the scale of the RMX-axis is multiplied by 10 while the noise is multiplied by 100. The same thing stands for RVX. And since the curves have exactly the same shape in both cases, the r.m.s. position and velocity errors are proportional to $\sqrt{N}$ at a given time.

The general result for the free system can be stated in the following way:

for model 1, the r.m.s. errors are given by

$$\begin{cases} RMX = (N/3)^{1/2} \quad t^{3/2} \\ RVX = N^{1/2} \quad t^{1/2} \end{cases} \qquad (5\text{-}1)$$

At any time, there is no difference between model 2 and 3. The 3 models are equivalent up to 14 minutes for position error and 8 minutes for velocity error.

5.2 Continuous filtering

The curve representing this mode is the lowest one on graphs 5 through 12. It is the only straight line on all these figures. Because the steady state error in continuous mode is very small and not easily readable on the plots, the results are given on page A-4, with the precision obtained on the computer. For all the cases that have been studied, the 3 values of RMX (in nautical mile) corresponding to the 3 models and the 3 values of RVX (in feet/sec) corresponding to the 3 models are shown.

The largest difference between the 3 models appears to be .04 percent and is indistinguishable from the truncation errors in the results.

Therefore, and this conclusion is an important one, <u>the steady state position and velocity r.m.s. errors do not depend on the model chosen to represent the I.N.S. in the case of continuous filtering</u>.

It could be expected from part 5.1 (free system), where models 2 and 3 appeared to yield the same error, that there were no difference between the 2 models. The important fact is now that model 1 yields also the same result.

The steady state errors for model 1 were shown in part 4.2.1 to obey equations (4-3). To argue the model independence, the same formula can be found from graphs 13 - 16 where the results for model 2 are plotted.

Graphs 13 and 14 show the variation of RMX with R for different values of N and with N for different values of R. On logarithmic paper, the curves appear to be straight parallel lines. Measuring the slopes yields the following dependence formula:

$$\log RMX = \frac{1}{8} \log N + \frac{3}{8} \log R + d$$

and the constant d turns out to be $\frac{1}{4} \log 2$

Thus:

$$RMX = 2^{1/4} N^{1/8} R^{3/8} \qquad (5\text{-}2)$$

A similar analysis for RVX yields:

$$RVX = 2^{1/4} N^{3/8} R^{1/8} \qquad (5\text{-}3)$$

In these equations the units are:

RMX in feet ; RVX in feet/sec

N in $feet^2/sec^3$; R in $feet^2.sec$

These results could also have been found by a dimensional analysis, N and R beeing the only parameters that can influence the errors.

The last thing to be said about this continuous filtering is that, as it was expected from figure 6 in model 1, the time necessary to reach the steady state is small and does not exceed 5 minutes in the worst case presented on graph 3.

Graph 15 gives the position and velocity r.m.s. errors in feet and feet/sec for given values of N (I.N.S. noise) and R (radar noise).

5.3 Discrete filtering

Let us recall that this case was studied with 3 different operating times: OPT = 30 seconds, 3 minutes and 18 minutes, which seem to cover a good part of the permissible range.

The corresponding curves are the 3 upper curves on graphs 5 through 12, and it is obvious that the larger the operating time is, the higher the corresponding curve goes.

Let us note that, especially for the lower two OPT, the curves are not quite representative of what happens. Indeed we have assumed that updating after each measurement was an instantaneous operation, so that the curve should drop with an infinite slope at each measurement time. This is not the case because, in order to save some computer time, we limited ourselves to 3 output values per operating time, and the 3 values could not be chosen to be just prior to and just after the measurement.

Furthermore, it is very difficult to analyse these results because the "mean value" for each case is uneasy to obtain, either from the graphs or from the printed outputs.

Therefore, let us see if it is possible to relate any discrete filtering problem to what will be called its "continuous approximation".

Let us start with a discrete measurement process obeying the usual equations:

$$\begin{cases} \dot{\underline{x}}(t) = F(t)\ \underline{x}(t) + G(t)\ \underline{u}(t) \\ \underline{m}(t_n) = H(t_n)\ \underline{x}(t_n) + \underline{v}(t_n) \end{cases}$$

where $\underline{u}(t)$ is a white noise and $\underline{v}(t_n)$ an independent Gaussian process such that:

$$\begin{cases} E\left[\underline{u}(t)\ \underline{u}'(t+s)\right] = Q(t)\ \delta(s) \\ E\left[\underline{v}(t_n)\ \underline{v}'(t_n)\right] = V = \text{constant} \\ E\left[\underline{v}(t_n)\ \underline{v}'(t_n+s)\right] = 0 \text{ if } s \text{ is not } 0. \end{cases}$$

Since the noises are zero-meaned, V is the variance of the radar noise.

This discrete measurement process can be approximated with an equivalent continuous one defined by

$$\underline{m}(t) = H(t)\ \underline{x}(t) + \underline{v}(t)$$

where now $\underline{v}(t)$ is a white noise obeying:

$$E\left[\underline{v}(t)\ \underline{v}'(t+s)\right] = R\ \delta(s) \qquad \text{with } R \text{ constant.}$$

Furthermore let us assume

$$R = V \cdot OPT \tag{5-4}$$

The units are right since V is a variance and R a power spectral density for a white noise. This really means that if the measurements are taken twice faster, the measurement error covariance must be twice larger to be approximated by the same continuous process.

Then it can be shown that, provided that the operating time is not too large, any discrete measurement process can be approached by a continuous one which appears as the mean of the previous one. This can be understood by deriving the continuous case variance equation from the discrete case scheme.

Between two measurements times t_{n-1} and t_n, equation (2-10) can be written with the state transition matrix $\Phi(t_1,t_2)$ as:

$$P(t_n^-) = \Phi(t_n,t_{n-1})\ P(t_{n-1})\ \Phi'(t_n,t_{n-1}) + \int_{t_{n-1}}^{t_n} \Phi(t_n,s)\ G(s)\ Q(s)\ G'(s)\ \Phi'(t_n,s)\ ds \tag{5-5}$$

If and F are continuous, Taylor expansion yields:

$$\Phi(t_n,t_{n-1}) = I + F(t_{n-1})\Delta t + \text{higher order terms in } \Delta t$$

Δt is here the operating time $t_n - t_{n-1}$.

Then, using the mean value theorem with $t_{n-1} \leqslant z \leqslant t_n$

(5-5) becomes:

$$P(t_n^-) = P(t_{n-1}) + F(t_{n-1})\, P(t_{n-1})\, \Delta t + P(t_{n-1})\, F'(t_{n-1})\, \Delta t + G(z)\, Q(z)\, G'(z)\, \Delta t + \text{higher order terms} \tag{5-6}$$

At measurement time we update $P(t_n^-)$ through

$$P(t_n^+) = P(t_n^-) - P(t_n^-)\, H' \left[H\, P(t_n^-)\, H' + \frac{R}{\Delta t} \right]^{-1} H\, P(t_n^-)$$

This yields finally the equation:

$$P(t_n) = P(t_{n-1}) + \Big[F(t_{n-1})\, P(t_{n-1}) + P(t_{n-1})\, F'(t_{n-1}) + G(z)\, Q(z)\, G'(z) - P(t_n^-)\, H' \left[H\, P(t_n^-) H' \Delta t + V \right]^{-1} H\, P(t_n^-) \Big] \Delta t \tag{5-7}$$

And in the limit when Δt approaches 0, we get

$$\dot{P}(t) = F\,P + P\,F' + G\,Q\,G' - P\,H'\,R^{-1}\,H\,P$$

which is the continuous filtering variance equation.

Thus, under assumption (5-4), any discrete problem can be approximated by a continuous one since $\frac{R}{\Delta t}$ has the same effect as V. It happens that, for not too large an OPT, this continuous approximation looks as the mean of the discrete process and is therefore the best measure of its accuracy.

From our point of view, the accuracy of the discrete filtering system with operating time OPT and error variance V is the same as the accuracy of a continuous filtering system with noise power spectral density $\bar{R} = V \cdot OPT$

Let us see now the significance of the limitation on the time between the measurements. Two reasons can be thought of:

1) if the discrete process yields the same results as the continuous one, this means that the r.m.s. errors do not depend on the model in the discrete process (since this was shown in the continuous case). But it was pointed out in part 5.1 that the 3 models are completely equivalent in free mode only during a time lower than 14 minutes for position error and 8 minutes for velocity error. Since this free mode is precisely used between the resets, it seems consistent to take as time limits for this continuous approximation of the discrete process the 2 numbers: 14 minutes for position and 8 minutes for velocity.

2) what is really implied by the relation (5-4) is to replace, on a plot of the autocorrelation function of the radar noise as a function of time, an impulse (rectangular shape) of area R by a triangular curve of hight V and area V . OPT = R . Although these two curves are not the same, they can yield the same final result if the system is unable to distinguish between both shapes. And this happens if the spread of the autocorrelation function is small by comparison with the time constant of the system. The practical limitations appear to be the same as before: 14 minutes for position and 8 minutes for velocity, as long as the Sculer oscillations do not change the response too

much.

Under these limits, from (5-2), the steady state error of the continuous process does not depend on the model. Thus the accuracy of a discrete process does not depend on the model. This, of course, stands as long as we can speak of a mean for this process; that is to say that the operating time must be small by comparison with the mission time.

Graphs 18 and 19 give the r.m.s. errors in position and velocity for discrete processes of given operating time and steady state errors of the corresponding (not equivalent) continuous process. They are to be used in conjunction with graph 17 in the following way:

for any I.N.S. and radar noise powers N and R, graph 17 gives the resulting position and velocity r.m.s. errors in the steady state. With these values of RMX and RVX, graphs 18 and 19 give the corresponding r.m.s. errors of a discrete filter of given operating time OPT (in seconds) using the same I.N.S. and radar.

The accuracy of the continuous approximation can be checked on graphs 6 and 11 where the mean values of the 30 seconds, 3 minutes and 18 minutes discrete processes have been plotted. It is to be remembered that the shapes of the curves are not exact and the error increases with the operating time.

Anyway, it appears that the approximations are quite good for the 30 seconds and 3 minutes cases. As for the 18 minutes process, while the position error curve is still acceptable, this is no longer true for the velocity error curve. This fact is in accordance with the operating time limitations previously introduced. It can be found convenient to consider 10 minutes as the limit on the operating time for both position and velocity informations.

5.4 Summary of the results ; Conclusion

1. For a use of the inertial navigator alone, without external position information, the 3 models are equivalent during a time not exceeding 10 minutes when they start from the same perfect initial state. After 10 minutes, the Sculer oscillations attenuate the errors in models 2 and 3 which are always equivalent. The errors in model 1 are given by equations (5-1).

2. When use of a position information is possible in a continuous way, the model chosen in the Kalman filter to represent the I.N.S. is of no importance. This means that a simple model consisting of 2 accelerometers kept roughly aligned with the north and east axes yields the same accuracy as a more sophisticated one(but becomes very bad if the external information happens to be lost).

For any model, the steady state r.m.s. errors are given by equations (5-2) and (5-3).

3. As long as the correlation time of the external device (radar) is small by comparison with the time constant of the system, any discrete measurement process can be approximated by a continuous one in the sense that the continuous process appears as the mean of the discrete one.

As long as the operating time is smaller than the time necessary for the Sculer oscillations to attenuate the errors, the 3 models yield the same final error.

Since this last constraint is less drastic than the first one, the result can be stated in the following way: as long as the operating time does not exceed 10 minutes, the position and velocity r.m.s. errors in a discrete measurement process do not depend on the model and can be approximated by the errors in a continuous scheme related to the discrete one by the equation (5-4).

The final equations are, with N in $feet^2/sec^3$, R in $feet^2$, OPT in seconds , RMX in feet and RVX in feet/sec :

$$\begin{cases} RMX = 2^{1/4} N^{1/8} R^{3/8} OPT^{3/8} \\ RVX = 2^{1/4} N^{3/8} R^{1/8} OPT^{1/8} \end{cases}$$

APPENDIX A

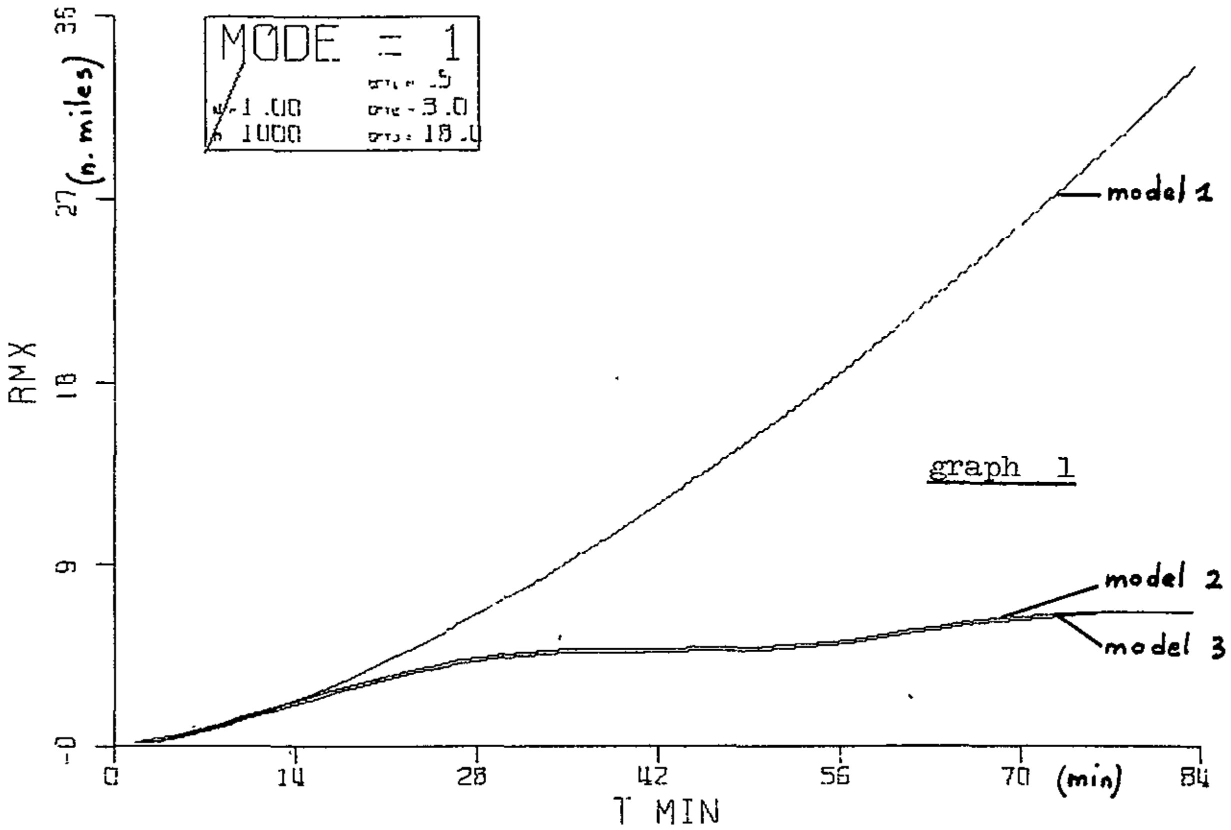
GRAPHS

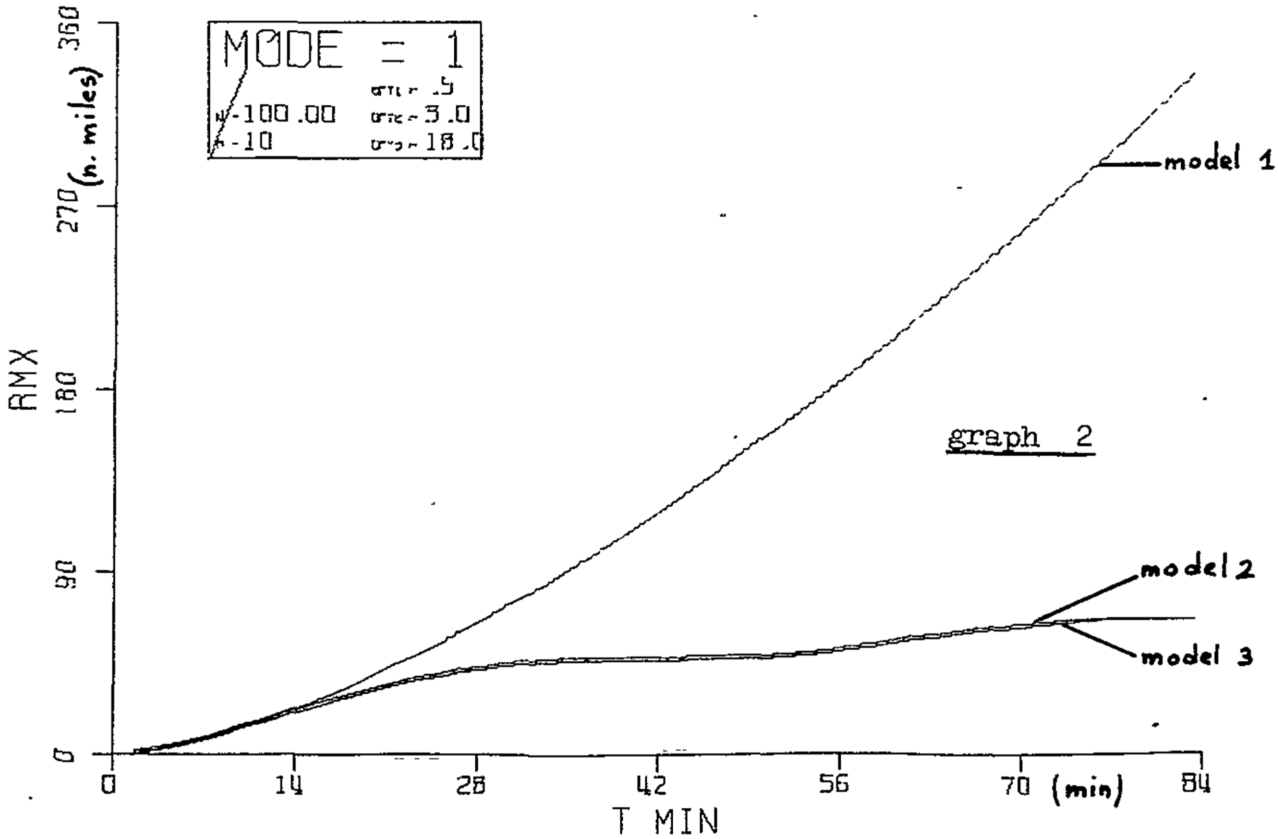
The units in the following graphs are the following:

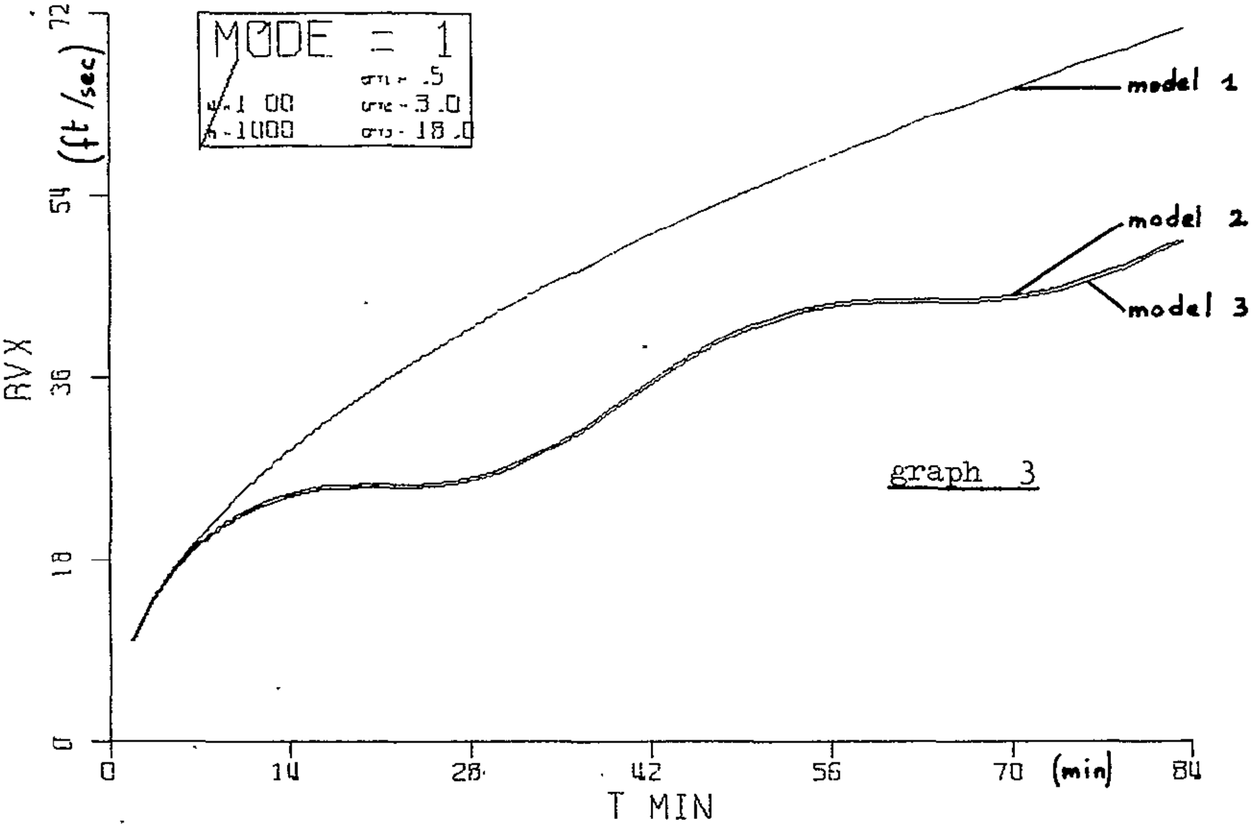
(feet)/sec. for velocity error RVX

nautical miles in graphs 1 through 12 as well as on page A-3 and feet otherwise for position error RMX.

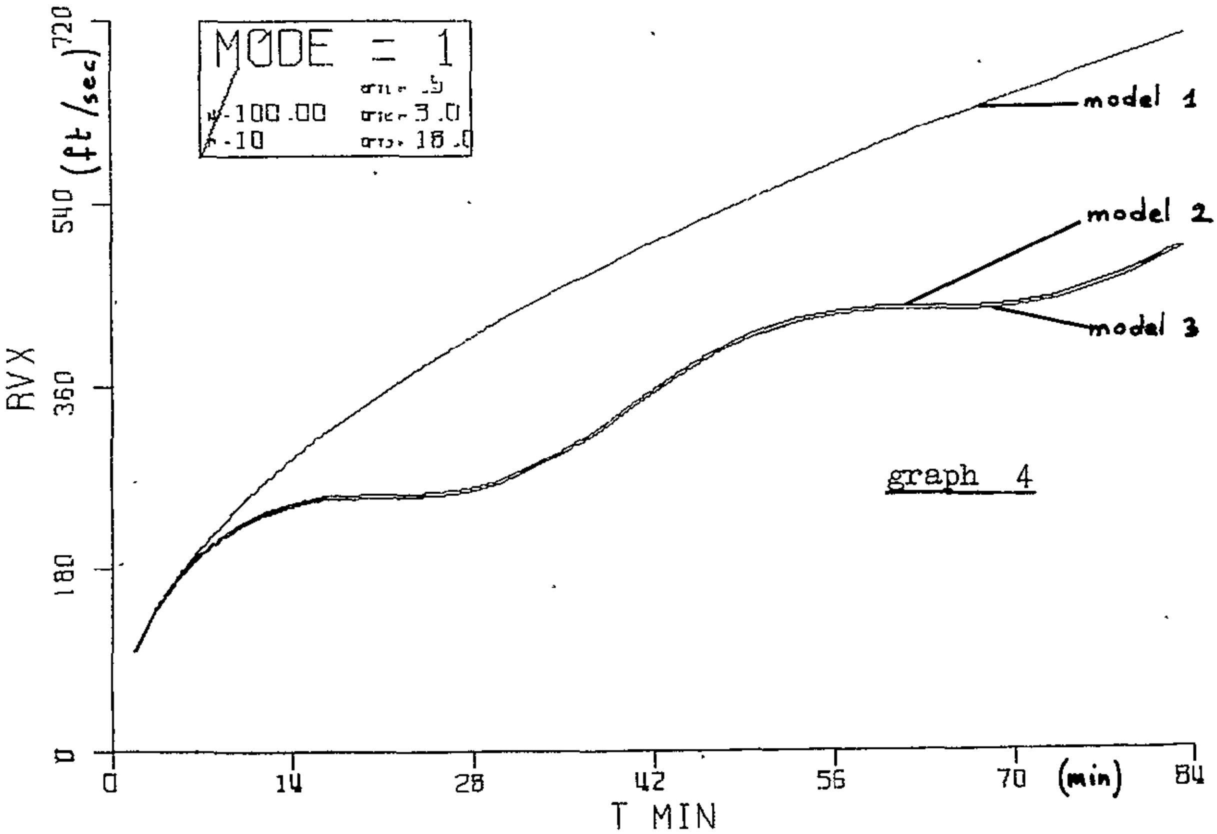
On graphs 1 through 12 which are output from the computer, some indications appear inside the box in upper left corner: the mode (1 for free system and 3 for discrete filtering); whenever the model does not appear, 3 curves corresponding to the 3 models are plotted. The other values are N in $(\text{feet})^2/(\text{sec})^3$, R in $(\text{feet})^2.\text{sec}$ and the 3 operating times in minutes.


MODE = 1
.5
-1.00
-3.0
1000
18.0
RMX (n. miles)
-0
9
18
27
35
0
14
28
42
56
70
84
(min)
T MIN
model 1
model 2
model 3
graph 1


MODE = 1
.5
-100.00
-3.0
-10
-18.0
RMX (n. miles)
0
90
180
270
360
0
14
28
42
56
70
84
(min)
T MIN
model 1
model 2
model 3
graph 2



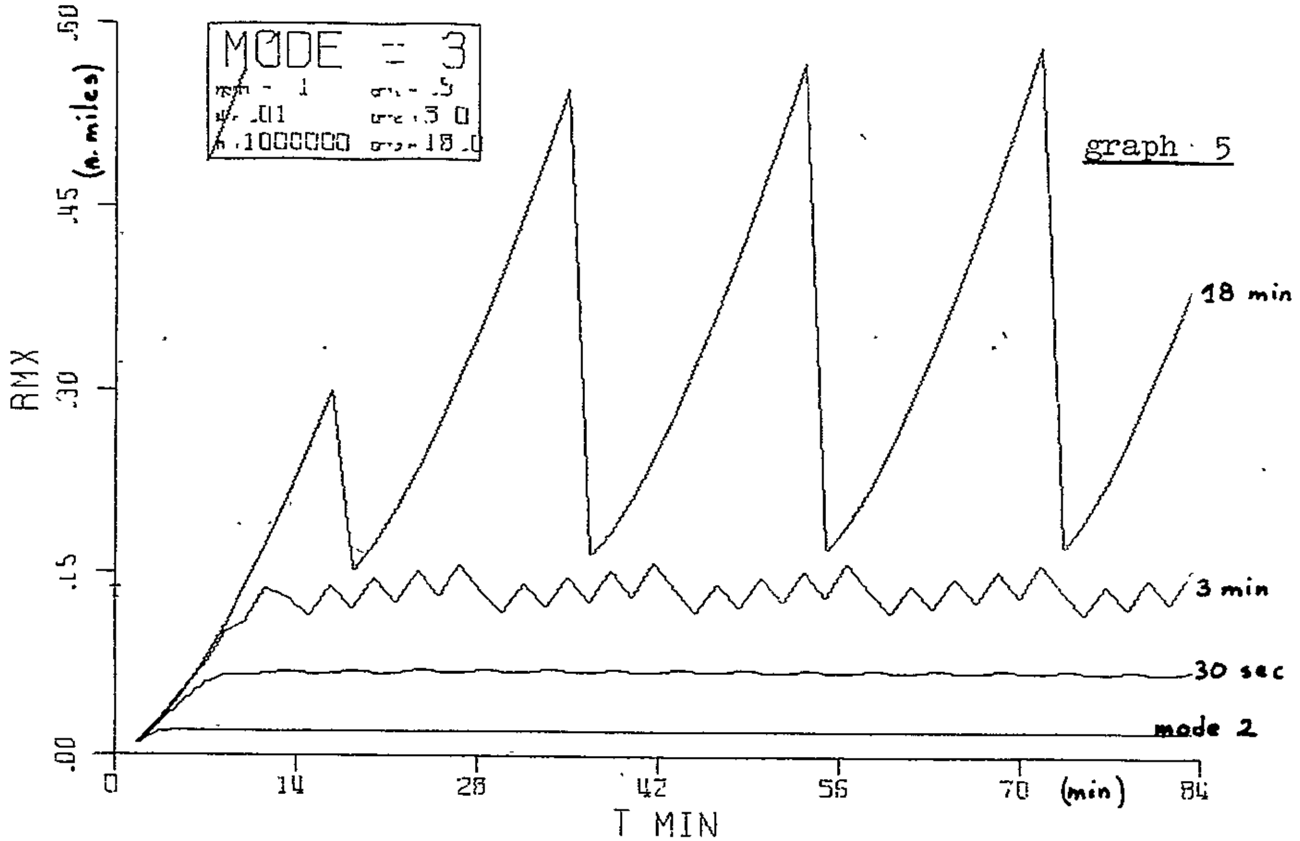
graph 3

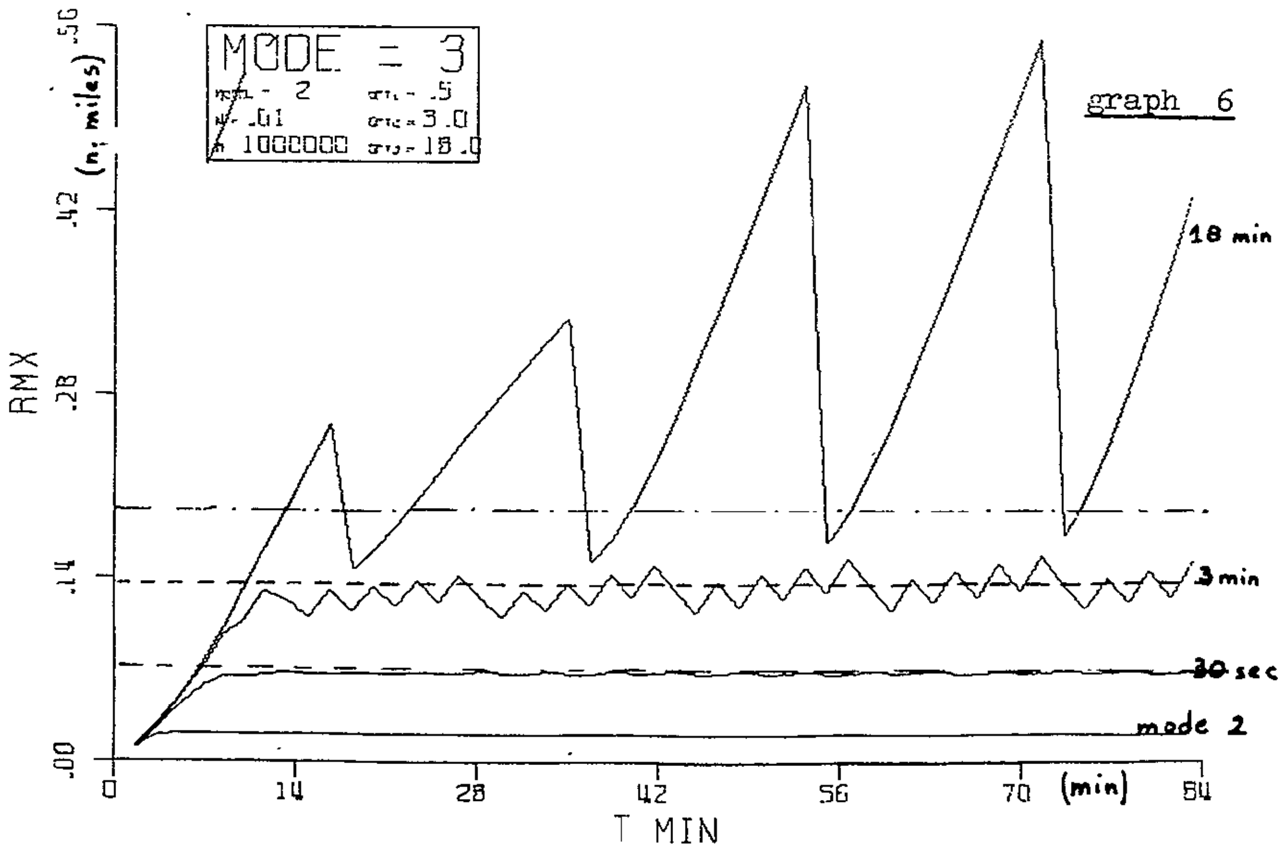


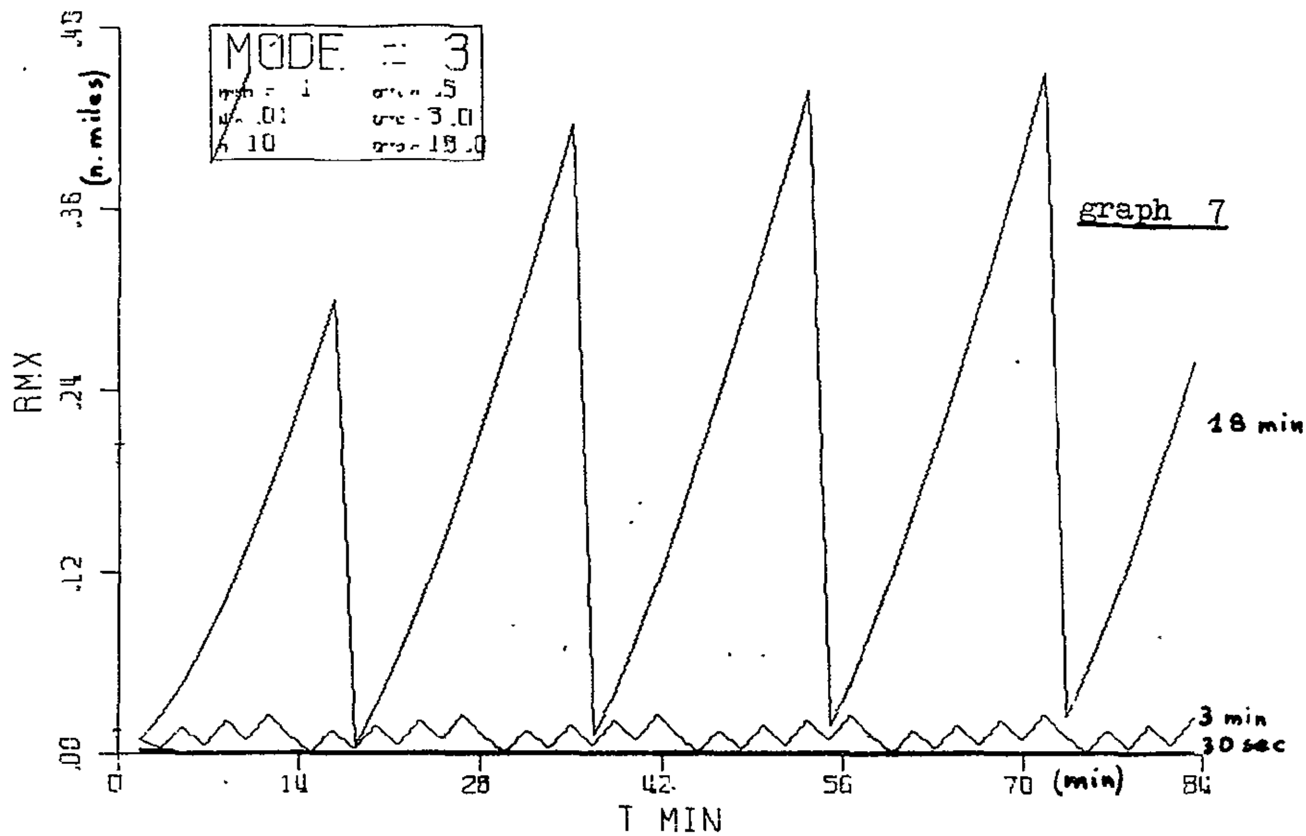
graph 4

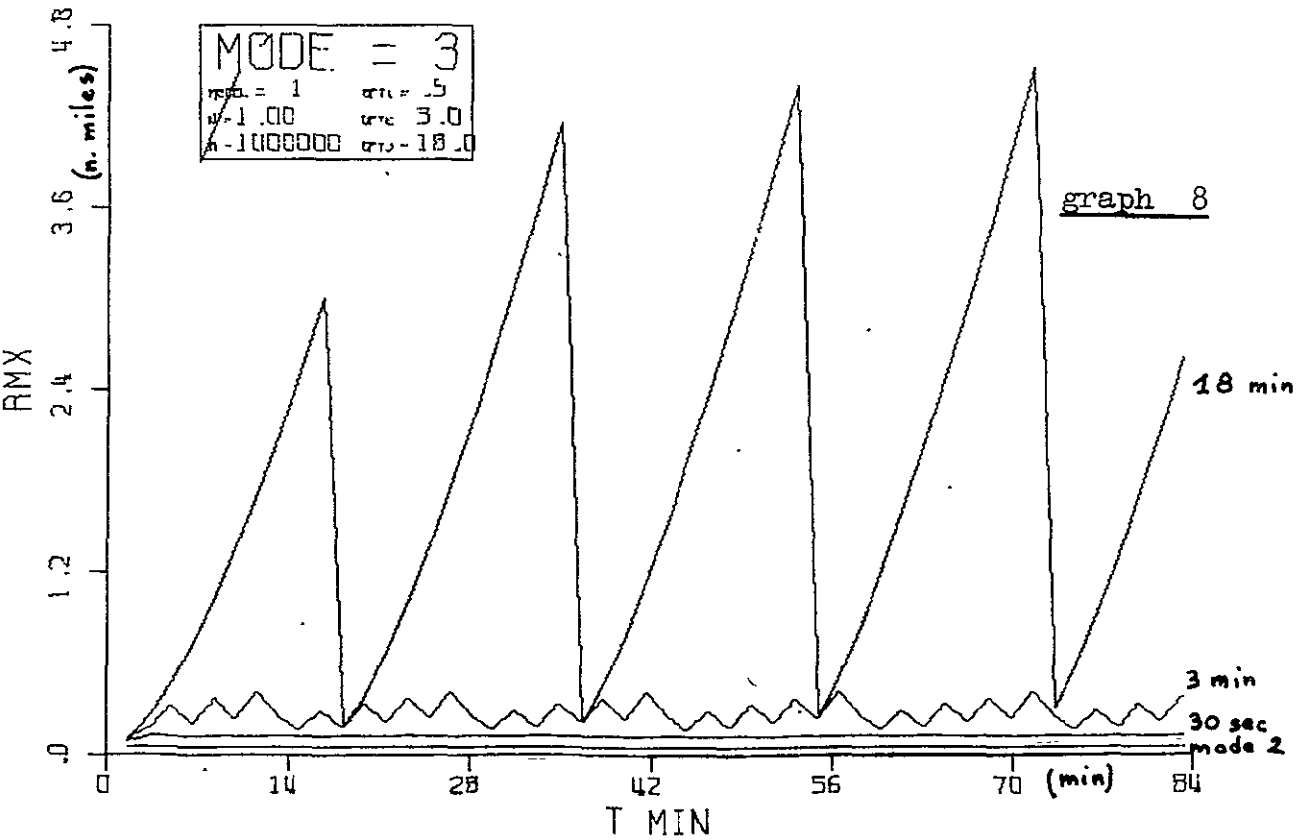
N		model	R = 10	model	R = 10^3	model	R = 10^4	model	R = 10^6
10^{-2}	RMX	1	2.610 10^{-4}	1	1.468 10^{-3}	1	3.480 10^{-3}	1	1.957 10^{-2}
		2	2.610 10^{-4}	2	1.468 10^{-3}	2	3.470 10^{-3}	2	1.950 10^{-1}
		3	2.610 10^{-4}	3	1.468 10^{-3}	3	3.470 10^{-3}	3	1.950 10^{-1}
	RVX	1	2.820 10^{-1}	1	5.015 10^{-1}	1	6.687 10^{-1}	1	1.189
		2	2.820 10^{-1}	2	5.014 10^{-1}	2	6.685 10^{-1}	2	1.185
		3	2.820 10^{-1}	3	5.014 10^{-1}	3	6.685 10^{-1}	3	1.185
1	RMX	1	4.641 10^{-4}	1	2.610 10^{-3}	1	6.189 10^{-3}	1	3.480 10^{-2}
		2	4.641 10^{-4}	2	2.610 10^{-3}	2	6.189 10^{-3}	2	3.479 10^{-2}
		3	4.641 10^{-4}	3	2.610 10^{-3}	3	6.189 10^{-3}	3	3.479 10^{-2}
	RVX	1	1.586	1	2.820	1	3.761	1	6.687
		2	1.586	2	2.820	2	3.760	2	6.685
		3	1.586	3	2.820	3	3.760	3	6.685
10	RMX	1	6.189 10^{-4}	1	3.480 10^{-3}	1	8.253 10^{-3}	1	4.641 10^{-2}
		2	6.189 10^{-4}	2	3.480 10^{-3}	2	8.253 10^{-3}	2	4.641 10^{-2}
		3	6.189 10^{-4}	3	3.480 10^{-3}	3	8.253 10^{-3}	3	4.641 10^{-2}
	RVX	1	3.761	1	6.687	1	8.918	1	15.86
		2	3.761	2	6.687	2	8.918	2	15.86
		3	3.761	3	6.687	3	8.918	3	15.86
10^2	RMX	1	8.253 10^{-4}	1	4.641 10^{-3}	1	1.101 10^{-2}	1	6.189 10^{-2}
		2	8.253 10^{-4}	2	4.641 10^{-3}	2	1.101 10^{-2}	2	6.189 10^{-2}
		3	8.253 10^{-4}	3	4.641 10^{-3}	3	1.101 10^{-2}	3	6.189 10^{-2}
	RVX	1	8.918	1	1.586 10^{1}	1	21.15	1	37.61
		2	8.918	2	15.86	2	21.15	2	37.60
		3	8.918	3	15.86	3	21.15	3	37.60

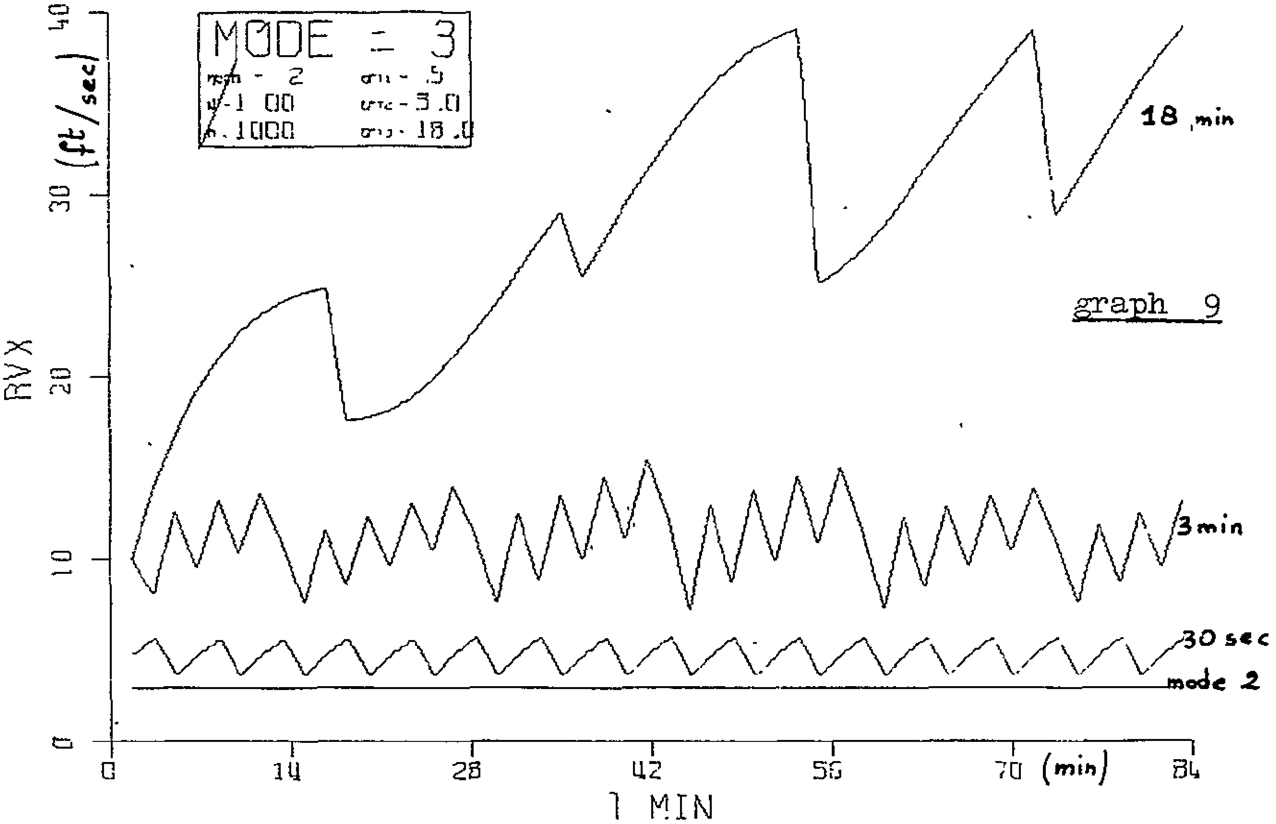
continuous filtering : steady-state errors (n.miles and ft/sec)

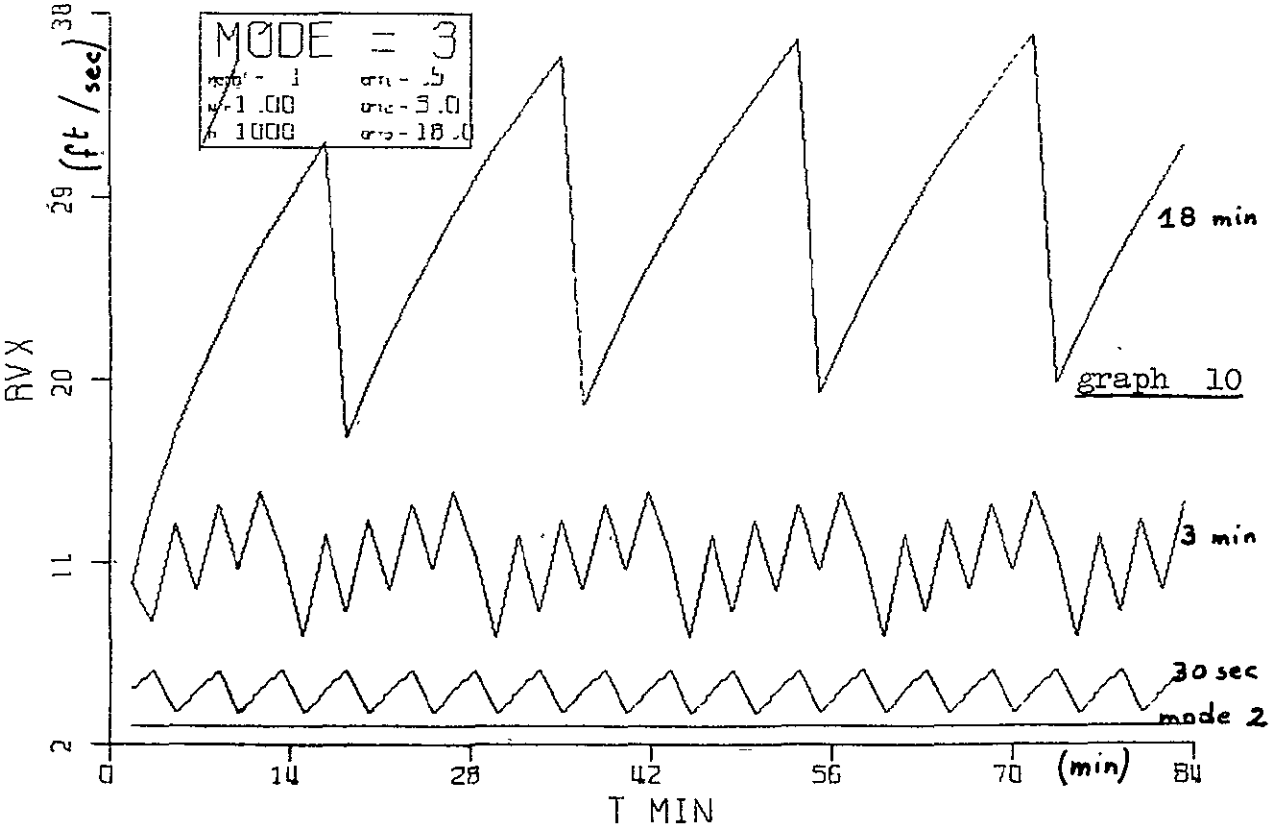

MODE = 3
graph 5
18 min
3 min
30 sec
mode 2
RMX (n. miles)
.00
.15
.30
.45
.60
0
14
28
42
56
70
84
(min)
T MIN

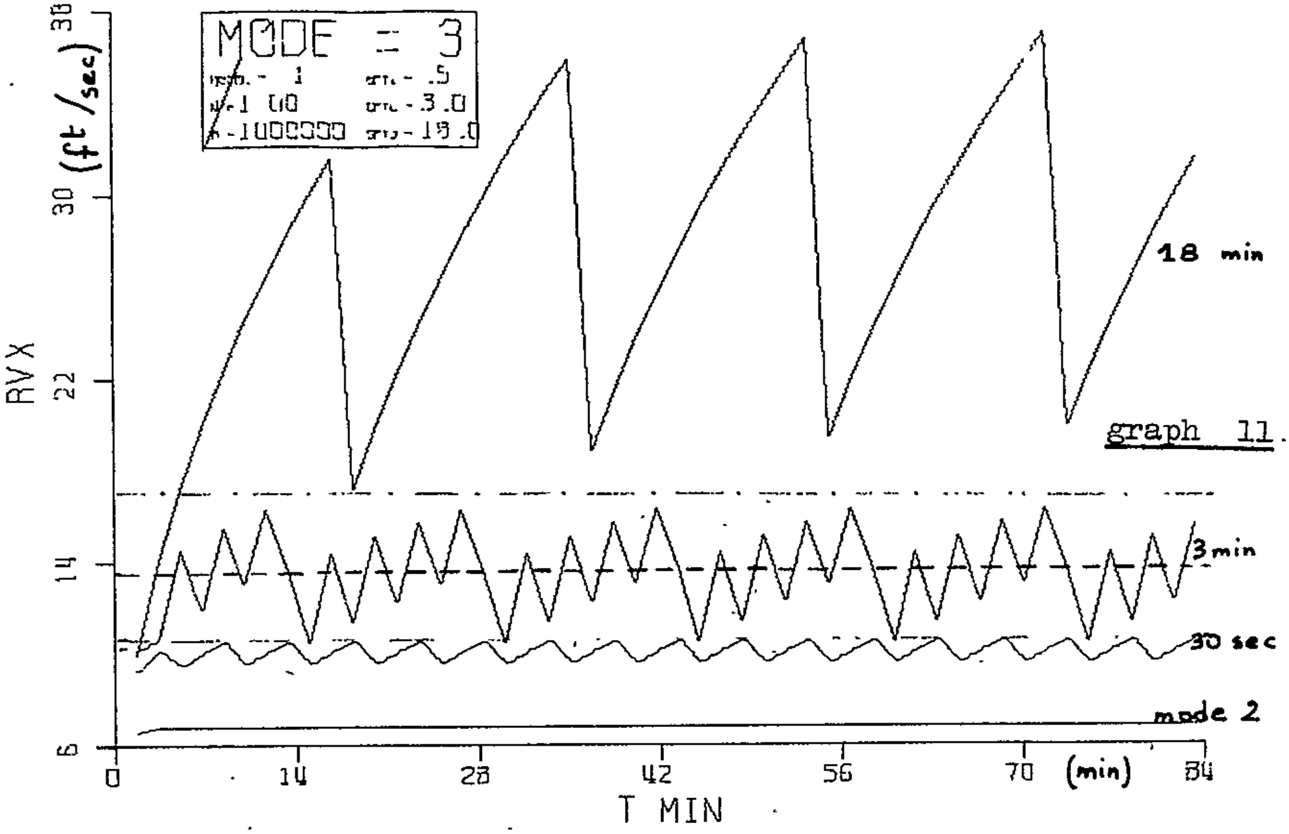

MODE = 3
graph 6
18 min
3 min
30 sec
mode 2
RMX (n. miles)
.00
.14
.28
.42
.56
0
14
28
42
56
70
84
(min)
T MIN

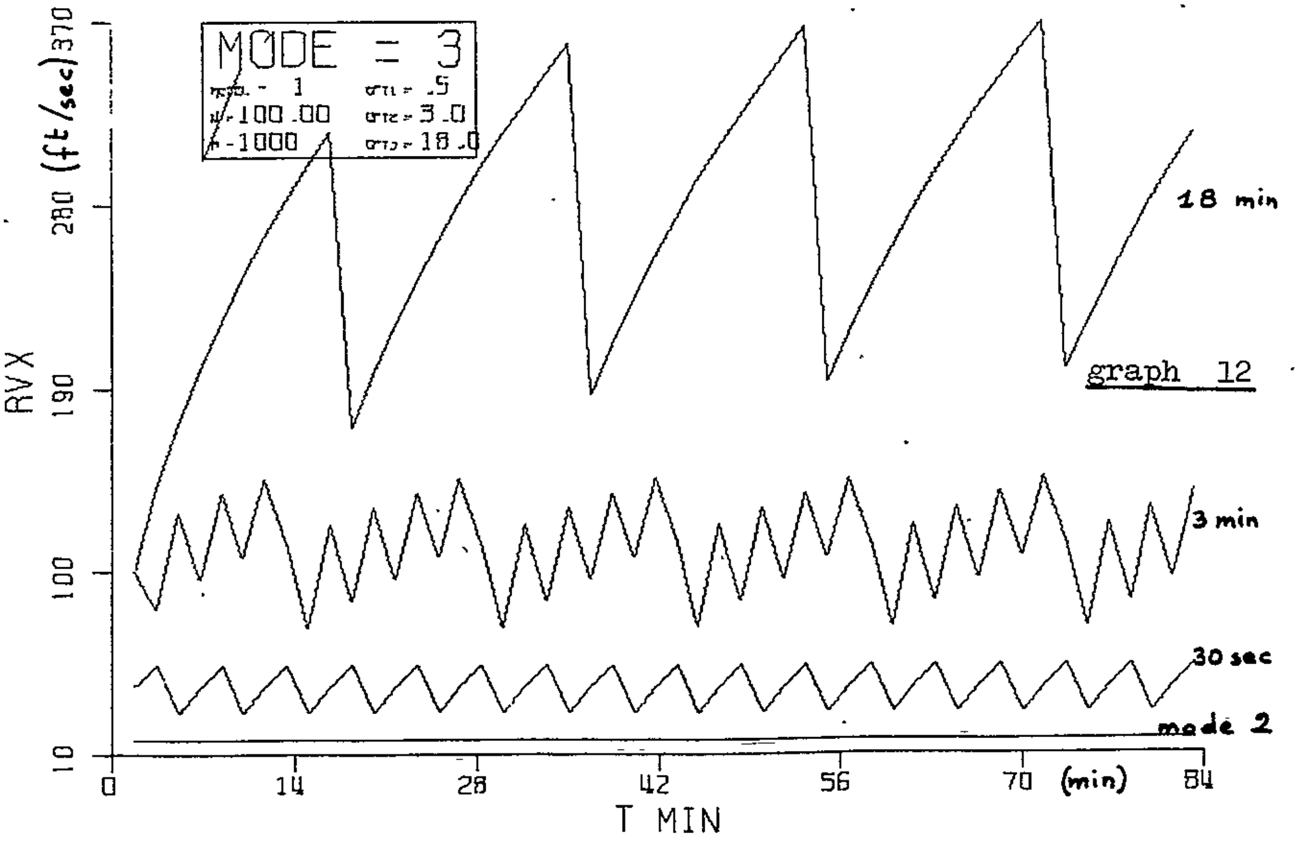

MØDE = 3
graph 7
RMX (n. miles)
.00
.12
.24
.36
.48
T MIN
0
14
28
42
56
70
84
(min)
18 min
3 min
30 sec

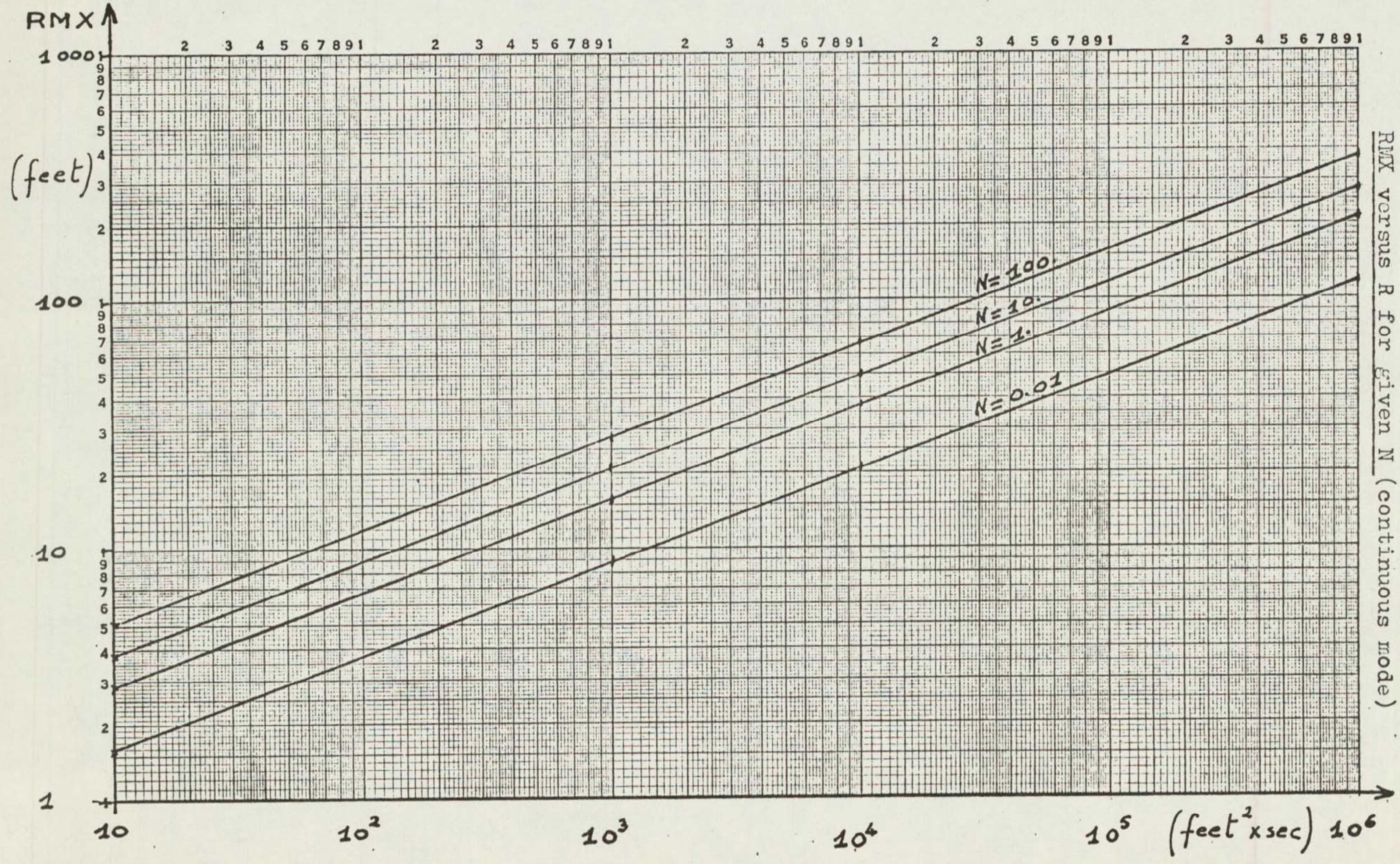

MØDE = 3
graph 8
RMX (n. miles)
.0
1.2
2.4
3.6
4.8
T MIN
0
14
28
42
56
70
84
(min)
18 min
3 min
30 sec
mode 2


MODE = 3
RVX (ft/sec)
18 min
3 min
30 sec
mode 2
graph 9
(min)
T MIN


MODE = 3
RVX (ft/sec)
18 min
3 min
30 sec
mode 2
graph 10
(min)
T MIN

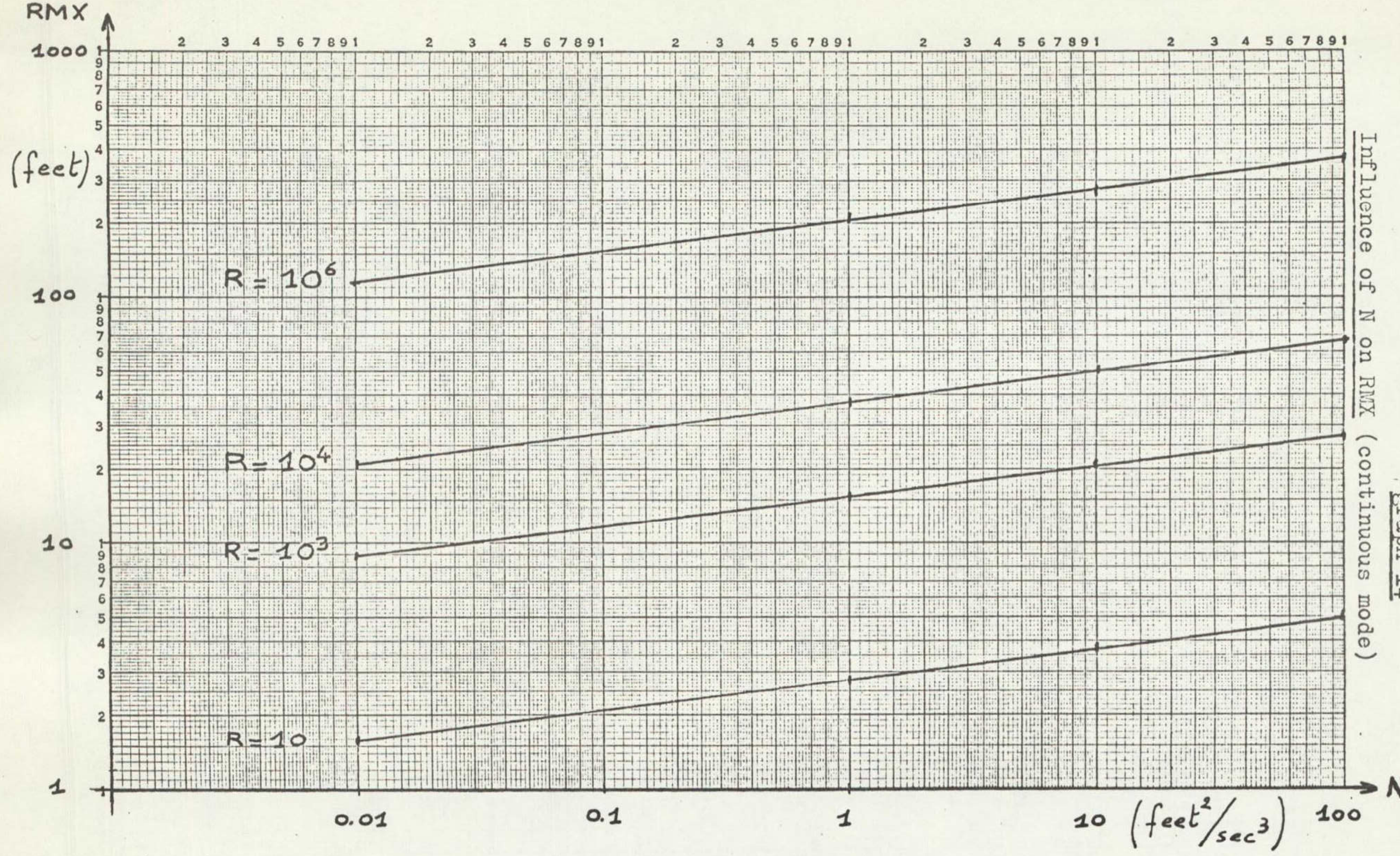

MODE = 3
RVX (ft/sec)
30
22
14
6
18 min
graph 11
3 min
30 sec
mode 2
0
14
28
42
56
70 (min)
84
T MIN


MODE = 3
RVX (ft/sec)
370
280
190
100
10
18 min
graph 12
3 min
30 sec
mode 2
0
14
28
42
56
70 (min)
84
T MIN



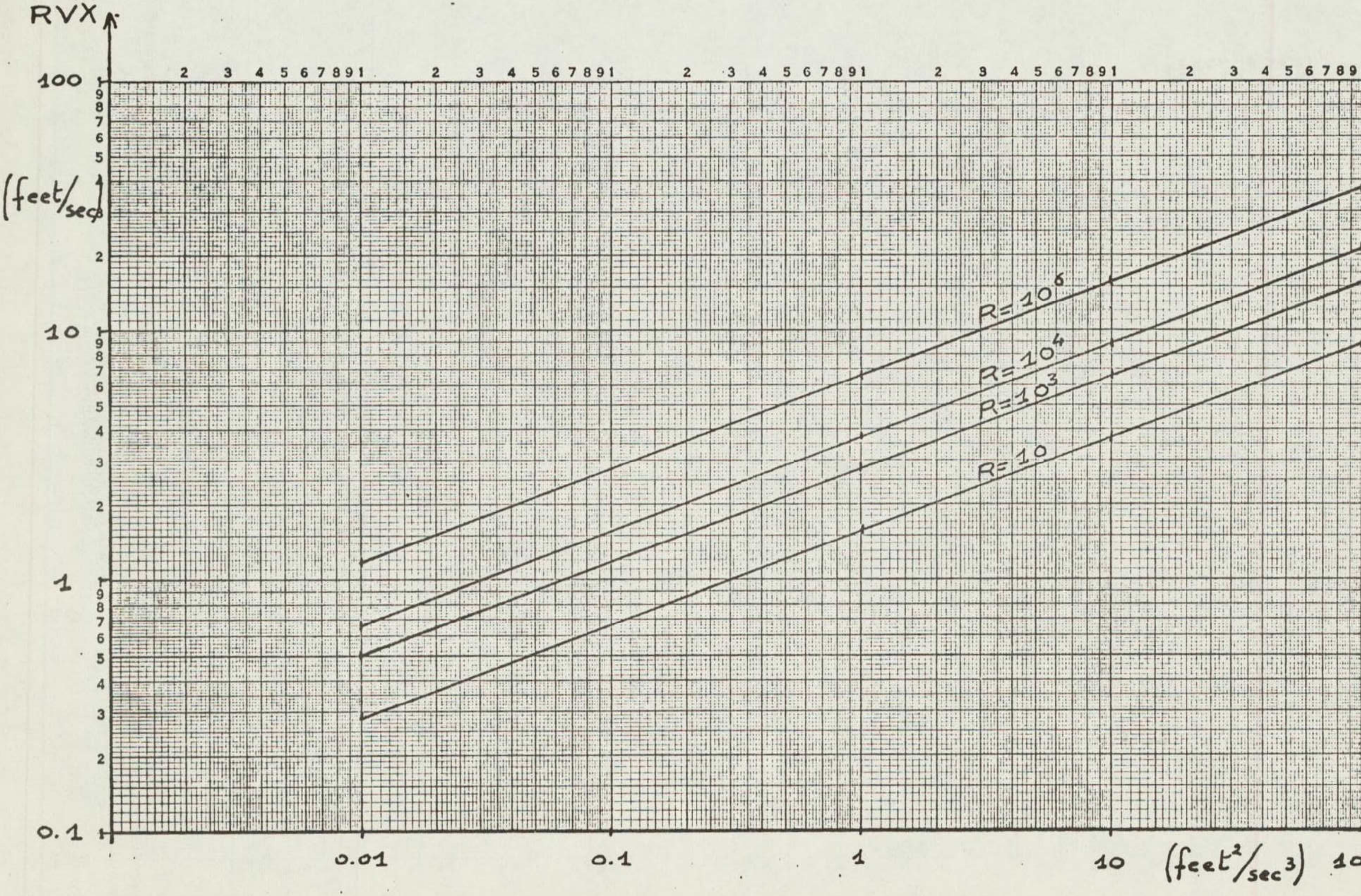
graph 13

RMX versus R for given N (continuous mode)



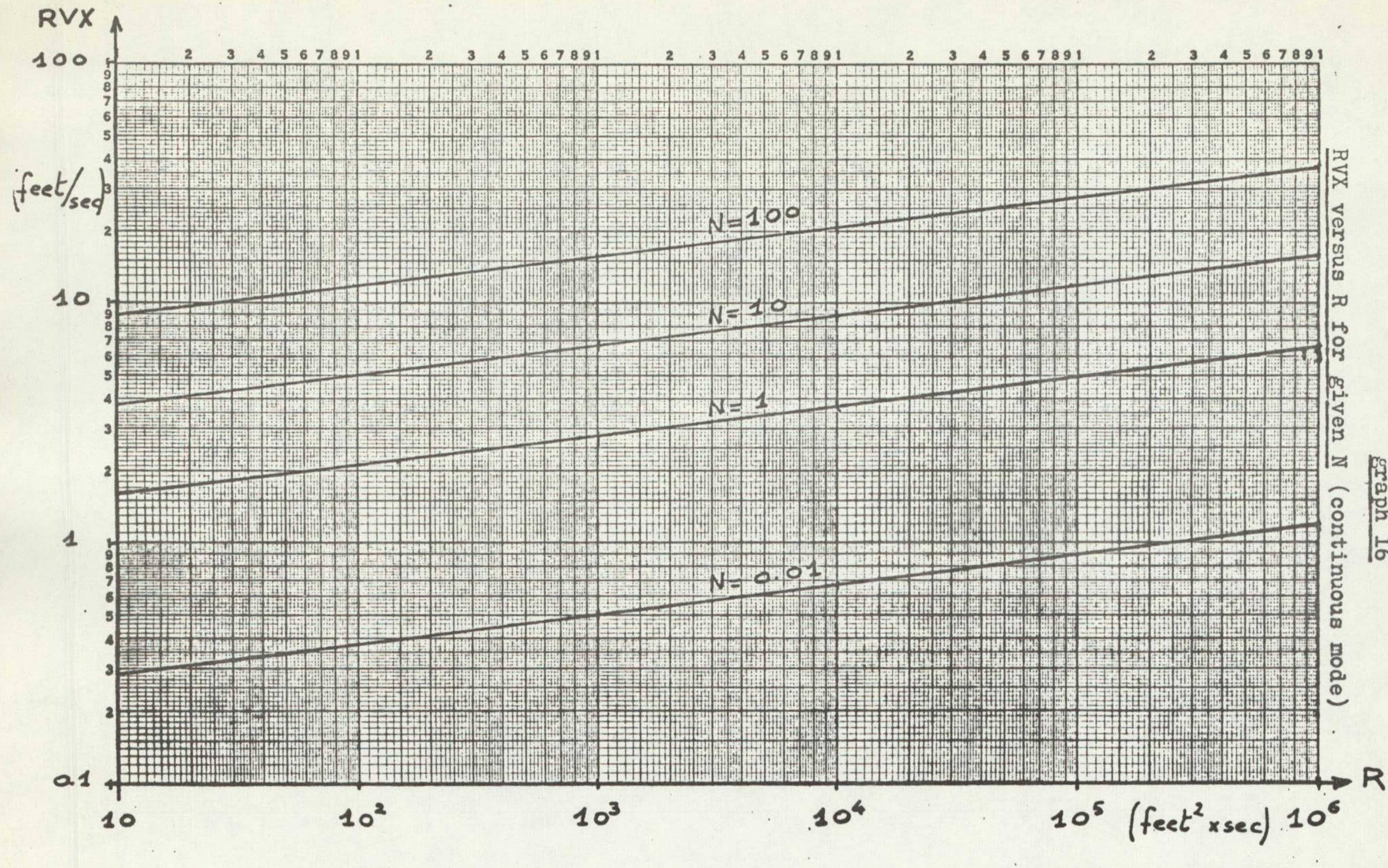
Graph 14

Influence of N on RMX (continuous mode)



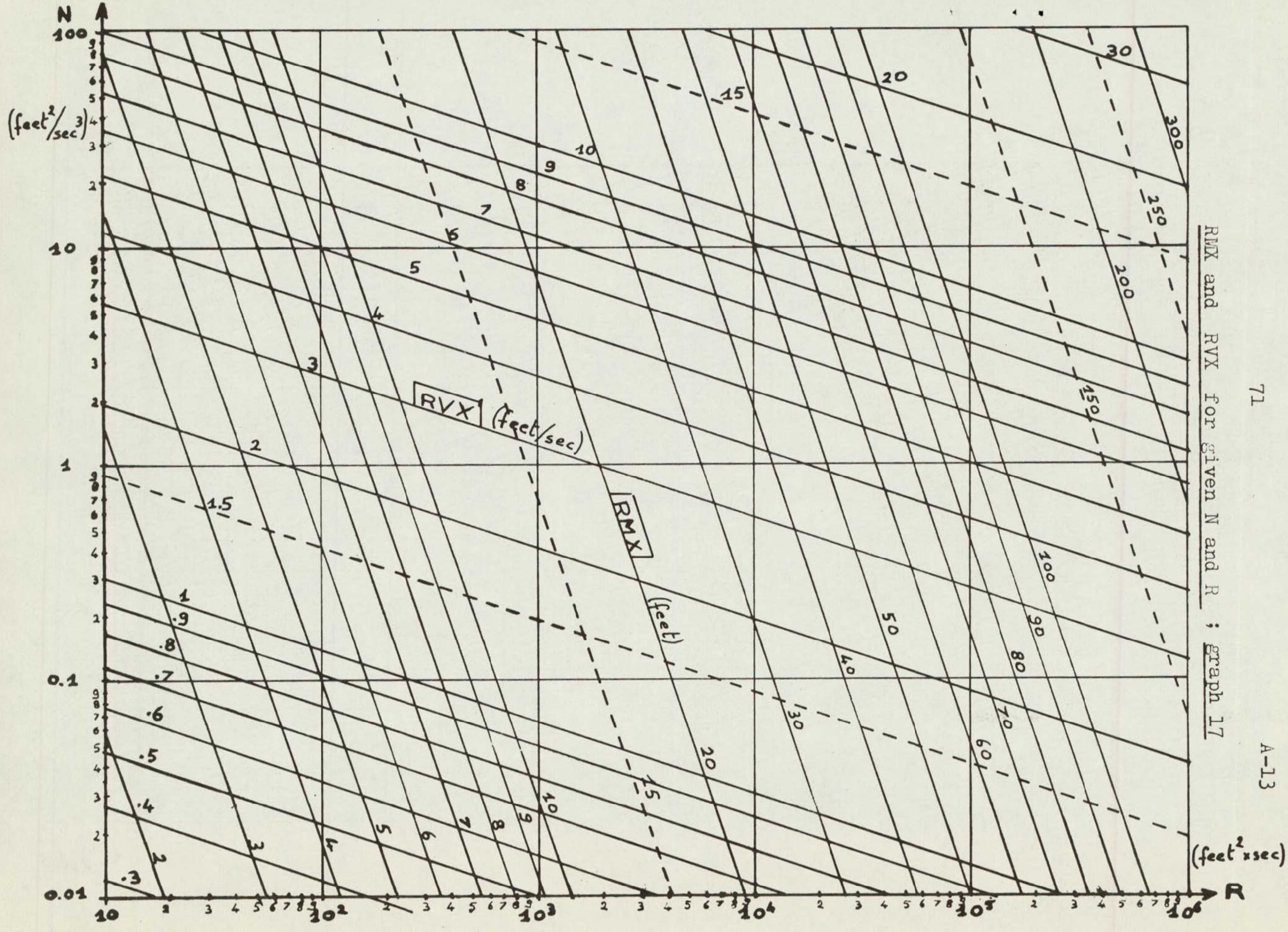
RVX versus N for given R (continuous mode)

graph 15

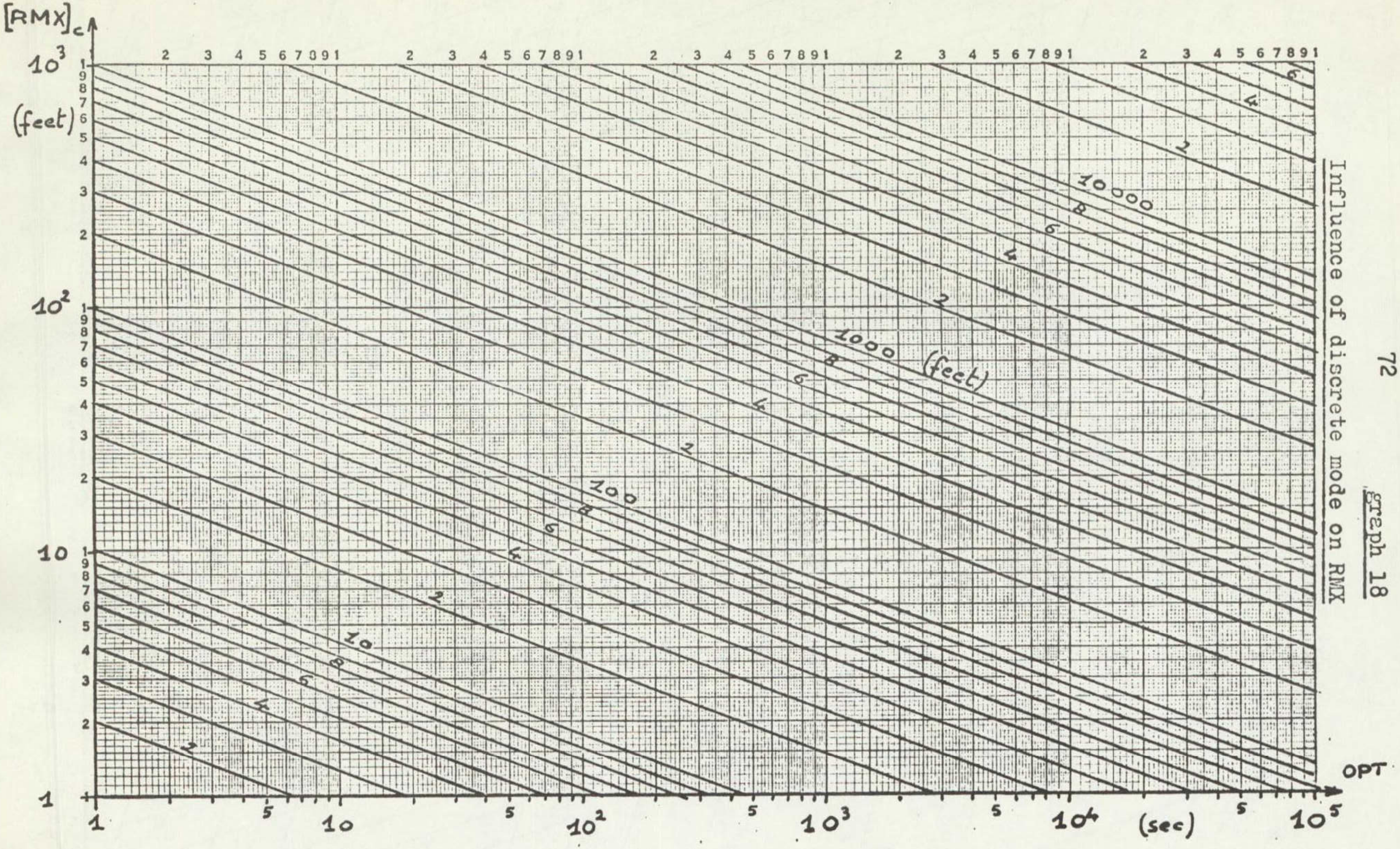


graph 16

RVX versus R for given N (continuous mode)



RMX and RVX for given N and R ; graph 17



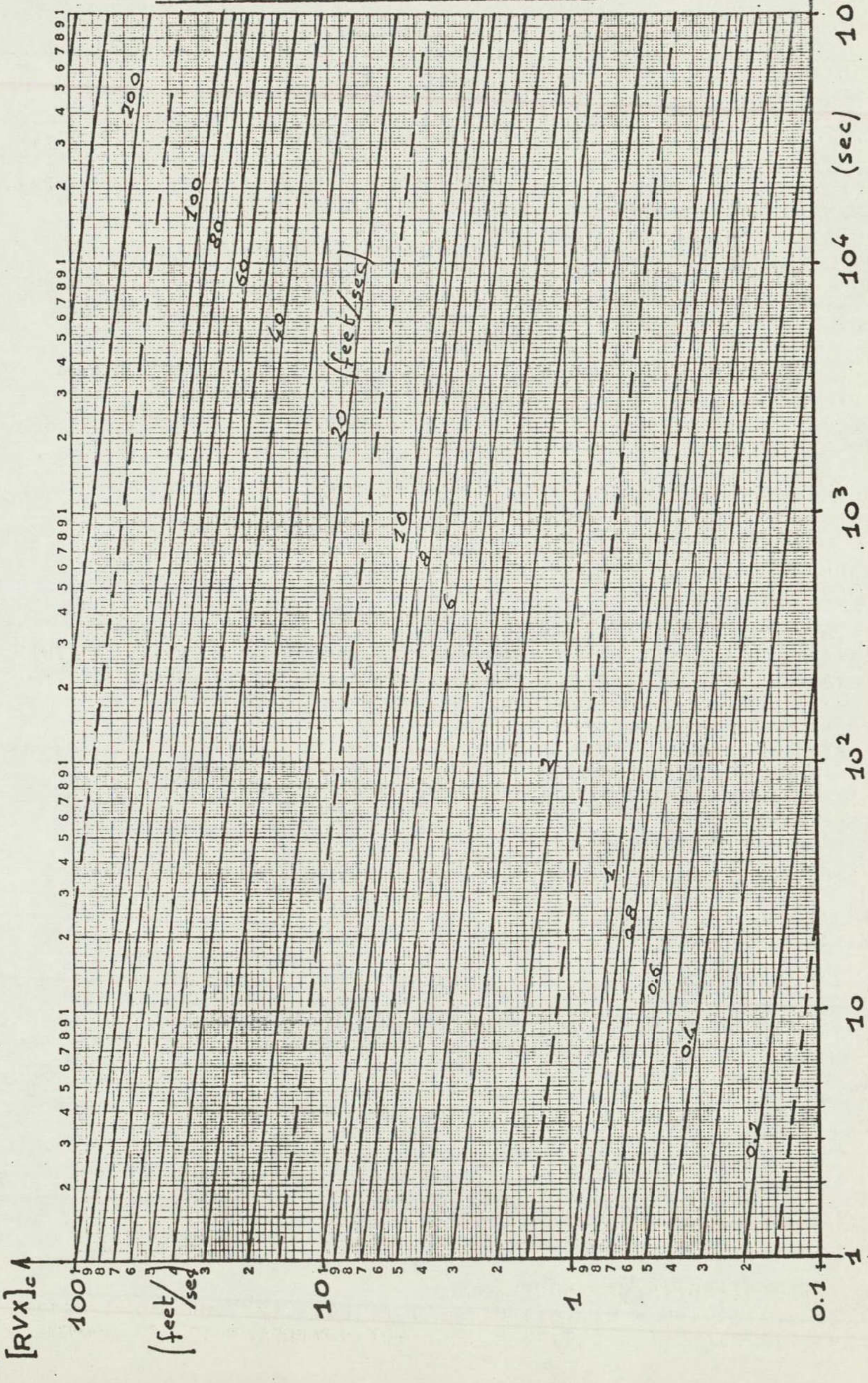
graph 18

Influence of discrete mode on RMX

72

graph 19

Influence of discrete mode on RVX



APPENDIX B : COMPUTER PROGRAM

```
-----------------------------------------------------------------------
      PROGRAM FOR COMPUTING POSITION AND VELOCITY R.M.S.
        ADDITIONAL OUTPUT : MISALIGNMENT ANGLES R.M.S.
-----------------------------------------------------------------------
      DIMENSION TI(50),X11(50),X21(50),X31(50),X12(50),X22(50),X32(50),V
     111(50),V21(50),V31(50),V12(50),V22(50),V32(50),X131(50),X132(50),X
     2133(50),X231(50),X232(50),X233(50),V131(50),V132(50),V133(50),V231
     3(50),V232(50),V233(50)
      COMMON A,B,Y(28),YP(28),R,N,J,NS
      COMMON G,RE,OX,OZ,ORX,ORZ,TR,Q,MODE,PAS,EPSI,SEUIL(28),T
      COMMON CK(14)
     *********************************
         READ AND WRITE DATA

      READ (5,1001) ALATI,EPSI,PA1,TF,DTW,SE,NPU,TPU1,K,MODEMI,MODEMA
 1001 FORMAT(F5.1,E8.3,E8.3,E8.3,E8.3,E8.3,I2,E8.3,I3,I2,I2)
      RE=6366000.
      G=9.8067
      OIE=3.1416/43200.
      FC1=3.28084
      FC2=1852.
      ALAT=(3.1416/180.)*ALAT1
    1 READ (5,1002) AQ,AR,OPT1,OPT2,OPT3
 1002 FORMAT(5E8.3)
      K=K+1
      WRITE (6,1003) ALAT1,EPSI,PA1,TF,DTW
 1003 FORMAT(1H1,12X,14HOPTIMUM MIXING///12X,11HLATITUDE = F5.1,12X,7HEP
     1SI = E11.4,12X,6HPAS = E11.4,12X,5HTF = E11.4//20X,6HDTW = E11.4)
      IF (ALAT.EQ.90.) GO TO 22
      ******************************
            COMPUTE USEFUL TERMS

      ALAT=(3.1416/180.)*ALAT
      OX=OIE*COS(ALAT)
      OZ=-OIE*SIN(ALAT)
      ORX=OX/RE
      ORZ=OZ/RE
      TR=SIN(ALAT)/(RE*COS(ALAT))
      Q=AQ/(FC1*FC1)
      R=AR/(FC1*FC1)
     *********************************
         BEGIN EACH CASE ; WRITE INITIAL CONDITIONS

      DO 35 MODEL=1,3
      DO 21 MODE=MODEMI,MODEMA
      OPT=OPT1
    2 PAS=PA1
      A=1.
      B=1.
      TW=DTW
      TM=OPT
      TPU=TPU1
      T=0.
      WRITE (6,1004) AQ,AR,OPT,MODEL,MODE
 1004 FORMAT(1H1//12X,4HN = E11.4,14H(FEET)2/(SEC)3,12X,4HR = E11.4,11H(
     1FEET)2*SEC,12X,6HOPT = E11.4,4H SEC//20X,8HMODEL = I2,20X,7HMODE =
     2 I2)
```

```
      GO TO (3,4,5),MODEL
    3 N=6
      A=0.
      GO TO 6
    4 N=12
      B=0.
      GO TO 6
    5 N=28
    6 DO 7 I=1,28
      Y(I)=0.
    7 SEUIL(I)=SE
      GO TO (9,10,11),MODEL
    9 WRITE (6,1005)
 1005 FORMAT(/20X,4HT = ,20X,6HRMX = ,20X,6HRVX = //)
      WRITE (6,1009) T,Y(1),Y(2)
      GO TO 12
   10 WRITE (6,1006)
 1006 FORMAT(10X,4HT = ,20X,6HRMX = ,20X,6HRVX = ,20X,6HRCY = /)
      WRITE (6,1010) T,Y(1),Y(5),Y(12)
      GO TO 12
   11 WRITE (6,1007)
 1007 FORMAT(6X,4HT = ,10X,6HRMX = ,8X,6HRMY = ,8X,6HRVX = ,8X,6HRVY = ,
     18X,6HRCX = ,8X,6HRCY = ,8X,6HRCZ = /)
      WRITE (6,1008)  T,Y(1),Y(4),Y(2),Y(5),Y(9),Y(12),Y(28)
     ******************************
           CALL DIFF. EQUA. FOR INITIAL DERIV.

   12 M=0
      CALL DAUX
     ******************************
           CALL SUBROUTINE OF INTEGRATION

   13 CALL KUTAM
      IF (M.GE.50) GO TO 210
     ******************************
            IF TEST S1 CALLED TOO MANY TIMES ,NEXT CASE

      IF (NS.GE.50) GO TO 1
     ******************************
            IF INTEGRATION STEP SIZE TOO LARGE,INITIAL VALUE

      IF (PAS.GT.PA1) PAS=PA1
     ******************************
            TEST ON FINAL TIME

      IF (T-TF) 14,210,210
     ******************************
            TEST ON TIME AND MODE TO CALL DISCRETE CASE COMPENSATION

   14 IF(T-TM.GE.0..AND.MODE.EQ.3) GO TO 36
      GO TO 16
   36 CALL UPDAT
      IF (J.GE.5) GO TO 23
      TM=TM+OPT
     ******************************
            AUX. CALCULUS ;WRITE RESULTS
```

```
 16 RMX=SQRT(Y(1))/FC2
    RVX=SQRT(Y(2))*FC1
    RMY=SQRT(Y(4))/FC2
    RVY=SQRT(Y(5))*FC1
    RCX=SQRT(Y(9))*1000.
    RCY=SQRT(Y(12))*1000.
    RCZ=SQRT(ABS(Y(28)))*1000.
   ******************************
         IF NPU=1 AND T.GE.TPU , PREPARE PUNCH

    IF (NPU.NE.1) GO TO 33
    IF (T.LT.TPU) GO TO 33
    M=M+1
    TPU=TPU+TPU1
    TI(M)=T
    GO TO (40,50,60),MODEL
 40 GO TO (41,42,43),MODE
 41 X11(M)=RMX
    V11(M)=RVX
    GO TO 33
 42 X12(M)=RMX
    V12(M)=RVX
    GO TO 33
 43 IF (OPT-OPT2) 431,432,433
431 X131(M)=RMX
    V131(M)=RVX
    GO TO 33
432 X132(M)=RMX
    V132(M)=RVX
    GO TO 33
433 X133(M)=RMX
    V133(M)=RVX
    GO TO 33
 50 GO TO (51,52,53),MODE
 51 X21(M)=RMX
    V21(M)=RVX
    GO TO 33
 52 X22(M)=RMX
    V22(M)=RVX
    GO TO 33
 53 IF (OPT-OPT2) 531,532,533
531 X231(M)=RMX
    V231(M)=RVX
    GO TO 33
532 X232(M)=RMX
    V232(M)=RVX
    GO TO 33
533 X233(M)=RMX
    V233(M)=RVX
    GO TO 33
 60 GO TO (61,62,33),MODE
 61 X31(M)=RMX
    V31(M)=RVX
    GO TO 33
 62 X32(M)=RMX
    V32(M)=RVX
   ******************************
         IF T.GE.TW , PRINT RESULTS
```

```
  33 IF(T.LT.TW) GO TO 13
     GO TO (18,19,34),MODEL
  34 WRITE (6,1008) T,RMX,RMY,RVX,RVY,RCX,RCY,RCZ
1008 FORMAT(4X,E11.4,3X,E11.4,3X,E11.4,3X,E11.4,3X,E11.4,3X,E11.4,3X,
    1E11.4,3X,E11.4)
     TW=TW+DTW
     GO TO 13
  18 IF (RMX.NE.RMY.OR.RVX.NE.RVY) GO TO 20
     WRITE (6,1009) T,RMX,RVX
1009 FORMAT(18X,E11.4,13X,E11.4,15X,E11.4)
     TW=TW+DTW
     GO TO 13
  19 IF (RMX.NE.RMY.OR.RVX.NE.RVY) GO TO 20
     WRITE (6,1010) T,RMX,RVX,RCY
1010 FORMAT(8X,E11.4,13X,E11.4,15X,E11.4,15X,E11.4)
     TW=TW+DTW
     GO TO 13
  20 WRITE (6,1011)
1011 FORMAT(50X,24HCHANNELS NOT INDEPENDENT)
 210 WRITE (6,2100) M
2100 FORMAT(2X,4HM = I3)
     IF (MODE-2) 21,21,214
 214 IF (OPT-OPT2) 211,212,213
 211 OPT=OPT2
     GO TO 2
 212 OPT=OPT3
     GO TO 2
 213 GO TO 35
  21 CONTINUE
  35 CONTINUE
   ********************************
          IF NPU = 1 , NORMALIZE RESULTS OF EACH STEP

     IF (NPU.NE.1) GO TO 71
     XM1=0.
     XM2=0.
     VM1=0.
     VM2=0.
     XM13=0.
     XM23=0.
     VM13=0.
     VM23=0.
     DO 100 M=1,50
     EX1=AMAX1(X11(M),X21(M),X31(M))
     EX2=AMAX1(X12(M),X22(M),X32(M))
     EV1=AMAX1(V11(M),V21(M),V31(M))
     EV2=AMAX1(V12(M),V22(M),V32(M))
     EX13=AMAX1(X131(M),X132(M),X133(M))
     EX23=AMAX1(X231(M),X232(M),X233(M))
     EV13=AMAX1(V131(M),V132(M),V133(M))
     EV23=AMAX1(V231(M),V232(M),V233(M))
     IF (EX1.GT.XM1) XM1=EX1
     IF (EX2.GT.XM2) XM2=EX2
     IF (EV1.GT.VM1) VM1=EV1
     IF (EV2.GT.VM2) VM2=EV2
     IF (EX13.GT.XM13) XM13=EX13
     IF (EX23.GT.XM23) XM23=EX23
```

```
      IF (EV13.GT.VM13) VM13=EV13
      IF (EV23.GT.VM23) VM23=EV23
  100 CONTINUE
      DO 101 M=1,50
      X11(M)=X11(M)/XM1
      X21(M)=X21(M)/XM1
      X31(M)=X31(M)/XM1
      X12(M)=X12(M)/XM2
      X22(M)=X22(M)/XM2
      X32(M)=X32(M)/XM2
      V11(M)=V11(M)/VM1
      V21(M)=V21(M)/VM1
      V31(M)=V31(M)/VM1
      V12(M)=V12(M)/VM2
      V22(M)=V22(M)/VM2
      V32(M)=V32(M)/VM2
      X131(M)=X131(M)/XM13
      X132(M)=X132(M)/XM13
      X133(M)=X133(M)/XM13
      X231(M)=X231(M)/XM23
      X232(M)=X232(M)/XM23
      X233(M)=X233(M)/XM23
      V131(M)=V131(M)/VM13
      V132(M)=V132(M)/VM13
      V133(M)=V133(M)/VM13
      V231(M)=V231(M)/VM23
      V232(M)=V232(M)/VM23
      V233(M)=V233(M)/VM23
  101 CONTINUE
     ********************************
            IF NPU = 1 , PUNCH RESULTS OF EACH STEP AND NORMALIZATION CONSTANTS

      PUNCH 1020,K,AQ,AR
 1020 FORMAT(5HCASE I2,2X,4HN = E11.4,3X,4HR = E11.4)
      PUNCH 1021,OPT1,OPT2,OPT3
 1021 FORMAT(7HOPT1 = E11.4,3X,7HOPT2 = E11.4,3X,7HOPT3 = E11.4)
      PUNCH 1022,XM1,XM2,VM1,VM2
 1022 FORMAT(6HXM1 = E11.4,6HXM2 = E11.4,6HVM1 = E11.4,6HVM2 = E11.4)
      PUNCH 1023,XM13,XM23,VM13,VM23
 1023 FORMAT(7HXM13 = E11.4,7HXM23 = E11.4,7HVM13 = E11.4,7HVM23 = E11.4
     1)
      DO 102 M=1,50
      PUNCH 1024,TI(M),X11(M),X21(M),X31(M),X12(M),X22(M),X32(M),V11(M),
     1V21(M),V31(M),V12(M),V22(M),V32(M),K,M
      PUNCH 1024,TI(M),X131(M),X132(M),X133(M),X231(M),X232(M),X233(M),V
     1131(M),V132(M),V133(M),V231(M),V232(M),V233(M),K,M
 1024 FORMAT(F6.0,12F5.3,6X,I2,2X,I3)
  102 CONTINUE
   71 GO TO 1
   22 WRITE (6,1012)
 1012 FORMAT(25X,20HLATITUDE NOT ALLOWED)
   23 WRITE (6,1013) MODEL,MODE
 1013 FORMAT(10X,7HMODEL  ,I2,7H MODE  ,I2,10HIMPOSSIBLE)
      STOP
      END
```

```
      SUBRCUTINE CAUX

      DIMENSICN CT(28)
      CCNNCN A,B,Y(28),YP(28),R,N,J,NS
      CCNNCN G,RE,CX,CZ,GRX,CRZ,TR,G,MCDE,PAS,EPSI,SEUIL(28),T
      COMMCN CK(14)
      C=A*B
C*******************************
C           FREE SYSTEM EQUA.

      YP(1)=2.*Y(3)
      YP(2)=2.*A*G*Y(11)+Q
      YP(3)=Y(2)+A*G*Y(10)
      YP(4)=2.*Y(6)
      YP(5)=-2.*A*G*Y(8)+Q
      YP(6)=Y(5)-A*G*Y(7)
      IF (A.EQ.C.) GC TC 1C1
      YP(7)=Y(8)+B*ORZ*Y(13)+Y(6)/RE+B*OZ*Y(19)
      YP(8)=-G*Y(9)+B*CRZ*Y(15)+Y(5)/RE+B*CZ*Y(20)
      YP(9)=2.*B*ORZ*Y(17)+2.*Y(8)/RE+2.*B*CZ*Y(21)
      YP(10)=Y(11)-Y(3)/RE-B*OZ*Y(17)+B*OX*Y(22)
      YP(11)=G*Y(12)-Y(2)/RE-B*CZ*Y(18)+B*CX*Y(24)
      YP(12)=-2.*Y(11)/RE-2.*B*OZ*Y(21)+2.*B*OX*Y(27)
      IF (B.EC.0.) GC TC 101
      YP(13)=Y(14)+Y(15)
      YP(14)=Y(16)+G*Y(19)
      YP(15)=Y(16)-G*Y(17)
      YP(16)=G*Y(2C)-G*Y(18)
      YP(17)=Y(18)+ORZ*Y(1)+Y(15)/RE+OZ*Y(10)
      YP(18)=G*Y(21)+CRZ*Y(3)+Y(16)/RE+CZ*Y(11)
      YP(19)=Y(20)-Y(14)/RE-OZ*Y(7)+OX*Y(23)
      YP(2C)=-G*Y(21)-Y(16)/RE-CZ*Y(8)+OX*Y(25)
      YP(21)=ORZ*Y(1C)+Y(2C)/RE+OZ*Y(12)-Y(18)/RE-CZ*Y(9)+CX*Y(26)
      YP(22)=Y(24)-ORX*Y(1)-TR*Y(15)-OX*Y(1C)
      YP(23)=Y(25)-CRX*Y(13)-TR*Y(6)-OX*Y(19)
      YP(24)=G*Y(27)-ORX*Y(3)-TR*Y(16)-CX*Y(11)
      YP(25)=-G*Y(26)-ORX*Y(15)-TR*Y(5)-CX*Y(2C)
      YP(26)=ORZ*Y(22)+Y(25)/RE+OZ*Y(27)-CRX*Y(17)-TR*Y(8)-CX*Y(21)
      YP(27)=-Y(24)/RE-OZ*Y(26)+OX*Y(28)-ORX*Y(1C)-TR*Y(2C)-CX*Y(12)
      YP(28)=-2.*CRX*Y(22)-2.*TR*Y(25)-2.*OX*Y(27)
  101 IF (MCDE-2) 1C5,1C2,1C5
C*******************************
C          CONTINUOUS FILTER GAINS

  102 CK(1)=Y(1)/R
      CK(2)=Y(4)/R
      CK(3)=Y(3)/R
      CK(4)=Y(6)/R
      IF (A.EQ.C.) GO TC 1C6
      CK(5)=Y(7)/R
      CK(6)=Y(10)/R
      IF(B.EQ.0.) GO TO 106
      CK(7)=Y(13)/R
      CK(8)=CK(7)
      CK(9)=Y(14)/R
      CK(10)=Y(15)/R
      CK(11)=Y(17)/R
      CK(12)=Y(19)/R
```

```
      CK(13)=Y(22)/R
      CK(14)=Y(23)/R
C********************************
C           CONTINUOUS FILTER COMPENSATION TERMS

  106 CT(1)=(Y(1)*Y(1)+C*Y(13)*Y(13))/R
      CT(2)=(Y(3)*Y(3)+C*Y(14)*Y(14))/R
      CT(3)=(Y(1)*Y(3)+C*Y(13)*Y(14))/R
      CT(4)=(C*Y(13)*Y(13)+Y(4)*Y(4))/R
      CT(5)=(C*Y(15)*Y(15)+Y(6)*Y(6))/R
      CT(6)=(C*Y(13)*Y(15)+Y(4)*Y(6))/R
      IF (A.EQ.0.) GO TO 103
      CT(7)=(B*Y(13)*Y(17)+Y(4)*Y(7))/R
      CT(8)=(B*Y(15)*Y(17)+Y(6)*Y(7))/R
      CT(9)=(B*Y(17)*Y(17)+Y(7)*Y(7))/R
      CT(10)=(Y(1)*Y(10)+B*Y(13)*Y(19))/R
      CT(11)=(Y(3)*Y(10)+B*Y(14)*Y(19))/R
      CT(12)=(Y(10)*Y(10)+B*Y(19)*Y(19))/R
      IF (B.EQ.0.) GO TO 103
      CT(13)=(Y(1)*Y(13)+Y(13)*Y(4))/R
      CT(14)=(Y(13)*Y(3)+Y(4)*Y(14))/R
      CT(15)=(Y(1)*Y(15)+Y(13)*Y(6))/R
      CT(16)=(Y(3)*Y(15)+Y(14)*Y(6))/R
      CT(17)=(Y(1)*Y(17)+Y(13)*Y(7))/R
      CT(18)=(Y(3)*Y(17)+Y(14)*Y(7))/R
      CT(19)=(Y(13)*Y(10)+Y(4)*Y(19))/R
      CT(20)=(Y(15)*Y(10)+Y(6)*Y(19))/R
      CT(21)=(Y(17)*Y(10)+Y(7)*Y(19))/R
      CT(22)=(Y(1)*Y(22)+Y(13)*Y(23))/R
      CT(23)=(Y(13)*Y(22)+Y(4)*Y(23))/R
      CT(24)=(Y(3)*Y(22)+Y(14)*Y(23))/R
      CT(25)=(Y(15)*Y(22)+Y(6)*Y(23))/R
      CT(26)=(Y(17)*Y(22)+Y(7)*Y(23))/R
      CT(27)=(Y(10)*Y(22)+Y(19)*Y(23))/R
      CT(28)=(Y(22)*Y(22)+Y(23)*Y(23))/R
  103 DO 104 I=1,N
  104 YP(I)=YP(I)-CT(I)
  105 RETURN
      END
```

```
      SUBROUTINE KUTAM
C********************************
C           INTEGRATION SUBROUTINE; R.KUTTA 4 METHOD,CONTROLLED PRECISION

      DIMENSION Y0(28), Y1(28), YP2(28), YP3(28), Y4(28),Y5(28), DERIV(2
     18)
      COMMON A,B,Y(28),YP(28),R,N,J,NS
      COMMON G,RE,OX,OZ,ORX,ORZ,TR,Q,MODE,PAS,EPSI, SEUIL(28),T
      LOGICAL S2
      LOGICAL S1
      NS=0
      NIT=1
```

```
      DO 25 I=1,N
      YO(I)=Y(I)
   25 DERIV(I)=YP(I)
      S2=.FALSE.
      TO=T
    5  H8=PAS/8.
      H38=PAS*3./8.
      H15=PAS*1.5
      H23=PAS*2./3.
      HD=PAS*2.
      H3=PAS/3.
      H6=PAS/6.
      H2=PAS/2.
     ******************************
            INTEGRATION STEPS ; TEST ON POSITIVITY OF MEAN-SQUARED VALUES

      DO 15 I=1,N
      Y1(I)=YO(I)+DERIV(I)*PAS/3.
      Y(I)=Y1(I)
   15 CONTINUE
      IF(S1(Y(1),Y(4),Y(2),Y(5),Y(9),Y(12),Y(28))) GO TO 27
      T=TO+H3
      CALL DAUX
      DO 2 I=1,N
    2 Y(I)=YO(I)+H6*(DERIV(I)+YP(I))
      IF(S1(Y(1),Y(4),Y(2),Y(5),Y(9),Y(12),Y(28))) GO TO 27
      CALL DAUX
      DO 4 I=1,N
    4 Y(I)=YO(I)+H8*DERIV(I)+H38*YP(I)
      DO 6 I=1,N
    6 YP2(I)=YP(I)
      IF(S1(Y(1),Y(4),Y(2),Y(5),Y(9),Y(12),Y(28))) GO TO 27
      T=TO+H2
      CALL DAUX
      DO 7 I=1,N
      YP3(I)=YP(I)
      Y4(I)=YO(I)+H2*DERIV(I)-H15*YP2(I)+YP3(I)*HD
    7 Y(I)=Y4(I)
      IF(S1(Y(1),Y(4),Y(2),Y(5),Y(9),Y(12),Y(28))) GO TO 27
      T=TO+PAS
      CALL DAUX
      DO 8 I=1,N
    8 Y5(I)=YO(I)+H6*DERIV(I)+YP3(I)*H23+H6*YP(I)
      IF(S1(Y5(1),Y5(4),Y5(2),Y5(5),Y5(9),Y5(12),Y5(28))) GO TO 27
      NIT=NIT+1
      IF (NIT-6) 22,23,23
   23 WRITE (6,24) ER,Y5(1),Y4(1)
   24 FORMAT(15X,6HP N R ,2X,3E11.4)
      GO TO 19
     ******************************
            TEST ON PRECISION

   22 E=0.
       DO 9 I=1,N
      ER=ABS(0.2*(Y5(I)-Y4(I))/AMAX1(ABS(Y5(I)),SEUIL(I)))
      IF (ER.GT.E) E=ER
    9 CONTINUE
```

```
      IF (E-EPSI) 13,13,16
C*******************************
C          IF PRECISION NOT REACHED,STEP SIZE HALVED
C
   16 PAS=H2
      GO TO 5
   13 IF (64.*E-EPSI) 18,19,19
   19 S2=.TRUE.
   18 DO 29 I=1,N
   29 Y(I)=Y5(I)
      CALL DAUX
      GO TO 26
   27 NS=NS+1
      WRITE (6,30) NS,T,Y(1),Y(2),Y(4),Y(5),Y(9),Y(12),Y(28)
   30 FORMAT(I4,2X,8E11.4)
C*******************************
C          IF POSITIVITY TEST CALLED TOO MANY TIMES,STOP
C
      IF (NS.GE.50) GO TO 21
      PAS=PAS/2.
      T=TO
       GO TO 5
C*******************************
C          IF PRECISION EXCESSIVE,STEP SIZE DOUBLED
C
   26 IF(S2) GO TO 21
      PAS=HD
   21 RETURN
      END
```

```
      SUBROUTINE UPDAT
C*******************************
C        TO COMPUTE DISCRETE FILTER GAINS AND UPDATE COVARIANCE MATRIX
C
      DIMENSION HK(14)
      COMMON A,B,Y(28),YP(28),R,N,J,NS
      DO 200 I=1,14
  200 HK(I)=0.
      J=1
      C=A*B
      AK1=Y(1)+R
      AK2=Y(4)+R
      D=AK1*AK2-C*Y(13)*Y(13)
      IF (D) 201,201,202
  201 PRINT 1201
 1201 FORMAT (25X,15HINFINITE  GAINS)
      J=J+1
      GO TO 204
C*******************************
C          DISCRETE FILTER GAINS
C
  202 HK(1)=(Y(1)*AK2-C*Y(13)*Y(13))/D
      HK(2)=(-C*Y(13)*Y(13)+Y(4)*AK1)/D
      HK(3)=(Y(3)*AK2-C*Y(13)*Y(14))/D
      HK(4)=(-C*Y(13)*Y(15)+Y(6)*AK1)/D
```

```
      IF (A.EQ.0.) GO TO 203
      HK(5)=(-B*Y(13)*Y(17)+Y(7)*AK1)/D
      HK(6)=(Y(10)*AK2-B*Y(13)*Y(19))/D
      IF (B.EQ.0.) GO TO 203
      HK(7)=(-Y(1)*Y(13)+Y(13)*AK1)/D
      HK(8)=(Y(13)*AK2-Y(13)*Y(4))/D
      HK(9)=(-Y(13)*Y(3)+Y(14)*AK1)/D
      HK(10)=(Y(15)*AK2-Y(13)*Y(6))/D
      HK(11)=(Y(17)*AK2-Y(13)*Y(7))/D
      HK(12)=(-Y(13)*Y(10)+Y(19)*AK1)/D
      HK(13)=(Y(22)*AK2-Y(13)*Y(23))/D
      HK(14)=(-Y(13)*Y(22)+Y(23)*AK1)/D
C******************************
C           UPDATE COVARIANCE MATRIX
C
  203 Y(1)=Y(1)-HK(1)*Y(1)-C*HK(7)*Y(13)
      Y(2)=Y(2)-HK(3)*Y(3)-C*HK(9)*Y(14)
      Y(3)=Y(3)-HK(1)*Y(3)-C*HK(7)*Y(14)
      Y(4)=Y(4)-HK(2)*Y(4)-C*HK(8)*Y(13)
      Y(5)=Y(5)-HK(4)*Y(6)-C*HK(10)*Y(15)
      Y(6)=Y(6)-HK(2)*Y(6)-C*HK(8)*Y(15)
      IF (A.EQ.0.) GO TO 204
      Y(7)=Y(7)-HK(2)*Y(7)-B*HK(8)*Y(17)
      Y(8)=Y(8)-HK(4)*Y(7)-B*HK(10)*Y(17)
      Y(9)=Y(9)-HK(5)*Y(7)-B*HK(11)*Y(17)
      Y(10)=Y(10)-HK(1)*Y(10)-B*HK(7)*Y(19)
      Y(11)=Y(11)-HK(3)*Y(10)-B*HK(9)*Y(19*
      Y(12)=Y(12)-HK(6)*Y(10)-B*HK(12)*Y(19)
      IF (B.EQ.0.) GO TO 204
      Y(13)=Y(13)-HK(1)*Y(13)-HK(7)*Y(4)
      Y(14)=Y(14)-HK(2)*Y(14)-HK(8)*Y(3)
      Y(15)=Y(15)-HK(1)*Y(15)-HK(7)*Y(6)
      Y(16)=Y(16)-HK(3)*Y(15)-HK(9)*Y(6)
      Y(17)=Y(17)-HK(1)*Y(17)-HK(7)*Y(7)
      Y(18)=Y(18)-HK(3)*Y(17)-HK(9)*Y(7)
      Y(19)=Y(19)-HK(2)*Y(19)-HK(8)*Y(10)
      Y(20)=Y(20)-HK(4)*Y(19)-HK(10)*Y(10)
      Y(21)=Y(21)-HK(5)*Y(19)-HK(11)*Y(10)
      Y(22)=Y(22)-HK(1)*Y(22)-HK(7)*Y(23)
      Y(23)=Y(23)-HK(2)*Y(23)-HK(8)*Y(22)
      Y(24)=Y(24)-HK(3)*Y(22)-HK(9)*Y(23)
      Y(25)=Y(25)-HK(4)*Y(23)-HK(10)*Y(22)
      Y(26)=Y(26)-HK(5)*Y(23)-HK(11)*Y(22)
      Y(27)=Y(27)-HK(6)*Y(22)-HK(12)*Y(23)
      Y(28)=Y(28)-HK(13)*Y(22)-HK(14)*Y(23)
  204 CALL DAUX
      RETURN
      END
```

```
      LOGICAL FUNCTION S1(A,B,C,D,E,F,G)
C******************************
C          TO CHECK IF THE DIAGONAL TERMS OF COV. MATRIX ARE POSITIVE
C
      IF (G.LT.0..AND.G.GT.-1.E-18) G=+0.0
      S1=.FALSE.
      IF(A.LT.0..OR.B.LT.0..OR.C.LT.0..OR.D.LT.0..OR.E.LT.0..OR.F.LT.0..
     1OR.G.LT.0.) S1=.TRUE.
      RETURN
      END
```

```
        TO PLOT RMX AND RVX IN MODES 1,2 AND 3 WITH MODELS 1 ,2
                 AND 3 DIFFERENT  OPT

      DIMENSION TI(50),X11(50),X21(50),X31(50),X12(50),X22(50),X32(50),V
     111(50),V21(50),V31(50),V12(50),V22(50),V32(50),X131(50),X132(50),X
     2133(50),X231(50),X232(70),X233(50),V131(50),V132(50),V133(50),V231
     3(50),V232(50),V233(50)
      COMMON AQ,AR,OPT1,OPT2,OPT3
      CALL NEWPLT('M6175','7312','WHITE ','BLACK')
     ******************************
         READ THE MAX. NUMBER OF CASES TO BE PLOTTED

1010  FORMAT(I3)
      READ (5,1010) K2
     ******************************
           READ RESULTS OF THE PREVIOUS PROGRAM ON CARDS
              CHECK THE ORDER OF THE CARDS

    1 READ (5,1020) K,AQ,AR
1020  FORMAT(5HCASE I2,2X,4HN = E11.4,3X,4HR = E11.4)
      K1=K
      READ (5,1021) OPT1,OPT2,OPT3
1021  FORMAT(7HOPT1 = E11.4,3X,7HOPT2 = E11.4,3X,7HOPT3 = E11.4)
      OPT1=OPT1/60.
      OPT2=OPT2/60.
      OPT3=OPT3/60.
      READ (5,1022) XM1,XM2,VM1,VM2
1022  FORMAT(6HXM1 = E11.4,6HXM2 = E11.4,6HVM1 = E11.4,6HVM2 = E11.4)
      READ (5,1023) XM13,XM23,VM13,VM23
1023  FORMAT(7HXM13 = E11.4,7HXM23 = E11.4,7HVM13 = E11.4,7HVM23 = E11.4
     1)
      DO 100 M=1,50
      READ (5,1024) TI(M),X11(M),X21(M),X31(M),X12(M),X22(M),X32(M),V11(
     1M),V21(M),V31(M),V12(M),V22(M),V32(M),K,M1
      IF (K.NE.K1.OR.M1.NE.M) GO TO 1
      READ (5,1024) TI(M),X131(M),X132(M),X133(M),X231(M),X232(M),X233(M
     1),V131(M),V132(M),V133(M),V231(M),V232(M),V233(M),K,M1
      IF (K.NE.K1.OR.M1.NE.M) GO TO 1
1024  FORMAT(F6.0,12F5.3,6X,I2,2X,I3)
  100 CONTINUE
     ******************************
         COMPUTE REAL VALUES WITH THE NORMALIZATION CONSTANTS

      DO 101 M=1,5C
      IF(X21(M).LT.XM23) NP1=M+5
      IF(V21(M).LT.VM23) NP2=M+5
      TI(M)=TI(M)/60.
      X11(M)=X11(M)*XM1
      X21(M)=X21(M)*XM1
      X31(M)=X31(M)*XM1
      X12(M)=X12(M)*XM2
      X22(M)=X22(M)*XM2
```

```
      X32(M)=X32(M)*XM2
      V11(M)=V11(M)*VM1
      V21(M)=V21(M)*VM1
      V31(M)=V31(M)*VM1
      V12(M)=V12(M)*VM2
      V22(M)=V22(M)*VM2
      V32(M)=V32(M)*VM2
      X131(M)=X131(M)*XM13
      X132(M)=X132(M)*XM13
      X133(M)=X133(M)*XM13
      X231(M)=X231(M)*XM23
      X232(M)=X232(M)*XM23
      X233(M)=X233(M)*XM23
      V131(M)=V131(M)*VM13
      V132(M)=V132(M)*VM13
      V133(M)=V133(M)*VM13
      V231(M)=V231(M)*VM23
      V232(M)=V232(M)*VM23
  101 V233(M)=V233(M)*VM23
C     ********************************
C             PLOT ALL THESE RESULTS
C
      CALL IDENPL(1,0)
      CALL PICTUR(6.,4.,'T MIN',5,'RMX',3,TI,X11,50,0.,KS,TI,X21,50,0.,K
     1S,TI,X31,50,0.,KS)
      CALL IDENPL(1,0)
      CALL PICTUR(6.,4.,'T MIN',5,'RVX',3,TI,V11,50,0.,KS,TI,V21,50,0.,K
     1S,TI,V31,50,0.,KS)
      CALL IDENPL(3,1)
      CALL PICTUR(6.,4.,'T MIN',5,'RMX',3,TI,X12,50,0.,KS,TI,X131,50,0.,
     1KS,TI,X132,50,0.,KS,TI,X133,50,0.,KS)
      CALL IDENPL(3,1)
      CALL PICTUR(6.,4.,'T MIN',5,'RVX',3,TI,V12,50,0.,KS,TI,V131,50,0.,
     1KS,TI,V132,50,0.,KS,TI,V133,50,0.,KS)
      CALL IDENPL(3,2)
      CALL PICTUR(6.,4.,'T MIN',5,'RMX',3,TI,X22,50,0.,KS,TI,X231,50,0.,
     1KS,TI,X232,50,0.,KS,TI,X233,50,0.,KS)
      CALL IDENPL(3,2)
      CALL PICTUR(6.,4.,'T MIN',5,'RVX',3,TI,V22,50,0.,KS,TI,V231,50,0.,
     1KS,TI,V232,50,0.,KS,TI,V233,50,0.,KS)
      DO 102 M=NP1,50
  102 X21(M)=0.
      DO 103 M=NP2,50
  103 V21(M)=0.
      CALL IDENPL(0,2)
      CALL PICTUR(6.,4.,'T MIN',5,'RMX',3,TI,X21,50,0.,KS,TI,X22,50,0.,K
     1S,TI,X233,50,0.,KS,TI,X231,50,0.,KS)
      CALL IDENPL(0,2)
      CALL PICTUR(6.,4.,'T MIN',5,'RVX',3,TI,V21,50,0.,KS,TI,V22,50,0.,K
     1S,TI,V233,50,0.,KS,TI,V231,50,0.,KS)
C     ********************************
C       IF NUMB. OF CASES PLOTTED = NUMB. DESIRED , STOP
C
      IF (K1.EQ.K2) GO TO 2
      GO TO 1
    2 CALL ENDPLT
      STOP
      END
```

```
      SUBROUTINE IDENPL(J,L)
C*******************************
C        TO DRAW SOME IDENTIFICATION ON TOP OF EACH PLOT
C
      COMMON AQ,AR,OPT1,OPT2,OPT3
      CALL PLOT1(.5,3.25,-3)
      CALL PLOT1(0.,0.75,-2)
      CALL PLOT1(1.5,0.,-2)
      CALL PLOT1(0.,-0.75,-2)
      CALL PLOT1(-1.5,0.,-2)
      IF(J.NE.0) GO TO 2
      CALL SYMBL5(0.1,0.5,.2,'MODEL= ',0.,+7)
      CALL NUMBR1(1.3,0.5,.2,L,0.,-1)
      GO TO 3
    2 CALL SYMBL5(0.1,0.5,.2,'MODE = ',0.,+7)
      CALL NUMBR1(1.3,0.5,.2,J,0.,-1)
    3 CALL SYMBL5(0.05,0.2,.05,'N = ',0.,+4)
      CALL NUMBR1(0.2,0.2,.10,AQ,0.,+2)
      CALL SYMBL5(0.05,.05,.05,'R = ',0.,+4)
      CALL NUMBR1(0.2,.05,.10,AR,0.,+0)
      IF(J.EQ.0) GO TO 4
      CALL SYMBL5(0.9,0.35,.05,'OPT1 = ',0.,+7)
      CALL NUMBR1(1.2,0.35,.10,OPT1,0.,+1)
      CALL SYMBL5(0.9,0.2,.05,'OPT2 = ',0.,+7)
      CALL NUMBR1(1.2,0.2,.10,OPT2,0.,+1)
    4 CALL SYMBL5(0.9,0.05,.05,'OPT3 = ',0.,+7)
      CALL NUMBR1(1.2,0.05,.10,OPT3,0.,+1)
      IF (L.EQ.0) GO TO 1
      IF(J.EQ.0) GO TO 5
      CALL SYMBL5(.05,.35,.05,'MODEL = ',0.,+8)
      CALL NUMBR1(0.50,.35,.10,L,0.,-1)
      GO TO 1
    5 CALL SYMBL5(0.05,.35,.10,'MODE 1 2 3',0.,+10)
    1 CALL PLOT1(-0.5,-3.25,-3)
      RETURN
      END
```

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