Option Pricing under Stochastic Interest Rates: An Empirical Investigation

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Abstract

Using daily data of Nikkei 225 index, call option prices, and call money rates in Japanese financial market, we compare the pricing performances of stock option pricing models under several stochastic interest rate processes proposed by existing term structure literature. The results show that (1) any option pricing model under a specific stochastic interest rate does not dominantly outperform another option pricing model under alternative stochastic interest rate, and (2) incorporating stochastic interest rates into stock option pricing does not contribute to the performance improvement of the original Black-Scholes pricing formula.

JEL Classification Numbers: G12, G13, C22.

Key words: Option Pricing, Stochastic Interest Rate, Nikkei 225 Index Option, Overnight Call Money Rate

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1 Introduction

There have been many attempts to fill the gap between the celebrated Black-Scholes (1973) model and market data. It has been argued that stock option pricing model under stochastic volatility has provided the improvement in pricing performance. Some researchers have suggested that option pricing model capturing the jump behavior of stock return also could contribute to its pricing performance. Incorporating stochastic interest rate into stock option pricing model is another line of extension. At the earliest Merton (1973) has discussed option pricing under stochastic interest rate.

However the empirical analysis for option pricing under stochastic interest rate has received little attention. To the author's knowledge, Rindell (1995) is the first one who has paid attention to this strand of research. Rindell examined the explanatory power of the stock option pricing model of Amin and Jarrow (1992) which is the closed form stock option pricing formula under Merton type interest rate based on Heath, Jarrow and Morton (1992) framework. Using Swedish option market data during 1992, Rindell concluded that stock option pricing model of Amin and Jarrow (1992) performs better than the original Black-Scholes model in the sense that the pricing error of the former is significantly small. It is noted that Rindell has used panel data of options with maturities of up to two years.

Bakshi, Cao, and Chen (hereafter, BCC) (1997) observed that, compared with Black-Scholes model, incorporating square-root interest rate process gives only marginal improvement in pricing performance of S&P 500 options which have up to one year to expiration (See footnote 6 in BCC (1997)). In addition, using LEAPS (Long-term Equity Anticipation Securities in S&P 500 options) options data which usually expire two to three years from the date of listing, BCC (2000) also concluded that once the model has accounted for stochastically varying volatility, allowing interest rates to be stochastic does not improve pricing performance, even for long-term options. We note that BCC (2000) have not compared option model under stochastic interest rate with Black-Scholes model which we will use as a benchmark in this study. However BCC (2000) reported that for long-term options, incorporating stochastic interest rates can nonetheless enhance hedging performance in certain cases.

The above empirical findings allow us to infer that stochastic interest rates may not be important for pricing of short-term options, at least. Since the short term interest rate (equivalently, short rate or instantaneous spot rate) is the endogenously determined fundamental variable in economy, the stock option value might be less affected by the change of interest rate within a relatively short time to expiration.

This paper re-examine these empirical observations by using Japanese Nikkei 225 options data. Nikkei 225 options have maturities up to four months and long-term options such as LEAPS options are not available in Japanese market. This market structure may allow the agents participating in Japanese security market to perceive the duration of time to maturities differently, comparing with the participants in other countries market. In particular, this study will also investigate the explanatory power of option pricing models under some alternative stochastic interest rate processes proposed by existing studies associated with term structure models. The model comparison is new feature in this empirical field. We will estimate the parameters of option pricing models for each trade day by adopting rolling estimation procedure and compare their pricing errors over test period.

When the short rate is assumed to be stochastic, the closed form expression of stock option value is hardly obtained, except for the case of Gaussian interest rate process. Kim and Kunitomo (1999) developed a simple stock option pricing model in an approximate sense when the short rate follows Ito process. We shall adopt this approach in evaluating option value under non-Gaussian short rates. [1]

[1] Although some researches including Bakshi and Chen (1997) have derived closed form expressions of stock option value under square-root process of interest rate and volatility, including jump behavior of stock price, those models need to evaluate Fourier inversion formula for the

The organization of this paper is as follows. In section 2, we examine option pricing models under diverse stochastic interest rates. Section 3 discusses test methodology. Section 4 explains data set. Section 5 provides empirical results and finally, section 6 concludes.

2 Stock Option Values under Stochastic Interest Rates

Let us consider Black-Scholes economy when the short term interest rate changes randomly. The stock price process, S_t, is given by

$$dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dW_{1,t}, \tag{1}$$

where μ and σ are constants and $\{W_{1,t}\}$ is the standard Brownian motion. The short rate r_t, is assumed to follow

$$dr_t = (\alpha + \beta\, r_t)\, dt + \delta\, r_t^{\gamma} dW_{2,t}, \tag{2}$$

where α, β, δ, and γ are constants and $\{W_{2,t}\}$ is the standard Brownian motion. The equation (2) defines a broad class of interest rate processes which nests many well-known instantaneous spot interest rate models. In this study, we are concerned with the case of $\gamma \in \{0, 0.5, 1\}$. The instantaneous correlation between the stock price process and short rate process, ρ, is assumed to be given.

The today's value of European stock call option with time to expiration T under stochastic interest rate (2), denoted V_0, is determined by

$$V_0 = E_0^Q \left[\exp\left(- \int_0^T r_t\, dt \right) \max(S_T - K, 0) \right],$$

where the expectation is taken with respect to the equivalent martingale measure Q, and K is the exercise price of option.

In this section we shall give the expressions for V_0 under four representative stochastic interest rate models. It is useful that we classify (2) into Gaussian and

probability distribution functions, numerically.

non-Gaussian process because the exact expression of V_0 can be obtained in the former case.

2.1 Option Pricing under Gaussian Interest Rate Process

The Gaussian short rate process subsumes Vasicek type interest rate which in itself nests Merton-Ho-Lee (hereafter, MHL) type interest rate. The closed form option pricing formulae under Gaussian interest rate was suggested by Merton (1973), Rabinovitch (1989), and Amin and Jarrow (1992). [2] These results can be recovered by adopting relatively simple algebra as shown below.

2.1.1 Option Pricing under Vasicek Type Interest Rate

Vasicek (1977) derived the term structure of interest rate when the short rate is given by (2) with $\gamma = 0$:

$$dr_t = \kappa(\theta - r_t)\,dt + \delta\,dW_{2,t}, \tag{3}$$

where κ, δ, and θ are positive constants under the original (or observed) probability. If we assume that the market price of risk, *i.e.* the increase in the expected instantaneous rate of return on a bond per an additional unit of risk, $\lambda(r_t, t)$ is a constant λ, the standard no-arbitrage argument states that (3) is now expressed as

$$dr_t = \kappa(\theta_* - r_t)\,dt + \delta\,d\tilde{W}_{2,t}, \tag{4}$$

where $\theta_* = \theta - \frac{\delta}{\kappa}\lambda$ under the equivalent martingale measure Q.

If we set $\alpha = \kappa\,\theta_*$ and $\beta = -\kappa$, the option value V_0 can be obtained as a special case of (A.7) in Appendix. That is, assuming $\sigma_s = \sigma$, $\alpha_s = \alpha$, $\beta_s = \beta$, and $\delta_s = \delta$ in (A.1) and (A.2) gives the option value under Vasicek type interest

[2] Goldstein and Zapatero (1996) also derived an endogenous Vasicek type interest rate, and obtained stock option pricing formula under the above interest rate as a simple exchange economy equilibrium.

rate, denoted $V_0(Vas)$, as follows:

$$V_0(Vas) = S_0\Phi(d_1) - K\,P(0,T)\Phi(d_2). \qquad (5)$$

In valuation equation (5), the current price of pure discount bond with time to maturity T, $P(0,T)$, is given by

$$P(0,T) = \exp\left(\frac{1}{2}\Sigma_{22}^T - B(T)\right),$$

where

$$\Sigma_{22}^T = \frac{\delta^2}{\kappa^2}\left[T - \frac{3 + e^{-\kappa T}(e^{-\kappa T} - 4)}{2\,\kappa}\right]$$

and

$$B(T) = -\frac{1}{\kappa}\left[\left(r_0 - \frac{\kappa\,\theta - \delta\lambda}{\kappa}\right)\left(e^{-\kappa T} - 1\right) - (\kappa\,\theta - \delta\,\lambda)T\right].$$

In addition, d_1 in (5) is represented by $d_1 = (\Sigma_{11}^T + \Sigma_{12}^T - C(T))/\sqrt{D}$, where $C(T) = \Sigma_{11}^T/2 - B(T) + \log(K/S_0)$, $D = \Sigma_{11}^T + 2\Sigma_{12}^T + \Sigma_{22}^T$, $\Sigma_{11}^T = \sigma^2 T$, and

$$\Sigma_{12}^T = \frac{\sigma\,\delta\,\rho}{\kappa}\left[\frac{e^{-\kappa T} - 1}{\kappa} + T\right].$$

Finally, $d_2 = d_1 - \sqrt{D}$.

The equation (5) is equivalent to equation (8) in Rabinovitch (1989) which he derived by using the equation (38) of Merton (1973).

2.1.2 Option Pricing under MHL Type Interest Rate

Merton (1973) had already considered the stock option pricing model when the short rate is given by (2) with $\gamma = 0$ and $\beta = 0$, *i.e.*,

$$dr_t = \alpha\,dt + \delta\,dW_{2,t}, \qquad (6)$$

where α is a constant and δ is positive constant. This equation is also the continuous time equivalent of Ho and Lee (1986) model with constant drift. If we assume that $\lambda(r_t, t)$ is λ, a constant, (6) can be written as

$$dr_t = \alpha_* dt + \delta\,d\tilde{W}_{2,t}, \qquad (7)$$

where $\alpha_* = \alpha - \delta\lambda$. By assuming $\sigma_s = \sigma$, $\alpha_s = \alpha_*$, $\beta_s = 0$, and $\delta_s = \delta$ in (A.1) and (A.2), we obtain the option value, denoted $V_0(MHL)$, as,

$$V_0(MHL) = S_0\Phi(d_1) - K\,P(0,T)\Phi(d_2), \tag{8}$$

where $P(0,T)$, d_1, d_2, and $C(T)$ are defined in the same way as in (5), replaced by $\Sigma_{11}^T = \sigma^2 T$, $\Sigma_{22}^T = \delta^2\,T^3/3$, $\Sigma_{12}^T = \sigma\,\delta\,\rho\,T^2/2$, and $B(T) = r_0\,T + (\alpha - \delta\lambda)T^2/2$.

The above formula (8) corresponds to the equation (3.22) in Amin and Jarrow (1992) model which starts from the assumed forward interest rate process.

2.2 Option Pricing under Level-dependent Volatility Interest Rate Process

Let us assume that the volatility of the short rate is the function of the short rate itself. In this case we can utilize Kim and Kunitomo (1999) and Kunitomo and Kim (2000) to obtain option values in some asymptotic sense. We deal with the stock option pricing formula under CIR type ($\gamma = 0.5$) and Brennan-Schwartz type ($\gamma = 1.0$) short rate.

2.2.1 Option Pricing under CIR Type Interest Rate

Cox, Ingersoll and Ross (1985) presented a logarithmic utility general equilibrium model in which the endogenous equilibrium interest rate dynamics is expressed as

$$dr_t = \kappa(\theta - r_t)\,dt + \delta\sqrt{r_t}\,dW_{2,t}, \tag{9}$$

where κ, θ, and δ are positive constants. Note that the risk neutral version of the short rate process (9) is described as

$$\begin{aligned} dr_t &= (\kappa(\theta - r_t) - \delta\,\lambda\,r_t)dt + \delta\sqrt{r_t}d\tilde{W}_{2,t} \\ &= \kappa_*(\theta_* - r_t)\,dt + \delta\sqrt{r_t}\,d\tilde{W}_{2,t}, \end{aligned} \tag{10}$$

where $\kappa_* = \kappa + \delta\lambda$ and $\theta_* = (\theta\,\kappa)/(\kappa + \delta\lambda)$ for a constant λ. [3]

[3] It should be noted that the endogenous general equilibrium interest rate of CIR (1985) economy follows the mean reverting square-root process and the associated risk neutral version

The value of stock option under CIR type interest rate, denoted $V_0(CIR)$, can be represented by

$$
\begin{aligned}
V_0(CIR) &= \left[S_0\,\Phi(d_1) - K\,\exp\left(-\int_0^T r_t^*\,dt\right)\Phi(d_2)\right] \\
&+ \delta\,C_0\left[S_0\,\phi(d_1) - K\exp\left(-\int_0^T r_t^* dt\right)(\phi(d_2) - \sigma\sqrt{T}\,\Phi(d_2))\right] \\
&+ \delta\,C_1\left[d_2\,S_0\,\phi(d_1) - d_1\,K\,\exp\left(-\int_0^T r_t^* dt\right)\phi(d_2)\right] + o(\delta), \qquad (11)
\end{aligned}
$$

where $\Phi(\cdot)$ is the cumulative probability distribution for a standard normal variable, $\phi(\cdot)$ is its density function. r_t^* is the deterministic version of the short rate and given by

$$
r_t^* = r_0\,e^{-\kappa\,t} + \theta\,(1 - e^{-\kappa\,t}),
$$

and the deterministic version of discount factor, therefore, is described by

$$
\exp\left(-\int_0^T r_t^* dt\right) = \exp\left(-\frac{r_0 - \theta}{\kappa}\left(1 - e^{-\kappa\,T}\right) - \theta\,T\right).
$$

d_1 is given by

$$
d_1 = \frac{1}{\sigma\sqrt{T}}\left[\log\frac{S_0}{K} + \frac{r_0 - \theta}{\kappa}\left(1 - e^{-\kappa\,T}\right) + \theta\,T + \frac{\sigma^2}{2}\,T\right],
$$

---

of the interest rate process also has the same form of the former, although with different drift coefficients. Following the exposition of chapter 10 in Duffie (1996), we have $\theta = b(h - \epsilon^2)/\kappa$ and $\delta = k\sqrt{h - \epsilon^2}$, where h and ϵ are the constant parameters of capital-stock process and b, κ, and k are the constant parameters of a shock process. Utilizing the relation between the state-price deflator and the equivalent martingale measure allows the short rate process to be represented under the equivalent martingale measure Q as follows:

$$
\begin{aligned}
dr_t &= [(b(h - \epsilon^2) - \kappa r_t - k\,\epsilon\,r_t]dt + \delta\sqrt{r_t}\,d\tilde{W}_{2,t} \\
&= [(b(h - \epsilon^2) - \kappa r_t - k\sqrt{h - \epsilon^2}\,\epsilon\,r_t/\sqrt{h - \epsilon^2}]dt + \delta\sqrt{r_t}\,d\tilde{W}_{2,t} \\
&= (\kappa\theta - \kappa r_t - \delta\lambda r_t)dt + \delta\sqrt{r_t}\,d\tilde{W}_{2,t},
\end{aligned}
$$

where λ is denoted by $\epsilon/\sqrt{h - \epsilon^2}$. Thus CIR type short rate process keeps the form of square-root process after the change of measure has occurred. In this sense the characterization of risk premium (or drift adjustment) as above is compatible with CIR economy.

and $d_2 = d_1 - \sigma\sqrt{T}$. C_0 is represented by

$$C_0 = \frac{1}{\sigma\sqrt{T}}\left[\frac{\lambda(r_0-\theta)}{\kappa}\left(\frac{1-e^{-\kappa T}}{\kappa} - T\,e^{-\kappa\,T}\right) + \frac{\lambda\,\theta\,T}{\kappa}\left(1-\frac{1-e^{-\kappa\,T}}{\kappa}\right)\right].$$

Finally, C_1 is given as follows:

$$C_1 = -\frac{\rho}{\sigma\,T}\cdot C_{11},$$

where

$$C_{11} = \frac{2\sqrt{\theta}\left((1+2e^{\kappa\,T})\sqrt{r_0} - 3\,e^{\frac{\kappa\,T}{2}}\sqrt{r_0-\theta\,(1-e^{\kappa\,T})}\right) + \left(\theta(1+2e^{\kappa\,T}) - r_0\right)\psi}{2e^{\kappa\,T}\kappa^2\sqrt{\theta}}$$

and

$$\psi = \log\left[\frac{\theta(2e^{\kappa\,T}-1) + r_0 + 2\,e^{\frac{\kappa\,T}{2}}\sqrt{\theta^2(e^{\kappa\,T}-1)+\theta\,r_0}}{(\sqrt{r_0}+\sqrt{\theta})^2}\right].$$

2.2.2 Option Pricing under Brennan-Schwartz Type Interest Rate

Brennan and Schwartz (1980) suggested the short rate process,

$$dr_t = \kappa(\theta - r_t)\,dt + \delta\,r_t\,dW_{2,t}, \tag{12}$$

where κ, δ, and θ are positive constants in deriving model for convertible bond prices. We assume that the risk premium for varying interest rate, $\lambda(r_t,t)$ is λ, a constant. The standard no-arbitrage argument states that under the equivalent martingale measure Q, (12) is written by

$$\begin{aligned} dr_t &= (\kappa(\theta-r_t) - \delta\lambda\,r_t)\,dt + \delta\,r_t\,d\tilde{W}_{2,t} \\ &= \kappa_*(\theta_* - r_t)\,dt + \delta\,r_t\,d\tilde{W}_{2,t}, \end{aligned} \tag{13}$$

where $\kappa_* = \kappa + \delta\,\lambda$ and $\theta_* = (\theta\,\kappa)/(\kappa+\delta\lambda)$.

The option value when the short rate is given by (13), denoted $V_0(BrS)$, is the same as $V_0(CIR)$ in (11) except that C_1 is now replaced as follows:

$$C_1 = -\frac{\rho}{\sigma\,T}\cdot C_{11}$$

where

$$C_{11} = \frac{(1-e^{\kappa\,T})(2\theta - r_0) + \kappa\,\theta\,T(1+e^{\kappa\,T}) - \kappa\,r_0\,T}{e^{\kappa\,T}\kappa^2}.$$

3 Methodology

We estimate the parameters for each trade day using time series data and compare pricing errors of option pricing models by utilizing the estimates of parameters, over test period.

3.1 Estimation of Parameters

For notational convenience, let us denote t also by each trade day and $\triangle$ by time interval of data observation. Since the daily data will be used in this study and the market is opened about 247 days per year, we set $\triangle$ to be 1/247.

Which time span do the investors take into account in estimating the parameters of stock option pricing models? In this study, we set the estimation period to be 494 trade days (two calendar years) up to today and estimate the parameters by adopting rolling estimation procedure. For the so called *stochastic volatility model* in financial econometrics, the parameters have been estimated by using relatively long period (at least several years) stock returns data. On the other hand, the estimation time span of historical volatility in Black-Scholes model usually ranges from 20 to 180 trading days in practice and there is no established rule. For the stochastic short rate process, several decades data has been used to estimate parameters in general. However from the practical viewpoint, if the investors utilize the rolling procedure to estimate the parameters of option value, we expect that the parameter estimation period might not be long since the time to expiration of our options data is relatively short. In this respect 494 trade days (2 years in calendar day) as the estimation period might be a reasonable compromise.

Let us consider a discrete version of (1) as follows.

$$\log \frac{S_{t+\triangle}}{S_t} = \mu_* \triangle + \varepsilon_{1,t+\triangle}, \tag{14}$$

where $\mu_* = \mu - \frac{\sigma^2}{2}$ and $\varepsilon_{1,t+\triangle}$ is normally distributed with conditional mean $E_t = 0$ and variance $V_t = \sigma^2 \triangle$.

Then the so called historical volatility of stock return, σ can be estimated simply by ordinary least squares regression of (14). [4]

For the estimation of parameters of short rate process, we adopt the New Local Linearization Method (hereafter NLLM) of Shoji and Ozaki (1997). They investigated the finite sample performances of several estimation methods for a continuous time stochastic process from discrete observations and advocated the NLLM by Monte Carlo experiments. The NLLM linearizes the drift of stochastic differential equation and obtains the discrete version of original SDE by solving Langevine equation. Following Shoji and Ozaki (1997), we discretize short rate models as follows. [5]

- MHL model

  The discretization scheme collapses to the Euler approximation in this case, *i.e.*,

$$r_{t+\triangle} = r_t + \alpha \triangle + \varepsilon_{2,t+\triangle}, \tag{15}$$

  where $\varepsilon_{2,t+\triangle}$ follows normal distribution with conditional mean $E_t = 0$ and variance $V_t = \delta^2 \triangle$.

- Vasicek model

  Let $f(r_t) = \alpha + \beta r_t$ and $L_t = \beta$. Then we have

$$r_{t+\triangle} = r_t + \frac{f(r_t)}{L_t}(\exp(L_t\triangle) - 1) + \varepsilon_{2,t+\triangle}, \tag{16}$$

  where $\varepsilon_{2,t+\triangle}$ follows normal distribution with $E_t = 0$ and $V_t = \delta^2 \frac{\exp(2L_t\triangle)-1}{2L_t}$.

- CIR model

  Transform r_t by $y_t = \sqrt{r_t}$. Let $f(y_t) = \frac{4\alpha-\delta^2}{8y_t} - \frac{\beta y_t}{2}$, $L_t = \frac{\beta}{2} - \frac{4\alpha-\delta^2}{8y_t^2}$, and

[4] Of course the estimate of μ_* will not be used in this study. It is well known that using implied volatility rather than historical volatility shows better pricing performances. However since the primary purpose of this study is to investigate the effects of stochastic interest rates on option pricing and it is needed to estimate correlation coefficients between stock returns and interest rates, we will use historical volatility.

[5] It is noted that for Vasicek, CIR, and Brennan-Schwartz model, $\alpha = \kappa\,\theta$ and $\beta = -\kappa$.

$M_t = \frac{\delta^2(4\alpha - \delta^2)}{4y_t^3}$. Then we have

$$y_{t+\triangle} \quad = \quad y_t + \frac{f(y_t)}{L_t}(\exp(L_t\triangle) - 1) + \frac{M_t}{L_t^2}\left[(\exp(L_t\triangle) - 1) - L_t\triangle\right] + \varepsilon_{2,t+\triangle}, \tag{17}$$

where $\varepsilon_{2,t+\triangle}$ follows normal distribution with $E_t = 0$ and $V_t = \delta^2\frac{\exp(2L_t\triangle)-1}{2L_t}$.

- Brennan-Schwartz model

  Transform r_t by $y_t = \log r_t$. Let $f(y_t) = -\frac{\delta^2}{2} + \alpha\exp(-y_t) + \beta$, $L_t = -\alpha\exp(-y_t)$, $M_t = \frac{\alpha\delta^2}{2}\exp(-y_t)$. Then we have

$$y_{t+\triangle} = y_t + \frac{f(y_t)}{L_t}(\exp(L_t\triangle) - 1) + \frac{M_t}{L_t^2}\left[(\exp(L_t\triangle) - 1) - L_t\triangle\right] + \varepsilon_{2,t+\triangle}, \tag{18}$$

  where $\varepsilon_{2,t+\triangle}$ follows normal distribution with $E_t = 0$ and $V_t = \delta^2\frac{\exp(2L_t\triangle)-1}{2L_t}$.

We estimate α, κ, θ, and δ by maximizing the log-likelihood function of (15)-(18). It should be noted that the log-likelihood functions of CIR and Brennan-Schwartz model should incorporate the Jacobian terms since data transformation has occurred. Each correlation coefficient ρ can be estimated from pair residuals of (14) and (15)-(18). [6]

The only remaining parameter λ can be estimated by utilizing the obtained estimates of short rate parameters. Let us consider the discount bond price at time t with remaining time to maturity τ, $P(t,\tau)$, where τ is longer than that of the discount bond associated with the 'proxy' short rate r_t. Then from the relation $Y(r_t, t) = -\frac{1}{\tau}\log P(t,\tau) + \eta_t$, where $Y(r_t, t)$ is the yield at day t and η_t is simply assumed to be i.i.d. pricing error, λ can be estimated by minimizing sum of squared error. In the case of MHL and Vasicek model, each $P(t,\tau)$ is given in the previous section and also well known for CIR case. However for Brennan-Schwartz model, some approximate values of term structure model are

[6] It is easily verified that the discretized process (14) is identical to one obtained by the NLLM if we apply Ito's lemma to $y_t = \log S_t$.

needed. In this study, by applying proposition 3 of Chapman, Long and Pearson (1999), we approximate the discount bond price of Brennan-Schwartz model by

$$P(t,\tau) = \sum_{n=0}^{N} \frac{1}{n!} g_n(r_t)\tau^n,$$

for $N \geq 2$, where

$$g_{n+1}(r_t) = ((\alpha + \beta r_t) - \delta\lambda r_t)\frac{dg_n(r_t)}{dr_t} + \frac{1}{2}\delta^2 r_t^2 \frac{d^2 g_n(r_t)}{d^2 r_t} - r_t g_n(r_t),$$

for $n \geq 0$ and $g_0(r_t) = 1$. We calculate $P(t,\tau)$ when $N = 3$.

3.2 Performance Measure

Let $\hat{\Theta}_t$ be the estimates of parameters in stock option pricing model for each day t. If we set $V_t^i(term : \hat{\Theta}_t)$ to be the theoretical value of stock option and P_t^i the observed option price, the pricing error $\epsilon_t^i(term : \hat{\Theta}_t)$ for each day t, is defined as

$$\epsilon_t^i(term : \hat{\Theta}_t) = P_t^i - V_t^i(term : \hat{\Theta}_t),$$

where the superscript $i \in \{1, \cdots, N_t\}$ denotes the kind of option, which represents different exercise price and time to expiration, and *term* represents each short rate model.

For each option pricing model, we calculate the mean of absolute pricing error for each day and obtain daily-average over the sample period (we call it DAPE). We compare DAPEs of option pricing models under alternative short rate processes.

4 Data

As options data, we use daily closing prices of Nikkei 225 index call option which is the most traded European style stock option in Japan. The Nikkei 225 index option market was introduced as the market of near American style option on June 12, 1989 and has completely shifted to European style option market on

June 12, 1992. Therefore options data before June 12, 1992 are excluded from samples. The short rate level in Japanese financial market has gone down since March 1991. And as of January 2001, it has kept below 1 percent in annual base from midyear of 1995. Because this study focuses primarily on the effects of interest rate on option pricing, we also cut out samples after June 30, 1995. Our options data covers from June 12, 1992 to June 30, 1995. The sample size is 16518. [7]

We need to adjust dividend flow of the Nikkei 225 index as no dividends are assumed in the previous sections. For each option contract with remaining time to expiration, T, from current time t, We obtain the present value of the daily dividends PVD_t as follows:

$$PVD_t = d_t \cdot S^o_{t-1} \cdot \frac{T}{365},$$

where d_t is the predicted average dividend yield (*Yoso-kijun Heikin Rimawari*, which is announced by *Nihon Keizai Shimbun* on every trading day t) and S^o_{t-1} is the observed closing index level of previous day. The dividend-adjusted stock price, S_t, is obtained by

$$S_t = S^o_t - PVD_t.$$

In addition to options data, we use 1248 daily closing level of Nikkei 225 index. [8] As the proxy for the unobservable short rate, we use overnight call money rate (hereafter, O/N call rate). O/N call rate keeps the literal sense of an instantaneous spot rate and is the short rate with the shortest time to maturity available in Japanese financial market. Although there might be some microstructure effects associated with overnight rates, we shall use O/N call rate because O/N call money market is the most traded interbank market in Japan.

[7] The option data are kindly provided by the Osaka Securities Exchange.

[8] The index levels are adjusted simply by

$$S_t = S^o_t - d_t \cdot S^o_{t-1}.$$

Furthermore O/N call rate is the interest rate of pure discount bond. During the sample period, O/N call rate shows two spikes on the last trade day of March and September each year, which reflects the bank's deposit sales competition for the fiscal-year-ended firms. We replace the rates of the last trade day on March and September by the average of previous two trade days rates, each year. To estimate λ, we add one month call rate to our data set. [9]

The data of dividend-adjusted Nikkei 225 index and call rates has 1248 time series from June 12, 1990 through June 30, 1995. The standard deviation of the dividend adjusted Nikkei 225 index return is 24.991%. The standard deviation of O/N call rate difference is 1.188%. The correlation coefficient between index returns and call rate change is -0.036. See also the Figure 1.

[Figure 1]

We divide options data into several categories according to either moneyness and/or time to expiration. At-the-money sample is assumed to satisfy $0.97 < S/K < 1.03$. Out-of-the-money (resp. in-the-money) sample is set to satisfy $S/K \leq 0.97$ (resp. $S/K \geq 1.03$). The longest time to maturity of Nikkei 225 index option is four months in calendar day. It is relatively short in comparison with those of US and other European countries. Hence we divide the entire samples into short and medium term options according to time to expiration. The short term option has maturity time less or equal to 60 days. The medium term option takes 60 days to four months to mature. The sample statistics of options data are given in Table 1.

[Table 1]

[9] One month call rate was selected mainly because of liquidity. Nevertheless, from June 12, 1992 to June 30, 1995, there are 22 missing values of one month call rate due to non-trade. We replace missing observations by the previous day's quoted rates.

5 Results

The estimates of parameters are given in Table 2. These estimates are the daily average of each day estimates over the test period (from June 12, 1992 through June 30, 1995, 755 time series). First, the daily average of historical volatility of stock returns is 24.38% and is close to its time series counterpart. Secondly, the correlations between the stock return and the short rate change are very small as it might be expected in time series counterpart. Thirdly, the estimates of volatility parameter of short rate process are very small except the case of Brennan-Schwartz model. Finally, the estimates of risk premiums have the negative values as it might be expected. ¿From Figure 2 to Figure 6, we also observe that some estimates of short rate model parameters change abruptly.

[Table 2]

[Figure 2] to [Figure 6]

Table 3 provides the daily average of mean absolute pricing errors over the test period. For the total 16518 sample case, test period covers 755 trading days and the number of Nikkei 225 index call prices which are available on a given trade day, ranges from 11 to 39, with the mean 21.878 and the standard deviation 4.181.

¿From Table 3, it is seen that any option pricing model under a specific stochastic interest rate does not unilaterally outperform another option pricing model under alternative stochastic interest rates and the original Black-Scholes model as well, overall. For the total sample, DAPE of option pricing model under Brennan-Schwartz (res. CIR) type short rate is 113.078 yen (res. 113.714 yen), which is slightly smaller than that of original Black-Scholes, 113.751 yen. On the other hand, DAPEs for Vasicek and MHL model are 114.114 yen and 113.815 yen. They are slightly bigger than that of the original Black-Scholes. Nevertheless, the differences among them are negligible. In addition, although the option pricing models incorporating non-Gaussian interest rate process such as CIR and Brennan-Schwartz model outperform those of Gaussian interest rates, they also show only marginal improvement. [10]

[Table 3]

The empirical results show that an option pricing model under a specific stochastic interest rate does not dominantly outperform another option pricing model under alternative stochastic interest rate process. Also, incorporating stochastic interest rates into stock option pricing does not contribute to the performance improvement of the original Black-Scholes pricing formula. This outcome coincides with the observations of BCC (1997), where they assumed the CIR type short rate and concluded that incorporating stochastic interest into stock option pricing is not important for pricing performance. As mentioned before, Rindell (1995) showed that the option pricing model incorporating the MHL type short rate process performs better than the original BS model. However, since Rindell's data included long-term options we could not conclude that our results contradict Rindell's observations.

Finally, the investigation based on other moderate estimation time spans and other interest rate quotes such as one week call rate and one month CD rate provides also no drastic changes of our results, although we did not report them.

6 Concluding Remarks

Based on Japanese market data on stock returns, option prices, and short rates, we have compared the pricing performances of option models under alternative stochastic interest rate processes as well as Black-Scholes model. Our findings show that incorporating stochastic interest rates into option pricing models has little impact on pricing performances in Japanese market where long term dated options are not available.

However, our conclusions should be understood with some cautions. Some

[10] However it is worth noting that for medium term samples, the option pricing model under Brennan-Schwartz type interest rate shows the good pricing performance relative to other models.

judiciously selected interest rate models which we did not adopt, in particular multi-factor models may contribute to improving pricing performances of options. Option pricing model incorporating stochastic interest rates and stochastic volatility, for example, may outperform option pricing under stochastic volatility. Comparing hedging performances of alternative stochastic interest rate models may provide another story. These considerations remain to be examined.

A Appendix

We recover the stock option pricing formula under Gaussian type short rate process by simple algebra. Let us consider the Black-Scholes economy under the equivalent martingale measure Q. The stock price is described by

$$S_t = S_0 + \int_0^t r_s S_s ds + \int_0^t \sigma_s S_s d\tilde{W}_{1,s} \tag{A.1}$$

and the short rate is given by

$$r_t = r_0 + \int_0^t (\alpha_s + \beta_s r_s) ds + \int_0^t \delta_s d\tilde{W}_{2,s}, \tag{A.2}$$

where σ_s, α_s, and β_s are the deterministic functions of time. Assume that the covariation between two Brownian motions is captured by ρ, constant.

The solutions of (A.1) and (A.2) are represented by

$$S_t = S_0 \exp\left(\int_0^t (r_s - \frac{\sigma_s^2}{2}) ds + \int_0^t \sigma_s d\tilde{W}_{1,s} \right) \tag{A.3}$$

and

$$r_t = \exp\left(\int_0^t \beta_s ds \right) \left[r_0 + \int_0^t \exp\left(- \int_0^s \beta_u du \right) \left(\alpha_s ds + \delta_s d\tilde{W}_{2,s} \right) \right]. \tag{A.4}$$

Define $\exp\left(-\int_0^T r_s ds\right)[S_T - K]$ by Z_T and $\exp\left(\int_0^t \beta_s ds\right)$ by $\beta(t)$. Then inserting (A.3) and (A.4) into Z_T, gives the following expression for Z_T.

$$Z_T = S_0 \exp\left(-\frac{1}{2} \int_0^t \sigma_s^2 ds + X_{1T} \right) - K \exp\left(-B(T) - X_{2T} \right), \tag{A.5}$$

where

$$B(T) = \int_0^T \beta(t) \left[r_0 + \int_0^t \beta(s)^{-1} \alpha_s ds \right] dt,$$

$$X_{1T} = \int_0^T \sigma_s d\tilde{W}_{1,s},$$

and

$$X_{2T} = \int_0^T \left(\int_s^T \beta(t)dt \right) \beta(s)^{-1} \delta_s d\tilde{W}_{2,s}.$$

Using the notations such as

$$\Sigma_{11}^T \equiv \int_0^T \sigma_s^2 ds,$$

$$\Sigma_{22}^T \equiv \int_0^T \left(\int_s^T \beta(t)dt \right)^2 \beta(s)^{-2} \delta_s^2 ds,$$

and

$$\Sigma_{12}^T \equiv \int_0^T \left(\int_s^T \beta(t)dt \right) \beta(s)^{-1} \delta_s \sigma_s \rho \, ds,$$

we can also express (A.5) as

$$Z_T = S_0 \exp\left(-\frac{1}{2}\Sigma_{11}^T + X_{1T} \right) - K \exp\left(-B(T) - X_{2T} \right),$$

where

$$\begin{pmatrix} X_{1T} \\ X_{2T} \end{pmatrix} \sim N_2 \left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_{11}^T & \Sigma_{12}^T \\ \Sigma_{12}^T & \Sigma_{22}^T \end{pmatrix} \right].$$

If it is noted that $Z_T \geq 0$ is equivalent to $X_{1T} + X_{2T} \geq C(T)$, where $C(T) = \frac{1}{2}\Sigma_{11}^T - B(T) + \log \frac{K}{S_0}$, the time T maturing European call option value at the initial time, V_0, can be expressed by

$$\begin{aligned} V_0 &= E_0^Q \left[\max(Z_T, 0) \right] \\ &= E_0^Q \Big[S_0 \exp\left(X_{1T} - \frac{\Sigma_{11}^T}{2} \right) I(X_{1T} + X_{2T} \geq C(T)) \Big] \\ &\quad - E_0^Q \Big[K \exp\left(-X_{2T} - B(T) \right) I(X_{1T} + X_{2T} \geq C(T)) \Big], \end{aligned} \tag{A.6}$$

where $I(\cdot)$ is 1 if $(\cdot)$ is true and 0, otherwise.

We set up the following lemma.

Lemma 1 (Kunitomo and Takahashi (1992)) *Let* $\mathbf{x} \sim N_2(\mu, \mathbf{\Sigma})$*, where* N_2 *is the 2-dimensional Gaussian distribution function. For arbitrary 2-dimensional vector* $\mathbf{a}$ *and scalar* b *and* c*, the following relationship exists.*

$$\iint_{(1,-b)\mathbf{x} \geq c} \exp(\mathbf{a}'\mathbf{x}) \, n_2(\mathbf{x}|\mu, \mathbf{\Sigma}) \, dx = \exp\left(\mathbf{a}'\mu + \frac{1}{2}\mathbf{a}'\mathbf{\Sigma}\mathbf{a} \right) \Phi \left[\frac{(1,-b)(\mu + \mathbf{\Sigma}\,\mathbf{a}) - c}{\sqrt{(1,-b)'\mathbf{\Sigma}\,(1,-b)}} \right],$$

where n_2 is the 2-dimensional Gaussian density function and Φ is the standard Gaussian distribution function.

By applying for $b = -1$ and $\mathbf{a} = (1,0)'$ in the first term and $\mathbf{a} = (0,-1)'$ in the second term of (A.6) to the above lemma, we have the expression of V_0,

$$V_0 = S_0 \Phi(d_1) - K\, P(0,T)\, \Phi(d_2), \tag{A.7}$$

where $P(0,T) = \exp\left(\Sigma_{22}^T/2 - B(T)\right)$ is the price of pure discount bond with time to maturity T, $d_1 = (\Sigma_{11}^T + \Sigma_{12}^T - C(T))/\sqrt{D}$, in which $D = \Sigma_{11}^T + 2\Sigma_{12}^T + \Sigma_{22}^T$, and $d_2 = d_1 - \sqrt{D}$.

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Table 1: **Summary Statistics of Option Data**

P, K and T are the prices of Nikkei 225 call option, exercise price and the remaining time-to-maturity, respectively. S is the dividend-adjusted value of Nikkei 225 stock index and r is overnight call money rate. Total is the entire sample covering from June 12, 1992 through June 30, 1995. Short is the options with time to expiration less or equal to 60 days and Medium is the options with time to expiration from 60 days through four months. ATM is at-the-money sample which satisfies $0.97 < S/K < 1.03$. OTM is out-of-the-money sample which satisfies $S/K \leq 0.97$. ITM is in-the-money sample which satisfies $S/K \geq 1.03$. All is the sum of ATM, OTM, and ITM. The numbers are the mean values and those in parentheses are the standard deviations.

		Obs.	S/K	P	T	r(%)
Total	ALL	16518	0.989(0.079)	771.090(819.183)	46.073(30.702)	2.816(0.882)
	ATM	4966	0.998(0.017)	665.314(300.764)	49.638(32.132)	2.759(0.819)
	OTM	7281	0.924(0.036)	213.690(210.311)	50.987(30.570)	2.814(0.933)
	ITM	4271	1.092(0.059)	1844.308(848.713)	33.551(25.289)	2.886(0.857)
Short	ALL	11757	0.997(0.085)	796.201(903.706)	29.793(17.029)	2.824(0.890)
	ATM	3286	0.999(0.017)	572.850(272.988)	30.344(17.619)	2.757(0.839)
	OTM	4780	0.921(0.038)	131.390(144.800)	32.403(16.498)	2.825(0.951)
	ITM	3691	1.094(0.060)	1856.002(876.326)	25.922(16.458)	2.881(0.848)
Medium	ALL	4761	0.970(0.057)	709.080(553.306)	86.276(16.833)	2.797(0.860)
	ATM	1680	0.995(0.016)	846.170(268.760)	87.375(17.099)	2.764(0.778)
	OTM	2501	0.929(0.032)	370.984(225.871)	86.505(16.690)	2.792(0.898)
	ITM	580	1.074(0.045)	1769.888(641.534)	82.103(16.055)	2.915(0.908)

Table 2: **The Estimates of Parameters**

The values are the daily averages of each day estimates over test period (755 trade days, from June 12, 1992 through June 30, 1995) . For each day t, we used the time series data of 494 trade days up to t to estimate these parameters. The numbers in parentheses are the standard deviations of the estimates. Vas (res. MHL,CIR,BrS) is concerned with the option pricing model under Vasicek (res. MHL, CIR, and Brennan-Schwartz) type interest rate process.

Parameter	Vas	MHL	CIR	BrS
σ	0.2438	0.2438	0.2438	0.2438
	(0.0305)	(0.0305)	(0.0305)	(0.0305)
κ	1.1903		1.2188	1.2212
	(0.7208)		(0.7291)	(0.7638)
θ	0.0177		0.0183	0.0191
	(0.0104)		(0.0100)	(0.0101)
α		-0.0159		
		(0.0061)		
δ	0.0119	0.0119	0.0289	0.2950
	(0.0029)	(0.0029)	(0.0028)	(0.0313)
ρ	0.0057	0.0050	0.00789	0.0111
	(0.0492)	(0.0491)	(0.0454)	(0.0415)
λ	-1.8476	-1.8389	-0.5726	-1.8834
	(0.3490)	(0.3184)	(0.1463)	(0.3645)

Table 3: **Daily average Absolute Pricing Error of Nikkei 225 Index Call Option**

The values are the daily averages of mean absolute pricing errors for each day over test period (DAPE). The numbers in parentheses are the standard deviations of DAPE. Total is the entire option samples. Short (res. Medium) is the options with short (res. medium) time to expiration. The sample is the number of observed call option prices and the numbers in square brackets are the valid number of day which we average over. BS is the original Black-Scholes option pricing model. Vas (res. MHL,CIR,BrS) is the option pricing model under Vasicek (res. MHL, CIR, and Brennan-Schwartz) type interest rate process. ALL is the entire sample and OTM (res. ATM, ITM) is out-of-the-money (res. at-the-money, in-the-money) sample.

		sample	BS	Vas	MHL	CIR	BrS
	ALL	16518	113.751	114.114	113.815	113.714	113.078
		[755]	(53.663)	(53.489)	(53.599)	(53.801)	(54.518)
	ATM	4966	126.563	127.110	126.622	126.456	124.466
		[755]	(78.662)	(78.062)	(78.439)	(78.884)	(81.148)
Total	OTM	7281	82.615	83.269	82.768	82.462	79.634
		[749]	(49.734)	(50.030)	(49.824)	(49.790)	(48.963)
	ITM	4271	167.464	166.913	167.289	167.703	173.029
		[714]	(94.183)	(93.826)	(94.038)	(94.328)	(95.269)
	ALL	11757	104.439	104.353	104.411	104.507	105.716
		[755]	(55.002)	(54.863)	(54.963)	(55.073)	(55.985)
	ATM	3286	112.462	112.449	112.437	112.500	112.696
		[755]	(80.025)	(79.715)	(79.908)	(80.133)	(81.858)
Short	OTM	4780	58.229	58.364	58.267	58.192	57.056
		[748]	(39.695)	(39.706)	(39.703)	(39.730)	(39.746)
	ITM	3691	170.311	169.760	170.138	170.544	176.158
		[713]	(95.369)	(95.242)	(95.323)	(95.431)	(95.844)
	ALL	4761	133.746	135.222	134.031	133.441	128.050
		[730]	(78.283)	(78.909)	(78.393)	(78.381)	(76.720)
	ATM	1680	150.251	151.936	150.500	149.884	143.269
		[669]	(98.769)	(98.620)	(98.581)	(99.016)	(99.975)
Medium	OTM	2501	126.434	128.025	126.813	126.076	120.219
		[702]	(78.204)	(79.060)	(78.457)	(78.191)	(75.964)
	ITM	580	151.147	150.522	150.906	151.464	154.403
		[320]	(124.599)	123.736	(124.063)	(125.152)	(126.574)

Figure 1: Dividend-Adjusted Nikkei 225 Stock Index and Overnight Call Money Rate

The sample data covers from June 12, 1990 to June 30, 1995 and has 1248 time series. (a) Dividend-Adjusted Nikkei 225 Stock Index; (b) Overnight Call Money Rate; (c)Spikes-Adjusted Overnight Call Money Rate; (d) One Month Call Money Rate.

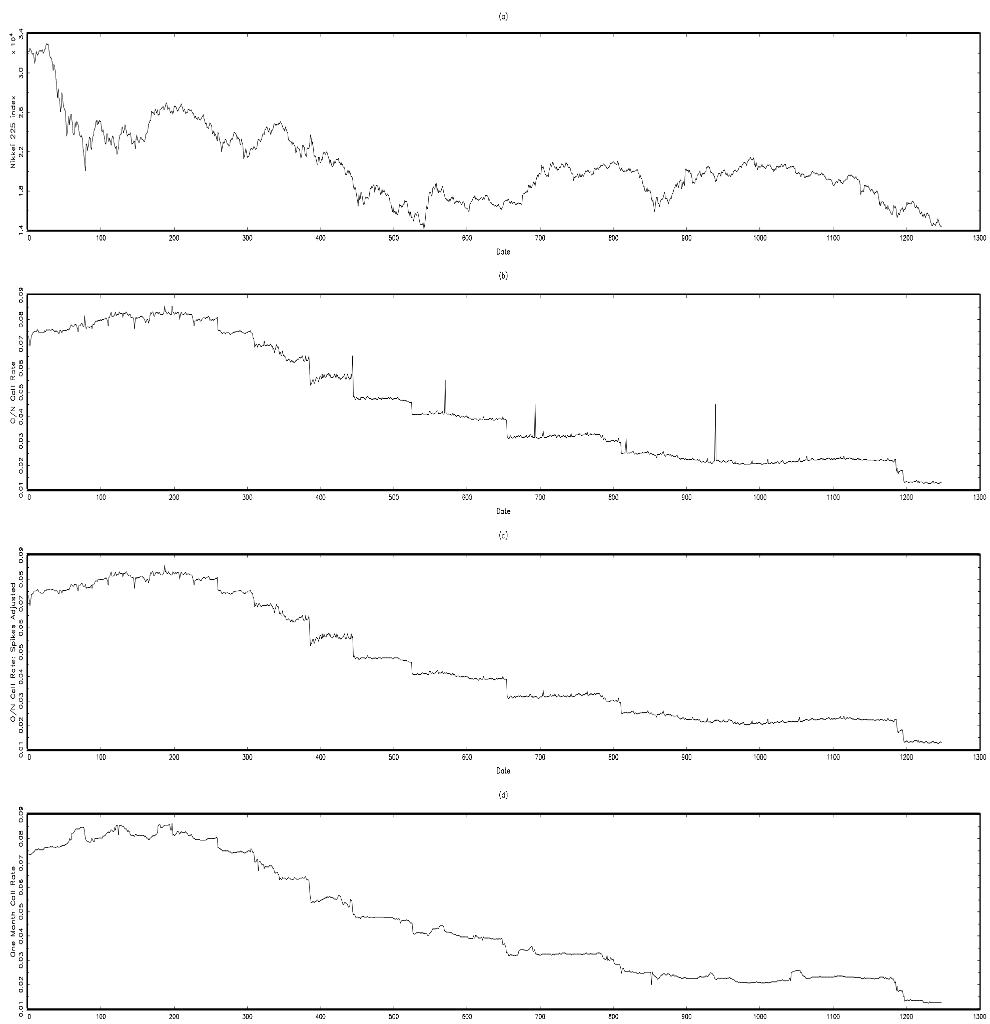

Figure 2: Historical Volatility

The estimates are the values of 755 trade days covering from June 12, 1992 to June 30, 1995. The estimation period for each volatility is 494 trade days.

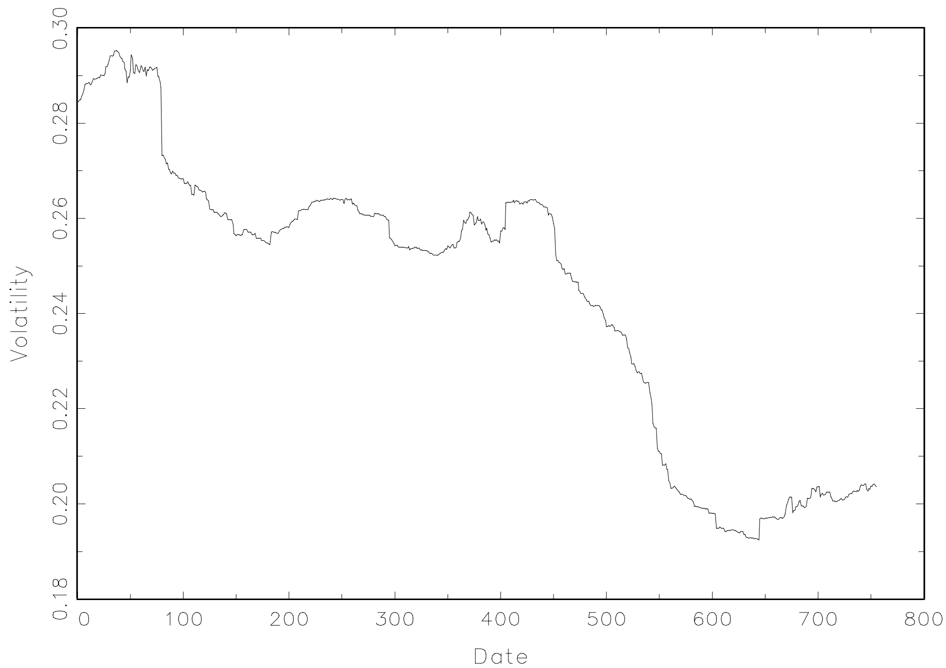

Figure 3: The Estimates of Vasicek Type Interest Rate

The estimates are the values of 755 trade days covering from June 12, 1992 to June 30, 1995. The estimation period for each parameter is 494 trade days. (a) κ; (b) θ; (c) δ; (d) ρ; (e) λ.

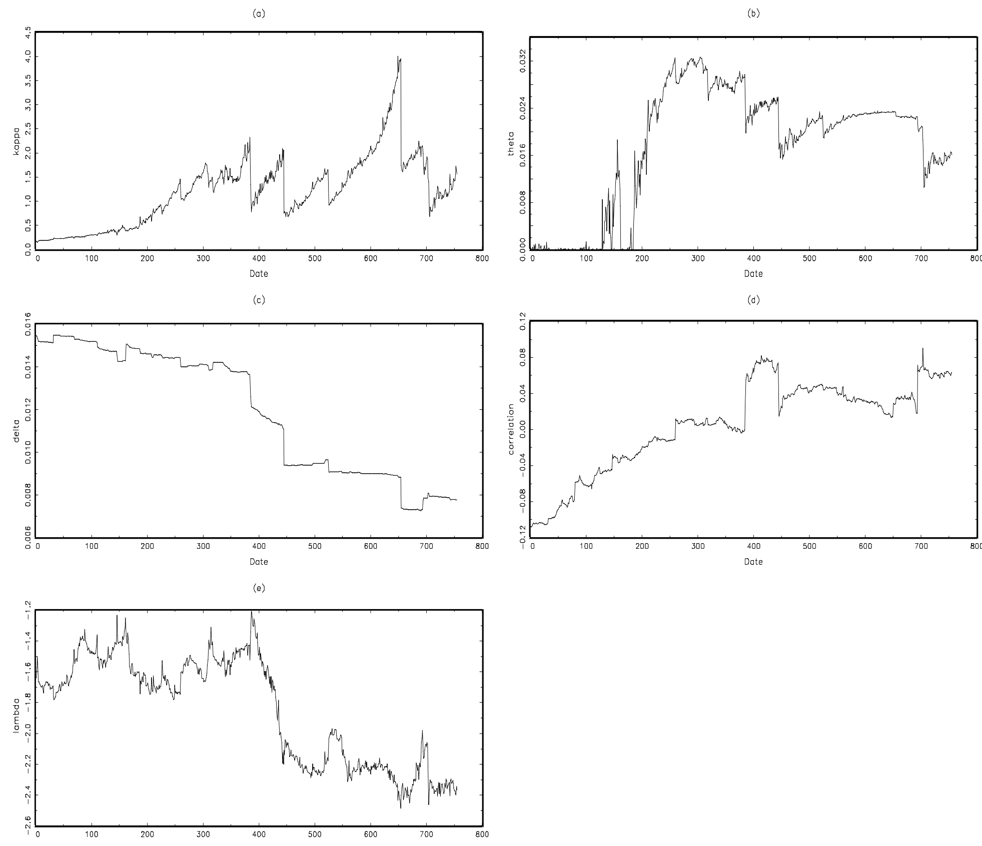

Figure 4: The Estimates of MHL Type Interest Rate

The estimates are the values of 755 trade days covering from June 12, 1992 to June 30, 1995. The estimation period for each parameter is 494 trade days. (a)α; (b)δ; (c)ρ; (d)λ.

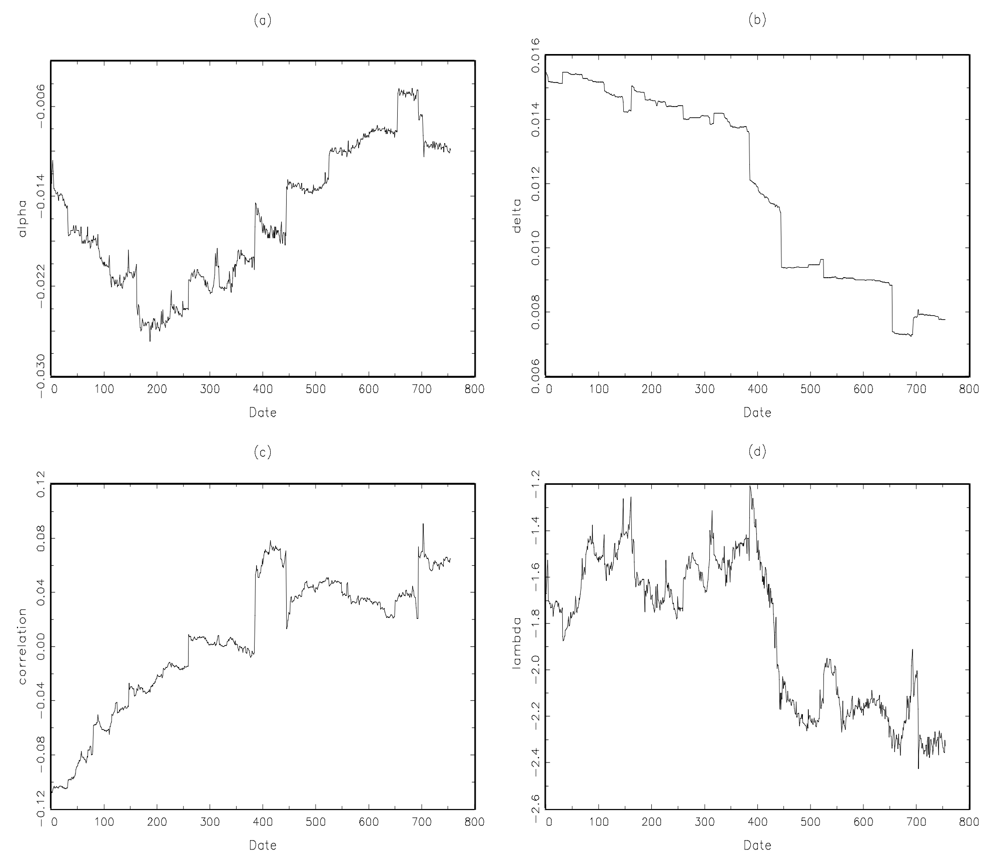

Figure 5: The Estimates of CIR Type Interest Rate

The estimates are the values of 755 trade days covering from June 12, 1992 to June 30, 1995. The estimation period for each parameter is 494 trade days. (a)κ; (b)θ; (c)δ; (d)ρ; (e)λ.

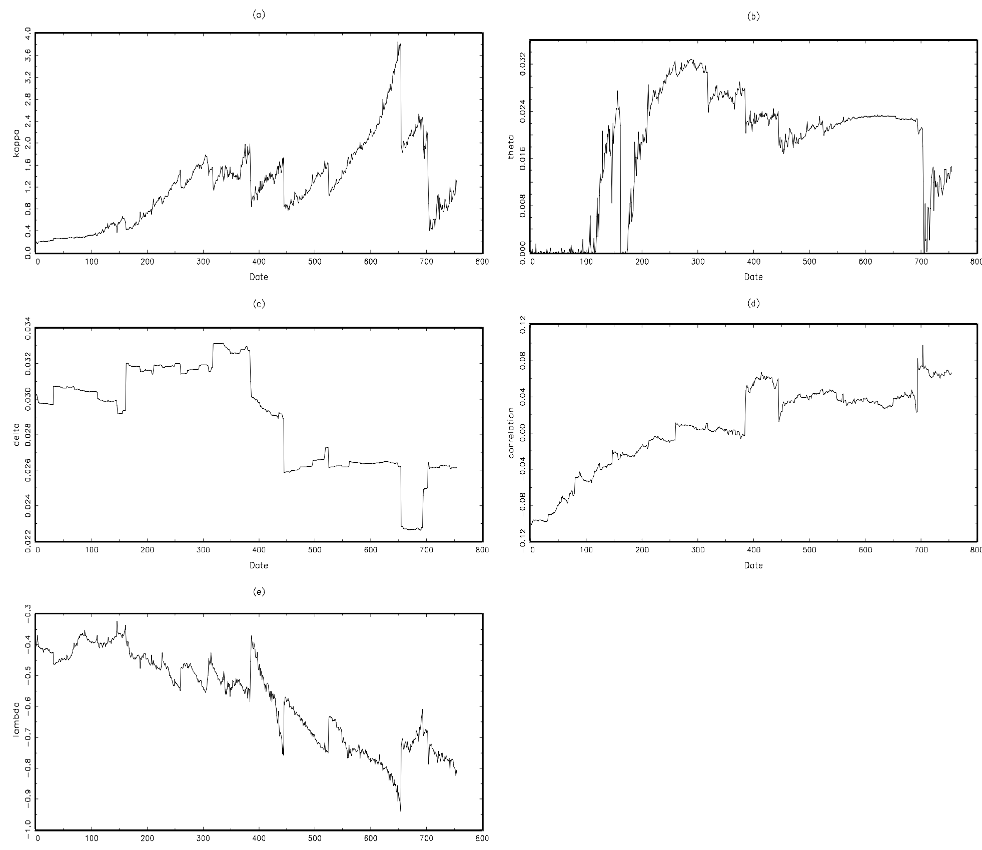

Figure 6: The Estimates of Brennan-Schwartz Type Interest Rate

The estimates are the values of 755 trade days covering from June 12, 1992 to June 30, 1995. The estimation period for each parameter is 494 trade days. (a)κ; (b)θ; (c)δ; (d)ρ; (e)λ.