

50th Anniversary Article

Option Pricing: Valuation Models and Applications

Mark Broadie

Graduate School of Business, Columbia University, 3022 Broadway, New York, New York 10027-6902, mnb2@columbia.edu

Jerome B. Detemple

School of Management, Boston University, 595 Commonwealth Avenue, Boston, Massachusetts 02215, detemple@bu.edu

This paper surveys the literature on option pricing from its origins to the present. An extensive review of valuation methods for European- and American-style claims is provided. Applications to complex securities and numerical methods are surveyed. Emphasis is placed on recent trends and developments in methodology and modeling.

Key words: option pricing; American options; risk-neutral valuation; jump and stochastic volatility models

1. Derivatives Markets: Introduction and Definitions

Management Science has a long tradition of publishing important research in the finance area, including significant contributions to portfolio optimization; asset-liability management; utility theory and stochastic dominance; and empirical finance and derivative securities. Within the derivatives area, contributions in the journal have advanced our understanding of the pricing, hedging, and risk management of derivative securities in a wide variety of financial markets, including equity, fixed income, commodity, and credit markets. In addition to theoretical advances, several articles have focused on practical applications through the design of efficient numerical procedures for valuing and hedging derivative securities. In this paper we survey the option pricing literature over the last four decades, including many articles that have appeared in the pages of *Management Science*. We begin with a description of derivative securities and their properties.

A *derivative security* is a financial asset whose payoff depends on the value of some underlying variable. The underlying variable can be a traded asset, such as a stock; an index portfolio; a futures price; a currency; or some measurable state variable, such as the temperature at some location or the volatility of an index. The payoff can involve various patterns of cash flows. Payments can be spread evenly through time, occur at specific dates, or a combination of the two. Derivatives are also referred to as contingent claims.

An *option* is a derivative security that gives the right to buy or sell the underlying asset, at or before some

maturity date T , for a prespecified price K , called the *strike* or *exercise price*. A call (put) option is a right to buy (sell). Because exercise is a right and not an obligation, the exercise payoff is $(S - K)^+ \equiv \max\{S - K, 0\}$ for a call option and $(K - S)^+ \equiv \max\{K - S, 0\}$ for a put option, where S denotes the price of the underlying asset. Options can be European style, which can only be exercised at the maturity date, or American style, where exercise is at the discretion of the holder, at any time before or at the maturity date.

Plain vanilla options, such as those described above, were introduced on organized option exchanges such as the Chicago Board of Options Exchange (CBOE) in 1973. Since then, innovation has led to the creation of numerous products designed to fill the needs of various types of investors. Path-dependent options, such as barrier options, Asian options, and lookbacks are examples of contractual forms that have emerged since and are now routinely traded in markets or quoted by financial institutions, or both. Even more exotic types of contracts, whose payoffs depend on multiple underlying assets or on occupation times of predetermined regions, have emerged in recent years and have drawn interest.

This paper surveys past contributions in the field and provides an overview of recent trends. We first review the fundamental valuation principles for European-style (§2) and American-style (§3) derivatives. Next, we outline extensions of the basic model (§4) and survey numerical methods (§5). An assessment of the future and comments on open problems are formulated in the concluding section (§6). The appendix treats derivatives in the fixed income

and credit markets, real options, path-dependent contracts, and derivatives written on multiple underlying assets.

2. European-Style Derivatives

The focus of this section is on the fundamentals of derivatives' valuation and applications to plain vanilla European options. We first review the no-arbitrage (§2.1) and the risk-neutral (§2.2) approaches to valuation for European-style derivatives. We then turn our attention to valuation in the presence of unhedgeable risks (§2.3).

2.1. No-Arbitrage Valuation

The no-arbitrage principle is central to the valuation of derivative securities. One of the most important insights of the seminal papers by Black and Scholes (1972) and Merton (1973) was to show how the principle can be used to characterize the price of an arbitrary derivative asset.

2.1.1. The Black-Scholes-Merton Framework. The Black-Scholes-Merton (BSM) analysis starts from the premise that the underlying asset price S follows a geometric Brownian motion process

$$\frac{dS_t}{S_t} = (\mu - \delta) dt + \sigma dW_t, \tag{1}$$

where μ , δ , and σ are constants, which represent the expected (total) return on the asset, the dividend rate, and the return volatility, respectively. The process W is a standard Brownian motion, under the physical (or statistical) probability measure P , that captures the underlying uncertainty in this market. Trading in this asset is unrestricted, i.e., no taxes, transactions costs, constraints, or other frictions. Likewise, investors can invest without restrictions, at the constant risk-free rate r .¹ The asset paying interest at the riskless rate and the underlying risky asset are sometimes called the primary assets.

In the sequel we refer to this model as the BSM framework or setting.

2.1.2. The Fundamental Valuation Equation. Suppose that we seek to value a European-style derivative that pays off $g(S_T)$ at a given maturity date T . Assuming that the current price $V_t \equiv V(S, t)$ of the security has suitable differentiability properties (specifically $V(S, t) \in \mathcal{C}^{2,1}$ on the domain $\mathbb{D} \equiv \mathbb{R}_+ \times [0, T)$) one can apply Itô's lemma (see Karatzas and Shreve 1988, p. 149) to show that

$$dV_t = \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} S_t (\mu - \delta) + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} S_t^2 \sigma^2 \right) dt + \frac{\partial V}{\partial S} S_t \sigma dW_t$$

¹ Trading strategies are subject to a mild regularity condition. To ensure a well-defined value process the quadratic variation of a portfolio value must be finite P -a.s.

on \mathbb{D} . Letting $\mu^V(t, S)$ and $\sigma^V(t, S)$ be the mean and volatility of the derivative's return enables us to write $dV_t = V_t [\mu^V(t, S_t) dt + \sigma^V(t, S_t) dW_t]$.

A self-financing portfolio of the risk-free asset, the underlying asset, and the derivative contract, however, has value X evolving according to

$$dX_t = rX_t dt + X_t \pi_t [(\mu - r) dt + \sigma dW_t] + X_t \pi_t^V [(\mu^V(t, S_t) - r) dt + \sigma^V(t, S_t) dW_t],$$

with initial value $X_0 = x$, the cost of the portfolio at initiation. Here $X\pi$ represents the (dollar) amount invested in the underlying asset, $X\pi^V$ the amount in the derivative, and $X(1 - \pi - \pi^V)$ the balance in the riskless asset (π , π^V and $1 - \pi - \pi^V$ are the fractions of the portfolio's value invested in the three assets). With the choices $X_t \pi_t^V = V(S_t, t)$ and $X_t \pi_t = -S_t \times \partial V(S_t, t) / \partial S$ we obtain

$$dX_t = \left(rX_t - \frac{\partial V}{\partial S} S_t (\mu - r) + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} S_t (\mu - \delta) + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} S_t^2 \sigma^2 - rV \right) dt.$$

Note that this portfolio is locally riskless. Because the initial capital could also have been invested risklessly, to preclude the existence of arbitrage opportunities the portfolio return must be equal to the risk-free rate. Equivalently, the derivative's price V must satisfy

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} S_t (r - \delta) + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} S_t^2 \sigma^2 - rV = 0 \tag{2}$$

on \mathbb{D} . This partial differential equation, along with the boundary conditions

$$\begin{cases} V(S_T, T) = g(S_T) & \text{on } \mathbb{R}_+ \\ V(0, t) = g(0)e^{-r(T-t)} & \text{on } [0, T) \\ \lim_{S \rightarrow \infty} V(S, t) = g(\infty)e^{-r(T-t)} & \text{on } [0, T), \end{cases} \tag{3}$$

characterizes the derivative's price. Equation (2) is known as the fundamental valuation equation for derivatives' prices. The equation is called fundamental because it applies to any derivative security, independently of its payoff structure. What changes across securities are the relevant Boundary Conditions (3).

One can also introduce the new variables (x, τ) and the function $u(x, \tau)$ such that

$$S = K \exp(x), \quad t = T - \frac{\tau}{\frac{1}{2}\sigma^2}, \quad V(S, t) = e^{-\alpha x - \beta \tau} u(x, \tau),$$

where $\alpha \equiv (r - \delta - \frac{1}{2}\sigma^2) / \sigma^2$, $\beta \equiv \alpha^2 + 2r / \sigma^2$, and where K is an arbitrary positive constant (in the special case of options, K represents the strike). The function $u(x, \tau)$ solves the modified valuation equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{4}$$

subject to the boundary conditions

$$\begin{cases} e^{-\alpha x} u(x, 0) = g(K \exp(x)) & \text{on } \mathbb{R} \\ \lim_{x \rightarrow -\infty} e^{-\alpha x - \beta \tau} u(x, \tau) = g(0) e^{-2r\tau/\sigma^2} & \text{on } \tau \in [0, \frac{1}{2}\sigma^2 T] \\ \lim_{x \rightarrow \infty} e^{-\alpha x - \beta \tau} u(x, \tau) = g(\infty) e^{-2r\tau/\sigma^2} & \text{on } \tau \in [0, \frac{1}{2}\sigma^2 T]. \end{cases} \quad (5)$$

Condition (4) is known as the heat equation. This partial differential equation, which characterizes the propagation of heat in a continuous medium, has been extensively studied in physics. Its fundamental solution (subject to the boundary condition $\lim_{x \rightarrow \pm\infty} u(x, \tau) = 0$) is the Gaussian density function $u(x, t) = (2\sqrt{\pi t})^{-1} \exp(-x^2/4t)$ with mean 0 and standard deviation $\sqrt{2t}$ (see Wilmott et al. 1993, p. 81).

2.1.3. The Black-Scholes Formula. When the payoff is specialized to a call option $g(S) = (S - K)^+$ the valuation equation and boundary conditions admit the solution

$$c(S_t, t; K) = S_t e^{-\delta(T-t)} N(d(S_t; K, T-t)) - K e^{-r(T-t)} N(d(S_t; K, T-t) - \sigma\sqrt{T-t}), \quad (6)$$

where $N(\cdot)$ is the cumulative standard normal distribution and

$$d(S_t; K, T-t) = \frac{1}{\sigma\sqrt{T-t}} \left[\log\left(\frac{S_t}{K}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)(T-t) \right]. \quad (7)$$

This expression is the celebrated Black-Scholes formula for the price of a European call option with strike K and maturity date T . That the Black-Scholes Formula (6)–(7) satisfies all the pricing conditions can be verified by computing the relevant derivatives and limits, and substituting in (2)–(3). Alternatively, one can work with the fundamental solution of the heat equation to deduce $C(S, t)$.

Similar arguments lead to the value of a European put option,

$$p(S_t, t; K) = K e^{-r(T-t)} N(-d(S_t; K, T-t) + \sigma\sqrt{T-t}) - S_t e^{-\delta(T-t)} N(-d(S_t; K, T-t)), \quad (8)$$

with $d(S_t; K, T-t)$ as defined in (7). The put formula can also be derived by a straightforward application of the put-call parity relationship for European options on dividend-paying assets. This no-arbitrage condition, which relates call and put options with identical maturities and strike prices, states that

$$c(S_t, t; K) + K e^{-r(T-t)} = p(S_t, t; K) + S_t e^{-\delta(T-t)} \quad (9)$$

for all $t \in [0, T]$. The condition reflects the fact that the portfolios on the left- and right-hand sides of the equation have the same payoff $(S_T - K)^+ + K = (K - S_T)^+ + S_T$ at the maturity date: Because terminal payoffs are identical, their prices must also be identical, at all times prior to maturity, to preclude arbitrages. Solving for the put price from (9), substituting the call Formula (6), and simplifying establishes (8).

Extensions of the model to settings with trading costs, short-sales constraints, and other frictions have been developed. See, for example, Leland (1985), Hodges and Neuberger (1989), Bensaid et al. (1992), Boyle and Vorst (1992), Karatzas and Kou (1996), and Broadie et al. (1998).

2.1.4. Delta Hedging. An important function of options and, more generally, of derivatives, is to provide the means to hedge given exposures to the underlying sources of risk or given positions in the underlying assets. Conversely, firms dealing in options often wish to hedge against the fluctuations inherent in their derivatives positions. The key to immunizing these positions is the delta of the option: The *delta* is the sensitivity of the option price with respect to the underlying asset price. Simple differentiation shows that the delta of a call is

$$\Delta^c(S_t, t) \equiv \frac{\partial C(S_t, t)}{\partial S} = e^{-\delta(T-t)} N(d(S_t; K, T-t)).$$

Thus, immunizing a call position against fluctuations in the value of the underlying asset entails shorting $\Delta^c(S_t, t)$ units of the underlying asset. Because delta depends on the asset price and time to maturity, delta hedging is inherently a dynamic process that requires continuous monitoring and rebalancing of the hedge over time.

Similar derivations establish the formula for the delta of a put: $\Delta^p(S_t, t) = -e^{-\delta(T-t)} N(-d(S_t; K, T-t))$. Immunizing a put position entails a long position consisting of $\Delta^p(S_t, t)$ units of the underlying asset.

Comparison with the formulas of §2.1.2 shows that the delta of an option is closely related to the self-financing portfolio used to synthesize a riskless position. Simple analysis reveals that delta also represents the number of shares to be held in order to replicate the option, i.e., the number of shares in the replicating portfolio. Further perspective on the relation between delta hedging and the self-financing condition can be found in Ingersoll (1987, p. 363). Carr and Jarrow (1990) provide a nice analysis of a trading strategy, the stop-loss start-gain strategy; it appears to replicate the option's payoff but in fact does not.

The sensitivities of option prices with respect to other parameters of the model are also of interest to traders. In particular, the second derivative of the option's price with respect to the underlying price

(gamma) and the derivatives with respect to time (theta), volatility (vega), and the interest rate (rho) have received attention. Formulas for these Greeks are easy to derive from (6) and (8) and can be found in standard textbooks such as Hull (2003).

2.2. Risk-Neutral Valuation

The fundamental valuation equations (2)–(3) reveal that the only market parameters relevant for pricing derivatives are the interest rate r , the volatility σ , and the dividend yield δ of the underlying asset. Surprisingly, the expected return μ does not matter. This property lies at the heart of the second approach to option valuation, the so-called risk-neutralization procedure. This method was discovered by Cox and Ross (1976). Its formalization and the identification of some fundamental underlying principles can be found in Harrison and Kreps (1979) and Harrison and Pliska (1981).

2.2.1. A Risk-Neutral Pricing Formula. Suppose that the financial market behaves in a risk-neutral manner with underlying asset price evolving according to

$$\frac{dS_t}{S_t} = (r - \delta) dt + \sigma dW_t^*, \quad (10)$$

where W^* is a Brownian motion process. Risk neutrality implies that the expected total return on the asset equals the risk-free rate: $E_t^*[dS_t/S_t + \delta dt] = r dt$, where $E_t^*[\cdot]$ is the conditional expectation at t with respect to the Brownian motion W^* . This property is reflected in the dynamics (10) of the asset price. Also assume that, in other respects, the market structure remains the same (securities are freely traded and a riskless asset is available).

In this environment, the no-arbitrage analysis of §2.1 applies and, as can be verified, leads to the same valuation equation. Derivative security prices in our risk-neutral economy and in the original risk-averse economy will therefore coincide! Moreover, an application of the Feynman-Kac formula (see Karatzas and Shreve 1988, p. 366) shows that

$$V(S_t, t) = e^{-r(T-t)} E_t^*[g(S_T)] \quad (11)$$

solves (2)–(3). It follows that the derivative's price is simply the expected payoff discounted at the risk-free rate, as should be the case with risk neutrality.

These observations motivate the second approach to the valuation of derivatives, the risk-neutralization procedure, which involves two steps. The first step consists of risk neutralizing the underlying price process, as in (10), by replacing the expected return μ with the risk-free rate r . The second step consists of computing the price from (10) by taking the expectation over the distribution of the underlying

price implied by (11), or equivalently over the distribution of the Brownian motion W^* .

The risk-neutral Formula (11) seems to emerge as an alternative to the standard present value rule, which states that the expected payoff of a security ought to be discounted at a risk-adjusted rate. Risk adjustment follows from equilibrium considerations linking the riskiness of an asset to its expected return (or risk premium). Somewhat surprisingly, Formula (11) appears to obviate the need for adjusting discount rates for the purpose of valuing derivatives.

2.2.2. The Equivalent Martingale Measure: Some Definitions. Resolution of this puzzle follows by taking a closer look at the relationship between the no-arbitrage and risk-neutralization approaches.

Consider again the price dynamics of the underlying security (1) in our risk-averse economy and note that it can be expressed in the form

$$\frac{dS_t}{S_t} = (r - \delta) dt + \sigma(dW_t + \theta dt),$$

where $\theta \equiv \sigma^{-1}(\mu - r)$. The variable θ is known as the market price of risk or the Sharpe ratio. It measures the reward (i.e., the risk premium) per unit risk. Defining $dW_t^* \equiv dW_t + \theta dt$ then enables us to write the price evolution in precisely the form (10). Of course, W^* is a Brownian motion process with drift θ in this original economy with physical probability measure P .

At this point, however, one can appeal to the Girsanov change of measure (see Karatzas and Shreve 1998, p. 191) to construct a new measure Q under which W^* has the (standard) Brownian motion property. Specifically, define the process

$$\eta_t \equiv \exp\left(-\frac{1}{2}\theta^2 t - \theta W_t\right), \quad t \in [0, T]$$

and construct the measure $dQ \equiv \eta_T dP$. An application of Itô's lemma shows that η is a P -martingale with initial value $\eta_0 = 1$.² Combining this with the fact that $\eta_T > 0$ (P -a.s.) enables us to conclude that Q is a probability measure, and that Q is equivalent to P .³ It is referred to as the equivalent martingale measure (EMM). The random variable η_T represents the Radon-Nikodym derivative of Q with respect to P (i.e., $\eta_T = dQ/dP$). Because η_T depends on the market price of risk, the measure Q can be interpreted as a risk-adjusted probability measure. Girsanov's theorem can then be invoked to assert that W^* is a Q -Brownian motion.⁴

² The process η is a P -martingale if and only if $E_t[\eta_v] = \eta_t$ for all $t \leq v$, where $E_t[\cdot]$ is the expectation with respect to P .

³ The measures Q and P are equivalent if they have the same null sets.

⁴ Alternatively, this can also be verified by computing the characteristic function of W^* under Q , given by

$$E^*[e^{i\lambda W_t^*}] = E[e^{i\lambda W_t} \eta_t] = E[e^{i\lambda W_t} \eta_t]$$

2.2.3. The Equivalent Martingale Measure: Asset Price Representations. The developments above show that W^* is a Brownian motion under Q . Let us now describe the behavior of prices under the new measure. Another straightforward application of Itô's lemma shows that

$$e^{-r(T-t)}S_T + \int_t^T e^{-r(v-t)}S_v\delta dv = S_t + \int_t^T e^{-r(v-t)}S_v\sigma dW_v^*$$

for all $t \in [0, T]$. Taking expectations with respect to Q , on both sides of this equality, establishes the formula

$$S_t = E_t^* \left[e^{-r(T-t)}S_T + \int_t^T e^{-r(v-t)}S_v\delta dv \right]. \quad (12)$$

This expression shows that the asset price equals the expected value of the discounted dividends augmented by the expected value of the discounted terminal price. Expectations are computed under the EMM Q . The discount factor is evaluated using the risk-free rate. The formula therefore suggests that the asset is priced, under Q , as if the market were risk neutral. This is why the EMM is often called the risk-neutral measure. This label should be used with some caution, however, because (as pointed out above) the EMM is effectively adjusted for risk. Recall that the standard present-value formula computes expected cash flows under the statistical (i.e., real-world) measure P and discounts those at a risk-adjusted discount rate. Our alternative Formula (12) proceeds the other way: It first corrects probabilities for risk and calculates a risk-adjusted expected cash flow, then discounts those at the risk-free rate.

It is also important to note that the discounted stock price augmented by the discounted value of dividends is a martingale under the EMM. This can easily be seen by rewriting (12) in the following manner:

$$e^{-rt}S_t + \int_0^t e^{-rv}S_v\delta dv = E_t^* \left[e^{-rT}S_T + \int_0^T e^{-rv}S_v\delta dv \right].$$

This property explains why Q is referred to as an equivalent martingale measure.

$$\begin{aligned} &= E[e^{(i\lambda - \theta)W_t + (i\lambda\theta - \frac{1}{2}\theta^2)t}] \\ &= e^{\frac{1}{2}(i\lambda - \theta)^2t + (i\lambda\theta - \frac{1}{2}\theta^2)t} = e^{-\frac{1}{2}\lambda^2t} \end{aligned}$$

for all $t \in [0, T]$. In this sequence of equalities the first follows from the passage to the measure P , the second uses the martingale property of η , and the fourth the distributional properties of the Brownian motion W under P and the moment-generating function of a normal distribution. Other equalities involve standard manipulations. The final expression corresponds to the characteristic function of a normal distribution with mean zero and variance t . Combining this distributional property with the continuity of the sample path of W^* , the initial condition $W_0^* = 0$, and the independence of increments establishes the Brownian motion behavior of W^* under the risk-adjusted probability measure Q .

Finally, it bears pointing out that the risk-neutral valuation Formula (12) for the underlying asset can also be restated in terms of expectations with respect to P . Indeed, changing the measure yields the alternative representation

$$S_t = E_t \left[\xi_{t,T}S_T + \int_t^T \xi_{t,v}S_v\delta dv \right], \quad (13)$$

where $\xi_{t,T} \equiv e^{-r(T-t)}\eta_T/\eta_t$. The quantity $\xi_v \equiv \xi_{0,v}$ is known as the state price density. The Arrow-Debreu price at date 0 of a dollar received at date v in state ω equals $\xi_v dP(\omega)$. Conditional Arrow-Debreu prices at date t for cash flows received at v are given by $\xi_{t,v} dP_t(\omega)$. Arrow-Debreu prices are also known as state prices (see Debreu 1959). The present value of the underlying asset (13) is the sum of cash flows multiplied by state prices.

2.2.4. Martingale Representation and No-Arbitrage Pricing. Let $\mathcal{F}_{(\cdot)}$ be the filtration generated by the Brownian motion W .⁵ The martingale representation theorem—hereafter MRT—(see Karatzas and Shreve 1988, p. 188) states that any random variable B that is measurable with respect to \mathcal{F}_T and is almost surely finite can be written as a sum of Brownian increments. More formally,

$$B = E[B] + \int_0^T \phi_s dW_s \quad (14)$$

for some process ϕ that is adapted and square integrable (P -a.s.).⁶ If the stochastic integral in (14) is a martingale we can also take conditional expectations and get $E_t[B] = E[B] + \int_0^t \phi_s dW_s$ for all $t \in [0, T]$. This holds, in particular, if B has a finite second moment, $E[B^2] < \infty$ (i.e., $B \in L^2$). In that instance the stochastic integral also has finite second moment and

$$E \left[\left(\int_0^t \phi_s dW_s \right)^2 \right] = E \left[\int_0^t \phi_s^2 ds \right] < \infty;$$

in addition, ϕ is unique (see Karatzas and Shreve 1988, p. 185 and p. 189).

When applied to the deflated payoff of a derivative security such that $\xi_T g(S_T) \in L^2$ the MRT gives

$$E_t[\xi_T g(S_T)] = E[\xi_T g(S_T)] + \int_0^t \phi_s dW_s \quad (15)$$

for some adapted process ϕ such that $E[\int_0^t \phi_s^2 ds] < \infty$.

Suppose now that we invest an initial amount $x \equiv E[\xi_T g(S_T)]$ and follow the portfolio policy consisting

⁵ The filtration generated by W is the increasing collection of sigma-algebras $\{\mathcal{F}_t: t \in [0, T]\}$ where $\mathcal{F}_t = \sigma - \{W_s: s \in [0, t]\}$, i.e., \mathcal{F}_t is the collection of possible trajectories of the Brownian motion at t .

⁶ The process ϕ is said to be adapted to the filtration $\mathcal{F}_{(\cdot)}$ if ϕ_t is \mathcal{F}_t -measurable for all $t \in [0, T]$. The process ϕ is square integrable (P -a.s.) if $\int_0^T \phi_s^2 ds < \infty$, P -a.s.

of an investment $X_t \pi_t = X_t \sigma^{-1} \theta + \xi_t^{-1} \sigma^{-1} \phi_t$ in the risky asset and the complement $X_t(1 - \pi_t)$ in the riskless asset, for $t \in [0, T]$. Then, by Itô's lemma, we obtain⁷

$$d(\xi_t X_t) = \xi_t X_t (\pi_t \sigma - \theta) dW_t = \phi_t dW_t;$$

subject to $\xi_0 X_0 = X_0 = x$.

Writing this differential equation in integral form yields

$$\xi_t X_t = x + \int_0^t \phi_s dW_s = E_t[\xi_T g(S_T)],$$

where the second equality follows from the definition $x \equiv E[\xi_T g(S_T)]$ and the martingale representation of the discounted payoff (15). In particular, for $t = T$ we see that $\xi_T X_T = \xi_T g(S_T)$, which means $X_T = g(S_T)$: Our selected portfolio duplicates the derivative's payoff at maturity. The no-arbitrage principle can then be invoked to conclude that the claim's price must be the same as the portfolio value. That is,

$$V(S_t, t) = X_t = \xi_t^{-1} E_t[\xi_T g(S_T)] = E_t[\xi_{t,T} g(S_T)]$$

for all $t \in [0, T]$. Any breakdown in this relation, at any point in time, means that two traded assets with identical payoffs have different prices, which implies the existence of an arbitrage opportunity.

These arguments establish that derivative securities, in the BSM market, satisfy the same type of valuation formulas as those characterizing the underlying asset. In effect, we can write

$$V(S_t, t) = E_t[\xi_{t,T} g(S_T)] = e^{-(T-t)} E_t^*[g(S_T)],$$

where we recall that $E_t^*[\cdot]$ is the expectation under the risk-neutral measure Q . As before, we see that discounted prices are Q -martingales, $e^{-rt} V(S_t, t) = e^{-rT} E_t^*[g(S_T)]$. We also observe that deflated prices are P -martingales, $\xi_t V(S_t, t) = E_t[\xi_T g(S_T)]$.

2.2.5. Risk-Neutral Pricing with Itô Price Processes: Some Generalizations. The principles outlined above, namely the change of measure and the representations of underlying and derivatives' prices, remain valid in the context of more general markets with stochastic interest rate and Itô price processes.

Specifically, suppose that the interest rate r is a progressively measurable process and that the market is comprised of d risky securities whose evolution is governed by the stochastic differential equation (SDE)

$$\frac{dS_t}{S_t} = (\mu_t - \delta_t) dt + \sigma_t dW_t, \tag{16}$$

⁷ Itô's lemma gives $d(\xi_t X_t) = \xi_t dX_t + X_t d\xi_t + d[\xi, X]_t$, where $d[\xi, X]_t$ is the cross-variation of the processes ξ and X . Substituting $dX_t = \sigma_t^{-1} (\theta + \phi_t) dt + \sigma_t^{-1} dW_t$ and $d[\xi, X]_t = -\xi_t X_t \pi_t \sigma \theta dt$, and rearranging gives the result stated.

where μ, δ, σ are progressively measurable processes and σ is invertible.⁸ In this setting W is a d -dimensional Brownian motion, μ is a $d \times 1$ vector of appreciation rates (expected total returns), δ is a $d \times 1$ vector of dividend rates, and σ is a $d \times d$ matrix of volatility coefficients. This class of models includes Markovian diffusion models with stochastic interest rates, stochastic dividends, and or stochastic volatilities. It includes deterministic volatility function models proposed in Dupire (1994), Derman and Kani (1994), and Rubinstein (1994). An important limitation is that all processes are driven by the hedgeable multidimensional Brownian motion W ; therefore, models with nonhedgeable factors are excluded.

Under the assumption that the market price of risk $\theta_t \equiv \sigma_t^{-1}(\mu_t - r_t \mathbf{1})$ is bounded, the density process

$$\eta_t \equiv \exp\left(-\frac{1}{2} \int_0^t \theta_v' \theta_v dv - \int_0^t \theta_v' dW_v\right), \quad t \in [0, T] \tag{17}$$

is a P -martingale with initial value $\eta_0 = 1$. The measure Q , such that $dQ \equiv \eta_T dP$, is well defined and has the same properties as in §2.2.3. It represents the EMM or risk-neutral measure in our more general market. The process

$$W_t^* \equiv W_t + \int_0^t \theta_v dv, \quad t \in [0, T]$$

is a d -dimensional Q -Brownian motion and the asset price S has the representations

$$\begin{aligned} S_t &= E_t^* \left[b_{t,T} S_T + \int_t^T b_{t,v} S_v \delta_v dv \right] \\ &= E_t \left[\xi_{t,T} S_T + \int_t^T \xi_{t,v} S_v \delta_v dv \right] \end{aligned} \tag{18}$$

under the measures Q and P , respectively. The quantity $b_{t,v} = \exp(-\int_t^v r_u du)$ is the discount factor at the locally riskless rate, and $\xi_t = b_{0,t} \eta_t \equiv b_t \eta_t$ is the applicable state price density.

A multidimensional version of the arguments in §2.2.4 can again be invoked to yield no-arbitrage representations of derivatives prices. For a derivative security generating a continuous payment f_t , per unit time, at date t and a terminal cash flow Y_T we obtain the valuation formula

$$\begin{aligned} V_t(f, Y) &= E_t^* \left[b_{t,T} Y_T + \int_t^T b_{t,v} f_v dv \right] \\ &= E_t \left[\xi_{t,T} Y_T + \int_t^T \xi_{t,v} f_v dv \right], \end{aligned} \tag{19}$$

⁸ A process x is said to be progressively measurable if x_t is measurable with respect to the product sigma-algebra $\mathcal{F}_t \otimes \mathcal{B}([0, t])$ for all $t \in [0, T]$, where $\mathcal{B}([0, t])$ is the Borel sigma-field on $[0, t]$.

where the expectations are taken with respect to Q and P , respectively.

The principles presented in this section and §2.2.4 are either special cases or bear relationship to the fundamental theorem of asset pricing (FTAP). In essence, the FTAP states that a market is arbitrage free if and only if there exists a probability measure Q under which (discounted) security prices are martingales. The probability measure Q is the equivalent martingale measure. Harrison and Kreps (1979) show that market completeness, as defined there (see also §2.3), is equivalent to the existence of a unique EMM. The EMM clearly relates to a change of numeraire. When the EMM as defined above exists, prices expressed in units of the bond (i.e., in the bond numeraire) are Q -martingales. As discussed in more detail in §3.6, prices can also be expressed in other numeraires and will retain the martingale property relative to a suitably adjusted probability measure.

Original versions of the theorem were established by Harrison and Kreps (1979), Harrison and Pliska (1981), and Kreps (1981). Versions of the theorem, as well as extensions to more-general contingent claims spaces or to more-general models with frictions can be found in a variety of papers. References include Dybvig and Ross (1987), Delbaen and Schachermayer (1994), Jacod and Shiryaev (1998), Pham and Touzi (1999), and Kabanov and Stricker (2001).

2.3. Incomplete Markets

A critical aspect of the models described in prior sections is the ability to hedge all risks with the existing menu of assets. This property is called market completeness. When the underlying source of risk consists of a d -dimensional Brownian motion, market completeness can be ensured if $d + 1$ primary securities, namely d risky assets and one locally riskless asset, are freely traded (see Duffie 1986). In §2.3 and its subsections we examine valuation in market settings where not all risks can be eliminated.

2.3.1. Effectively Complete Markets. A simple approach to valuation, when risks cannot all be hedged, is to resort to a general equilibrium analysis. Pioneered by Lucas (1978) and Cox et al. (1985) this approach rests on the notion that the economy, in the aggregate, behaves like a single, well-defined agent. Analysis of this representative agent's behavior combined with market-clearing conditions lead to equilibrium values for the interest rate, the market price of risk, and the prices of primary and derivative assets.

For exposition purposes we sketch a continuous time version of the Lucas pure exchange economy model. The economy has a finite time horizon $[0, T]$: All processes described below are understood to live

on that interval. The fundamental quantity, aggregate consumption, is assumed to follow an Itô process

$$dC_t = C_t[\mu_t^c dt + \sigma_t^c dW_t],$$

where W is a d -dimensional Brownian motion, μ^c is a progressively measurable process representing the expected consumption growth rate, and σ^c is a $1 \times d$ vector of progressively measurable processes capturing the volatility exposures of the consumption growth rate with respect to the various sources of risk.

Financial markets are composed of primary and derivative assets. Primary assets include n risky stocks and one riskless asset. The riskless asset is in zero net supply and pays interest at the rate r , a progressively measurable process. Each stock is in unit supply (one share outstanding) and pays dividends. Stock j pays D^j per unit time, where $dD_t^j = D_t^j[\mu_t^j dt + \sigma_t^j dW_t]$ with (μ^j, σ^j) progressively measurable. Assuming that dividends are the only source of consumption mandates the consistency condition $\sum_{j=1}^n D_t^j = C_t$. Derivative assets are in zero net supply and consist of k securities with payoffs $\{(f^j, Y_T^j): j = 1, \dots, k\}$, where f^j is a continuous, progressively measurable payment and Y_T^j a terminal, measurable cash flow. No specific restrictions are placed on n , d , and k . Hence $n + k < d$ is an admissible market structure.

Stock prices $\{S^j: j = 1, \dots, n\}$ and derivative prices $\{V^j: j = 1, \dots, k\}$ are conjectured to satisfy Itô processes whose evolution is described by

$$dS_t^j + D_t^j dt = S_t^j[\mu_t^{s,j} dt + \sigma_t^{s,j} dW_t], \quad j = 1, \dots, n \quad (20)$$

$$dV_t^j + f_t^j dt = V_t^j[\mu_t^{d,j} dt + \sigma_t^{d,j} dW_t],$$

$$V_T^j = Y_T^j; \quad j = 1, \dots, k \quad (21)$$

with progressively measurable coefficients $(\mu^{s,j}, \sigma^{s,j})_{j=1,\dots,n}$ and $(\mu^{d,j}, \sigma^{d,j})_{j=1,\dots,k}$. These coefficients, as well as the rate of interest, are endogenous in equilibrium. All assets, primary and derivatives, are freely traded at the relevant prices.

The economy has a representative agent who maximizes welfare by consuming and investing in the assets available. The consumption space is the space of progressively measurable, nonnegative processes. Preferences over consumption processes, denoted by $c \equiv \{c_t: t \in [0, T]\}$, are represented by the von Neumann–Morgenstern utility

$$U(c) \equiv E\left[\int_0^T u(c_t, t) dt\right], \quad (22)$$

where $u(c_t, t)$ is the utility of consumption at time t . The utility function is assumed to be strictly increasing and concave in consumption and to satisfy the limiting (Inada) conditions $\lim_{c \rightarrow 0} u'(c, t) = \infty$, $\lim_{c \rightarrow \infty} u'(c, t) = 0$, for all $t \in [0, T]$, where $u'(c, t)$ is

the derivative with respect to consumption. Investment policies are progressively measurable processes denoted by $\pi \equiv (\pi^s, \pi^d) \equiv \{(\pi_t^s, \pi_t^d): t \in [0, T]\}$ where π^s represents the vector of fractions of wealth invested in stocks and π^d the vector of fractions in derivatives. The complement $1 - (\pi^s + \pi^d)\mathbf{1}$ is the fraction in the risk-free asset ($\mathbf{1}$ is a vector of ones with suitable dimension). For a given consumption-portfolio policy (c, π) , wealth X evolves according to the dynamic budget constraint

$$dX_t = (rX_t - c_t) dt + X_t(\pi_t^s)'[(\mu_t^s - r_t)\mathbf{1} dt + \sigma_t^s dW_t] + X_t(\pi_t^d)'[(\mu_t^d - r_t)\mathbf{1} dt + \sigma_t^d dW_t], \quad (23)$$

subject to some initial condition $X_0 = x$. The representative agent maximizes (22) over policies (c, π) satisfying the budget constraint (23) and the nonnegativity constraint $X_t \geq 0$, for all $t \in [0, T]$. Policies that solve this maximization problem are said to be optimal for U at the given price processes (S, V, r) .

To close this model we need to specify a notion of equilibrium. A competitive rational expectations equilibrium is a collection of price processes $\{(S, \mu^s, \sigma^s), (V, \mu^d, \sigma^d), r\}$ and consumption-portfolio policies (c, π) such that (c, π) is optimal for U at the price processes (S, V, r) and markets clear. Market clearing mandates $c = C$ (consumption good market), $X\pi^s = S$ (stock market), $X\pi^d = 0$ (derivatives market), and $X(1 - (\pi^s + \pi^d)\mathbf{1}) = 0$ (riskless asset market).

Under suitable conditions, standard arguments can be invoked (see, for instance, Karatzas and Shreve 1998) to show that optimal consumption satisfies the first-order condition $u'(c_t, t) = y\xi_t$, where y is a constant and ξ is the relevant state price density implied by the given price processes (S, V, r) (recall that $\xi_t = b_t\eta_t$ with η as in (17)). Equilibrium in the goods market then gives the condition $u'(C_t, t) = y\xi_t$. Since $\xi_0 = 1$ it must be that $y = u'(C_0, 0)$ and therefore

$$\xi_t = \frac{u'(C_t, t)}{u'(C_0, 0)}. \quad (24)$$

The equilibrium state price density is therefore equal to the marginal rate of substitution between consumptions at time t and time 0. Moreover, the same arguments that were used in prior sections can be applied in this context to establish the price representations

$$S_t^j = E_t^* \left[\int_t^T b_{t,v} D_v^j dv \right] = E_t \left[\int_t^T \xi_{t,v} D_v^j dv \right] \quad (25)$$

$$\begin{aligned} V_t^j(f, Y) &= E_t^* \left[b_{t,T} Y_T^j + \int_t^T b_{t,v} f_v^j dv \right] \\ &= E_t \left[\xi_{t,T} Y_T^j + \int_t^T \xi_{t,v} f_v^j dv \right], \end{aligned} \quad (26)$$

where $b_{t,v} = \exp(-\int_t^v r_u du)$ (by convention $S_T^j = 0$ because the economy lives on $[0, T]$ and ceases to pay dividends at T). The equilibrium interest rate r and the market price of risk θ are obtained by applying

Itô's lemma to both sides of (24). This yields

$$\begin{aligned} r_t &= -\frac{\partial u'(C_t, t)/\partial t}{u'(C_t, t)} + R(C_t, t)\mu_t^c \\ &\quad - \frac{1}{2}R(C_t, t)P(C_t, t)\sigma_t^c(\sigma_t^c)' \end{aligned} \quad (27)$$

$$\theta_t' = R(C_t, t)\sigma_t^c, \quad (28)$$

where $R(C, t) = -u''(C, t)C/u'(C, t)$ is the relative risk-aversion coefficient and $P(C, t) = -u'''(C, t)C_t/u''(C, t)$ the relative-prudence coefficient.

The valuation Formulas (25)–(26) issued from this general equilibrium analysis have the same structure as those prevailing in the complete market settings of earlier sections. In light of this structural property, the representative agent model is said to have effectively complete markets.

Formulas (24)–(28) and their extensions to settings with discontinuous price processes (see, e.g., Naik and Lee 1990), have been used in a variety of contexts involving the valuation of primitive as well as derivative assets. The formulas are especially useful in settings with stochastic volatility, stochastic interest rates, and jump risk, where all the risks cannot be hedged away. In those instances pricing based on the representative agent paradigm provides a simple and often attractive approach to valuation.

2.3.2. Incomplete Markets: Private Values and Certainty-Equivalents. There are situations, however, in which a representative agent analysis might not be adequate. A good example is the valuation of executive stock options (ESOs). These securities are granted to executives as part of their compensation packages, to provide incentives for them to act in the best interest of shareholders. ESOs cannot be traded in organized markets and carry restrictions on the ability to trade the underlying asset (grantees, typically, cannot short the stock of the firm). In essence they represent nontraded assets that are in positive supply and cannot be fully hedged.

For those types of situations valuation becomes an individual matter. An ESO, for instance, has value for the executive holding it because it affects his or her private consumption in the future. To the extent that it does not materially impact the rest of the market, it will have little relevance for prices in general and for the policies pursued by other investors. The executive's trading strategy and consumption choices, however, could be significantly affected.

In order to value securities that are privately held one can resort to the notion of certainty-equivalent (or private value) discussed by Pratt (1964). The certainty-equivalent of a claim (f, Y) is the sure cash amount $\widehat{V}(f, Y)$ that leaves the individual indifferent between holding the claim to maturity or receiving the said cash amount.

Before proceeding with the continuous time market setting, it is useful to visualize the notion in a simple one-period context. Accordingly, suppose that the investor cares about terminal wealth, which consists exclusively of the risky payoff \tilde{Y} . By definition the certainty-equivalent of the risky cash flow \tilde{Y} is the sure amount $\widehat{V}(Y)$ such that

$$u(\widehat{V}(Y)) = E[u(\tilde{Y})].$$

Equivalently, $\widehat{V}(Y) = u^{-1}(E[u(\tilde{Y})])$. Simple properties of this certainty-equivalent can be readily established. Under the standard assumption of a concave and increasing utility function, $\widehat{V}(Y)$ is increasing in the expected cash flow and decreasing in the riskiness of the cash flow.

In a continuous time setting, the investor holding the nontraded asset consumes and invests in traded securities. Assume the Itô market of §2.2.5 and suppose that the investor derives utility from consumption over time as well as from terminal wealth. Also suppose that the nontraded asset produces cash flows (f, Y) adapted to the filtration generated by (W, Z) , where Z is a Brownian motion orthogonal to W .

The individual's optimization problem becomes

$$\max_{c, \pi} E \left[\int_0^T u(c_t, t) dt + B(X_T) \right]$$

subject to a dynamic budget constraint

$$dX_t = (rX_t + f_t - c_t) dt + X_t \pi_t [(\mu_t - r_t) dt + \sigma_t dW_t], \\ t \in [0, T]; \quad X_0 = x, \quad X_T = X_{T-} + Y_T,$$

where $u(\cdot, t)$ is utility of consumption at time t and $B(\cdot)$ is the utility of terminal wealth (bequest function). As usual, this optimization is subject to the liquidity constraint $X \geq 0$. This constraint mandates a nonnegative portfolio value, i.e., the individual cannot borrow against the future cash-flows generated by the nontraded asset.⁹ Let $J(x, f, Y)$ be the value function associated with this optimization problem.

Alternatively, suppose that (f, Y) is exchanged, at date zero, against a sure cash amount $\widehat{V}(f, Y)$. The new optimization problem faced by the agent has the same structure as the one just described, but with $(f, Y) = (0, 0)$ and $X_0 = x + \widehat{V}(f, Y)$. The associated value function is $J(x + \widehat{V}(f, Y), 0, 0)$.

The certainty-equivalent $\widehat{V}(f, Y)$ is now easily defined. It is the cash amount that leaves the individual indifferent between holding on to the claim or giving it up, i.e., the solution of the problem

$J(x + \widehat{V}(f, Y), 0, 0) = J(x, f, Y)$. If the value function $J(\cdot, 0, 0)$ is invertible the solution is

$$\widehat{V}(f, Y) = J^{-1}(J(x, f, Y), 0, 0) - x.$$

The certainty-equivalent value is a private value to the extent that it depends on the preferences of the individual holding the asset and exposed to the nontraded risks. It represents the ask price for this particular individual.¹⁰

The practical computation of the certainty equivalent is arduous due to the difficulty in resolving the optimization problem with the nontraded asset (a constrained optimization problem). Problems of this nature are analyzed by Duffie et al. (1997), Cuoco (1997), Cvitanic et al. (2001), Karatzas and Zitkovic (2003), and Hugonnier and Kramkov (2004), in continuous time settings. Implementation of the certainty-equivalent method is easier in some discrete time settings. For results in the binomial framework and an application to ESO valuation, see Detemple and Sundaresan (1999).

Several other approaches have been proposed to deal with incomplete markets and unspanned risks. A popular notion, which appears in several contexts, is to use the minimal martingale measure ξ^o for pricing purposes. In essence, this measure assigns null market price to risks that cannot be hedged with the menu of traded assets. Derivative prices satisfy the usual present value Formula (19), substituting ξ^o for ξ . The method is consistent with the dynamic consumption-portfolio choice problem of a myopic individual, with logarithmic utility function. The resulting price can be interpreted as a price at the margin calculated using the marginal rate of substitution of that particular individual. Originally proposed by Föllmer and Sondermann (1986) and Föllmer and Schweizer (1991), the method has since been generalized or applied in a number of papers (e.g., Schweizer 1992, 1995a). Other pricing measures, such as the minimal variance measure (Schweizer 1995b, Delbaen and Schachermayer 1996) and the minimal entropy measure (Frittelli 2000) have also been studied.

3. American Options

American-style derivatives can be exercised at any point in time before maturity. As a result, part of the valuation problem consists of identifying the optimal exercise policy, i.e., the exercise time that maximizes value for the holder of the security.

⁹ One could weaken this restriction by allowing for borrowings against the hedgeable components of these cash flows.

¹⁰ This definition assumes that the claim is held to maturity. Variations of the concept could allow for liquidation at a selected set of dates and for divisibility of the claim in a finite number of lots.

This section reviews the various approaches to value American-style contracts. We survey the free-boundary approach (§3.1), the variational inequalities method (§3.2), and the risk-neutralization procedure (§3.3). Useful representation formulas for the values of those derivatives are reviewed last (§3.4).

3.1. The Free-Boundary Approach

The free-boundary method goes back to Samuelson (1965), McKean (1965), Taylor (1967), and Merton (1973). It applies to Markovian settings such as the Black-Scholes framework of §2.1. An important element is the observation that the arguments of §2.1.2 apply, even when the claim under consideration is American style. The fundamental valuation equation therefore characterizes its price, provided the contract is alive. In the complementary event, where the claim is exercised, its value must equal the exercise payoff.

Let $V(S, t)$ be the price of an American derivative with payoff $g(S)$ at exercise. Because exercise is at the option of the holder we know that it is suboptimal to exercise if $g(S) < 0$. Attention can therefore be restricted to securities with nonnegative payoffs, $g(S) \geq 0$. Assume that the payoff function $g(\cdot)$ is continuous, continuously differentiable almost everywhere, and twice continuously differentiable on $\{S: g(S) > 0\}$.¹¹ The continuation region, $\mathcal{C} \equiv \{(S, t) \in \mathbb{R}_+ \times [0, T]: V(S, t) > g(S)\}$, is the set of point (S, t) at which the derivative is worth more alive. Its complement is the exercise region, $\mathcal{E} \equiv \{(S, t) \in \mathbb{R}_+ \times [0, T]: V(S, t) = g(S)\}$. Under the weak assumption that the derivative’s price is continuous with respect to its arguments, the continuation region is an open set and the exercise region a closed set. The boundary of \mathcal{E} , denoted by B , is then part of the exercise region (i.e., $B \in \mathcal{E}$) and serves to distinguish points where exercise is optimal from those where continuation is best. This set is called the immediate exercise boundary. Its t -section, $B(t) \equiv \{S \in \mathbb{R}_+: (S, t) \in B\}$, is the set of boundary points at time t . Note that $B(t)$, for general payoff function $g(\cdot)$, might not be single valued: The exercise region can, in principle, have lower and upper boundaries, or can even be the union of disconnected intervals. The immediate exercise boundary is the collection of its t -sections: $B = \{B(t): t \in [0, T]\}$. The identification of B is key to the resolution of the valuation problem. Because B is not known a priori it is often referred to as a free boundary. The American claim valuation problem is a free-boundary problem.

When the option is alive, in \mathcal{C} , it makes intuitive sense to assume that the price is smooth, precisely that $V(S, t) \in \mathcal{C}^{2,1}$. The same assumption was invoked for European claims. Under this assumption

Itô’s lemma applies and the arguments leading to (2) can be invoked. The derivative’s price satisfies (2) in \mathcal{C} .

Characterizing the immediate exercise boundary is more difficult. Of course, for $(S, t) \in B$ immediate exercise is optimal, so $V(B(t), t) = g(B(t))$. This condition, however, is not enough to fully determine B . Indeed, one could specify any arbitrary function $B(t)$ and simply solve the valuation partial differential equation (PDE) subject to that boundary condition. This would only produce different functions V corresponding to different contractual specifications (i.e., exercise policies). To solve the pricing problem under consideration one must therefore add a condition reflecting the optimality of exercise along B .

No-arbitrage arguments can be used for that purpose. Suppose that the derivative of the option price, at $B(t)$, does not coincide with that of the payoff, i.e., $\partial V(B(t), t)/\partial S \neq g'(B(t))$. If so, one can show that an arbitrage opportunity exists. Indeed, consider a sequence $S^n \in \mathcal{C}$ that converges to $B(t)$ and such that $\lim_{n \rightarrow \infty} |\partial V(S^n, t)/\partial S| > |g'(B(t))|$. Then, for n large enough (i.e., $S^n \in \mathcal{C}$ and close to $B(t)$), it must be that $V(S^n, t) < g(S)$, in contradiction with the definition of \mathcal{C} (equivalently, an arbitrage opportunity). The existence of an arbitrage opportunity can also be established if $|g'(B(t))| > \lim_{n \rightarrow \infty} |\partial V(S^n, t)/\partial S|$. An informal argument is provided by Wilmott et al. (1993).¹² The combination of these two results gives the condition $\partial V(S, t)/\partial S = g'(S)$ at the point $S = B(t)$. This condition, mandating that the price and payoff derivatives match along the immediate exercise boundary, is known as the high contact or smooth pasting condition (see, e.g., Ingersoll 1987, p. 374 or Dixit and Pindyck 1994, p. 130).

In summary, the price of the derivative solves the fundamental valuation Equation (2) subject to the boundary conditions

$$\begin{cases} V(S, T) = g(S) & \text{on } \mathbb{R}_+ \\ V(0, t) = g(0) & \text{for } t \in [0, T] \\ V(S, t) = g(S) & \text{for } (S, t) = (B(t), t) \\ \frac{\partial V(S, t)}{\partial S} = g'(S) & \text{for } (S, t) = (B(t), t). \end{cases} \quad (29)$$

The system (2), (29) is the free-boundary characterization of the pricing function.

3.2. Variational Inequalities

Free-boundary problems are often difficult to solve because of the need to identify the unknown boundary in the resolution process. An alternative is to

¹¹ These are weak assumptions, satisfied by standard option contracts such as puts and calls.

¹² Another informal proof, based on the notion that $B(t)$ is optimal, is found in Merton (1992).

recast the problem in terms of a variational inequality. This transformation is attractive because the inequality does not explicitly depend on the boundary. The pricing function can be obtained directly from this characterization. The boundary can then be identified from this solution.

To understand the variational inequality formulation it might be useful to first express the pricing problem in linear complementarity form. Recall that the pricing function can be recovered from the solution, denoted by u , of the heat Equation (4) and its corresponding boundary condition (5). To simplify notation, write

$$\mathcal{A}u(x, \tau) \equiv \frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2},$$

and note that \mathcal{A} is the Itô differential operator associated with a time-reversed Brownian motion $x \equiv \{x_\tau: \tau \in [0, \frac{1}{2}\sigma^2 T]\}$: It represents the drift of the function $u(x, \tau)$ as time moves backward (i.e., as the rescaled time-to-maturity measure $\tau = \frac{1}{2}\sigma^2(T - t)$ increases). In the continuation region where $u(x, \tau) > e^{\alpha x + \beta \tau} g(K \exp(x)) \equiv \hat{g}(x, \tau)$ the price satisfies $\mathcal{A}u(x, \tau) = 0$. In the exercise region $u(x, \tau) = \hat{g}(x, \tau)$, and therefore

$$\mathcal{A}u(x, \tau) = \mathcal{A}\hat{g}(x, \tau).$$

Moreover, because $\mathcal{A}\hat{g}(x, \tau)$ represents the local gains from exercising, it cannot be negative when exercise is optimal: This gives $\mathcal{A}u(x, \tau) = \mathcal{A}\hat{g}(x, \tau) \geq 0$. In summary,

$$\mathcal{A}u(x, \tau)(u(x, \tau) - \hat{g}(x, \tau)) = 0 \quad (30)$$

$$u(x, \tau) - \hat{g}(x, \tau) \geq 0 \quad \text{and} \quad \mathcal{A}u(x, \tau) \geq 0. \quad (31)$$

The problem of finding a continuous and almost everywhere differentiable function $u(x, \tau)$ that satisfies (30)–(31) subject to the boundary conditions

$$\begin{cases} u(x, 0) = e^{\alpha x} g(K \exp(x)) & \text{for } x \in \mathbb{R} \\ \lim_{x \rightarrow -\infty} e^{-\alpha x - \beta \tau} u(x, \tau) = g(0) & \text{for } \tau \in [0, \frac{1}{2}\sigma^2 T] \\ \lim_{x \rightarrow \infty} e^{-\alpha x - \beta \tau} u(x, \tau) = g(\infty) & \text{for } \tau \in [0, \frac{1}{2}\sigma^2 T], \end{cases} \quad (32)$$

is called the linear complementarity problem (LCP). Note that the unknown exercise boundary corresponding to the problem expressed in terms of x does not appear in (30)–(32). The LCP can be interpreted as an obstacle problem where a function, whose unconstrained evolution is governed by the fundamental valuation equation, must exceed an obstacle given by the exercise payoff and meet conditions at the boundaries of its domain.

The variational inequality formulation is obtained by integrating and relaxing the previous conditions (30)–(31). Specifically, let \mathcal{V} be a set of test functions consisting of functions $v(x, \tau)$ that are continuous, continuously differentiable in τ , differentiable almost everywhere in x , and that satisfy the boundary conditions (32) and the inequality constraint $v(x, \tau) \geq \hat{g}(x, \tau)$ for all $(x, \tau) \in \mathbb{R} \times [0, \frac{1}{2}\sigma^2 T]$. Note that the solution of our problem, the function u , belongs to \mathcal{V} . Moreover, for any $v \in \mathcal{V}$ we have $v(x, \tau) \geq \hat{g}(x, \tau)$ and therefore $\mathcal{A}u(x, \tau)(v(x, \tau) - \hat{g}(x, \tau)) \geq 0$, with equality if $v(x, \tau) = \hat{g}(x, \tau)$. Integrating over x yields¹³

$$\int_{-\infty}^{\infty} \mathcal{A}u(x, \tau)(v(x, \tau) - \hat{g}(x, \tau)) dx \geq 0 \quad (33)$$

$$\int_{-\infty}^{\infty} \mathcal{A}u(x, \tau)(u(x, \tau) - \hat{g}(x, \tau)) dx = 0. \quad (34)$$

Taking the difference between (33) and (34) eliminates the function $\hat{g}(x, \tau)$ and gives

$$\int_{-\infty}^{\infty} \mathcal{A}u(x, \tau)(v(x, \tau) - u(x, \tau)) dx \geq 0.$$

Integration by parts now enables us to get rid of second derivatives¹⁴

$$\begin{aligned} 0 &\leq \int_{-\infty}^{\infty} \frac{\partial u}{\partial \tau} (v(x, \tau) - u(x, \tau)) dx \\ &\quad - \frac{\partial u}{\partial x} (v(x, \tau) - u(x, \tau)) \Big|_{-\infty}^{\infty} \\ &\quad + \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} (v'(x, \tau) - u'(x, \tau)) dx. \end{aligned}$$

$v, u \in \mathcal{V}$, thus they satisfy the same boundary conditions at $x = \pm\infty$ and therefore, after collecting terms,

$$\begin{aligned} 0 &\leq \int_{-\infty}^{\infty} \frac{\partial u}{\partial \tau} (v(x, \tau) - u(x, \tau)) dx \\ &\quad + \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} (v'(x, \tau) - u'(x, \tau)) dx. \end{aligned} \quad (35)$$

The variational inequality problem is to find $u \in \mathcal{V}$ such that (35) holds for all $t \in [0, T]$, for all test functions $v \in \mathcal{V}$. One could also go an extra step by integrating over time, to produce a global variational inequality.

The variational inequality approach for pricing American claims is developed in Jaillet et al. (1990). Mathematical treatises on the method can be found in Bensoussan and Lions (1978) and Kinderlehrer and Stampacchia (1980).

¹³ Note (30)–(31) holds if and only if (33)–(34) holds for $u \in \mathcal{V}$ and for all $v \in \mathcal{V}$.

¹⁴ A consequence is that the existence of a second derivative is not needed. The variational inequality approach applies even if the payoff is only required to be differentiable almost everywhere.

3.3. The Risk-Neutralization Approach

The risk-neutralization principle described in §2.2 also applies to American claims. To develop this point let us adopt the general framework with Itô processes of §2.2.5 and suppose that the claim under consideration has an exercise payoff given by a progressively measurable process Y . In this context, let $\mathcal{S}([t, T])$ be the set of stopping times of the filtration with values in $[t, T]$ and define the process

$$Z_t \equiv \sup_{\tau \in \mathcal{S}([t, T])} E_t^*[b_\tau Y_\tau], \tag{36}$$

where we recall that $b_\tau \equiv \exp(-\int_0^\tau r_v dv)$ is the discount factor at the locally risk-free rate and $E_t^*[\cdot]$ is the conditional expectation at date t , under the risk-neutral measure Q defined earlier. The process Z is known as the Snell envelope of the discounted payoff $b_\tau Y_\tau$. It is the result of the maximization described on the right-hand side of (36). The solution of this maximization is a stopping time, indexed by time, denoted by $\tau(t)$. The Snell envelope is the value function of this maximization.

The Snell envelope has several attractive properties (see El Karoui 1981 and Karatzas and Shreve 1998 for details). For one, it is a Q -supermartingale because¹⁵

$$\begin{aligned} E_t^*[Z_s] &= E_t^*\left[\sup_{\tau \in \mathcal{S}([s, T])} E_s^*[b_\tau Y_\tau]\right] \\ &= E_t^*[b_{\tau(s)} Y_{\tau(s)}] \leq \sup_{\tau \in \mathcal{S}([t, T])} E_t^*[b_\tau Y_\tau] = Z_t, \end{aligned}$$

where the first equality on the second line follows from the definition of $\tau(s)$, i.e., the solution of (36) at time s , and the law of iterated expectations, and the inequality on the same line from the inclusion $\mathcal{S}([s, T]) \subset \mathcal{S}([t, T])$, for $t \leq s$. The last equality is simply the definition of Z .

It also has paths that are right continuous with left limits RCLL and is nonnegative because exercise is at the option of the holder. Immediate exercise at t is always feasible, thus it must also be that $Z_t \geq b_t Y_t$: The Snell envelope majorizes the discounted payoff. In fact, it is the smallest supermartingale majorant of the discounted payoff.

To see the relation with our pricing problem it suffices to invoke a no-arbitrage argument. Let $\tau(0)$ be the solution at $t = 0$. An application of the martingale representation theorem (assuming $\xi_{\tau(0)} Y_{\tau(0)}$ is almost surely finite) gives

$$E_t[\xi_{\tau(0)} Y_{\tau(0)}] = E[\xi_{\tau(0)} Y_{\tau(0)}] + \int_0^t \phi_v dW_v$$

¹⁵ A process $Z \equiv \{Z_t: t \in [0, T]\}$ is a supermartingale if $E_t[Z_s] \leq Z_t$, P -a.s., for all $s \geq t$ and $t \in [0, T]$. It is a Q -supermartingale if the expectation is taken with respect to the measure Q .

for all $t \in [0, \tau(0)]$ and for some progressively measurable process ϕ . The arguments in §2.2.4 can then be used to show that the portfolio $X\pi \equiv \{X_t \pi_t = X_t \sigma^{-1} \theta + \xi_t^{-1} \sigma^{-1} \phi_t: t \in [0, \tau(0)]\}$ has value $X_t = E_t[\xi_{t, \tau(0)} Y_{\tau(0)}]$. $X_{\tau(0)} = Y_{\tau(0)}$, therefore this policy replicates the claim's payoff at the exercise time $\tau(0)$. Its initial cost is $X_0 = E[\xi_{\tau(0)} Y_{\tau(0)}]$. A claim paying off $Y_{\tau(0)}$ at $\tau(0)$ must therefore be worth

$$V_0 = E[\xi_{\tau(0)} Y_{\tau(0)}]$$

at inception, to preclude arbitrage opportunities. But by definition $\tau(0)$ maximizes $E[\xi_\tau Y_\tau]$ (recall that $Z_0 \equiv \sup_{\tau \in \mathcal{S}([0, T])} E^*[b_\tau Y_\tau] = \sup_{\tau \in \mathcal{S}([0, T])} E[\xi_\tau Y_\tau]$). It can therefore not be improved upon: $\tau(0)$ represents the optimal exercise policy for the American claim issued at $t = 0$ and V_0 is its market (no-arbitrage) value. Similarly, $\tau(t)$ can be interpreted as the optimal exercise time for the American claim issued at time t ; its market value is $V_t = E_t[\xi_{t, \tau(t)} Y_{\tau(t)}]$, which can also be expressed as $V_t = E_t^*[b_{t, \tau(t)} Y_{\tau(t)}]$ under the risk-neutral measure.

These developments show that valuation principles for American claims are fully consistent with those established for European-style contracts. The value of a claim is simply the present value of its exercise payoff, where present value is calculated by discounting cash flows at the locally risk-free rate and taking expectations under the risk-neutral measure. In this calculation the exercise payoff is at the optimal exercise time, i.e., the stopping time that maximizes value. Alternatively, the price can be written as the expected deflated value of the exercise payoff where the expectation is under the original measure and the (stochastic) deflator is the state price density.

The theory of optimal stopping has a long history. The Markovian case is considered by several authors, including Fakeev (1971), Bismut and Skalli (1977), and Shiryaev (1978). An extensive treatment for general stochastic processes can be found in El Karoui (1981). Karatzas and Shreve (1998, Appendix D) contains detailed results for continuous time processes. The connection between the Snell envelope and no-arbitrage pricing is drawn by Bensoussan (1984) and Karatzas (1988).

3.4. Exercise Premium Representations

The arguments outlined in §3.3 provide a theoretical resolution of the valuation problem for American claims. Several transformations of the pricing formula, however, prove useful for interpretation purposes and practical implementation.

The first transformation leads to a formula known as the early exercise premium (EEP) representation of the claim's price. The key to this formula is the behavior of the Snell envelope Z . In the event $\{\tau(v) = v\}$,

where stopping is optimal, the Snell envelope is the discounted payoff $Z_{\tau(v)} = b_{\tau(v)} Y_{\tau(v)}$. Therefore, by Itô's lemma,

$$dZ_v = b_v(dY_v - r_v Y_v dv), \quad \text{for } \{\tau(v) = v\}.$$

On the complementary event $\{\tau(v) > v\}$, the formula $Z_v \equiv E_v^*[b_{\tau(v)} Y_{\tau(v)}]$ shows that Z is a Q -martingale, i.e., $dZ_v = \phi_v^* dW_v^*$ for some progressively measurable process ϕ^* . Combining these two elements shows that

$$\begin{aligned} Z_T &= Z_0 + \int_0^T dZ_v \\ &= Z_0 + \int_0^T 1_{\{\tau(v)=v\}} b_v(dY_v - r_v Y_v dv) \\ &\quad + \int_0^T 1_{\{\tau(v)>v\}} \phi_v^* dW_v^*. \end{aligned}$$

Taking expectations (under Q) and rearranging them yields

$$Z_0 = E^*[Z_T] + E^*\left[\int_0^T 1_{\{\tau(v)=v\}} b_v(r_v Y_v dv - dY_v)\right],$$

where $Z_T \equiv b_T Y_T$ is the discounted terminal payoff. Now note that the left-hand side of this expression is the initial price V_0 of the American claim. The first term on the right-hand side is the initial price of its European counterpart. The second term on the right is called the EEP. It captures the present value of the gains associated with exercise prior to the maturity date. These gains are interest-rate gains, realized by investing the exercise payoff at the risk-free rate, net of the opportunity cost associated with the forgone appreciation of the payoff.

The same argument yields, for any time $t \in [0, T]$,

$$V_t = E_t^*[b_{t,T} Y_T] + E_t^*\left[\int_t^T 1_{\{\tau(v)=v\}} b_{t,v}(r_v Y_v dv - dY_v)\right]. \quad (37)$$

The price of a newly issued American claim (or an unexercised claim, if $t < \tau(0)$) is the same as the European claim value plus an EEP given by the present value of the gains to be realized by optimally exercising prior to maturity.¹⁶

The second transformation, called the *delayed exercise premium (DEP) representation*, emphasizes the gains realized by deferring exercise. This formula follows almost immediately from the pricing formula for the claim. Indeed,

$$\begin{aligned} V_t &= E_t^*[b_{t,\tau(t)} Y_{\tau(t)}] = Y_t + E_t^*\left[\int_t^{\tau(t)} d(b_{t,v} Y_v)\right] \\ &= Y_t + E_t^*\left[\int_t^{\tau(t)} b_{t,v}(dY_v - r_v Y_v dv)\right]. \end{aligned}$$

¹⁶ Note that only the drift of dY under the risk-neutral measure matters in (37). Martingale components vanish because of the expectation in front.

The first term is the immediate exercise payoff. The second is the present value of the gains from delaying exercise, the negative of the gains from exercising (see the structure of the EEP above). The second component can also be interpreted as the time value of the American claim; the immediate exercise payoff is the intrinsic value.

The EEP representation corresponds to the Riesz decomposition of the Snell envelope. This decomposition states that any supermartingale is the sum of a martingale and a potential.¹⁷ The EEP representation essentially identifies the structure of the processes in this decomposition in terms of the underlying parameters of the model and the payoff process. The Riesz decomposition of the Snell envelope was established by El Karoui and Karatzas (1991) for a setting with Brownian motion. The EEP formula for American options, in the Black-Scholes framework with constant coefficients, is due to Kim (1990), Jacka (1991), and Carr et al. (1992). An extension to payoffs adapted to a general filtration is given in Rutkowski (1994) and to jump-diffusion processes in Gukhal (2001). The DEP formula above extends the corresponding formula in Carr et al. (1992) derived for the Black-Scholes framework.

3.5. Integral Equation for American Options

For American call options immediate exercise will only occur if $S > K$. $Y_v = S_v - K$, thus the local gains from exercise, in the exercise region, equal $r_v Y_v dv - E_v^*(dY_v) = (\delta_v S_v - r_v K) dv$ and the EEP formula for the call price is given by

$$\begin{aligned} C_t &= E_t^*[b_{t,T}(S_T - K)^+] \\ &\quad + E_t^*\left[\int_t^T 1_{\{\tau(v)=v\}} b_{t,v}(\delta_v S_v - r_v K) dv\right]. \end{aligned}$$

The net local benefits from exercise are now seen to consist of the dividends paid on the underlying asset net of the interest forgone by paying the strike.

In the Black-Scholes setting, with constant coefficients, the formula can be further refined. First, it is easy to verify that immediate exercise takes place, at date v , provided $S_v > B(v)$ where $B(v)$ is the value at v of a function of time $B(\cdot)$. Expectations can then be computed more explicitly to give

$$\begin{aligned} C(S_t, t; B(\cdot), K) &= c(S_t, t; K) + \int_t^T \delta S_t e^{-\delta(v-t)} N(d(S_t; B(v), v-t)) dv \\ &\quad - \int_t^T r K e^{-r(v-t)} N(d(S_t; B(v), v-t) - \sigma\sqrt{v-t}) dv, \end{aligned} \quad (38)$$

¹⁷ A process x is a *potential* if it is a right-continuous nonnegative supermartingale such that $\lim_{t \rightarrow \infty} E x_t = 0$.

where $c(S_t, t; K)$ is the Black-Scholes Formula (6) and $d(S_t; B, v - t)$ is defined in (7).

In order to implement (38) one still needs to identify the unknown exercise boundary B . A few additional steps provide the characterization needed. Indeed, recall that immediate exercise is optimal when $S = B$. It follows that

$$\begin{aligned}
 B(t) - K &= c(B(t), t; K) \\
 &+ \int_t^T \delta B(t) e^{-\delta(v-t)} N(d(B(t); B(v), v - t)) dv \\
 &- \int_t^T r K e^{-r(v-t)} N(d(B(t); B(v), v - t) \\
 &\quad - \sigma \sqrt{v - t}) dv. \tag{39}
 \end{aligned}$$

Moreover, because $t \rightarrow T$, it can be verified that $\lim_{t \rightarrow T} B(t) = \max(K, (r/\delta)K)$. Equation (39) is the integral equation for the exercise boundary. Solving the integral equation subject to the boundary condition indicated identifies the optimal exercise boundary. Substituting in (38) provides the American call option price.

This characterization of the exercise boundary and the resulting two-step valuation procedure have come to be known as the integral equation method. This approach to American option pricing can be traced to Kim (1990), Jacka (1991), and Carr et al. (1992). Variations of the method have been proposed in subsequent work. Little et al. (2000), in particular, point to an interesting reduction in dimensionality leading to an equation for the boundary with a single integral.

3.6. Put-Call Symmetry

The valuation of put options is a simple matter once call option prices are known. Indeed, a straightforward change of variables shows that a put is identical to a call in a suitably modified financial market. This result is known as put-call symmetry.

For exposition purposes we adopt the Itô financial market of §2.2.5 assuming a single asset and Brownian motion ($d = 1$) and focus on American-style claims (the symmetry relation for European claims is a special case). In this context the price of a put is given by $P_t = E_t^*[b_{t,\tau}(K - S_\tau)^+]$ where τ stands for the optimal exercise time. Simple transformations yield

$$\begin{aligned}
 P_t &= E_t^*[b_{t,\tau} S_{t,\tau} (K/S_{t,\tau} - S_t)^+] \\
 &= \widehat{E}_t \left[\exp \left(- \int_t^\tau \delta_v dv \right) (\widehat{S}_\tau - S_t)^+ \right], \tag{40}
 \end{aligned}$$

where the first equality uses $S_{t,\tau} \equiv S_\tau/S_t$ and the second one a change of measure to \widehat{Q} such that $d\widehat{Q} = \exp(-\frac{1}{2} \int_0^T \sigma_v^2 dv + \int_0^T \sigma_v dW_v^*) dQ$. The expectation $\widehat{E}_t[\cdot]$ is under \widehat{Q} and $\widehat{S}_\tau \equiv K/S_{t,\tau}$ is a new price whose evolution (under \widehat{Q}) is described by

$$d\widehat{S}_v = \widehat{S}_v [(\delta_v - r_v) dv + \sigma_v d\widehat{W}_v], \quad \widehat{S}_t = K, \tag{41}$$

where $d\widehat{W}_v \equiv -dW_v^* + \sigma_v dv$ is a \widehat{Q} -Brownian motion. Passage to the measure \widehat{Q} is akin to using the stock as a new numeraire. The right-hand side of (40) is the value of a call, exercised at τ , with strike S_t , written on an asset with price \widehat{S} and paying dividends at the rate r , and in a financial market with interest rate δ .

Although derived for the optimal put exercise time τ , the relation (40) holds for any stopping time of the filtration. It follows that optimal stopping times must be identical across the two markets, i.e., the optimal exercise time of the American put in the original market is the same as the optimal exercise time of the American call option, identified by the formula, in the modified financial market. The call option in the modified market is therefore symmetric to the put in the original market.

Early versions of the symmetry relation, in restricted settings, can be found in Grabbe (1983), Bjerksund and Stensland (1993), Chesney and Gibson (1995), McDonald and Schroder (1998). For general markets with semimartingale prices the symmetry relation is established by Schroder (1999); see also Kholodnyi and Price (1998), for an approach using group theory. This general result relies on a change of numeraire method introduced in Jamshidian (1989) and Geman et al. (1995). A review of these results and some extensions appear in Detemple (2001).

4. Beyond the BSM Model

The BSM Equation (6) is one of the most successful and widely used in financial economics. The underlying Model (1) and its related assumptions are simple and elegant. But as a tool for real-world use the model must be compared to financial data. Deviations between the model and empirical evidence offer opportunities for the development of more-realistic models, which in turn create new computational challenges for the pricing and hedging of derivative securities. Section 4 and its subsections present brief empirical results (§4.1), as well as extensions of the BSM model to include jumps in returns (§4.2) and stochastic volatility (§4.3).

4.1. Empirical Evidence

The BSM model can be tested using data from the underlying asset (or underlying variable) or options data. The first approach checks for consistency between the data and the assumed geometric Brownian motion process. The second approach examines the data for consistency between observed option prices and those implied by the model, and therefore jointly tests the model for the asset price process and additional assumptions required for deriving pricing equations (e.g., frictionless trading).

The BSM model in Equation (1) is an exactly solvable SDE with solution

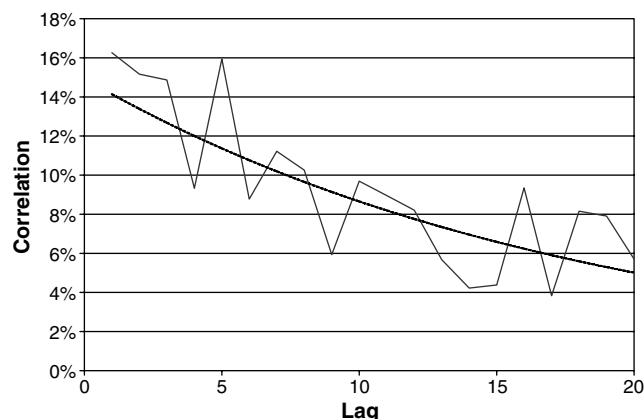
$$S_{t+h} = S_t e^{(\mu - \delta - \sigma^2/2)h + \sigma \sqrt{h}Z}, \quad (42)$$

where Z is a standard normal random variable. Equation (42) implies that log-returns $\ln(S_{t+h}/S_t)$ are normally distributed.

A quick examination of financial time series for almost any underlying asset (equities, commodities, bonds) indicates violations of the normality assumption for log-returns. To give an example, on October 19, 1987, the S&P 500 index closed at 224.84, which represented a one-day log-return of -22.9% from the previous business day's close of 282.70. Using reasonable values for the parameters μ , σ , and δ , the probability of a market drop of this magnitude or smaller is approximately 10^{-97} under the normality assumption of the BSM model. The expected time to observe an event as extreme as this is approximately 10^{84} billion years. Repeating the calculation for the less extreme one-day log return of -6.3% on October 13, 1989, gives a probability of approximately 10^{-9} . The expected time to observe an event with this probability is approximately 1 million years. The data are clearly not consistent with the normality assumption of log-returns. Statistical tests that are more formal show that the kurtosis (fourth moment) of log-returns is significantly larger than implied by normality, i.e., financial data have fatter tails than does the normal distribution. Extremely large returns that occur more frequently in financial data is one motivation for the addition of a jump process to the BSM diffusion model for asset prices.

The BSM model assumes that price changes over nonoverlapping time intervals are independent. The quantity $\rho \equiv \rho((\ln(S_{t+h}/S_t))^2, (\ln(S_{t+u+h}/S_{t+u}))^2)$ represents the autocorrelation of lagged squared log-returns. The independent increments assumption in the BSM model implies that ρ should be zero. Choosing the return interval h and time lag u both equal to one business day, and computing ρ using closing prices of the S&P 500 index from January 1997 through December 2002 gives an estimated autocorrelation value of $\hat{\rho} = 16.5\%$. If the assumptions of the BSM model hold, the standard error of the estimate is 2.5. In other words, the observed autocorrelation estimate cannot be explained by statistical error because the estimate is more than six standard errors from zero. Figure 1 shows that autocorrelations are also statistically significant at much longer time lags. Squared log-returns are a measure of volatility, thus the positive autocorrelation value indicates that high-volatility days (i.e., large positive or negative returns) are more likely to be followed by high-volatility days than by low-volatility days. This phenomenon of

Figure 1 Autocorrelation of S&P 500 Squared Log>Returns



Note. Raw results are given by the jagged line; the other line shows a fitted smooth curve.

volatility clustering is one motivation for the addition of a stochastic volatility process to the BSM model.

Although a diffusion process for volatility can generate autocorrelations in returns that are consistent with the data, such a process cannot easily generate the extreme returns (i.e., jumps) found in the data. These two departures from the BSM model are in a sense complementary.

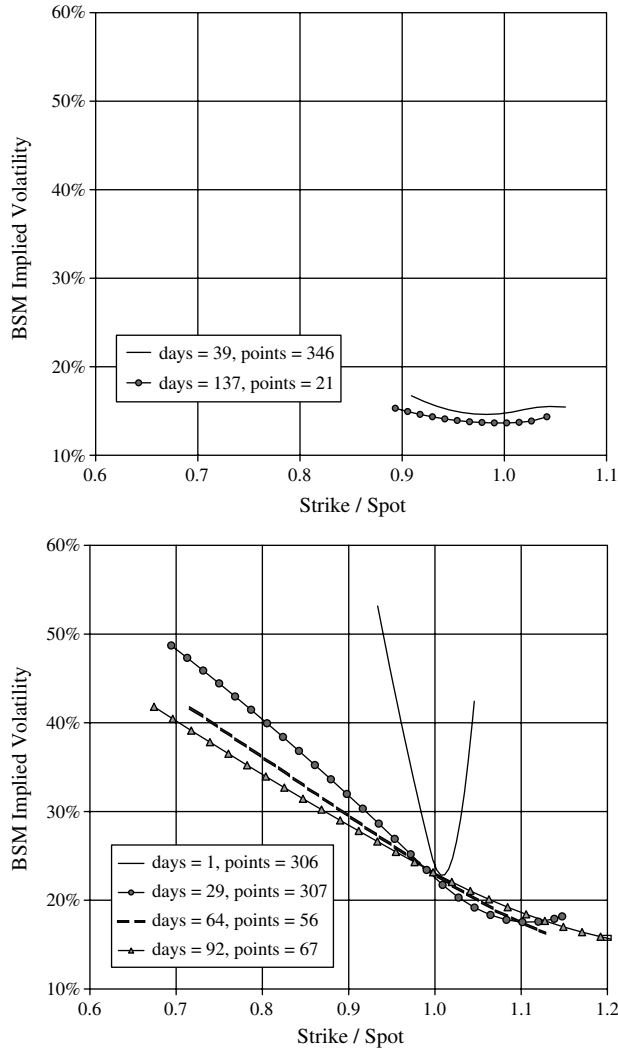
It is useful to measure option prices in units of volatility rather than dollars. The BSM implied volatility (or simply the implied volatility) of an option is the volatility parameter that equates the BSM model price with the market price. Suppose a European call option with a strike K and maturity T is traded in the market at the dollar price \hat{c} . Define the option's implied volatility $\hat{\sigma}$ to be the solution of the equation

$$\hat{c} = c(S_t, t; K, T, \hat{\sigma}), \quad (43)$$

where $c(S_t, t; K, T, \hat{\sigma})$ is computed using the BSM Formula (6). Because the right-hand side of (6) is a monotonically increasing function of $\hat{\sigma}$, a unique solution exists as long as the market price does not violate the no-arbitrage condition. Different options will have different implied volatilities, so $\hat{\sigma}$ can be viewed as a function of the option parameters, i.e., $\hat{\sigma} = \hat{\sigma}(S_t, t; K, T)$. Quoting an option price in terms of implied volatility is analogous to quoting a bond price in terms of yield to maturity. For a bond the price-yield equation allows a dollar bond price to be converted to a yield; for an option the BSM equation allows a dollar option price to be converted to an implied volatility.

Whether or not the BSM model is correct, option prices can be, and often are, quoted in units of implied volatility. In the BSM model, the constant volatility σ is a property of the underlying asset, and does not

Figure 2 BSM-Implied Volatility Curves, February 4, 1985 (Top Panel) and September 16, 1999 (Bottom Panel)

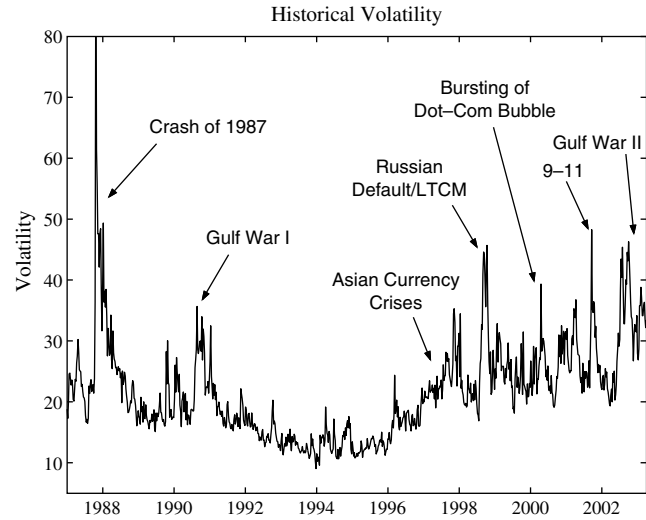


Note. Days indicate the number of days to maturity, and points indicate the number of trades used to create the curves.

depend on an option's strike or maturity. So if the BSM model is correct, then options of all strikes and maturities traded on the same underlying asset will have identical implied volatilities. In practice, option-implied volatilities depend in a systematic way on K and T and vary through time t .

Figure 2 shows BSM-implied volatilities for options on the S&P 500 (futures) on February 4, 1985, and on September 16, 1999. These options began trading on the Chicago Mercantile Exchange (CME) in 1985. For the two maturities shown on February 4, 1985, options traded in the narrow implied volatility range of 14% to 17%. Prior to the crash of October 1987, option-implied volatilities showed relatively little dependence on strike and maturity. After the crash, however, the situation changed dramatically. For example, on September 16, 1999, options traded in

Figure 3 Time-Series of One Month at-the-Money BSM-Implied Volatility



the implied volatility range of 16% to 53%. Restricted to the single shortest maturity, the implied volatilities ranged from 23% to 53%. For a fixed maturity, the implied-volatility curve $\hat{\sigma}(S_t, t; K, T)$ as a function of K is often called the implied volatility skew or the implied volatility smile (see, e.g., the curve for the shortest maturity on September 16, 1999). Similar departures from the BSM model are obtained for many other financial markets.

The systematic dependence of implied volatility on strike and maturity is further dramatic evidence of the inconsistency of the BSM model with market data. Higher implied volatilities for low-strike options means that these options are relatively more expensive than predicted by the BSM model. Expression (11) relates the option price to the expected discounted payoff under the risk-neutral measure. For a fixed-option payoff, e.g., a put payoff, a higher put price is generated by a higher risk-neutral probability of $S_T < K$. In other words, a downward sloping implied volatility curve (which associates higher volatilities with lower strikes) corresponds to a fatter left-tail of the risk-neutral distribution of S_T than would be implied by the BSM model. Thus market option data suggest the need for an asset price model where log-returns have fatter tails than occurs with the normal distribution.

Not only do implied volatility curves depend in a systematic way on K and T , but this dependence changes in an unpredictable way with the passage of calendar time t . Figure 3 shows a time series of BSM-implied volatility for 30-day maturity at-the-money options (i.e., the strike K_t equals S_t).¹⁸

¹⁸ An option with this exact strike and maturity cannot be traded, thus interpolation might be necessary to define this quantity.

Large changes in volatility are seen to occur in the data and these changes are often associated with significant economic or political events. If the BSM-constant volatility model assumption were correct, then the correlation of daily changes in implied volatility with daily log-returns should be zero. Computing this correlation using S&P 500 futures options data from January 1997 through December 2002 gives a value of -57% , which is again highly statistically significant. The negative correlation is interpreted to mean that options become more expensive when market returns are negative and is reflected in a skewed distribution of asset returns.

There is a vast literature on empirical tests and econometric estimation of option pricing models. An influential early paper is Engle (1982). Recent contributions include Bakshi et al. (1997), Bates (2000), Chernov and Ghysels (2000), Pan (2002), Chernov et al. (2003) and Eraker et al. (2003). For recent surveys see Garcia et al. (2004) and Bates (2003). Empirical price data for the underlying asset and associated options suggest significant departures from the assumptions of the BSM model. Researchers have proposed many extensions of the BSM model to incorporate these and other empirical features. Some of these extensions are described next.

4.2. Jump Models

For the underlying price, Merton (1976) proposed the following jump-diffusion model:

$$\frac{dS_t}{S_t} = (\mu - \delta) dt + \sigma dW_t + d\left(\sum_{n=1}^{N_t} (e^{Z_n} - 1)\right), \quad (44)$$

where N_t is a Poisson process with intensity λ and the jumps in returns are determined by $Z_n \sim N(\mu_s, \sigma_s^2)$, which are independent of W . This model has three additional parameters: λ determines the arrival rate of jumps; and μ_s and σ_s determine the mean and variance of the jumps in return. When the n th jump occurs at time t the stock price changes from S_{t-} to $e^{Z_n} S_{t-}$. The average jump size is then $E[e^{Z_n} - 1] = \exp(\mu_s + \sigma_s^2/2) - 1 \equiv \mu_1$. It is often more convenient to specify μ_1 and σ_s rather than μ_s and σ_s as primitive model parameters.

In this model, option payoffs cannot be replicated by trading in the primitive assets, i.e., the market is not complete, so option prices are not uniquely determined from arbitrage considerations (i.e., market prices of risk, implied by the asset structure, are not unique). In order to proceed, Merton (1976) assumed that jump risk was diversifiable, so the market price of jump risk is zero. This model can provide a better fit to empirical asset price data, it can produce a range of implied volatility curves, and maintains almost all of the analytical tractability of the BSM model. For a

discussion of the jump-risk premium, see Bates (1988, 1996) and Naik and Lee (1990).

To price call and put options in this model, note that the risk-neutralized model has the same form as Equation (44), but with the drift $\mu - \delta$ replaced by $r - \delta - \lambda\mu_1$. The SDE under the risk-neutral Q -measure has the explicit solution

$$S_T = S_0 e^{(r-\delta-\lambda\mu_1-\sigma^2/2)T+\sigma\sqrt{T}Z_0} \prod_{n=1}^{N_T} e^{Z_n}, \quad (45)$$

where Z_0 is a standard normal variate. Let $x_t = \ln(S_t)$. Then conditional on $x_0 = \ln(S_0)$ and $N_T = j$, the distribution of $X_T = \ln(S_T)$ is normal with mean $x_0 + (r - \delta - \lambda\mu_1 - \frac{\sigma^2}{2})T + j\mu_s$ and variance $\sigma^2 T + j\sigma_s^2$. Analytical tractability is maintained because S_T is a product of a (random number) of lognormal random variables.

Using the extension of Equation (11) to the setting with discontinuous prices under consideration, the time $t = 0$ price of a call option, $c_M(S_0, 0)$, can be written as

$$\begin{aligned} e^{-rT} E_0^*[(S_T - K)^+] \\ = e^{-rT} E_0^*[S_T \mathbf{1}_{\{x_T > \ln(K)\}}] - e^{-rT} K E_0^*[\mathbf{1}_{\{x_T > \ln(K)\}}]. \end{aligned} \quad (46)$$

The term $E_0^*[\mathbf{1}_{\{x_T > \ln(K)\}}] = Q(x_T > \ln(K))$ can be computed as follows. Let $f(\phi) = E_0^*[e^{i\phi x_T}]$ be the characteristic function of x_T . Using the Fourier inversion formula

$$Q(x_T > y) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left(f(\phi) \frac{e^{-i\phi y}}{i\phi} \right) d\phi, \quad (47)$$

the characteristic function can be computed explicitly as

$$\begin{aligned} f(\phi) &= E_0^*[e^{i\phi x_T}] = \sum_{j=0}^\infty \frac{(\lambda T)^j e^{-\lambda T}}{j!} E_0^*[e^{i\phi x_T} | x_0, N_T = j] \\ &= \sum_{j=0}^\infty \frac{(\lambda T)^j e^{-\lambda T}}{j!} \exp\left(i\phi m - \frac{\phi^2 \sigma^2 T}{2}\right) \\ &\quad \cdot \left(\exp\left(i\phi \mu_s - \frac{\phi^2 \sigma_s^2}{2}\right)\right)^j \\ &= \exp\left(i\phi m - \frac{\phi^2 \sigma^2 T}{2} - \lambda T\right) \\ &\quad \cdot \exp\left(\lambda T \exp\left(i\phi \mu_s - \frac{\phi^2 \sigma_s^2}{2}\right)\right), \end{aligned} \quad (48)$$

where $m \equiv x_0 + (r - \delta - \lambda\mu_1 - \sigma^2/2)T$.¹⁹

The term $E_0^*[S_T \mathbf{1}_{\{x_T > \ln(K)\}}]$ in Equation (46) can be transformed to a probability computation by using

¹⁹ The equality on the second line follows from $E(e^{i\phi Z}) = \exp(i\phi\mu - \phi^2\sigma^2/2)$ when $Z \sim N(\mu, \sigma^2)$ and the equality on the last line uses $e^y = \sum_{j=0}^\infty y^j/j!$ for all complex y .

the change of measure outlined in §3.6, adapted to the present context. Let $\eta_t \equiv e^{-(r-\delta)t} S_t/S_0$ and recall the measure $d\hat{Q} \equiv \eta_T dQ^*$. Then $E_0^*[\eta_T h(X_T)] = \hat{E}_0[h(X_T)]$ and so $E_0^*[S_T \mathbf{1}_{\{x_T > \ln(K)\}}] = e^{(r-\delta)T} S_0 \hat{Q}(x_T > \ln(K))$. The characteristic function under the new measure can be computed explicitly as

$$\begin{aligned} g(\phi) &= \hat{E}_0[e^{i\phi x_T}] = E_0^*[\eta_T e^{i\phi x_T}] \\ &= e^{-(r-\delta)T-x_0} E_0^*[e^{(1+i\phi)x_T}] = e^{-(r-\delta)T-x_0} f(-i + \phi) \end{aligned} \tag{49}$$

where $f(\cdot)$ is defined in (48). The probability $\hat{Q}(x_T > \ln(K))$ can be computed using the Fourier inversion Formula (47) with f replaced by g .

To summarize, the price of a European call in Merton’s jump-diffusion model is

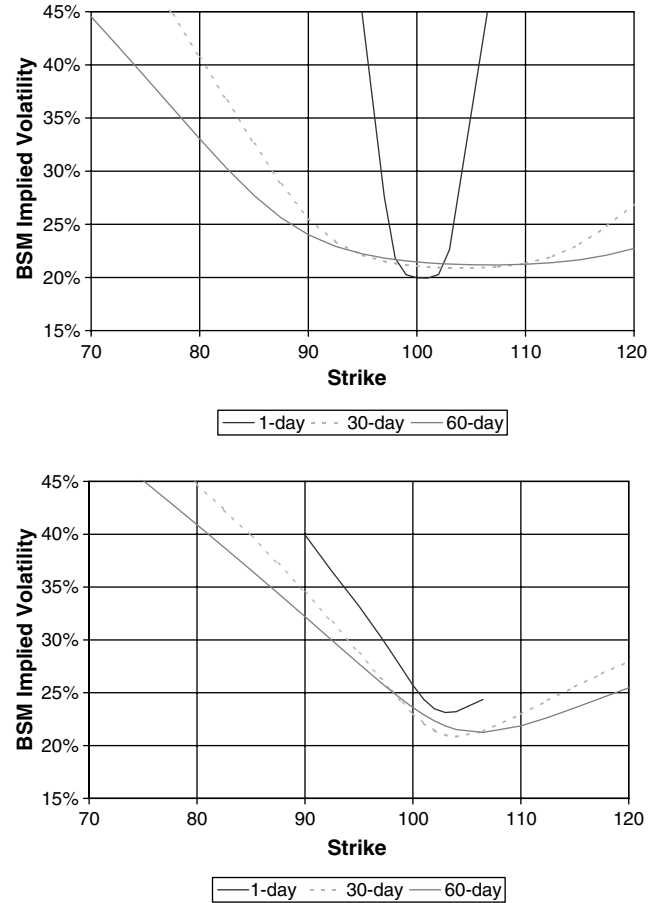
$$c_M(S_0, 0) = S_0 e^{-\delta T} \hat{Q}(x_T > \ln(K)) - e^{-rT} K Q(x_T > \ln(K)), \tag{50}$$

where the two probabilities can be computed using (47) and the explicit Formulas (48) and (49). The result is equivalent to Merton’s original formula that expressed the option price as an infinite weighted sum of BSM formulas. Although the expressions (48) and (49) are lengthy, the final Equation (50) involves two one-dimensional integrals that can be easily computed numerically. Put prices can be derived in a similar manner, or computed from call prices using the put-call parity relation (9).

The form of Equation (50) mirrors the BSM Formula (6). Indeed, the two formulas coincide for $\lambda = 0$, and so this Fourier inversion approach provides an alternate derivation and procedure for computing the BSM formula.

The Merton jump-diffusion model can provide a better fit to underlying asset price data and it can generate implied volatility curves that are more consistent with market option prices. To illustrate, Figure 4 shows implied volatility curves for the Merton model with the parameters $S_0 = 100$, $\sigma = 20\%$, $\lambda = 10\%$, $\mu_1 = -20\%$, $\sigma_s = 40\%$, $\delta = 1\%$, and $r = 5\%$. For example, for 30-day maturity options (i.e., $T = 30/365.25$ years) with strikes of $K = 80, 90$, and 100 , European put option prices under the Merton model are 0.103, 0.211, and 2.243, respectively. These dollar option prices correspond to BSM-implied volatilities of 40.7%, 25.6%, and 21.1%, respectively. The negative value of the average jump size parameter, μ_1 , generates a fatter left tail compared with the lognormal distribution of S_T under the BSM model, which in turn produces the declining implied volatilities mentioned. For the shortest one-day maturity options, the parameter σ_s is large enough to generate positive jumps, so the implied volatility curve becomes the smile shown in Figure 4.

Figure 4 Merton Jump-Diffusion Model Implied Volatility Curves (Top Panel); Heston SV Model Implied Volatility Curves (Bottom Panel)



Other models that feature jumps in returns include Bates (1996), Madan et al. (1998), Carr et al. (2002), and Carr and Wu (2003), Scott (1997), and Kou (2002). For a more complete treatment of jump models, see Cont and Tankov (2004). Although jump-diffusion and pure-jump models can generate jumps in returns and implied volatility skews consistent with empirical data, asset returns under these models produce zero autocorrelation of squared log-returns. Furthermore, implied volatility curves, for a fixed option maturity, would remain constant through time. To address these empirical observations, researchers have proposed modeling volatility as a random process rather than as a constant, as in the BSM model. An illustrative stochastic volatility model is discussed next.

4.3. Stochastic Volatility Models

Heston (1993) proposed the following two-factor asset price model:

$$\frac{dS_t}{S_t} = (\mu - \delta) dt + \sqrt{V_t} dW_t \tag{51}$$

$$dV_t = \kappa(\theta - V_t) dt + \sigma_v \sqrt{V_t} dW_t^v, \tag{52}$$

where W_t and W_t^v are two Brownian motion processes with $E[dW_t dW_t^v] = \rho dt$, κ is the speed of mean reversion in the variance process, and θ is the long-run variance mean. Equation (52) models variance as a mean-reverting process: When $V_t > \theta$ the drift is negative and variance is more likely to decrease than increase and, conversely, when $V_t < \theta$ variance is more likely to increase. Variance cannot become negative because the volatility of variance approaches zero as V_t decreases due to the $\sqrt{V_t}$ term. This same $\sqrt{V_t}$ term leads to an exponential affine structure in the solution, which affords a considerable degree of analytical tractability. Equation (52) generates volatility clustering—i.e., high-volatility periods are more likely to be followed by high-volatility periods—and also generates positive autocorrelation of squared log-returns. The parameter ρ allows correlation between log-returns and changes in variance to be incorporated in the model. The distribution of S_T can be viewed as an infinite mixture of lognormals with different volatility parameters. This mixing property generates excess kurtosis (compared with the BSM model), which produces implied volatility curves.

Heston (1993) showed that the price of a European call, in this setting, has the same form as Equation (50). The characteristic functions have explicit, exponential-affine forms and the Fourier inversion formula can again be used to numerically compute the probabilities required for the option price. Alternative inversion formulas are presented and analyzed in Lee (2004). For a discussion of the volatility risk premium and its impact on option prices, see Bates (1988, 1996) and Lewis (2000). As an illustration, Figure 4 shows implied volatility curves for the Heston model with parameters $S_0 = 100$, $V_0 = (0.26)^2$, $\kappa = 6$, $\theta = (0.35)^2$, $\sigma_v = 3$, $\rho = -50\%$, $\delta = 1\%$, $r = 5\%$, and the strikes and maturities indicated. Although the parameters were not chosen to optimally fit any given set of data, it is clear that the stochastic volatility (SV) model can generate implied-volatility curves that are similar to those observed in the data. Furthermore, the curves change randomly through time as V_t evolves, with correlated changes in volatility and returns. The model has a difficult time reproducing steep smiles typically observed for very short maturity options.

The Fourier inversion approach was first proposed in Stein and Stein (1991) and Heston (1993). Important early contributions to the SV literature include Cox (1975, 1996) and Hull and White (1987). Recent contributions include Duan (1995), Fouque et al. (2000), Davydov and Linetsky (2001), Detemple and Tian (2002), and Cvsa and Ritchken (2001). For an overview of SV models see Ghysels et al. (1996) and Lewis (2000).

The attractive features of jump-diffusion and SV models have been combined in the models of Bates (1996), Scott (1997), Bakshi et al. (1997), and the recent model in Carr et al. (2003). A general affine jump-diffusion framework that includes many of these models as special cases is provided in Duffie et al. (2000). Affine jump-diffusion interest-rate models are given in Chen and Scott (1992), Duffie and Kan (1996), and Dai and Singleton (2000).

5. Numerical Methods

The pricing of American, path-dependent, and multi-asset options in the BSM framework generally requires the use of numerical methods. More recent models incorporating jumps in returns, SV, jumps in volatility, stochastic interest rates, default risk, etc., pose computational challenges for European-style options and more-exotic structures. Calibration of models to market data typically involves an optimization procedure to determine a set of parameters that best fit the data, and each step in the optimization can require many price and derivative computations that can result in a tremendous computational challenge. Investment banks, mutual funds, and many other financial and nonfinancial institutions hold portfolios containing large numbers of derivative securities. Determining the risk of these positions typically requires portfolio revaluations under many potential future market outcomes, and this presents another difficult yet practically important computational task.²⁰

In theory, closed-form analytical solutions are quick to evaluate and perfectly accurate. In practice, however, even the BSM formula requires either numerical integration or the use of approximations for determining the normal probabilities in the formula. Although typically not a major issue for the BSM formula, it does illustrate the point that almost all derivative pricing problems ultimately require a numerical procedure, and the choice of methods involves trade-offs between speed and accuracy and between simplicity and generality.

The major numerical approaches can be classified in one of four categories: (i) formulas and approximations, (ii) lattice and finite difference methods, (iii) Monte Carlo simulation, or (iv) other specialized methods.

In the first category, the most significant advances have been in the application of transform methods and asymptotic expansion techniques. Fourier, Laplace, and generalized transform methods have

²⁰ Financial risk management is another important area of theoretical and practical interest. A theoretical foundation is provided in Artzner et al. (1999). A related early paper is Rudd and Schroeder (1982). Numerical algorithms are treated in Glasserman et al. (2000). For a comprehensive introduction to this field, see Jorion (2001) and the references therein.

been applied to SV models (Stein and Stein 1991, Heston 1993, Duffie et al. 2000), the pricing of Asian options (Reiner 1990, Geman and Yor 1993) and many other options pricing models. Asymptotic expansion and singular perturbation techniques have proved especially useful in developing analytical formulas and approximations for SV models (Hull and White 1987, Hagan and Woodward 1999, Fouque et al. 2000, Andersen et al. 2001).

5.1. Lattice and Finite Difference Methods

Lattice methods use discrete-time and discrete-state approximations to SDEs to compute derivative prices. Lattice approaches were first proposed in Parkinson (1977) and Cox et al. (CRR) (1979). Lattice methods are easy to explain and implement and they are described in virtually every textbook on derivatives. The triangular lattice obviates the need for spatial or side boundary conditions required for finite difference methods. These features make lattice methods attractive for pedagogical purposes and for the computation of derivative prices in simpler models. Finite difference methods provide numerical solutions to the fundamental pricing PDE, and are often the method of choice for models and securities that are more complicated.

The risk-neutral BSM model has an exact solution $S_{t+h} = S_t e^Z$, where $Z \sim \mathcal{N}((r - \delta - \sigma^2/2)h, \sigma^2 h)$. Over a discrete time step, a lattice method replaces Z with a discrete random variable X , where $X = x_i$ with probability p_i for $i = 1, 2, \dots, m$. Over multiple time steps this leads to a distribution of asset prices: binomial for $m = 2$, trinomial for $m = 3$, and multinomial for general m . There have been dozens of proposals for discrete approximations X ; the four widely used lattices of CRR (1979), Jarrow and Rudd (1983), Amin (1991), and Boyle (1986) are given in Table 1.

The discrete distribution $S_{t+h}(i) = S_t e^{x_i}$, for $i = 1, \dots, m$, is chosen to closely approximate the exact

continuous distribution. For $m = 2$, the points x_1 and x_2 and associated probabilities are chosen so that the first and second moments either match exactly or match in the limit as $h \rightarrow 0$. Additional lattices have been proposed with $m = 4$ or higher, and with those it is possible to match higher moments of the distribution (see, e.g., Heston and Zhou 2000, Alford and Webber 2001). Unfortunately, when applied to the pricing of American options, values of m greater than two have not resulted in better overall convergence when the additional computational time is taken into consideration.

Pricing options with the trinomial Lattice (4) requires three calculations at each node instead of two. Although the additional outcome produces American option prices that are more accurate, compared with the binomial method, the additional accuracy is almost exactly balanced by the additional computational cost, as shown in Broadie and Detemple (1996). An advantage of the Boyle (1986) trinomial approach is that the additional stretch parameter $\lambda \geq 1$ can be used to adjust a particular asset node to a convenient level, e.g., to coincide with a strike or barrier.

Lattices (1) and (4) have the advantage of maintaining constant asset price levels through time, so the number of distinct asset price levels S_T is linear in the number of time steps, n . This happens because up and down moves lead back to the same asset price level. For example, in Lattice (1), $S_t e^{x_1} e^{x_2} = S_t$ but $S_t e^{x_1} e^{x_2} \neq S_t$ in Lattices (2) and (3). In Lattices (1) and (4), the drift is captured by the probabilities, whereas in (2) and (3) the drift is captured in the stock price levels. When pricing options, the constant level property of Lattices (1) and (4) results in computational savings because $O(n)$ distinct asset price levels can be computed once and stored; in the other lattices $O(n^2)$ distinct price levels need to be computed.

Table 1 Standard Lattice Methods

Lattice	Outcome	Probability
CRR (1979) (1)	$x_1 = \sigma\sqrt{h}$	$p_1 = \frac{e^{(r-\delta)h} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}$
	$x_2 = -\sigma\sqrt{h}$	$p_2 = 1 - p_1$
Jarrow and Rudd (1983) (2)	$x_1 = (r - \delta - \sigma^2/2)h + \sigma\sqrt{h}$	$p_1 = \frac{e^{\sigma^2 h/2} - e^{-\sigma\sqrt{h}}}{e^{\sigma\sqrt{h}} - e^{-\sigma\sqrt{h}}}$
	$x_2 = (r - \delta - \sigma^2/2)h - \sigma\sqrt{h}$	$p_2 = 1 - p_1$
Amin (1991) (3)	$x_1 = (r - \delta - \ln(\cosh(\sigma\sqrt{h})))h + \sigma\sqrt{h}$	$p_1 = 1/2$
	$x_2 = (r - \delta - \ln(\cosh(\sigma\sqrt{h})))h - \sigma\sqrt{h}$	$p_2 = 1/2$
Boyle (1986) (4)	$x_1 = \lambda\sigma\sqrt{h}$	$p_1 = \frac{1}{2\lambda^2} + \frac{(r - \delta - \sigma^2/2)\sqrt{h}}{2\lambda\sigma}$
	$x_2 = 0$	$p_2 = 1 - 1/\lambda^2$
	$x_3 = -\lambda\sigma\sqrt{h}$	$p_3 = \frac{1}{2\lambda^2} - \frac{(r - \delta - \sigma^2/2)\sqrt{h}}{2\lambda\sigma}$

Related to lattice methods are numerical solutions of the PDE (2) with Boundary Conditions (3). The finite difference approach to option pricing was first proposed by Brennan and Schwartz (1977, 1978). A major advantage of this approach is the wealth of existing theory, algorithms, and numerical software that can be brought to bear on the problem. Issues of numerical consistency, convergence, and stability have been well studied. The generality of finite difference methods is especially important for models that extend beyond the constant coefficients BSM model. The finite difference method can handle processes with time-varying coefficients, Itô processes that are more general, jump and SV models, single and multifactor interest-rate models, etc. Generalizing lattice methods for these models requires separate developments, the resulting algorithms are often complicated to implement, and it is difficult to theoretically analyze their convergence properties. In addition to widely available numerical libraries of finite difference methods, a higher-level language for automatic code generation has also been developed and is described in Randall et al. (1997). The language enables users to concisely represent financial models and solution methods; these specifications are then automatically translated into source code in a standard programming language, e.g., C.

Lattice methods can be viewed as explicit finite difference methods, but the fundamental BSM PDE can also be solved with a number of other methods, including implicit, Crank-Nicolson, and finite element methods and alternating direction implicit (ADI) methods for higher-dimensional problems. The finite difference approach offers considerable flexibility in the choices of grids for the time and space dimensions, which is useful for dealing with discrete dividends, barriers, and other common features. Higher-order derivative approximations are possible in the finite difference approach, and these can lead to improved convergence. For a general treatment of numerical methods for PDEs, see Morton and Mayers (1994), and for applications to financial derivatives see Tavella and Randall (2000).

The computation time (or w for work) of lattice methods for pricing options in the BSM model is $O(mn)$, where m is the number of time steps and n is the number of asset price levels. Typically, m is $O(n)$, in which case the computation time is $O(n^2)$. The pricing error decreases as $O(1/n)$ (see Diener and Diener 2004), which implies that the error decreases as $O(1/\sqrt{w})$. Faster convergence for many numerical applications can be obtained using extrapolation techniques, but these techniques usually require a smoothly converging solution. Both lattice and finite difference methods are affected by option payoffs that are discontinuous or have discontinuous

derivatives (which includes standard put and call payoff structures as well as many others). This lack of smoothness typically results in oscillatory convergence of the solution as the grid size decreases (i.e., as the number of time steps increases), and this poses an obstacle to applying extrapolation to accelerate convergence. For the American option pricing problem, Broadie and Detemple (1996) propose smoothing the solution by applying the European formula for the continuation value at the penultimate time step in combination with Richardson extrapolation. This idea, which applies equally well to lattice and finite difference methods, can increase the convergence rate by an order of magnitude (the error decreases as $O(1/w)$) and is also often trivial to implement. Rannacher timestepping (Rannacher 1984) is an alternative smoothing procedure for finite difference methods when a closed-form solution or approximation is not available. For a general introduction to extrapolation methods see, e.g., Brezinski and Zaglia (1991). Richardson extrapolation was introduced in derivatives pricing in Geske and Johnson (1984). Accelerated convergence for American options in a finite difference method is given in Forsyth and Vetzal (2002).

Lattice and finite difference methods for interest-rate models are given in Ho and Lee (1986), Black et al. (1990), Hull and White (1994), and Pelsser (2000). Fitting a model to a given initial term structure involves numerically solving for a time-dependent drift function. The forward induction idea of Jamshidian (1991) has proved to be particularly useful in these applications.

5.1.1. Path-Dependent Options. Many options have payoff structures that depend not just on the current asset price, but also on the history of the asset price through time. Pricing options with these general payoff structures require keeping track of the asset price through time, and this leads to nonrecombining lattice or bushy tree algorithms. A nonrecombining lattice algorithm for the pricing of interest-rate derivatives in the general Heath-Jarrow-Morton (HJM) framework is proposed in Heath et al. (1990). A nonrecombining lattice with branches determined through Monte Carlo sampling, i.e., a simulated tree algorithm, is proposed in Broadie and Glasserman (1997) for the pricing of path-dependent (and higher-dimensional) American options. The computational time of these methods is $O(m^n)$, where n is the number of branch points in the time dimension and m is the number of branches at each node. The memory requirement, however, is only $O(mn)$ (see Broadie et al. 1997 for details). The exponential dependence on the number of branch points renders these methods infeasible for all but small values of n . Nevertheless, it is feasible to use n up to 30 or 40 with small

values of m (e.g., two or three) on today's personal computers (PC). Applications with a small number of exercise opportunities, e.g., some Bermudan options, are natural for this approach. Combined with extrapolation techniques (see, e.g., Broadie et al. 1997) the nonrecombining lattice approach can lead to surprisingly accurate option price approximations in some applications.

Fortunately, most path-dependent payoff structures seen in practice do not depend in an arbitrary way on the path of asset prices. Mildly path-dependent payoff structures depend on the current asset price and a single sufficient statistic that summarizes the relevant information from the path. For example, for Asian options whose payoff depends on an average asset price, it suffices to track the *current* average price (also called the *running* average or the *average to date*). In the case of barrier and lookback options, it suffices to track the current maximum or minimum asset price, or both. Mildly path-dependent options can be priced in an enlarged state space, which includes the current price as well information related to the sufficient statistic. For example, in the case of *in* barrier options, the state space includes the current asset price and whether or not the option has been *knocked in* (i.e., the asset price has breached the knock-in level). In lattice and finite difference methods, the placement of the barrier relative to the discrete asset price levels can have a large effect on the accuracy of the method. Boyle and Lau (1994) first noticed and proposed a solution to the problem. In the case of Asian options, the state space can be expanded to include the current asset price and the current average price. Ingersoll (1987) shows that the Asian option price satisfies a two-dimensional PDE, and finite difference methods are given in Zvan et al. (1998). Alternatively, for each node in the lattice or finite difference grid, it is possible to keep track of option values corresponding to each value of the average. Unfortunately, the number of potential average prices is typically exponential in the number of time steps. So, rather than storing information for each distinct average price, a range of possible averages are grouped into bins or buckets, and interpolation is used in a backwards pricing procedure. The convergence of lattice and finite difference methods for pricing mildly path-dependent options using interpolation is investigated in Forsyth et al. (2002).

In some cases significant computational savings when pricing mildly path-dependent options can be obtained by a change of measure, also called the change of numeraire technique (see §3.6). Babbs (1992) applies this idea to the lattice pricing of lookback options and Rogers and Shi (1995), Andreasen (1998), and Vecer (2001) apply it to the pricing of Asian options. Significant computational savings are

possible using the related dimensionality-reduction idea presented in Hilliard et al. (1995).

5.1.2. Jump Processes and SV Models. When jumps are added to the BSM model as in Equation (44), the fundamental pricing PDE (2) becomes a partial integro-differential equation (PIDE). Numerical methods to solve the pricing equation are more complicated, because the computation of the continuation value at any node now requires an integral over the jump distribution. Straightforward extensions of lattice or finite difference methods take an order-of-magnitude more computation time (i.e., the CPU time is $O(n^3)$, where n is the number of time steps) to achieve comparable accuracy to models without jumps (see, e.g., Amin 1993). Andersen and Andreasen (2000) develop a finite difference approach using a combination of fast Fourier transform and ADI methods; this approach is stable and reduces the computational time requirement to $O(n^2 \log(n))$. Their method applies to a jump-diffusion and local volatility function model that enables exact calibration to a set of market option prices. For the standard Merton jump-diffusion model in (44), Broadie and Yamamoto (2003) develop a method based on the fast Gauss transform that is easy to implement and further reduces the computational time requirement to $O(n^2)$. Hirta and Madan (2004) develop a finite difference method for the solution of the PIDE that arises in pricing American options in the variance gamma model.

SV models are treated in detail in Lewis (2000). Most of the explicit formulas provided there are derived through transform analysis (e.g., generalized Fourier transforms) and apply to path-independent European-style options. American and path-dependent options require numerical solutions of the pricing PDE. Examples include Forsyth et al. (1999), who present a finite element approach to the pricing of lookback options with SV; Kurpiel and Roncalli (2000), who develop hopscotch methods for SV and other two-state models; and Apel et al. (2002), who develop a finite element method for an SV model.

5.1.3. Multiasset Options. A multidimensional extension of the BSM model is given in the SDE (16). Formulas for the prices of European derivatives usually involve multidimensional integrals, so numerical methods are needed even in this comparatively simple case. As in the single-asset case, two related approaches are multidimensional lattices and multidimensional finite difference methods. Suppose that the time dimension is divided in m time steps and that each of d assets is divided into n asset price levels. Then a multidimensional lattice or finite difference grid will have $O(mn^d)$ nodes and the computational

time will be of the same order. Pricing algorithms do not need to store information at all nodes simultaneously, but typically store information at the nodes of two adjacent time steps, which leads to an $O(n^d)$ storage (i.e., memory) requirement. Both memory and computation time are exponential in the number of assets d . Given typical processor speeds and memory configurations on today's PCs, it turns out that memory is the limiting factor on the size of problems that can be solved. Problems with three to four assets can be feasibly solved. For example, the storage requirement for a four-asset finite difference grid with 50 nodes for each asset and two time steps in memory is 95 megabytes (MB). This assumes an 8-byte double precision number is stored for each of the $50^4 \times 2$ nodes, where each double requires $8/1,024^2$ MB of memory. A similar five-asset problem would require 4,768 MB of memory, which is not currently feasible.

A lattice for an arbitrary number of assets in the multidimensional BSM model was proposed in Boyle et al. (BEG) (1989). At each node in the lattice a discrete approximation to a multidimensional log-normal distribution is required. Their lattice uses a discrete approximation with 2^d outcomes, i.e., there are 2^d branches at each node. Kamrad and Ritchken (KR) (1991) generalize the BEG lattice by adding an additional outcome together with a stretch parameter (analogous to the parameter in the Boyle 1986 single-asset trinomial lattice). The additional flexibility in the placement of nodes afforded by the stretch parameter is useful, for example, in the pricing of barrier options. An advantage of the BEG and KR lattices over finite difference approaches is that spatial boundary conditions do not need to be specified. Finite difference approaches, however, offer considerable flexibility in the choice of grid (i.e., node placement) and a variety of approaches for multidimensional PDEs, e.g., ADI methods or improvements based on Krylov subspace reduction as presented in Druskin et al. (1997).

Relatively little work has been done to date on computational methods for multiasset options in the presence of SV or jumps, or both.

5.2. Monte Carlo Methods

As illustrated in (19), derivative pricing often reduces to the computation of an expected value. In some cases this calculation can be done explicitly, e.g., in the case of the BSM formula, whereas in other cases lattice or finite difference methods prove useful for numerical evaluation. Monte Carlo simulation is a natural approach for expected value computations. Consider, for example the calculation of the price of a derivative security represented as

$$V(S_0, t_0) = E^*[h(S_{t_0}, S_{t_1}, \dots, S_{t_m})], \quad (53)$$

where h represents the discounted derivative payoff that depends on the path of underlying (and possibly vector-valued) state variables $S_{t_0}, S_{t_1}, \dots, S_{t_m}$. The Monte Carlo method consists of three main steps. The first step is to generate n random paths of the underlying state variables. The next step is to compute the corresponding n discounted option payoffs. The final step is to average the results to estimate the expected value; usually a standard error of the estimate is also computed. In this situation, the convergence rate of the Monte Carlo method is often $O(1/\sqrt{n})$ by the central limit theorem, independent of the problem dimension, which makes it especially attractive for pricing high-dimensional derivatives. This forward path generation approach also makes simulation well suited for pricing complex path-dependent derivatives. Much of the focus of recent research has been on (i) path simulation methods, especially when there are nonlinearities in the financial SDEs; (ii) computational improvements through variance reduction techniques; and (iii) extending the Monte Carlo method to calculate price derivatives and American option values. For a survey of the literature in this area, see Boyle et al. (1997), and for a more extensive treatment see Glasserman (2004), which contains over 350 references.

An extensive treatment of the numerical solution of nonlinear SDEs is given in Kloeden and Platen (1999). In the area of efficiency improvements, much attention has been given to quasi-Monte Carlo or low-discrepancy methods (Niederreiter 1992, Birge 1994, Paskov and Traub 1995, Joy et al. 1996, Owen 1997). Variance reduction techniques based on moment matching are presented and analyzed in Baraquand (1995), Duan and Simonato (1998), and Duan et al. (2001). Algorithms for computing security price derivatives using simulation are given in Broadie and Glasserman (1996).

The determination of optimal exercise strategies in the pricing of American options requires a backwards dynamic programming algorithm that appears to be incompatible with the forward nature of Monte Carlo simulation. Much research was focused on the development of fast methods to compute approximations to the optimal exercise policy. In this context, *fast* refers to a method whose computation time requirement is a multiple of the time to price the corresponding European option using simulation. Notable examples include the functional optimization approach in Andersen (2000) and the regression-based approaches of Carriere (1996), Longstaff and Schwartz (2001), and Tsitsiklis and Van Roy (1999, 2001).

These methods often incur unknown approximation errors and are limited by a lack of error bounds. Broadie and Glasserman (1997) propose a method based on simulated trees that generates error bounds

(in the form of confidence intervals) and converges to the correct value under broadly applicable conditions. The simulated tree method, while able to handle high-dimensional problems, has a computation time requirement that is exponential in the number of potential exercise dates, making it practical only for problems with a small number of potential exercise dates. The stochastic mesh method of Broadie and Glasserman (2004) avoids this exponential dependence on the number of exercise opportunities, is a provably convergent method, and also generates error bounds. This method has a work requirement that is quadratic in the number of simulated paths; experimental results suggest a square-root convergence in the error, leading to an overall complexity in which the error decreases as the fourth-root of the work, which is rather slow. Glasserman (2004, §8.6) shows that the regression-based approaches can be viewed as special cases of the stochastic mesh method.

All of the fast approximation methods mentioned above share the same limitation: the lack of known error bounds. Each method, though, by virtue of explicitly or implicitly providing a suboptimal exercise strategy, can be used to generate a lower bound on the true price, because any suboptimal policy trivially delivers less value than the value-maximizing optimal stopping rule. A complete bound on the true price would be available if an upper bound could be generated from any given exercise policy.

A dual formulation that represents the American option price as the solution to a minimization problem was independently developed by Haugh and Kogan (2004) and Rogers (2002). In addition to its theoretical interest, a computationally effective method to compute upper bounds based on this dual formulation was introduced in Andersen and Broadie (2004).

Let $h(x, t_k)$ represent the discounted option payoff at time t_k when the state $S_{t_k} = x$ is a d -dimensional vector. As in §3.3, the discrete-time American option price is given by

$$V(S_0, t_0) = \sup_{\tau} E^*[h(S_{\tau}, \tau)], \quad (54)$$

where τ is a stopping time taking values in $\mathcal{T} = \{t_1, t_2, \dots, t_m\}$. Let M_{t_k} be any adapted martingale. Then

$$\begin{aligned} V(S_0, t_0) &= \sup_{\tau \in \mathcal{T}} E^*[h(S_{\tau}, \tau) + M_{\tau} - M_{t_0}] \\ &= M_{t_0} + \sup_{\tau \in \mathcal{T}} E^*[h(S_{\tau}, \tau) - M_{\tau}] \\ &\leq M_{t_0} + E^*\left[\max_{k=1, \dots, m} (h(S_k, t_k) - M_{t_k})\right], \quad (55) \end{aligned}$$

where the second equality follows from the optional sampling property of martingales. The upper bound given above is especially useful because, for an

appropriate choice of the martingale M , the inequality holds with equality, i.e., the duality gap is zero. To see this, recall that the American option value process $V_k \equiv V(S_k, t_k)$ is a Q -supermartingale (see §3.3) and so admits a Doob-Meyer decomposition $V_k = M'_k - A_k$, where M'_k is a martingale, $A_0 = 0$, and A_k is a nondecreasing process. Now set $M_k = M'_k$, i.e., use the martingale component of the true American option price to get

$$\begin{aligned} V(S_0, t_0) &\leq M'_0 + E^*\left[\max_{k=1, \dots, m} (h(S_k, t_k) - M'_k)\right] \\ &= V(S_0, t_0) + E^*\left[\max_{k=1, \dots, m} (h(S_k, t_k) - V(S_k, t_k) - A_k)\right] \\ &\leq V(S_0, t_0), \quad (56) \end{aligned}$$

where the last inequality follows from $V(S_k, t_k) \geq h(S_k, t_k)$ and $A_k \geq 0$.

There are two challenges to developing tight upper bounds with this approach. First, because the true value process V is unknown, the optimal martingale is also unknown. Second, the determination of a nearly optimal martingale and a corresponding upper bound must be done in a computationally practical manner. The approach taken in Andersen and Broadie (2004) is to base the upper bound computation on any exercise policy by taking M to be the martingale component of that policy. Their algorithm is based on a simulation within a simulation. The inner simulation is used to compute the martingale M based on any approximate exercise policy, and the outer simulation is used to compute the expectation $E^*[\max_{k=1, \dots, m} (h(X_k, t_k) - M_k)]$ to determine an upper bound. Although this approach sounds computationally intensive, much of the work in the inner simulation can be avoided by using the properties of the chosen martingale M . The resulting algorithm can generate practically useful lower and upper bounds on many problems of financial importance in nearly real time (e.g., within a few minutes on a current PC) and further improvements are possible with specially designed variance reduction techniques.

6. Conclusions

Valuation methods for derivative securities have reached a fair degree of maturity. Nevertheless, challenges remain on several fronts. It is sure to be a continuing goal to develop asset price models that better match observed data, while maintaining as much parsimony and tractability as possible. Multi-factor SV, stochastic correlation, and regime-switching models are potential avenues for further investigation. Models that are more detailed and that incorporate information arrival; liquidity; trading volume

and open interest; market incompleteness; and interactions between multiple securities or multiple agents might prove useful. Models that integrate market and credit risk in a general equilibrium framework await development.

Ever-improving computing technology will expand the amounts and types of data that can be analyzed; the types of financial models that can be feasibly studied and numerically solved will be similarly expanded. Added computing power alone, however, is of fairly limited usefulness. Many problems are effectively exponential in their memory or computation time requirement, and these will require algorithmic advances. For example, it is not currently feasible to calibrate model parameters to a large set of securities that are priced by simulation (a problem of simulation within optimization). Efficient and convergent methods for pricing high-dimensional and path-dependent American securities depend on the development of new algorithms, not faster computers.

New markets and products continue to be introduced. Recent examples include passport options, variance swaps, and volatility futures, which began trading on the CBOE in 2004.²¹ Businesses could run markets to better predict the success of a new product or service, to predict the market share of existing products, or to predict other factors of interest. Markets with payoffs tied to political events already exist, and trading in the outcomes of other events has been proposed. Environmental markets and the trading of emissions permits might expand on a global scale. New markets might develop to protect an individual's home value or a country's gross domestic product, as envisioned in Shiller (2003). In each case, new models, valuation theory, and numerical methods might need to be developed. What is certain is that there will be surprising developments and continuing challenges in this exciting and expanding field.

Acknowledgments

The authors thank the editor, David Hsieh, as well as Mike Chernov, and especially Leif Andersen for many comments that improved the paper. This work was supported in part by the National Science Foundation under Grant No. DMS-0074637.

Appendix. Securities and Markets

Since 1973, when plain vanilla equity options were first introduced on the CBOE, numerous products have been created to fill various needs of investors. In this appendix we briefly review the vast literature on fixed income derivatives, credit derivatives, real options (§A.1), path-dependent contracts (§A.2), and derivatives with multiple underlying assets (§A.3).

²¹ For a description and analysis of passport and related options see Andersen et al. (1998) and Shreve and Vecer (2000). Volatility derivatives are treated in Neuberger (1994), Detemple and Osakwe (2000), Carr and Madan (1998), and Demeterfi et al. (1999).

A.1. Fixed Income, Credit, and Other Derivatives

The same principles for pricing and hedging derivatives apply to many markets. However, each market has important and unique features that need to be modeled, and these in turn raise new analytical and computational issues. The rise of equity derivatives in the 1970s was followed by the development of derivatives in fixed income, foreign exchange, commodity, and credit markets. More recently, energy, weather, catastrophe (e.g., earthquake and hurricane), and other derivatives have traded. The literature is far too vast to do any justice in this brief space, so we settle for mentioning a few selected highlights, representative pieces of research, and pointers to more extensive references.

A central challenge in the fixed income market is the consistent modeling of a set of related securities, in this case bonds of different maturities that make up a yield curve. James and Webber (2000) provide over 500 references in this area. The seminal paper of Heath et al. (1992) unifies and substantially extends previous interest-rate models, including those of Vasicek (1977), Ho and Lee (1986), and Black et al. (1990). Working with discretely rather than continuously compounded rates, Brace et al. (1997) and Miltersen et al. (1997) propose the Libor market model, which generates prices consistent with the Black (1976) formula for European options on a market interest rate. Significant analytical developments were enabled with the change of measure approach proposed in Jamshidian (1989) and Geman et al. (1995). Extensions of interest-rate models to include jumps in rates are given in Duffie and Kan (1996) and Glasserman and Kou (2003). An extension to SV is given in Andersen and Brotherton-Ratcliffe (2001). Empirical evidence supporting SV and jumps in interest rates is given in Collin-Dufresne and Goldstein (2002) and Johannes (2004).

An important application of interest-rate modeling is to the pricing of mortgage-backed securities, asset-backed securities (e.g., securities backed by credit card receivables), and related derivative securities. Relevant papers in this area include Dunn and Spatt (1985), Schwartz and Torous (1989), Kau et al. (1990), McConnell and Singh (1993), Zipkin (1993), and Akesson and Lehoczky (2000). See the textbook by Sundaresan (2002) for more discussion and references.

Credit risk and the pricing of securities in the presence of default risk have been investigated since at least the 1970s. A commonly traded credit derivative is the credit default swap, or CDS. This is an agreement between two parties, where party *A* makes a fixed periodic payment to party *B*, until either the maturity of the CDS, or until there is a default by the reference entity (e.g., a publicly traded company). In the event of a default, party *B* pays party *A* a contractually specified amount, which can depend on the market value of a specific bond, (or in some cases, party *B* delivers to party *A* a specific bond issued by the reference entity). Credit risk models are often classified as structural or reduced form. In the structural approach (sometimes called the firm-value approach) default occurs based on a model of an issuer's ability to meet its liabilities. In reduced-form models (sometimes called intensity-based or hazard rate models) default is an exogenous

process that is calibrated to observed historical or current market data. The seminal contributions given in Black and Scholes (1973), Merton (1974), and Black and Cox (1976), and the more recent models of Leland (1994) and Anderson and Sundaresan (1996) fall in the first category. Important reduced-form models include Jarrow and Turnbull (1995) and Duffie and Singleton (1999). In recent years the markets for convertible bonds and credit derivatives (including default swaps, collateralized debt obligations, multiline credit derivatives, and others) has grown substantially and there has been a corresponding growth in related research. Representative papers in this area include Hull and White (1995), Das and Sundaram (2000), Li (2000), and Andersen and Buffum (2004). Excellent sources for more information include the books by Bielecki and Rutkowski (2002), Duffie and Singleton (2003), and Schonbucher (2003).

Government deregulation is one factor that has fueled the expansion of existing markets for energy and the development of new markets for energy derivatives. Electricity markets present new challenges primarily because of electricity's limited ability to be stored. Supply and demand must be balanced through time, whereas demand varies significantly between day and evening hours and between warm and cold months. These factors together with recent instances of enormous price spikes illustrate the need for models that are specifically tailored to these markets. Recent work in this area includes Schwartz and Smith (2000), Lucia and Schwartz (2002), Barlow (2002), and the books by Pilipović (1998) and Clewlow and Strickland (2000).

Weather affects the revenues and expenses of many energy and nonenergy businesses, and weather derivatives offer a means of hedging the associated risks. Securities with payoffs linked to the occurrence or nonoccurrence of earthquake, hurricane, or other events are called *catastrophe derivatives*. References in these areas include Litzenberger et al. (1996), Canter et al. (1997) and Geman (1999).

The term *real options* refers to the opportunity to initiate, expand, or contract a capital investment in a real asset, such as land, physical plant, or equipment. Difficulties arise because market prices might not be continuously observable. Investment decisions can be discrete rather than continuous; decisions might be reversible or irreversible; and current decisions could lead to different potential future decisions. A seminal paper in this area is Brennan and Schwartz (1985), and representative research includes Cortazar et al. (1998), Kulatilaka and Perotti (1998), Bollen (1999), and Schwartz and Zozaya-Gorostiza (2003). Books on the subject include Dixit and Pindyck (1994), Trigeorgis (1996), Amram and Kulatilaka (1999), and Copeland and Antikarov (2001).

A.2. Path-Dependent Options

A *path-dependent option* is an option whose payoff depends on prices not only at, but also prior to the exercise time. Some of the most popular path-dependent contracts, reviewed in §§A.2.1–A.2.3, are barrier options, capped options, and Asian options. Occupation time derivatives are examples of recent products that are experiencing a surge of interest in the academic community. Definitions and references for those contracts will be provided in §A.2.4.

A.2.1. Barrier Options. A *barrier option* is an option whose payoff depends on the hitting time of some prespecified barrier. Standard barrier options have a constant barrier and come into existence (in-options) or expire (out-options) when the barrier is breached. Examples include down-and-out, down-and-in, up-and-out, and up-and-in options. A down-and-out option, for instance, expires when the asset price breaches a barrier whose level is below the price of the underlying asset at inception of the contract. Out-options sometimes involve a rebate, such as a flat payment, in the event of early termination.

In- and out-barrier options with constant barriers are easy to price in the BSM setting. A comprehensive set of formulas can be found in Rubinstein and Reiner (1991).

Symmetry relations also tie pairs of barrier options together. For example, an up-and-out put can be priced from a down-and-out call by using a change of variables. Indeed, suppose that H is the put barrier and that $\tau(H) = \inf\{v \in [t, T]: S_v = H\}$ represents the first hitting time of H . Performing the transformations described in §3.6 yields $\tau(H) = \tau(\hat{H})$, where $\hat{H} = KS_t/H$ and

$$\tau(\hat{H}) = \inf\{v \in [t, T]: \hat{H} = KS_t/S_v = \hat{S}_v\},$$

with \hat{S} described in (41). It is then apparent that an up-and-out put on S , with barrier H , strike K , and maturity date T has the same price as a down-and-out call on \hat{S} , with barrier $\hat{H} = KS_t/H$, strike S_t , and maturity date T in a modified financial market with interest rate δ and dividend rate r . Equivalently, $p^{uo}(S, t; K, H, T, r, \delta) = c^{do}(K, t; S, \hat{H}, T, \delta, r)$, where the last two arguments have been added to capture the market structure (interest rate and dividend rate).

In-options can be valued by complementarity. Simple inspection reveals that a standard option is the sum of an in-option plus an out-option with identical characteristics. For instance, for calls we have

$$\begin{aligned} c^{do}(S, t; K, H, T) + c^{di}(S, t; K, H, T) &= c(S, t; K, T) \\ c^{uo}(S, t; K, H, T) + c^{ui}(S, t; K, H, T) &= c(S, t; K, T), \end{aligned}$$

where the expressions on the right-hand side are the prices of standard call options. Similar relations hold across puts.

Valuation formulas for European barrier options, in the BSM setting, are found in Rubinstein and Reiner (1991). American barrier options are priced by Rubinstein and Reiner (1991) in the binomial model and Gao et al. (2000) in the BSM setting. The latter paper establishes an EEP decomposition of the price and derives an integral equation for the exercise boundary.

A.2.2. Capped Options. Capped options were introduced on the CBOE in 1991 (see the capped-style index options (CAPS) contract) and have also appeared as components of hybrid securities issued by firms for financing purposes (such as the Mexican index-linked euro security (MILES) contract). Capped options can be European or American style, and can include various types of automatic exercise provisions in the event that the underlying price breaches the cap. The caps involved can be constant, time-dependent functions, or stochastic processes.

A typical capped option involves a limit on the amount realized upon exercise. A standard capped call option, for instance, pays $(\min\{S, L\} - K)^+$ upon exercise, where $L > K$ is a constant cap on the underlying price ($L - K$ is the cap on the payoff). European-style capped options, with payoff at the maturity date, are easily valued. A European capped call is identical to a bull spread and is therefore simply the difference between two call option prices: $c(S, t; K, L, T) = c(S, t; K, T) - c(S, t; L, T)$.

American-style contracts are usually more difficult to price. A relatively simple case is when the cap is a constant. In that case it is clear that immediate exercise is optimal at the cap, because the maximal value of the payoff, $L - K$, is attained. This is true under the relatively weak assumption of a positive interest rate. Of course exercise can also be optimal prior to hitting the cap, but the optimality of exercise at the cap enables us to conclude that the capped-call option is in fact an up-and-out barrier option with barrier L and rebate $L - K$. Valuation methods for barrier options can then be applied to this contract.

In the context of the BSM model one can also use simple dominance arguments to show that the exercise boundary of the capped option, $B(\cdot; L, T)$, is the minimum of the cap L and of the exercise boundary of the corresponding uncapped option $B(\cdot; T)$. That is $B(t; L, T) = \min\{L, B(t; T)\}$. The price of the claim is the present value of the payoff stopped at this first hitting time of this boundary.

Some of these insights and extensions to growing caps can be found in Broadie and Detemple (1995) for the BSM framework. Results for some diffusion models are in Detemple and Tian (2002).

A.2.3. Asian Options. The payoff of an Asian option depends on an average of prices over time. An Asian call option has the payoff $(A - K)^+$, where A is the underlying average price and K the strike price. An average strike call has payoff $(S - A)^+$ based on the difference between the current and average prices. Asian puts are defined in a symmetric fashion. Arithmetic averages are common, although geometric averages are sometimes used. Continuously averaged prices are of great theoretical interest, whereas averages at discrete time intervals are typically used in practice.

Asian options were first proposed in Boyle and Emanuel (1980) and have become very popular in practice. For firms with periodic revenues or expenses tied to an underlying price or index level an Asian option is a better hedging instrument than a European option whose payoff depends on a price at a single point in time. The pricing of Asian options in the BSM model is quite challenging. A numerical approach based on Monte Carlo simulation for the pricing of Asian options was developed in Kemna and Vorst (1990), and additional numerical approaches (including the case of American Asian options) were presented in Ritchken et al. (1993), Curran (1994), Hansen and Jorgensen (2000), and Ben-Ameur et al. (2002). An analytical formula for continuously averaged Asian options was first published by Geman and Yor (1993).²² Series solutions for continuously averaged Asians have been developed in Dufresne (2000), Schröder (2002), and Linetsky (2004).

²² A similar formula was presented in Reiner (1990).

A.2.4. Occupation Time Derivatives. Occupation time derivatives are fairly new products attracting some attention from investors and researchers. A defining characteristic of these contracts is an exercise payoff that depends on the time spent by the underlying asset in some predetermined region(region). Typical specifications of the occupation region involve barriers, thus an occupation time derivative can be thought of as a type of barrier option.

Various claims with features of this type have been studied. One example is the quantile option. A European α -quantile call pays off $(M(\alpha, T) - K)^+$ at exercise where $M(\alpha, T)$ is the smallest (constant) barrier such that the fraction of time spent by the underlying price at or below $M(\alpha, T)$, during $[0, T]$, exceeds α . Formally,

$$M(\alpha, T) \equiv \inf \left\{ x: \int_0^T 1_{\{S_v \leq x\}} dv > \alpha T \right\},$$

where $\int_0^T 1_{\{S_v \leq x\}} dv$ is the occupation time of the set $\{S \leq x\}$ during the period $[0, T]$.

Quantile options were suggested by Miura (1992) as an alternative to standard barrier options, which have the drawback of losing all value at the first touch of the barrier. Quantile options lose value more gradually. The barrier implied by the quantile α decreases as the underlying asset spends more time at lower levels. The option loses all value only if $M(\alpha, T) \leq K$. Pricing formulas are provided by Akahori (1995) and Dassios (1995). Formulas for the distribution of the quantiles of a Brownian motion with drift can be found in Embrecht et al. (1995) and Yor (1995).

A second example is the Parisian option. A Parisian out option with window D , barrier L , and maturity date T will lose all value if the underlying price has an excursion of duration D above or below the barrier L during the option's life. If the loss of value is prompted by an excursion above (below) the barrier, the option is said to be an up-and-out (down-and-out) Parisian option. A Parisian in option with window D , barrier L , and maturity date T comes alive if the underlying price has an excursion of duration D before maturity. If specification involves an excursion above (below) the barrier the option is an up-and-in (down-and-in) Parisian option.

Parisian contractual forms were introduced and studied by Chesney et al. (1997). The motivation parallels the motivation for quantile options. Parisian options are more sturdy, because they lose value more gradually than do standard barrier options. Contracts of this type are more robust to eventual price manipulations. The pricing formulas in Chesney et al. (1997) involve inverse Laplace transforms. Implementation issues are examined by Chesney et al. (1997). A variation of the Parisian option is the cumulative Parisian option, which pays off based on the cumulative amount of time above or below a barrier. Pricing formulas are provided by Chesney et al. (1997) and Hugonnier (1999).

Our last example is the step option. This derivative's payoff is discounted at some rate that depends on the amount of time spent above or below a barrier. Various discounting schemes can be used. For instance, a proportional step call, with strike K and barrier L , pays off

$$\exp(-\rho_0 t)(S_t - K)^+$$

at exercise, for some constant $\rho > 0$, where $o_t(S, L)$ is the amount of time spent by time t above (or below) the barrier L . The discount rate ρ is the knock-out rate, and $\exp(-\rho o_t(S, L))$ is the knock-out factor. Proportional step options are also referred to as geometric or exponential step options. Simple step options use a piecewise linear amortization scheme instead of an exponential one. A down-and-out simple step call, with strike K and barrier L , pays off

$$\max(1 - \rho o_t(S, L), 0)(S_t - K)^+.$$

This option is knocked out at the first time at which $o_v(S, L) \geq 1/\rho$. This stopping time is the knock-out-time.

Step options were studied by Linetsky (1999). Laplace transforms and their inverses are central to the valuation formulas provided.

A.3. Multiasset Options

Derivatives written on multiple underlying assets have long been of interest to investors. Contracts such as options on the maximum (max-options) or the minimum (min-options) of several underlying prices have long been quoted over the counter. Options on the spread between two prices (spread options) can now be traded on organized exchanges such as the New York Mercantile Exchange (NYMEX—an example is the Gasoline Crack Spread Option). Options to exchange one asset for another (exchange options) appear in various financial contexts including convertible securities and takeover attempts.

Pricing formulas for European contracts were provided early on. Margrabe (1978), for instance, prices exchange options, whereas Johnson (1981) and Stulz (1982) deal with max- and min-options. Results for American options are more recent. Broadie and Detemple (1997) study max-call options, spread options, and related contracts (dual max-calls, min-put options, capped exchange options, etc.) written on dividend-paying assets. Min-call options are examined by Villeneuve (1999) for non-dividend-paying assets and Detemple et al. (2003) for dividend-paying assets.

The main difference between options on a single underlying price and those written on multiple prices rests with the structure of the exercise region. In the single-asset case optimal exercise can be described in terms of a single exercise boundary. This simple structure fails in the multiasset case: Multiple exercise boundaries are typically needed in order to describe the optimal exercise decision. This section illustrates the complexity of the exercise policy for the special case of a call on the maximum of two prices. The properties described were established in the BSM framework.

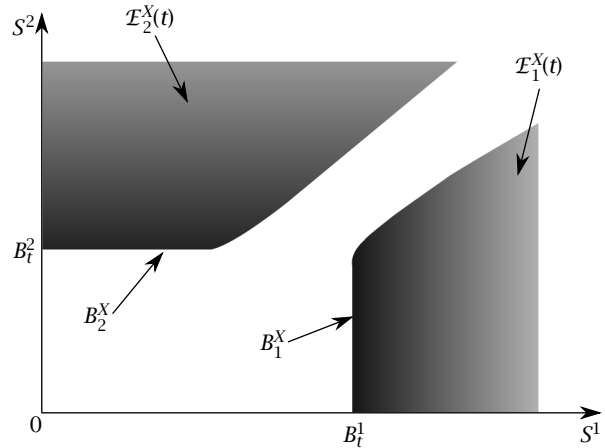
A max-call option written on two underlying prices pays off $(S^1 \vee S^2 - K)^+$ at exercise. The function $S^1 \vee S^2$ is not differentiable along the diagonal $S^1 = S^2$, thus this payoff does not have the usual smoothness properties. This feature is a source of surprising exercise properties, described next.

Figure A1 illustrates the structure of the exercise region for the contract. The most notable aspect is that this region is the union of two subregions, corresponding to sets where Asset 1 is the maximum or Asset 2 is the maximum, with disjoint time sections for $t < T$. That is, $\mathcal{E}^{\max} = \mathcal{E}_1^{\max} \cup \mathcal{E}_2^{\max}$ where

$$\mathcal{E}_1^{\max} = \{(S^1, S^2, t) \in \mathbb{R}_+^2 \times [0, T): S^1 = S^1 \vee S^2 \text{ and } S^1 \geq B^1(S^2, t)\}$$

$$\mathcal{E}_2^{\max} = \{(S^1, S^2, t) \in \mathbb{R}_+^2 \times [0, T): S^2 = S^1 \vee S^2 \text{ and } S^2 \geq B^2(S^1, t)\}$$

Figure A1 Exercise Region of a Max-Call Option



for some boundaries $B^1(S^2, t)$ and $B^2(S^1, t)$, such that $B^1(S^2, t) > S^2$ for all $(S^2, t) \in \mathbb{R}_+ \times [0, T)$ and $B^2(S^1, t) > S^1$ for all $(S^1, t) \in \mathbb{R}_+ \times [0, T)$. Indeed, for max-calls immediate exercise along the diagonal, where $S^1 = S^2$, is never optimal prior to maturity; this is true independently of the values taken by underlying prices, i.e., even if they become very large, $S^1 = S^2 \rightarrow \infty!$ Intuition for this property can perhaps be gained by considering the case of independent and symmetric price processes. Consider a point along the diagonal $S^1 = S^2$ and suppose that the holder of the security must decide whether to exercise or wait. By independence the probability of an increase in the maximum of the two prices, over the next time increment, is roughly equal to 3/4. With identical price volatilities value increases if the decision to exercise is postponed by an infinitesimal amount of time. This argument also prevails for prices off the diagonal but sufficiently close to it. At those points immediate exercise can be improved on by capitalizing on likely increases in the max of the two prices over the next short time period.

The pricing methods presented for single-asset options also apply to multiasset options. The price of the max-call can be characterized as the solution of a free-boundary problem, or a variational inequality problem. The EEP representation also applies. This representation gives rise to a pair of coupled integral equations for the boundary components $B^1(S^2, t)$ and $B^2(S^1, t)$.

Several contracts display exercise properties similar to those of the max-call. For a dual max-call, with payoff $((S^1 - K_1) \vee (S^2 - K_2))^+$, immediate exercise is suboptimal along the translated diagonal $S^1 = S^2 + K_1 - K_2$. The immediate exercise region can also be described by two subregions, each defined by its own boundary. A spread call option, with payoff $(S^1 - S^2 - K)^+$, can be viewed as a one-sided version of a dual max-call. Immediate exercise is suboptimal on or below the translated diagonal $S^1 = S^2 + K$; the exercise region is characterized by a single boundary, parametrized by (S^1, t) , that lies above this line. A min-put option, paying off $(K - S^1 \wedge S^2)^+$, has two exercise boundaries issued from the origin and lying either above or below the diagonal. Immediate exercise along the diagonal is also suboptimal for this contract.

The min-call option, with payoff $(S^1 \wedge S^2 - K)^+$, presents additional challenges for valuation. For this contract immediate exercise could be optimal along the diagonal. The exercise region has two exercise boundaries that meet and merge along the diagonal (see Detemple et al. 2003). In the absence of dividend payments it collapses to a subset of the diagonal (see Villeneuve 1999). Valuation can be performed using any of the methods previously discussed. One novel aspect is the appearance of a local time component in the EEP representation of the price. This term is related to the discontinuities in the derivatives of the payoff function in the exercise region.

References

- Akahi, J. 1995. Some formulae for a new type of path-dependent option. *Ann. Appl. Probab.* 5 383–388.
- Akesson, F., J. Lehoczky. 2000. Path generation for quasi-Monte Carlo simulation of mortgage-backed securities. *Management Sci.* 46(9) 1171–1187.
- Alford, J., N. Webber. 2001. Very high order lattice methods for one factor models. Working paper, University of Warwick, Coventry, U.K.
- Amin, K. 1991. On the computation of continuous time option prices using discrete approximations. *J. Financial Quant. Anal.* 26(4) 477–496.
- Amin, K. 1993. Jump-diffusion option valuation in discrete time. *J. Finance* 48(5) 1833–1863.
- Amram, M., N. Kulatilaka. 1999. *Real Options: Managing Strategic Investment in an Uncertain World*. Harvard Business School Press, Boston, MA.
- Andersen, A., J. Andreasen. 2000. Jump-diffusion processes: Volatility smile fitting and numerical methods for option pricing. *Rev. Derivatives Res.* 4 231–262.
- Andersen, L. 2000. A simple approach to the pricing of Bermudan swaptions in the multi-factor Libor market model. *J. Comput. Finance* 3(1) 5–32.
- Andersen, L., M. Broadie. 2004. A primal-dual simulation algorithm for pricing multi-dimensional American options. *Management Sci.* 50(9).
- Andersen, L., D. Buffum. 2004. Calibration and implementation of convertible bond models. *J. Computational Finance* 7(2) 1–34.
- Andersen, L., R. Brotherton-Ratcliffe. 2001. Extended Libor market models with stochastic volatility. Working paper, General Re Financial Products, New York.
- Andersen, L., J. Andreasen, R. Brotherton-Ratcliffe. 1998. The passport option. *J. Comput. Finance* 1(3) 15–36.
- Anderson, R., S. Sundaresan. 1996. Design and valuation of debt contracts. *Rev. Financial Stud.* 9 37–68.
- Andreasen, J. 1998. The pricing of discretely sampled Asian and lookback options: A change of numeraire approach. *J. Comput. Finance* 2(1) 5–30.
- Apel, T., G. Winkler, U. Wystup. 2002. Valuation of options in Heston's stochastic volatility model using finite element methods. J. Hakala, U. Wystup, eds. *Foreign Exchange Risk: Models, Instruments and Strategies*. Risk Books, London, U.K.
- Artzner, P., F. Delbaen, J.-M. Eber, D. Heath. 1999. Coherent measures of risk. *Math. Finance* 9(3) 203–228.
- Babbs, S. 1992. Binomial valuation of lookback options. Working paper, Midland Global Markets, London, U.K.
- Bakshi, G., C. Cao, Z. Chen. 1997. Empirical performance of alternative option pricing models. *J. Finance* 52 2003–2049.
- Barlow, M. T. 2002. A diffusion model for electricity prices. *Math. Finance* 12(4) 287–298.
- Barraquand, J. 1995. Numerical valuation of high dimensional multivariate European securities. *Management Sci.* 41(12) 1882–1891.
- Bates, D. 1988. Pricing options under jump-diffusion processes. Working paper, Rodney L. White Center for Financial Research, University of Pennsylvania, Philadelphia, PA.
- Bates, D. 1996. Jumps and stochastic volatility: Exchange rate processes implicit in Deutsche Mark options. *Rev. Financial Stud.* 9 69–107.
- Bates, D. 2000. Post-'87 crash fears in S&P 500 futures options. *J. Econometrics* 94 181–238.
- Bates, D. 2003. Empirical option pricing: A retrospection. *J. Econometrics* 116(1/2) 387–404.
- Ben-Ameur, H., M. Breton, P. L'Ecuyer. 2002. A dynamic programming procedure for pricing American-style Asian options. *Management Sci.* 48(5) 625–643.
- Bensaid, B., J. P. Lesne, H. Pagès, J. Scheinkman. 1992. Derivative asset pricing with transaction costs. *Math. Finance* 2(2) 63–86.
- Bensoussan, A. 1984. On the theory of option pricing. *Acta Appl. Math.* 2 139–158.
- Bensoussan, A., J. L. Lions. 1978. *Applications des Inéquations Variationnelles en Contrôle Stochastique*. Dunod, Paris, France.
- Bielecki, T., M. Rutkowski. 2002. *Credit Risk: Modeling, Valuation and Hedging*. Springer-Finance, Berlin, Germany.
- Birge, J. R. 1994. Quasi-Monte Carlo approaches to option pricing. Technical report 94-19, Department of Industrial and Operations Engineering, University of Michigan, Ann Arbor, MI.
- Bismut, J. M., B. Skalli. 1977. Temps d'arrêt, théorie générale de processus et processus de Markov. *Z. Wahrscheinlichkeitstheorie verwandte Gebiete* 39 301–313.
- Bjerkstrand, P., G. Stensland. 1993. American exchange options and a put-call transformation: A note. *J. Bus. Finance, Accounting* 20 761–764.
- Black, F. 1976. The pricing of commodity contracts. *J. Financial Econom.* 3 167–179.
- Black, F., J. Cox. 1976. Valuing corporate securities: Some effects of bond indenture provisions. *J. Finance* 31 351–367.
- Black, F., E. Derman, W. Toy. 1990. A one-factor model of interest rates and its applications to treasury bond options. *Financial Analysts J.* 46(1) 33–39.
- Black, F., M. Scholes. 1973. The pricing of options and corporate liabilities. *J. Political Econom.* 81 637–654.
- Bollen, N. 1999. Real options and product life cycles. *Management Sci.* 45 670–684.
- Boyle, P. P. 1986. Option valuation using a three-jump process. *Internat. Options J.* 3 7–12.
- Boyle, P. P., D. Emanuel. 1980. The pricing of options on the generalized mean. Working paper, University of British Columbia, Vancouver, BC, Canada.
- Boyle, P. P., J. Evnine, S. Gibbs. 1989. Numerical evaluation of multivariate contingent claims. *Rev. Financial Stud.* 2 241–250.
- Boyle, P. P., S. Lau. 1994. Bumping up against the barrier with the binomial method. *J. Derivatives* 1(4) 6–14.
- Boyle, P. P., T. Vorst. 1992. Option replication in discrete time with transaction costs. *J. Finance* 47(1) 271–294.
- Boyle, P. P., M. Broadie, P. Glasserman. 1997. Monte Carlo methods for security pricing. *J. Econom. Dynam. Control* 21(8–9) 1276–1321.
- Brace, A., D. Gatarek, M. Musiela. 1997. The market model of interest rate dynamics. *Math. Finance* 7 127–154.
- Brennan, M., E. Schwartz. 1977. The valuation of American put options. *J. Finance* 32 449–462.
- Brennan, M., E. Schwartz. 1978. Finite difference methods and jump processes arising in the pricing of contingent claims: A synthesis. *J. Financial Quant. Anal.* 13 461–474.

- Brennan, M., E. Schwartz. 1985. Evaluating natural resource investments. *J. Bus.* 58(2) 135–157.
- Brezinski, C., M. Zaglia. 1991. *Extrapolation Methods: Theory and Practice*. North-Holland, New York.
- Broadie, M., J. B. Detemple. 1995. American capped call options on dividend-paying assets. *Rev. Financial Stud.* 8 161–191.
- Broadie, M., J. B. Detemple. 1996. American option valuation: New bounds, approximations, and a comparison of existing methods. *Rev. Financial Stud.* 9 1211–1250.
- Broadie, M., J. B. Detemple. 1997. The valuation of American options on multiple assets. *Math. Finance* 7 241–285.
- Broadie, M., P. Glasserman. 1996. Estimating security price derivatives using simulation. *Management Sci.* 42(2) 269–285.
- Broadie, M., P. Glasserman. 1997. Pricing American-style securities using simulation. *J. Econom. Dynam. Control* 21(8–9) 1323–1352.
- Broadie, M., P. Glasserman. 2004. A stochastic mesh method for pricing high-dimensional American options. *J. Comput. Finance* 7(4) 35–72.
- Broadie, M., Y. Yamamoto. 2003. Application of the fast Gauss transform to option pricing. *Management Sci.* 49(8) 1071–1088.
- Broadie, M., J. Cvitanić, M. Soner. 1998. Optimal replication of contingent claims under portfolio constraints. *Rev. Financial Stud.* 11(1) 59–79.
- Broadie, M., P. Glasserman, G. Jain. 1997. Enhanced Monte Carlo estimates for American option prices. *J. Derivatives* 5(1) 25–44.
- Canter, M., J. Cole, R. Sandor. 1997. Insurance derivatives: A new asset class for the capital markets and a new hedging tool for the insurance industry. *J. Appl. Corporate Finance* 10(3) 69–83.
- Carr, P., R. Jarrow. 1990. The stop-loss start-gain paradox and option valuation: A new decomposition into intrinsic and time value. *Rev. Financial Stud.* 3 469–492.
- Carr, P., D. Madan. 1998. Towards a theory of volatility trading. R. A. Jarrow, ed. *Volatility: New Estimation Techniques for Pricing Derivatives*. Risk Books, London, U.K., 417–427.
- Carr, P., L. Wu. 2003. The finite moment logstable process and option pricing. *J. Finance* 58(2) 753–777.
- Carr, P., R. Jarrow, R. Myneni. 1992. Alternative characterizations of American put options. *Math. Finance* 2 87–106.
- Carr, P., H. Geman, D. Madan, M. Yor. 2002. The fine structure of asset returns: An empirical investigation. *J. Bus.* 75 305–332.
- Carr, P., H. Geman, D. Madan, M. Yor. 2003. Stochastic volatility for Lévy processes. *Math. Finance* 13(3) 345–382.
- Carriere, J. 1996. Valuation of the early-exercise price for options using simulations and nonparametric regression. *Insurance, Math., Econom.* 19 19–30.
- Chen, R.-R., L. Scott. 1992. Pricing interest rate options in a two-factor Cox-Ingersoll-Ross model of the term structure. *Rev. Financial Stud.* 5 613–636.
- Chernov, M., E. Ghysels. 2000. A study towards a unified approach to the joint estimation of objective and risk neutral measures for the purpose of options valuation. *J. Financial Econom.* 56 407–458.
- Chernov, M., E. Ghysels, A. Gallant, G. Tauchen. 2003. Alternative models for stock price dynamics. *J. Econometrics* 116 225–257.
- Chesney, M., R. Gibson. 1995. State space symmetry and two-factor option pricing models. *Adv. Futures Options Res.* 8 85–112.
- Chesney, M., M. Jeanblanc-Picqué, M. Yor. 1997. Brownian excursions and Parisian barrier options. *Adv. Appl. Probab.* 29 165–184.
- Chesney, M., J. Cornwall, M. Jeanblanc-Picqué, G. Kentwell, M. Yor. 1997. Parisian pricing. *Risk* 1 77–79.
- Clewlow, L., C. Strickland. 2000. *Energy Derivatives: Pricing and Risk Management*. Lacima Publications, London, U.K.
- Collin-Dufresne, P., R. Goldstein. 2002. Do bonds span fixed income markets? Theory and evidence for “unspanned” stochastic volatility. *J. Finance* 57(4) 1685–1730.
- Cont, R., P. Tankov. 2004. *Financial Modelling with Jump Processes*. Chapman & Hall/CRC, Boca Raton, FL.
- Copeland, T., V. Antikarov. 2001. *Real Options: A Practitioner's Guide*. Texere, New York.
- Cortazar, G., E. Schwartz, M. Salinas. 1998. Evaluating environmental investments: A real options approach. *Management Sci.* 44(8) 1059–1070.
- Cox, J. C. 1975. Notes on options I: Constant elasticity of variance diffusion. Working paper, Stanford University, Stanford, CA.
- Cox, J. C. 1996. The constant elasticity of variance option pricing model. *J. Portfolio Management* 22 15–17.
- Cox, J. C., S. Ross. 1976. The valuation of options for alternative stochastic processes. *J. Financial Econom.* 3 145–166.
- Cox, J. C., J. Ingersoll, S. Ross. 1985. A theory of the term structure of interest rates. *Econometrica* 53 385–407.
- Cox, J. C., S. A. Ross, M. Rubinstein. 1979. Option pricing: A simplified approach. *J. Financial Econom.* 7 229–263.
- Cuoco, D. 1997. Optimal consumption and equilibrium prices with portfolio constraints and stochastic income. *J. Econom. Theory* 72 33–73.
- Curran, M. 1994. Valuing Asian and portfolio options by conditioning on the geometric mean price. *Management Sci.* 40 1705–1711.
- Cvitanić, J., W. Schachermayer, H. Wang. 2001. Utility maximization in incomplete markets with random endowment. *Finance Stochastics* 5 259–272.
- Cvsa, V., P. Ritchken. 2001. Pricing claims under GARCH-level dependent interest rate processes. *Management Sci.* 47(12) 1693–1711.
- Dai, Q., K. Singleton. 2000. Specification analysis of affine term structure models. *J. Finance* 55(5) 1943–1978.
- Das, S., R. Sundaram. 2000. A discrete-time approach to arbitrage-free pricing of credit derivatives. *Management Sci.* 46(1) 46–62.
- Dassios, A. 1995. The distribution of the quantile of a Brownian motion with drift and the pricing of related path-dependent options. *Ann. Appl. Probab.* 5 389–398.
- Davydov, D., V. Linetsky. 2001. The valuation and hedging of barrier and lookback options under the CEV process. *Management Sci.* 47 949–965.
- Debreu, G. 1959. *Theory of Value*. Wiley, New York.
- Delbaen, F., W. Schachermayer. 1994. A general version of the fundamental theorem of asset pricing. *Math. Ann.* 300 463–520.
- Delbaen, F., W. Schachermayer. 1996. The variance-optimal martingale measure for continuous processes. *Bernoulli* 2 81–105.
- Demeterfi, K., E. Derman, M. Kamal, J. Zou. 1999. A guide to volatility and variance swaps. *J. Derivatives* 6 9–32.
- Derman, E., I. Kani. 1994. Riding on a smile. *Risk* 7 32–39.
- Detemple, J. B. 2001. American options: Symmetry properties. J. Cvitanić, E. Jouini, M. Musiela, eds. *Handbooks in Mathematical Finance: Topics in Option Pricing, Interest Rates and Risk Management*. Cambridge University Press, Cambridge, U.K., 67–104.
- Detemple, J., C. Osakwe. 2000. The valuation of volatility options. *Eur. Finance Rev.* 4(1) 21–50.
- Detemple, J. B., S. Sundaresan. 1999. Non-traded asset valuation with portfolio constraints: A binomial approach. *Rev. Financial Stud.* 12 835–872.
- Detemple, J., W. Tian. 2002. The valuation of American options for a class of diffusion processes. *Management Sci.* 48 917–937.
- Detemple, J. B., S. Feng, W. Tian. 2003. The valuation of American call options on the minimum of two dividend-paying assets. *Ann. Appl. Probab.* 13 953–983.
- Diener, F., M. Diener. 2004. Asymptotics of the price oscillations of a vanilla option in a tree model. *Math. Finance* 13(2) 271–294.
- Dixit, A., R. Pindyck. 1994. *Investment under Uncertainty*. Princeton University Press, Princeton, NJ.

- Druskin, V., L. Knizhnerman, T. Tamarchenko, S. Kostek. 1997. Krylov subspace reduction and its extensions for option pricing. *J. Comput. Finance* 1(1) 63–79.
- Duan, J. C. 1995. The GARCH option pricing model. *Math. Finance* 5 13–32.
- Duan, J.-C., J.-G. Simonato. 1998. Empirical martingale simulation for asset prices. *Management Sci.* 44(9) 1218–1233.
- Duan, J.-C., G. Gauthier, J.-G. Simonato. 2001. Asymptotic distribution of the EMS option price estimator. *Management Sci.* 47(8) 1122–1132.
- Duffie, D. 1986. Stochastic equilibria: Existence, spanning number, and the “no expected financial gain from trade” hypothesis. *Econometrica* 54 1161–1183.
- Duffie, D., R. Kan. 1996. A yield-factor model of interest rates. *Math. Finance* 6 379–406.
- Duffie, D., K. Singleton. 1999. Modeling term structures of defaultable bonds. *Rev. Financial Stud.* 12 687–720.
- Duffie, D., K. Singleton. 2003. *Credit Risk*. Princeton University Press, Princeton, NJ.
- Duffie, D., J. Pan, K. Singleton. 2000. Transform analysis and asset pricing for affine jump-diffusions. *Econometrica* 68 1343–1376.
- Duffie, D., W. Fleming, M. Soner, T. Zariphopoulou. 1997. Hedging in incomplete markets with Hara utility. *J. Econom. Dynam. Control* 21 753–782.
- Dufresne, D. 2000. Laguerre series for Asian and other options. *Math. Finance* 10 407–428.
- Dunn, K., C. Spatt. 1985. An analysis of mortgage contracting: Prepayment penalties and the due-on-sale clause. *J. Finance* 40 293–308.
- Dupire, B. 1994. Pricing with a smile. *Risk* 7 18–20.
- Dybvig, P. H., S. A. Ross. 1987. Arbitrage. J. Eatwell, M. Milgate, P. Newman, eds. *The New Palgrave: Finance*. Norton, New York.
- El Karoui, N. 1981. Les aspects probabilistes du contrôle stochastique. Lecture notes in Mathematics 876, Springer-Verlag, Berlin, Germany, 73–238.
- El Karoui, N., I. Karatzas. 1991. A new approach to the Skorohod problem and its applications. *Stochastics Stochastics Rep.* 34 57–82.
- Embrecht, P., L. C. G. Rogers, M. Yor. 1995. A proof of Dassios’ representation of the α -quantile of Brownian motion with drift. *Ann. Appl. Probab.* 5 757–767.
- Engle, R. F. 1982. Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica* 50(4) 987–1008.
- Eraker, B., M. Johannes, N. Polson. 2003. The impact of jumps in equity index volatility and returns. *J. Finance* 58 1269–1300.
- Fakeev, A. G. 1971. Optimal stopping of a Markov process. *Theory Probab. Appl.* 16 694–696.
- Föllmer, H., M. Schweizer. 1991. Hedging of contingent claims under incomplete information. M. H. A. Davis, R. J. Elliott, eds. *Applied Stochastic Analysis Stochastics Monographs*, Vol. 5. Gordon and Breach, London, U.K., 389–414.
- Föllmer, H., D. Sondermann. 1986. Hedging of non-redundant contingent claims. W. Hildenbrand, A. MasColell, eds. *Contributions to Mathematical Economics: Essays in Honor of G. Debreu*. North-Holland, Amsterdam, The Netherlands, 205–223.
- Forsyth, P., K. Vetzal. 2002. Quadratic convergence for valuing American options using a penalty method. *SIAM J. Sci. Comput.* 23(6) 2095–2122.
- Forsyth, P., K. Vetzal, R. Zvan. 1999. A finite element approach to the pricing of discrete lookbacks with stochastic volatility. *Appl. Math. Finance* 6 87–106.
- Forsyth, P., K. Vetzal, R. Zvan. 2002. Convergence of lattice and PDE methods for valuing path-dependent options using interpolation. *Rev. Derivatives Res.* 5 273–314.
- Fouque, J.-P., G. Papanicolaou, R. Sircar. 2000. *Derivatives in Financial Markets with Stochastic Volatility*. Cambridge University Press, Cambridge, U.K.
- Fritelli, M. 2000. The minimal entropy martingale measure and the valuation problem in incomplete markets. *Math. Finance* 10 39–52.
- Gao, B., J.-Z. Huang, M. Subrahmanyam. 2000. The valuation of American barrier options using the decomposition technique. *J. Econom. Dynam. Control* 24 1783–1827.
- Garcia, R., E. Ghysels, E. Renault. 2004. The econometrics of option pricing. Y. Ait-Sahalia, L. Hansen, eds. *Handbook of Financial Econometrics*. Elsevier-North Holland, Amsterdam, The Netherlands, Forthcoming.
- Geman, H., ed. 1999. *Insurance and Weather Derivatives From Exotic Options to Exotic Underlyings*. Risk Books, London, U.K.
- Geman, H., M. Yor. 1993. Bessel processes, Asian options and perpetuities. *Math. Finance* 3 349–375.
- Geman, H., N. El Karoui, J.-C. Rochet. 1995. Changes of numeraire, changes of probability measure and option pricing. *J. Appl. Probab.* 32 443–458.
- Geske, R., H. Johnson. 1984. The American put option valued analytically. *J. Finance* 39 1511–1524.
- Ghysels, E., A. Harvey, E. Renault. 1996. Stochastic volatility. G. S. Madal, ed. *Handbook of Statistics*, Vol. 14. *Statistical Methods in Finance*. North Holland, Amsterdam, The Netherlands.
- Glasserman, P. 2004. *Monte Carlo Methods in Financial Engineering*. Springer-Verlag, New York.
- Glasserman, P., S. G. Kou. 2003. The term structure of simple forward rates with jump risk. *Math. Finance* 13 383–410.
- Glasserman, P., P. Heidelberger, P. Shahabuddin. 2000. Variance reduction techniques for estimating value-at-risk. *Management Sci.* 46 1349–1364.
- Grabbe, O. 1983. The pricing of call and put options on foreign exchange. *J. Internat. Money Finance* 2 239–253.
- Gukhal, C. R. 2001. Analytical valuation of American options on jump-diffusion processes. *Math. Finance* 11(1) 97–115.
- Hagan, P., D. Woodward. 1999. Equivalent Black volatilities. *Appl. Math. Finance* 6(3) 147–157.
- Hansen, A., P. Jorgensen. 2000. Analytical valuation of American-style Asian options. *Management Sci.* 46(8) 1116–1136.
- Harrison, M., D. Kreps. 1979. Martingales and arbitrage in multi-period security markets. *J. Econom. Theory* 20 381–408.
- Harrison, M., S. Pliska. 1981. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes Appl.* 11 215–260.
- Haugh, M., L. Kogan. 2004. Approximating pricing and exercising of high-dimensional American options: A duality approach. *Oper. Res.* 52(2) 258–270.
- Heath, D., R. Jarrow, A. Morton. 1990. Bond pricing and the term structure of interest rates: A discrete time approximation. *J. Financial Quant. Anal.* 25(4) 419–440.
- Heath, D., R. Jarrow, A. Morton. 1992. Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica* 60 77–105.
- Heston, S. L. 1993. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Rev. Financial Stud.* 6 327–343.
- Heston, S., G. Zhou. 2000. On the rate of convergence of discrete-time contingent claims. *Math. Finance* 10(1) 53–75.
- Hilliard, J., J. Kau, D. Keenan, W. Muller III. 1995. Pricing a class of American and European path-dependent securities. *Management Sci.* 41(12) 1892–1889.
- Hirsa, A., D. Madan. 2004. Pricing American options under variance gamma. *J. Comput. Finance* 7(2) 63–80.
- Ho, T., S.-B. Lee. 1986. Term structure movements and the pricing of interest rate contingent claims. *J. Finance* 41 1011–1029.

- Hodges, S., A. Neuberger. 1989. Optimal replication of contingent claims under transaction costs. *Rev. Futures Markets* 8 222–239.
- Hugonnier, J. 1999. The Feynman-Kac formula and pricing occupation time derivatives. *Internat. J. Theoret. Appl. Finance* 2 153–178.
- Hugonnier, J., D. Kramkov. 2004. Optimal investment with random endowments in incomplete markets. *Ann. Appl. Probab.* 14(2) 845–864.
- Hull, J. 2003. *Options, Futures and Other Derivatives*, 5th ed. Prentice Hall, Englewood Cliffs, NJ.
- Hull, J., A. White. 1987. The pricing of options on assets with stochastic volatilities. *J. Finance* 42 281–300.
- Hull, J., A. White. 1994. Numerical procedures for implementing term structure models I: Single factor models. *J. Derivatives* 2(1) 7–16.
- Hull, J., A. White. 1995. The impact of default risk on the prices of options and other derivative securities. *J. Banking Finance* 19 299–322.
- Ingersoll, J. 1987. *Theory of Financial Decision Making*. Rowman and Littlefield Publishers, NJ.
- Jacka, S. D. 1991. Optimal stopping and the American put. *Math. Finance* 1 1–14.
- Jacod, J., A. N. Shiryaev. 1998. Local martingales and the fundamental asset pricing theorem in the discrete-time case. *Finance Stochastics* 2 259–273.
- Jaillet, P., D. Lamberton, B. Lapeyre. 1990. Variational inequalities and the pricing of American options. *Acta Appl. Math.* 21 263–289.
- James, J., N. Webber. 2000. *Interest Rate Modelling*. John Wiley and Sons, Chichester, U.K.
- Jamshidian, F. 1989. An exact bond option pricing formula. *J. Finance* 44 205–209.
- Jamshidian, F. 1991. Forward induction and construction of yield curve diffusion models. *J. Fixed Income* 1 62–74.
- Jarrow, R., A. Rudd. 1983. *Option Pricing*. Dow Jones-Irwin, Homewood, IL.
- Jarrow, R., S. Turnbull. 1995. Pricing options on financial securities subject to default risk. *J. Finance* 50 53–86.
- Johannes, M. 2004. The statistical and economic role of jumps in continuous-time interest rate models. *J. Finance* 59(1) 227–260.
- Johnson, H. 1981. The pricing of complex options. Working paper, Louisiana State University, LA.
- Jorion, P. 2001. *Value at Risk: The Benchmark for Controlling Market Risk*, 2nd ed. McGraw-Hill, New York.
- Joy, C., P. P. Boyle, K. S. Tan. 1996. Quasi-Monte Carlo methods in numerical finance. *Management Sci.* 42 926–938.
- Kabanov, Y., C. Stricker. 2001. The Harrison-Pliska arbitrage pricing theorem under transaction costs. *J. Math. Econom.* 35 185–196.
- Kamrad, B., P. Ritchken. 1991. Multinomial approximating models for options with k state variables. *Management Sci.* 37 1640–1652.
- Karatzas, I. 1988. On the pricing of American options. *Appl. Math. Optim.* 17 37–60.
- Karatzas, I., S. G. Kou. 1996. On the pricing of contingent claims under constraints. *Ann. Appl. Probab.* 6(2) 321–369.
- Karatzas, I., S. Shreve. 1988. *Brownian Motion and Stochastic Calculus*. Springer Verlag, New York.
- Karatzas, I., S. Shreve. 1998. *Methods of Mathematical Finance*. Springer Verlag, New York.
- Karatzas, I., G. Zitkovic. 2003. Optimal consumption from investment and random endowment in incomplete semimartingale markets. *Ann. Probab.* 31(4) 1821–1858.
- Kau, J., D. Keenan, W. Muller, III, J. Epperson. 1990. The valuation and analysis of adjustable rate mortgages. *Management Sci.* 36(12) 1417–1431.
- Kemna, A. G. Z., A. C. F. Vorst. 1990. A pricing method for options based on average asset values. *J. Banking Finance* 14 113–129.
- Kholodnyi, V. A., J. F. Price. 1998. *Foreign Exchange Option Symmetry*. World Scientific Publishing Co., London, U.K.
- Kim, I. J. 1990. The analytic valuation of American options. *Rev. Financial Stud.* 3 547–572.
- Kinderlehrer, D., G. Stampacchia. 1980. *An Introduction to Variational Inequalities and their Applications*. Academic Press, New York.
- Kloeden, P. E., E. Platen. 1999. *Numerical Solution of Stochastic Differential Equations*, 3rd ed. Springer-Verlag, Berlin, Germany.
- Kou, S. G. 2002. A jump-diffusion model for option pricing. *Management Sci.* 48 1086–1101.
- Kreps, D. M. 1981. Arbitrage and equilibrium in economies with infinitely many commodities. *J. Math. Econom.* 8 15–35.
- Kulatilaka, N., E. Perotti. 1998. Strategic growth options. *Management Sci.* 44(8) 1021–1031.
- Kurpiel, A., T. Roncalli. 2000. Hopscotch methods for two-state financial models. *J. Computational Finance* 3(2) 53–89.
- Lee, R. 2004. Option pricing by transform methods: Extensions, unification, and error control. *J. Computational Finance* 7(3) 51–86.
- Leland, H. 1985. Option pricing and replication with transaction costs. *J. Finance* 40(5) 1283–1301.
- Leland, H. 1994. Corporate debt value, bond covenants, and optimal capital structure. *J. Finance* 49 1213–1252.
- Lewis, A. 2000. *Option Valuation under Stochastic Volatility*. Finance Press, Newport Beach, CA.
- Li, D. X. 2000. On default correlation: A copula function approach. *J. Fixed Income* 9(4) 43–54.
- Linetsky, V. 1999. Step options. *Math. Finance* 9 55–96.
- Linetsky, V. 2004. Spectral expansions for Asian (average price) options. *Oper. Res.* Forthcoming.
- Little, T., V. Pant, C. Hou. 2000. A new integral representation of the early exercise boundary for American put options. *J. Comput. Finance* 3(3) 73–96.
- Litzenberger, R., D. Beaglehole, C. Reynolds. 1996. Assessing catastrophe reinsurance-linked securities as a new asset class. *J. Portfolio Management* (Winter Special Issue) 76–86.
- Longstaff, F., E. Schwartz. 2001. Valuing American options by simulation: A simple least squares approach. *Rev. Financial Stud.* 14 113–148.
- Lucas, R. E. 1978. Asset prices in an exchange economy. *Econometrica* 46 1429–1445.
- Lucia, J., E. Schwartz. 2002. Electricity prices and power derivatives: Evidence from the Nordic power exchange. *Rev. Derivatives Res.* 5(1) 5–50.
- Madan, D., P. Carr, E. Chang. 1998. The variance gamma process and option pricing. *Eur. Financial Rev.* 2 79–105.
- Margrabe, W. 1978. The value of an option to exchange one asset for another. *J. Finance* 33 177–186.
- McConnell, J., M. Singh. 1993. Valuation and analysis of collateralized mortgage obligations. *Management Sci.* 39(6) 692–709.
- McDonald, R., M. Schroder. 1998. A parity result for American options. *J. Comput. Finance* 1 5–13.
- McKean, H. P. 1965. A free boundary problem for the heat equation arising from a problem in mathematical economics. *Indust. Management Rev.* 6 32–39.
- Merton, R. C. 1973. Theory of rational option pricing. *Bell J. Econom. Management Sci.* 4 141–183.
- Merton, R. C. 1974. On the pricing of corporate debt: The risk structure of interest rates. *J. Finance* 29 449–470.
- Merton, R. C. 1976. Option pricing when underlying stock returns are discontinuous. *J. Financial Econom.* 3 125–144.
- Merton, R. C. 1992. *Continuous Time Finance*. Blackwell Publishing.

- Miltersen, K., K. Sandmann, D. Sondermann. 1997. Closed form solutions for term structure derivatives with lognormal interest rates. *J. Finance* **52** 409–430.
- Miura, R. 1992. A note on look-back option based on order statistics. *Hitosubashi J. Commerce Management* **27** 15–28.
- Morton, K. W., D. F. Mayers. 1994. *Numerical Solutions of Partial Differential Equations*. Cambridge University Press, Cambridge, U.K.
- Naik, V., M. Lee. 1990. General equilibrium pricing of options on the market portfolio with discontinuous returns. *Rev. Financial Stud.* **3**(4) 493–521.
- Neuberger, A. 1994. The log contract. *J. Portfolio Management* **20**(Winter) 74–80.
- Niederreiter, H. 1992. *Random Number Generation and Quasi-Monte Carlo Methods*. Society for Industrial and Applied Mathematics, Philadelphia, PA.
- Owen, A. B. 1997. Scrambled net variance for integrals of smooth functions. *Ann. Statist.* **25** 1541–1562.
- Pan, J. 2002. The jump-risk premia implicit in options: Evidence from an integrated time-series study. *J. Financial Econom.* **63** 3–50.
- Parkinson, M. 1977. Option pricing: The American put. *J. Bus.* **50** 21–36.
- Paskov, S., J. Traub. 1995. Faster valuation of financial derivatives. *J. Portfolio Management* **22** 113–120.
- Pelsser, A. 2000. *Efficient Methods for Valuing Interest Rate Derivatives*. Springer Finance, London, U.K.
- Pham, H., N. Touzi. 1999. The fundamental theorem of asset pricing under cone constraints. *J. Math. Econom.* **31** 265–280.
- Pilipović, D. 1998. *Energy Risk*. McGraw-Hill, New York.
- Pratt, J. W. 1964. Risk aversion in the small and the large. *Econometrica* **32** 122–136.
- Randall, C., E. Kant, A. Chhabra. 1997. Using program synthesis to price derivatives. *J. Comput. Finance* **1**(2) 97–129.
- Rannacher, R. 1984. Finite element solution of diffusion problems with irregular data. *Num. Math.* **43** 309–327.
- Reiner, E. 1990. Valuation of path-dependent options. Presentation at the Waterloo Options Conference, Waterloo, Ontario, Canada.
- Ritchken, P., L. Sankarasubramanian, A. Vijh. 1993. The valuation of path-dependent contracts on the average. *Management Sci.* **39**(10) 1202–1213.
- Rogers, C. 2002. Monte Carlo valuation of American options. *Math. Finance* **12** 271–286.
- Rogers, C., Z. Shi. 1995. The value of an Asian option. *J. Appl. Probab.* **32** 1077–1088.
- Rubinstein, M. 1994. Implied binomial trees. *J. Finance* **49** 771–818.
- Rubinstein, M., E. Reiner. 1991. Breaking down the barriers. *Risk* **4** 28–35.
- Rudd, A., M. Schroeder. 1982. The calculation of minimum margin. *Management Sci.* **28** 1369–1379.
- Rutkowski, M. 1994. The early exercise premium representation of foreign market American options. *Math. Finance* **4** 313–325.
- Samuelson, P. A. 1965. Rational theory of warrant pricing. *Indust. Management Rev.* **6** 13–31.
- Schönbucher, P. 2003. *Credit Derivatives Pricing Models: Model, Pricing and Implementation*. John Wiley and Sons, New York.
- Schröder, M. 1999. Changes of numeraire for pricing futures, forwards and options. *Rev. Financial Stud.* **12** 1143–1163.
- Schröder, M. 2002. Mathematical ramifications of option valuation: The case of Asian options. Habilitationsschrift Universität, Mannheim, Germany.
- Schwartz, E. S., C. Zozaya-Gorostiza. 2003. Investment under uncertainty in information technology: Acquisition and development. *Management Sci.* **49**(1) 57–70.
- Schwartz, E. S., J. Smith. 2000. Short-term variations and long-term dynamics in commodity prices. *Management Sci.* **46**(7) 893–911.
- Schwartz, E. S., W. Torous. 1989. Prepayment and the valuation of mortgage-backed securities. *J. Finance* **44** 375–392.
- Schweizer, M. 1992. Mean-variance hedging for general claims. *Ann. Appl. Probab.* **2** 171–179.
- Schweizer, M. 1995a. On the minimal martingale measure and the Föllmer-Schweizer decomposition. *Stochastic Anal. Appl.* **13** 573–599.
- Schweizer, M. 1995b. Variance-optimal hedging in discrete time. *Math. Oper. Res.* **20** 1–32.
- Scott, L. 1997. Pricing stock options in a jump-diffusion model with stochastic volatility and stochastic interest rates: Application of Fourier inversion methods. *Math. Finance* **7** 413–426.
- Shiller, R. 2003. *The New Financial Order: Risk in the 21st Century*. Princeton University Press, Princeton, NJ.
- Shiryayev, A. N. 1978. *Optimal Stopping Rules*. Springer-Verlag, New York.
- Shreve, S., J. Vecer. 2000. Options on a traded account: Vacation calls, vacation puts and passport options. *Finance Stochastics* **4** 255–274.
- Stein, E. M., J. C. Stein. 1991. Stock price distributions with stochastic volatility: An analytical approach. *Rev. Financial Stud.* **4**(4) 727–752.
- Stulz, R. M. 1982. Options on the minimum or the maximum of two risky assets. *J. Financial Econom.* **10** 161–185.
- Sundaresan, S. 2002. *Fixed Income Markets and Their Derivatives*, 2nd ed. South-Western, Cincinnati, OH.
- Tavella, D., C. Randall. 2000. *Pricing Financial Instruments: The Finite Difference Method*. John Wiley and Sons, New York.
- Taylor, H. M. 1967. Evaluating a call option and optimal timing strategy in the stock market. *Management Sci.* **14** 111–120.
- Trigeorgis, L. 1996. *Real Options: Managerial Flexibility and Strategy in Resource Allocation*. MIT Press, Cambridge, MA.
- Tsitsiklis, J., B. Van Roy. 1999. Optimal stopping of Markov processes: Hilbert space theory, approximation algorithms, and an application to pricing high-dimensional financial derivatives. *IEEE Trans. Automatic Control* **44** 1840–1851.
- Tsitsiklis, J., B. Van Roy. 2001. Regression methods for pricing complex American-style options. *IEEE Trans. Neural Networks* **12**(4) (Special issue on computational finance) 694–703.
- Vasicek, O. 1977. An equilibrium characterization of the term structure. *J. Financial Econom.* **5** 177–188.
- Vecer, J. 2001. A new PDE approach for pricing arithmetic average Asian options. *J. Comput. Finance* **4** 105–113.
- Villeneuve, S. 1999. Exercise regions of American options on several assets. *Finance Stochastics* **3**(3) 295–322.
- Wilmott, P., J. Dewynne, S. Howison. 1993. *Option Pricing: Mathematical Models and Computation*. Oxford Financial Press, Oxford, U.K.
- Yor, M. 1995. The distribution of Brownian quantiles. *J. Appl. Probab.* **32** 405–416.
- Zipkin, P. 1993. Mortgages and Markov chains: A simplified evaluation model. *Management Sci.* **39**(6) 683–691.
- Zvan, R., P. Forsyth, K. Vetzal. 1998. Robust numerical methods for PDE models of Asian options. *J. Comput. Finance* **2** 39–78.