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Orbifold Gromov-Witten Invariants and Topological Strings

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Abstract. In this contribution I propose a (hopefully) pedagogical approach to the computation of orbifold Gromov-Witten invariants using mirror symmetry and topological string theory, focusing on the orbifold $\mathbb{C}^3/\mathbb{Z}_3$. Recent B-model developments on the mirror side, which led to predictions for "open orbifold Gromov-Witten invariants" of $\mathbb{C}^3/\mathbb{Z}_3$, are also addressed. This contribution is based on the results of [1] and [9].

1 Introduction

1.1 General idea. Historically, mirror symmetry has proved to be very successful in addressing problems of enumerative geometry, starting with the seminal work of Candelas et al [13] on the number of rational curves in the quintic threefold. The leitmotiv of the physical approach to Gromov-Witten theory is to map the problem to the mirror side, where computationally efficient techniques such as special geometry and the holomorphic anomaly equations of [7] are available.

Our goal is to pursue this line of thought in the context of enumerative geometry of orbifolds. Since direct evaluation of the Hodge integrals entering into the definition of orbifold Gromov-Witten invariants is rather complex (see for instance [8]), we aim at using mirror symmetry to map the problem of computing orbifold Gromov-Witten invariants to the mirror side.

1.2 Geometry. More precisely, we consider orbifolds of the form $\overline{X} = \mathbb{C}^3/G$, where G is a finite abelian group, $G \subseteq SU(3)$; such orbifolds admit a toric Calabi-Yau crepant resolution X. The prototypical example that we follow throughout this paper is $\overline{X} = \mathbb{C}^3/\mathbb{Z}_3$, which has a unique crepant resolution given by the total space of the canonical bundle over $\mathbb{P}^2$. Another example that could be studied

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along similar lines is $\mathbb{C}^3/\mathbb{Z}_4$, which has a toric crepant resolution given by the total space of the canonical bundle over F_2 — the second Hirzebruch surface.

1.3 Strategy. The usual mirror symmetric approach consisted in mapping the local problem of computing A-model topological string amplitudes near the large radius point in the stringy Kähler moduli space $\mathcal{KM}(X)$ of X — that is, generating functions of Gromov-Witten invariants of X — to the often simpler problem of computing B-model topological string amplitudes near the mirror point in the (suitably compactified) complex structure moduli space $\mathcal{M}(Y)$ of the mirror threefold Y. The main ingredients were the mirror map near the large radius point, and a formalism to compute the B-model amplitudes.

When X is the crepant resolution of an orbifold $\overline{X}$, the stringy Kähler moduli space $\mathcal{KM}(X)$ contains both a large radius point, where the A-model amplitudes generate Gromov-Witten invariants of X, and an orbifold point, where the A-model amplitudes now generate orbifold Gromov-Witten invariants of $\overline{X}$. Since mirror symmetry provides a global isomorphism between $\mathcal{KM}(X)$ and $\mathcal{M}(Y)$, we should be able to parallel the large radius procedure near the orbifold point to compute orbifold Gromov-Witten invariants of $\overline{X}$ from mirror symmetry. In order to make such an attempt successful, we need to understand the mirror map near the orbifold point, and we must provide a formalism to compute the B-model amplitudes at the mirror point. This is precisely the aim of this paper.

1.4 Opening up the strategy. In fact, we go even further. From a physics perspective, closed topological string theory can hardly live without its best friend open topological string theory. The latter is related to enumerative problems involving open Gromov-Witten invariants, which are concerned with maps from Riemann surfaces with boundaries. Again, one can use (an open version of) mirror symmetry at large radius combined with the open B-model to compute these invariants.

Using the new B-model formalism developed in [9, 33], as in the closed case we are able to extend this procedure to the orbifold point: we obtain predictions for generating functions of "open orbifold Gromov-Witten invariants". While open Gromov-Witten invariants are well-defined for smooth target spaces (see [24] for a computational definition using localization), it is not clear whether and how they can be generalized to the orbifold setting. However, the physics calculation provides an incentive for looking for a mathematical definition of such invariants.

1.5 Outline. In this contribution we propose a (hopefully) pedagogical approach to these ideas, which were mostly developed in [1] with Mina Aganagic and Albrecht Klemm and in [9] with Albrecht Klemm, Marcos Mariño and Sara Pasquetti. In section 2, we review the usual mirror symmetry approach at large radius. In section 3 we propose a parallel procedure to map orbifold Gromov-Witten theory to the mirror side, and use it to compute (high genus) orbifold Gromov-Witten invariants of $\mathbb{C}^3/\mathbb{Z}_3$. These two sections follow relatively closely the physics presentation in [8]. Section 4 is then devoted to the study of open amplitudes, using the B-model formalism of [9, 33]. As an application, we compute the "disk amplitude" of the orbifold $\mathbb{C}^3/\mathbb{Z}_3$.

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2 Mirror symmetry at large radius

Let us start by reviewing general features of mirror symmetry at large radius, and how it can be used to compute Gromov-Witten invariants of smooth Calabi-Yau threefolds. Good references on mirror symmetry include the two books [19, 28], while topological string theory is discussed in the book [34].

2.1 General statement. The main characters are:

- (X, Y): a mirror pair of smooth Calabi-Yau threefolds, where X is a (non-compact) toric Calabi-Yau threefold which is the crepant resolution of an orbifold $\overline{X}$ of the form $\mathbb{C}^3/G$, with $G \subseteq SU(3)$ a finite abelian group;
- $\mathcal{M}(Y)$: a suitable compactification of the complex structure moduli space of Y;
- $\mathcal{KM}(X)$: a suitable compactification of the complexified Kähler moduli space of X (the so-called *enlarged* or *stringy* Kähler moduli space).

Mirror symmetry provides a local isomorphism, called the *mirror map*, between $\mathcal{KM}(X)$ and $\mathcal{M}(Y)$, which maps a neighborhood of a maximally unipotent boundary point $q_0 \in \mathcal{M}(Y)$ to a neighborhood of a corresponding large radius point $p_0 \in \mathcal{KM}(X)$. Moreover, mirror symmetry tells us that the mirror map lifts to an isomorphism between the *A-model* amplitudes at $p_0 \in \mathcal{KM}(X)$, and the *B-model* amplitudes at $q_0 \in \mathcal{M}(Y)$. Of course, we have not yet defined what the A- and the B-model amplitudes are; we will come back to that in a minute. Let us start by expanding a little more on how the mirror map is defined.

2.2 The mirror map. The isomorphism in the neighborhood of a large radius point $p_0 \in \mathcal{KM}(X)$ can be described as follows. $H^2(X, \mathbb{C})$ is spanned by

$$t_1 T_1 + \ldots + t_r T_r, \tag{2.1}$$

where $T_1, \ldots, T_r \in H^2(X, \mathbb{C})$ is a basis of generators for the cone σ containing the large radius point $p_0 \in \mathcal{KM}(X)$ corresponding to X. The complexified Kähler parameters t_i parameterize $\mathcal{KM}(X)$ near p_0.

On the mirror side, when Y is compact, $\mathcal{M}(Y)$ is projective special Kähler,[1] hence can be parameterized by the periods of the holomorphic volume-form Ω on Y. More precisely, choose a symplectic basis of three-cycles $A^I, B_J \in H_3(Y, \mathbb{Z})$, $I, J = 0, \ldots, r$, and define the periods

$$\omega^I = \oint_{A^I} \Omega, \qquad \frac{\partial \mathcal{F}}{\partial \omega^I} = \oint_{B_I} \Omega, \tag{2.2}$$

where $\mathcal{F}$ is the prepotential. The periods are solutions of the Picard-Fuchs equations, with the following properties. In terms of coordinates q_i, $i = 1, \ldots, r$, centered at the maximally unipotent boundary point $q_0 \in \mathcal{M}(Y)$, there is a unique period which is holomorphic, say ω^0, and r periods have logarithmic behavior,

$$\omega^i = \frac{\omega^0}{2\pi i} \log(q_i) + \mathcal{O}(q_i), \qquad i = 1, \ldots, r. \tag{2.3}$$

There are r other periods which are quadratic in the logarithm, and one is cubic.

[1] See [23] for a mathematical description of special geometry.

The mirror map then consists in choosing appropriate combinations of these periods which are mapped to the complexified Kähler parameters t_i. It turns out that the mirror map reads

$$(t_1, \dots, t_r) \mapsto \frac{1}{\omega^0}(\omega^1, \dots, \omega^r), \tag{2.4}$$

which are sometimes called the *flat coordinates* on the projective special Kähler manifold $\mathcal{M}(Y)$.

When X and Y are noncompact, as in the case of a toric threefold X, $\mathcal{M}(Y)$ is not projective special Kähler anymore. However, various properties of special geometry still hold, such as the existence of a prepotential $\mathcal{F}$, and the parameterization of $\mathcal{M}(Y)$ by "periods" — once those are properly defined in the noncompact case — of the holomorphic volume-form. This can be understood either by seeing Y as the limit of a compact threefold [16], or intrinsically for the noncompact geometry (see for instance [30] for the latter point of view). Furthermore, in the noncompact case the mirror map is simplified by the fact that $\omega^0 = 1$, hence the t^i are directly identified with the logarithmic periods ω^i.

2.3 A-model on X. We now describe the A-model amplitudes on X. Recall that X is a toric Calabi-Yau threefold, which can be described as a symplectic quotient

$$X = \{X_1, \dots, X_{3+k} \in (\mathbb{C}^{3+k} - Z) | \sum_{i=1}^{3+k} L_i^a |X_i|^2 = r^a \quad, a = 1, \dots, k\}/(S^1)^k, \tag{2.5}$$

where the k S^1's act as

$$X_i \mapsto e^{i L_i^a \theta_a} X_i. \tag{2.6}$$

The charge matrix L gives the toric data of X, and for X to be Calabi-Yau the charges must satisfy

$$\sum_{i=1}^{3+k} L_i^a = 0, a = 1, \dots, k. \tag{2.7}$$

Z is the union of the singular sets Z_a implied by the relations $\sum_{i=1}^{3+k} L_i^a |X_i|^2 = r^a$, and the $r^a \in \mathbb{R}^+$ are the Kähler parameters of X.

To describe the A-model amplitudes, start with a theory — a nonlinear sigma model — of maps $f : \Sigma \to M$ from Riemann surfaces Σ to a target Calabi-Yau threefold M. There are two ways of twisting this sigma model to obtain topological theories, namely the A- and the B-models. The A-model does not depend on complex moduli, while the B-model is independent of Kähler moduli.

The A-model on X becomes a theory of holomorphic maps $f : \Sigma \to X$, which can be reformulated in terms of Gromov-Witten invariants of the target space X. In the neighborhood of $p_0 \in \mathcal{KM}(X)$, the A-model genus g amplitudes F_g become generating functions for the unmarked genus g Gromov-Witten invariants $N_{g,\beta}$ of X, that is,

$$F_g = \sum_{\beta \in H_2(X)} N_{g,\beta} Q^\beta, \tag{2.8}$$

where

$$Q^\beta = \mathrm{e}^{2\pi i \int_\beta \omega}, \tag{2.9}$$

and ω is a complexified Kähler class of X. In terms of the basis of $H^2(X,\mathbb{C})$ introduced above, we can write

$$Q^{\beta} = \prod_{i=1}^{r} Q_i^{\int_{\beta} T_i}, \tag{2.10}$$

with the exponentiated complexified Kähler parameters

$$Q_i = \mathrm{e}^{2\pi i t_i}. \tag{2.11}$$

2.4 B-model on Y. Following the general procedure of Hori-Vafa [29], the mirror Y of a toric Calabi-Yau threefold X has the form

$$Y = \{ww' = G(x,y;q_1,\ldots,q_r)\}, \tag{2.12}$$

where $w, w' \in \mathbb{C}$, $x, y \in \mathbb{C}^*$, and $q_1,\ldots,q_r$ are coordinates on $\mathcal{M}(Y)$ centered around the corresponding maximally unipotent boundary point. $G(x,y;q_1,\ldots,q_r)$ is a polynomial in x and y which describes a family of punctured Riemann surfaces embedded in $\mathbb{C}^*\times\mathbb{C}^*$, parameterized by the q_i's. In other words, Y is given by a conic fibration over $\mathbb{C}^*\times\mathbb{C}^*$, where the fiber degenerates to two lines over the family of Riemann surfaces

$$\Sigma(q_1,\ldots,q_r) = \{G(x,y;q_1,\ldots,q_r) = 0\}. \tag{2.13}$$

In the following, for brevity we will often omit the dependence on the q_i's, and call Σ a Riemann surface, understanding implictly that it is in fact a family of Riemann surfaces parameterized by the q_i's. Σ is generally called the *mirror curve* of X.

The B-model on Y localizes on constant maps, and becomes a theory of variations of complex structures of the target space Y. In particular, the genus 0 amplitude F_0 is simply given by the prepotential $\mathcal{F}$ of special geometry introduced above. Let us be a little more specific on special geometry for the noncompact Y, following [2]. When Y has the form (2.12), we consider only complex structure deformations that only involve varying the mirror curve Σ. The periods of the holomorphic volume-form

$$\Omega = \frac{\mathrm{d}w\mathrm{d}x\mathrm{d}y}{wxy} \tag{2.14}$$

over three-cycles reduce to integrals

$$\int_D \frac{\mathrm{d}x}{x}\wedge\frac{\mathrm{d}y}{y}, \tag{2.15}$$

where $D\subset\mathbb{C}^*\times\mathbb{C}^*$ is a real two-dimensional domain such that $\partial D\subset\Sigma$. The above integral reduces, in a given coordinate patch, to

$$\int_{\gamma} \log y \frac{\mathrm{d}x}{x}, \tag{2.16}$$

where $\gamma\in\Sigma$ is a one-cycle. Thus, the periods become integrals of the one-form

$$\lambda = \log y \frac{\mathrm{d}x}{x} \tag{2.17}$$

over one-cycles on the Riemann surface Σ. These periods govern the complex structure deformations, and, as in the compact case, are annihilated by a system of Picard-Fuchs equations which can be determined following the work of Chiang, Klemm, Yau and Zaslow [16]. Moreover, they satisfy special geometric relations, which define a prepotential $\mathcal{F}$ as before.

The higher genus B-model amplitudes are harder to describe mathematically. The genus 1 amplitude F_1 can be defined in terms of Ray-Singer torsion of Y. For $g \geq 2$, one can use the holomorphic anomaly equations of [7] — which may be understood as some sort of higher genus generalization of special geometry — to reconstruct the amplitudes recursively in the neighborhood of $q_0 \in \mathcal{M}(Y)$, up to an unknown holomorphic function at each genus depending on a finite number of constants. External data, such as boundary conditions, must be used to fix these functions.

2.5 Computing Gromov-Witten invariants of X. The mirror symmetric isomorphism between the A-model amplitudes at a large radius point and the B-model amplitudes near a maximally unipotent boundary point provides a concrete way of computing Gromov-Witten invariants of X, which has proved very succesful historically. This was the approach first used by Candelas et al [13] to compute the number of rational curves in the quintic threefold, which was then extended to higher genus in [7]. The two main ingredients entering in the calculation are:

- the mirror map near the large radius point;
- a framework to compute the B-model amplitudes near q_0, such as special geometry and the holomorphic anomaly equations.

Recall however that the holomorphic anomaly equations, in themselves, do not provide a complete framework to compute the higher genus amplitudes F_g, due to the holomorphic ambiguity persisting at each genus. It must be supplemented by additional data. However, recently boundary conditions for the amplitudes have been found which fully fix the ambiguities in some local geometries (such as local $\mathbb{P}^2$), and allow computations of high (but finite) genus amplitudes for compact threefolds (for instance, $g = 51$ for the quintic) [25, 26, 27].

2.6 Example: local $\mathbb{P}^2$. The main example that we will study in this paper is the orbifold $\overline{X} = \mathbb{C}^3/\mathbb{Z}_3$. Its unique crepant resolution is the toric threefold $X = \mathcal{O}(-3) \to \mathbb{P}^2$, which is the total space of the canonical bundle over $\mathbb{P}^2$. It is usually called local $\mathbb{P}^2$ in the physics literature. Let us describe mirror symmetry at the large radius point in the Kähler cone of local $\mathbb{P}^2$.

X is toric; its fan is generated by the one-dimensional cones:

$$\{(0,0,1),(0,-1,1),(-1,0,1),(1,1,1)\}. \tag{2.18}$$

These satisfy the linear relation

$$-3(0,0,1)+(0,-1,1)+(-1,0,1)+(1,1,1)=(0,0,0), \tag{2.19}$$

hence the charge matrix is simply $L = (-3, 1, 1, 1)$.

The mirror threefold Y is given by

$$Y = \{ww' = y^2 + y(1+x) + qx^3\}, \tag{2.20}$$

where $w, w' \in \mathbb{C}$, $x, y \in \mathbb{C}^*$ and q is a coordinate on $\mathcal{M}(Y)$ centered at the maximally unipotent boundary point $q_0 := \{q = 0\} \in \mathcal{M}(Y)$. That is, the mirror curve reads

$$\Sigma = \{y^2 + y(1+x) + qx^3 = 0\}, \tag{2.21}$$

which has genus 1 and three punctures. It can be seen pictorially by fattening the toric diagram of local $\mathbb{P}^2$, as in figure 1.

Following [16], the Picard-Fuchs system in the coordinate q reads

$$\mathcal{D}_q = \Theta_q^3 + 3q(3\Theta_q + 2)(3\Theta_q + 1)\Theta_q, \tag{2.22}$$

Figure 1 The toric diagram of local $\mathbb{P}^2$ and the mirror curve $\Sigma \subset \mathbb{C}^* \times \mathbb{C}^*$.

where we introduced the logarithmic derivative $\Theta_q = q\partial_q$. Solving $\mathcal{D}_1\Pi = 0$, we get the solution vector

$$\Pi = (1, \omega(q), \pi(q)), \tag{2.23}$$

where 1 is just the constant solution, $\omega(q)$ is a logarithmic solution and $\pi(q)$ is doubly logarithmic. The mirror map is given by

$$t = \omega(q) = \frac{1}{2\pi i}\left(\log q + \sum_{n=1}^{\infty} \frac{(-1)^n}{n}\frac{(3n)!}{(n!)^3} q^n\right), \tag{2.24}$$

where t is the complexified Kähler parameter of local $\mathbb{P}^2$. The period t_D dual to t is given by the combination

$$t_D = \frac{1}{2}\pi(q) - \frac{1}{2}\omega(q) - \frac{1}{4} = -3\frac{\partial \mathcal{F}}{\partial t}, \tag{2.25}$$

which defines the prepotential $\mathcal{F}$.[2]

The prepotential $\mathcal{F}$ computes the genus 0 amplitude F_0 at the large radius point of local $\mathbb{P}^2$. To compute the higher genus amplitudes, one can solve the holomorphic anomaly equations and supplement them with boundary conditions to fix the holomorphic ambiguity at each genus [32].

3 Mirror symmetry at the orbifold point

So far we only gave a local description of mirror symmetry, near a large radius point of $\mathcal{KM}(X)$. However, from a physics point of view, mirror symmetry should be global, in the sense that $\mathcal{KM}(X)$ should be globally isomorphic to $\mathcal{M}(Y)$, and similarly for the A- and the B-model amplitudes.

3.1 A global formulation of mirror symmetry. To study this global description we need to understand better the enlarged Kähler moduli space $\mathcal{KM}(X)$. Generically, the compactification $\mathcal{KM}(X)$ of the Kähler moduli space has a rather complicated structure which goes beyond the Kähler cone of X. Roughly speaking, $\mathcal{KM}(X)$ is obtained by gluing along common walls the Kähler cones of threefolds

[2]The unusual factor of -3 here comes from the fact that since Y is noncompact, it is not possible to find a symplectic basis of three-cycles; instead, the A- and the B-cycles have intersection number -3.

birationally equivalent to X. Some of these cones correspond to smooth threefolds related to X by flops; each such cone then contains a large radius point, which is mapped to a corresponding maximally unipotent boundary point in $\mathcal{M}(Y)$ on the mirror side. However, some other patches correspond to "non-geometric phases", by which we mean that they are obtained from X by contracting some cycles.

When X is toric, it turns out that $\mathcal{KM}(X)$ is also toric and is easily described by the *secondary fan* associated to X (see for instance [19], section 3.4 and chapter 6, for a more precise discussion). In toric geometry, the secondary fan of a toric manifold X is the fan generated by the one-dimensional cones encoding the linear relations between the one-dimensional cones in the fan of X. In other words, the one-dimensional cones in the secondary fan of X are given by the columns of the charge matrix L of X.

In this paper, we are interested in toric Calabi-Yau threefolds X which are the crepant resolutions of orbifolds $\overline{X}$ of the form $\mathbb{C}^3/G$. In these cases, $\mathcal{KM}(X)$ comprises a patch which contains an *orbifold point* $p_{orb} \in \mathcal{KM}(X)$, where the cycles of X are contracted to yield the orbifold $\overline{X}$. On the mirror side, the orbifold point is mapped to the point $q_{orb} \in \mathcal{M}(Y)$ at the intersection of the *orbifold divisors* in $\mathcal{M}(Y)$, which are the divisors around which the periods have finite monodromy. We will call q_{orb} the point of *finite monodromy*. When $\mathcal{M}(Y)$ is one-dimensional, as for local $\mathbb{P}^2$, the point of finite monodromy is simply the orbifold divisor itself, around which the periods have finite mondoromy.

We studied mirror symmetry near the large radius point p_0 corresponding to X; we now want to understand mirror symmetry near the orbifold point p_{orb} corresponding to $\overline{X}$. Our aim is to use mirror symmetry in this neighborhood to compute orbifold Gromov-Witten invariants of $\overline{X}$. In order to do so, we first need to describe the mirror map near the orbifold point, and then propose a formalism to compute the B-model amplitudes near this point.

3.2 The orbifold mirror map. Recall that the mirror map at large radius was defined by providing a map between the parameters spanning $H^2(X, \mathbb{C})$ in a basis canonically defined at p_0 to the logarithmic solutions of the Picard-Fuchs equations near q_0. In fact, the "full" mirror map may be understood as mapping the parameters spanning the even cohomology groups $H^{2n}(X, \mathbb{C})$, $n = 0, 1, 2, 3$, to corresponding solutions of the Picard-Fuchs equations.

The correct notion of cohomology of an orbifold $\mathbb{C}^3/G$ consists in the Chen-Ruan orbifold cohomology ring [14, 15]. With respect to ordinary cohomology, orbifold cohomology contains extra classes incorporating combinations of geometric and representation theoretic data associated to the action of G on $\mathbb{C}^3$. In particular, although $\mathbb{C}^3/G$ has no compact cycles, its orbifold cohomology ring is not trivial. The extra classes correspond to *twisted sectors* in physics language, and are in correspondence with the even cohomology classes at the large radius point.

The mirror map at the orbifold point can then be defined by mapping the parameters spanning the orbifold cohomology ring of $\mathbb{C}^3/G$ in a basis canonically defined at p_{orb} to corresponding solutions of the Picard-Fuchs solutions near the point of finite monodromy q_{orb}. The burden of the work consists in finding a canonical basis for the orbifold cohomology ring, and a good basis of solutions for the Picard-Fuchs equations. We claim that for simple orbifolds such as $\mathbb{C}^3/\mathbb{Z}_3$ we can determine the orbifold mirror map uniquely, up to a scale factor, simply by requiring that the B-model genus 0 amplitude be invariant under the $\mathbb{Z}_3$-monodromy around

q_{orb}. An equivalent prescription will be to match the representation theoretic data in the orbifold cohomology ring, keeping track of the action of $\mathbb{Z}_3$, to the action of the $\mathbb{Z}_3$-monodromy on the periods.

3.3 B-model at the point of finite monodromy. The next step is to provide a framework to compute the B-model amplitudes at the point of finite monodromy $q_{orb} \in \mathcal{M}(Y)$. In section 4, we will propose a new formalism to compute unambiguously all open and closed B-model amplitudes at any point in the moduli space, which can then be used to calculate the closed B-model amplitudes at q_{orb}. However, for the purpose of computing only the closed B-model amplitudes, we do not need such a formalism, the holomorphic anomaly equations being sufficient. Let us now explain why.

3.3.1 *The holomorphic anomaly equations and modularity.* For simplicity, in the following we focus on the case where $\mathcal{M}(Y)$ is one-dimensional; the results are easily generalized to the multi-dimensional case. Parameterize $\mathcal{M}(Y)$ by a period t, for instance the logarithmic period canonically chosen by the mirror map near a maximally unipotent boundary point. It turns out that the B-model amplitudes $\widehat{F}_g$,[3] when expanded in terms of the period t, are *not* holomorphic. Their anti-holomorphic dependence is in fact encoded in the holomorphic anomaly equation, which reads, in the noncompact case,

$$\partial_{\bar{t}} \widehat{F}_g = \frac{1}{2} C_{\bar{t}}^{(0)tt} \left(D_t \partial_t \widehat{F}_{g-1} + \sum_{n=1}^{g-1} \partial_t \widehat{F}_n \partial_t \widehat{F}_{g-n} \right), \tag{3.1}$$

where both $C_{\bar{t}}^{(0)tt}$, which is related to the Yukawa coupling, and the covariant derivative D_t can be defined in terms of special geometric data. We refer the reader to [7, 25] for precise definitions of these objects.

The crucial point for us is that (3.1) can be integrated, either using an iterative Feynman graph procedure as in [7, 1], or by direct integration of (3.1) using modular properties of the amplitudes [25]. For $g > 1$, the result is [1, 25]

$$\widehat{F}_g(\tau, \bar{\tau}) = (\partial_t \tau)^{2g-2} \sum_{k=0}^{3g-3} \widehat{E}_2^k(\tau, \bar{\tau}) c_k^{(g)}(\tau), \tag{3.2}$$

where we now expressed the amplitudes in terms of the period matrix

$$\tau = -\frac{1}{4\pi} \frac{\partial t_D}{\partial t} = -\frac{1}{4\pi} \frac{\partial^2 \mathcal{F}}{\partial t^2}, \tag{3.3}$$

with t_D the period dual to t and $\mathcal{F}$ the prepotential, to make the modular properties of the amplitudes manifest. In (3.2), we introduced the standard non-holomorphic extension of the second Eisenstein series $E_2(\tau)$, defined by

$$\widehat{E}_2(\tau, \bar{\tau}) = E_2(\tau) - \frac{3}{\pi \mathrm{Im}\,(\tau)}. \tag{3.4}$$

The $c_k^{(g)}(\tau)$ are holomorphic modular forms of weight $6(g-1) - 2k$ with respect to the monodromy group of the periods — that is, the monodromy group of the Picard-Fuchs equations. It can be shown that $\partial_t \tau$ is a holomorphic modular form

[3] The reason for the hat here is that these amplitudes are the *physical* B-model topological string amplitudes, which are defined globally all over the moduli space $\mathcal{M}(Y)$. To make contact with the objects considered so far we will need to take their limits at special points in $\mathcal{M}(Y)$, such as maximally unipotent boundary points and points of finite monodromy.

of weight -3, while $\widehat{E}_2(\tau,\bar{\tau})$ is an almost holomorphic modular form of weight 2. As a result, we get that the $\widehat{F}_g$'s are almost holomorphic modular forms of weight 0. Their only anti-holomorphic dependence appears through $\widehat{E}_2(\tau,\bar{\tau})$.

Moreover, the $c_k^{(g)}(\tau)$, for $k \geq 1$ and $g > 1$, are entirely fixed by the holomorphic anomaly equation. On the one hand, by solving the equation iteratively using a Feynman procedure, as in [7, 1], one can show that the $c_k^{(g)}(\tau)$ are determined in terms of derivatives of lower genus amplitudes; see [1] for the explicit formulae. On the other hand, since we know that the $c_k^{(g)}(\tau)$ are modular forms of weight $6(g-1)-2k$, we can express them as polynomials in the generators of the ring of modular forms of the given weight, and find the coefficients by directly integrating the holomorphic anomaly equation. The second procedure, which was studied in [25], is a much more efficient way to fix the forms $c_k^{(g)}(\tau)$ for $k \geq 1, g > 1$.

However, at each genus the form $c_0^{(g)}(\tau)$, which is a holomorphic modular form of weight $6(g-1)$, is undetermined by the holomorphic anomaly equation. It corresponds to the so-called holomorphic ambiguity at each genus, which comes from the fact that the equation (3.1) only fixes the anti-holomorphic derivative of the $\widehat{F}_g$'s. This is the main hindrance when using the holomorphic anomaly equations to compute the amplitudes F_g; additional data must be used to calculate the $c_0^{(g)}(\tau)$'s.

3.3.2 *Going to special points in* $\mathcal{M}(Y)$. As mentioned above, so far we discussed the *physical* amplitudes $\widehat{F}_g$, which are defined globally all over the moduli space. To make contact with the rest of the paper, we must consider the limits of these amplitudes near special points in $\mathcal{M}(Y)$.

Recall that the holomorphic anomaly equation (3.1) was written in terms of a chosen period t. Suppose that t is the logarithmic period near a maximally unipotent boundary point, canonically chosen by the mirror map. Then, it was explained in [7] that the limit of the amplitudes at this point, which are mirror dual to the Gromov-Witten generating functions at large radius, is obtained by sending $\bar{t} \to \infty$. In terms of the period matrix τ, this corresponds to

$$F_g = \lim_{\mathrm{Im}(\tau)\to\infty} \widehat{F}_g. \tag{3.5}$$

In mathematical language, since $\widehat{F}_g$ is an almost holomorphic modular form, it can be written as a finite power series in $\mathrm{Im}(\tau)^{-1}$, starting with a constant term. This limit is given by keeping only the constant term; this is the usual isomorphism between the ring of almost holomorphic modular forms and the ring of quasi-modular forms; see [31]. The F_g's are then quasi-modular forms of weight 0 under the monodromy group.

Now suppose we consider another period, call it σ, instead of t, and rewrite the holomorphic anomaly equations (3.1) in terms of σ. For instance, let σ be the period canonically chosen by the mirror map at the point of finite monodromy. Again, the limit of the amplitudes at the orbifold point will be given by sending $\mathrm{Im}(\tau_\sigma) \to \infty$, where τ_σ is now defined by (3.3) in terms of the dual periods σ and σ_D. Taking this limit should give the B-model amplitudes F_g^{orb} at the point of finite monodromy, which should be mirror dual to the generating functionals of orbifold Gromov-Witten invariants.

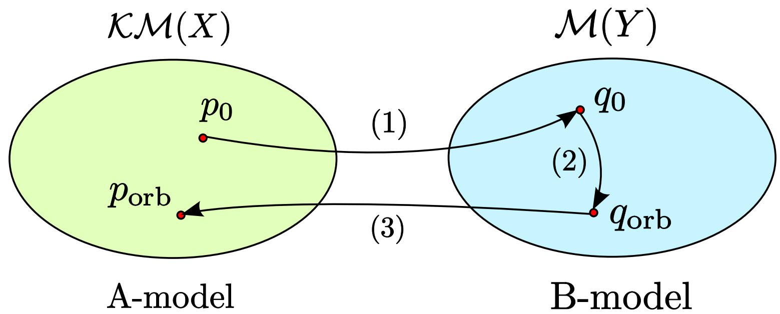

Figure 2 A schematic illustration of our strategy to compute orbifold Gromov-Witten invariants.

Hence, the holomorphic anomaly equation can be used to determine the F_g's at any point in the moduli space, including the point of finite monodromy, up to the holomorphic ambiguities $c_0^{(g)}$ at each genus.

3.4 Computing orbifold Gromov-Witten invariants of $\overline{X}$. Now, what kind of additional data can we use to fix the ambiguities at the point of finite monodromy? Well, the simple realization of [1] is that we in fact do not need any new data! Indeed, a crucial point is that the ambiguities at each genus, which correspond to the combinations

$$h_g = (\partial_t \tau)^{2g-2}\, c_0^{(g)}, \tag{3.6}$$

are globally defined holomorphic functions on the moduli space $\mathcal{M}(Y)$. Hence, if we know the amplitudes at a large radius point $q_0 \in \mathcal{M}(Y)$, we can fix the h_g, and use them, in conjunction with (3.2), to compute the amplitudes at the point of finite monodromy $q_{orb} \in \mathcal{M}(Y)$.

Our strategy to compute orbifold Gromov-Witten invariants should now be clear. Consider a toric Calabi-Yau threefold X which is the crepant resolution of an orbifold $\overline{X} = \mathbb{C}^3/G$; the enlarged Kähler moduli space $\mathcal{KM}(X)$ contains an orbifold point $p_{orb} \in \mathcal{KM}(X)$ corresponding to $\overline{X}$. We first determine the mirror maps near the large radius point $p_0 \in \mathcal{KM}(X)$ and the orbifold point $p_{orb} \in \mathcal{KM}(X)$. The calculation then proceeds in three steps, which are illustrated in figure 2.

1. We compute the generating functionals of Gromov-Witten invariants of X, using for instance the topological vertex [3], or localization of Hodge integrals. These are mapped by mirror symmetry at large radius to the B-model amplitudes near $q_0 \in \mathcal{M}(Y)$.
2. From these amplitudes we fix the holomorphic functions h_g, which are valid all over the moduli space and can be used to compute the B-model amplitudes at q_{orb} through (3.2).
3. Finally, we use the orbifold mirror map to extract the orbifold Gromov-Witten invariants of $\overline{X}$ from the B-model amplitudes at q_{orb}.

3.4.1 *An alternative strategy.* Note that there is a second method that can be used to compute the B-model amplitudes at q_{orb}, which was emphasized in [1]. The amplitudes at the maximally unipotent boundary point q_0, which we denote by

$F_g^\infty(\tau)$, can be written as

$$F_g^\infty(\tau) = (\partial_t \tau)^{2g-2} \sum_{k=0}^{3g-3} E_2^k(\tau) c_k^{(g)}(\tau), \tag{3.7}$$

since taking the limit $\mathrm{Im}(\tau) \to \infty$ simply corresponds to replacing $\widehat{E}_2(\tau, \bar{\tau})$ by $E_2(\tau)$. At the orbifold point q_{orb}, similarly the amplitudes read

$$F_g^{orb}(\tau_\sigma) = (\partial_\sigma \tau_\sigma)^{2g-2} \sum_{k=0}^{3g-3} E_2^k(\tau_\sigma) \tilde{c}_k^{(g)}(\tau_\sigma), \tag{3.8}$$

where now τ_σ is defined from the periods σ, σ_D canonically chosen at the point q_{orb}. By definition of the period matrix, τ and τ_σ must be related by a symplectic transformation, that is,

$$\tau_\sigma = \frac{a\tau + b}{c\tau + d} \quad \text{for some} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C}). \tag{3.9}$$

By using modular properties of the objects entering into (3.7) one can implement the symplectic transformation to calculate the amplitudes (3.8). That method relates directly, on the mirror side, the orbifold amplitudes to the amplitudes of its crepant resolution, in the spirit of the crepant resolution conjecture [11, 18, 35].

Note however that this procedure may seem a little bit strange from a modular forms point of view, since the transformation is generically in $SL(2, \mathbb{C})$ rather than in $SL(2, \mathbb{Z})$. But the transformation of the amplitudes from the point q_0 to the point q_{orb} can be made precise without direct reference to modular forms, as in [1], using wavefunction properties of the amplitudes. There it was also shown that this alternative approach is indeed equivalent to the first method proposed above.

3.5 Example: $\mathbb{C}^3/\mathbb{Z}_3$. Let us come back to our main example. Recall that $X = \mathcal{O}(-3) \to \mathbb{P}^2$ is the unique crepant resolution of the orbifold $\overline{X} = \mathbb{C}^3/\mathbb{Z}_3$. The enlarged Kähler moduli space $\mathcal{KM}(X)$ is described by the secondary fan associated to X. The secondary fan is generated by the two one-dimensional cones,

$$\{(-3), (1)\}. \tag{3.10}$$

Hence $\mathcal{KM}(X)$ is one-dimensional and has two patches, one of which contains the large radius point p_0 associated to X, and the other of which includes the orbifold point p_{orb} associated to $\overline{X}$.

The mirror threefold Y was described in (2.20), in terms of the coordinate q centered at the maximally unipotent boundary point $q_0 = \{q = 0\} \in \mathcal{M}(Y)$. The point of finite monodromy $q_{orb} \in \mathcal{M}(Y)$ is located at $q \to \infty$. A natural coordinate centered at q_{orb} is

$$\psi = q^{-1/3}, \tag{3.11}$$

as can be read off from the secondary fan. Note that under $\mathbb{Z}_3$-monodromy around $q_{orb} = \{\psi = 0\}$, ψ undergoes

$$\psi \mapsto \mathrm{e}^{2\pi i/3} \psi. \tag{3.12}$$

3.5.1 *The orbifold mirror map.* We first need to fix the mirror map near the orbifold point $p_{orb} \in \mathcal{KM}(X)$. Writing the Picard-Fuchs equations (2.22) in terms of the coordinate ψ, we get

$$\mathcal{D}_\psi = \psi^3\Theta_\psi^3 + 27(\Theta_\psi - 2)(\Theta_\psi - 1)\Theta_\psi, \tag{3.13}$$

with $\Theta_\psi = \psi\partial_\psi$. A solution vector to $\mathcal{D}_\psi \Pi^{orb} = 0$ is given by

$$\Pi^{orb} = (1, B_1(\psi), B_2(\psi)),$$

with

$$B_k(\psi) = \frac{(-1)^{k+1}\psi^k}{k}\ {}_3F_2\left(\frac{k}{3}, \frac{k}{3}, \frac{k}{3}; \frac{2k}{3}, 1 + \frac{k}{3}; \left(-\frac{\psi}{3}\right)^3\right). \tag{3.14}$$

Using the explicit expansion of the hypergeometric system we get

$$B_k(\psi) = \sum_{n\geq 0} \frac{(-1)^{3n+k+1}\psi^{3n+k}}{(3n+k)!}\left(\frac{\Gamma\left(n+\frac{k}{3}\right)}{\Gamma\left(\frac{k}{3}\right)}\right)^3. \tag{3.15}$$

As described in section 3.2, to get the orbifold mirror map we need to find linear combinations of the solutions above that are mapped to a basis for the orbifold cohomology of $\mathbb{C}^3/\mathbb{Z}_3$. The orbifold cohomology $H^*_{\text{orb}}(\mathbb{C}^3/\mathbb{Z}_3)$ has basis $\mathbf{1}_1$, $\mathbf{1}_\omega$ and $\mathbf{1}_{\omega^2}$, which are indexed by the elements $1, \omega, \omega^2 \in \mathbb{Z}_3$, with $\omega = \mathrm{e}^{2\pi i/3}$. The basis elements have degrees

$$\deg(\mathbf{1}_1) = 0, \qquad \deg(\mathbf{1}_\omega) = 2, \qquad \deg(\mathbf{1}_{\omega^2}) = 4. \tag{3.16}$$

Hence $H^*_{\text{orb}}(\mathbb{C}^3/\mathbb{Z}_3)$ is spanned by

$$\sigma_0\mathbf{1}_1 + \sigma_1\mathbf{1}_\omega + \sigma_2\mathbf{1}_{\omega^2}. \tag{3.17}$$

The orbifold mirror map will be given by mapping σ_1 to an appropriate combination of 1, $B_1(\psi)$ and $B_2(\psi)$.

Recall that monodromy around q_{orb} is given by $\psi \mapsto \mathrm{e}^{2\pi i/3}\psi$, which implies

$$(1, B_1(\psi), B_2(\psi)) \mapsto (1, \mathrm{e}^{2\pi i/3}B_1(\psi), \mathrm{e}^{4\pi i/3}B_2(\psi)). \tag{3.18}$$

But $\mathbf{1}_\omega$ corresponds to the element $\omega \in \mathbb{Z}_3$; thus, it is clear that σ_1 should be mapped to $B_1(\psi)$ directly, up to an overall scale factor. More precisely, we claim that the mirror map is given by

$$(\sigma_1, \sigma_2) = (B_1(\psi), B_2(\psi)). \tag{3.19}$$

Another way of arguing for this mirror map is by computing the genus 0 amplitude, as we do next. Up to scale, the above mirror map is the only map that yields a genus 0 amplitude which is invariant under orbifold monodromy. Note that this is also the mirror map that was proved in [17].

3.5.2 *Genus* 0 *amplitude.* Before computing the genus 0 amplitude, let us clarify the relation between the A-model amplitudes and Gromov-Witten theory at the orbifold point. At large radius, the genus g A-model amplitudes became generating functions for unmarked genus g Gromov-Witten invariants $N_{g,\beta}$ in homology classes $\beta \in H_2(X, \mathbb{Z})$. At the orbifold point, $\mathbb{C}^3/\mathbb{Z}_3$ contains no compact curve, hence the only invariants correspond to constant maps $\beta = 0$. However, the A-model amplitudes now become generating functions for marked orbifold Gromov-Witten invariants; more precisely

$$F_g^{orb} = \sum_{n=0}^{\infty} \frac{1}{(3n)!} N_{g,n}\sigma_1^{3n}, \tag{3.20}$$

where the invariants $N_{g,n}$ are given by $3n$ insertions of the orbifold cohomology class $\mathbf{1}_\omega$. Note that the unmarked ($n = 0$) invariants are only well-defined for $g \geq 2$. Moreover, we incorporated implicitly the fact that only invariants with $3n$ insertions are non-zero, which ensures that the amplitudes are invariant under orbifold monodromy.

The genus 0 amplitude is as usual given by the prepotential of special geometry, in the basis canonically chosen at q_{orb}. That is,

$$\sigma_2 = -3 \frac{\partial \mathcal{F}^{orb}}{\partial \sigma_1} = -3 \frac{\partial F_0^{orb}}{\partial \sigma_1}. \tag{3.21}$$

Integrating σ_2, we get:

$$N_{0,1} = \frac{1}{3}, \quad N_{0,2} = -\frac{1}{3^3}, \quad N_{0,3} = \frac{1}{3^2}, \quad N_{0,4} = -\frac{1093}{3^6}, \ldots \tag{3.22}$$

This potential has now been computed mathematically in three independent ways by Coates, Corti, Iritani and Tseng in [17], Bayer and Cadman in [6] and by Cadman and Cavalieri [12]. Agreement with these results for the genus 0 amplitude fixes the normalization of the mirror map (3.19).

3.5.3 *Higher genus amplitudes.* To extract the higher genus amplitudes of $\mathbb{C}^3/\mathbb{Z}_3$, we need to compute the holomorphic functions h_g at each genus g. This can be done easily at large radius, by first computing the A-model amplitudes through the topological vertex, and then mapping them to the B-model side using the usual mirror map at large radius. We obtain, for the marked invariants:

$$F_g^{orb} = \sum_{n=1}^{\infty} \frac{1}{(3n)!} N_{g,n} \sigma_1^{3n}, \tag{3.23}$$

with the numbers:[4]

g	$n=1$	2	3	4
0	$\frac{1}{3}$	$-\frac{1}{3^3}$	$\frac{1}{3^2}$	$-\frac{1093}{3^6}$
1	0	$\frac{1}{3^5}$	$-\frac{14}{3^5}$	$\frac{13007}{3^8}$
2	$\frac{1}{2^4 \cdot 3^4 \cdot 5}$	$-\frac{13}{2^4 \cdot 3^6}$	$\frac{20693}{2^4 \cdot 3^8 \cdot 5}$	$-\frac{12803923}{2^4 \cdot 3^{10} \cdot 5}$
3	$-\frac{31}{2^5 3^5 5 \cdot 7}$	$\frac{11569}{2^5 3^9 5 \cdot 7}$	$-\frac{2429003}{2^5 3^{10} 5 \cdot 7}$	$\frac{871749323}{2^4 3^{11} 5 \cdot 7}$
4	$\frac{313}{2^7 3^9 5^2}$	$-\frac{1889}{2^7 3^9}$	$\frac{115647179}{2^6 3^{13} 5^2}$	$-\frac{29321809247}{2^8 3^{12} 5^2}$
5	$-\frac{519961}{2^9 3^{11} 5^2 7 \cdot 11}$	$\frac{196898123}{2^9 3^{12} 5^2 7 \cdot 11}$	$-\frac{339157983781}{2^9 3^{14} 5^2 7 \cdot 11}$	$\frac{78658947782147}{2^9 3^{16} 5 \cdot 7}$
6	$\frac{14609730607}{2^{12} 3^{13} 5^3 7^2 11}$	$-\frac{258703053013}{2^{10} 3^{15} 5^1 7^2 11}$	$\frac{2453678654644313}{2^{12} 3^{14} 5^3 7^2 11}$	$-\frac{40015774193969601803}{2^{11} 3^{18} 5^3 7^2 11}$

The unmarked invariants ($n = 0$) for $g \geq 2$ (these are not well-defined for $g = 0, 1$) can also be calculated, and read

$$N_{2,0} = \frac{-1}{2160} + \frac{\chi(X)}{5760}, \quad N_{3,0} = \frac{1}{544320} - \frac{\chi(X)}{1451520}, \tag{3.24}$$
$$N_{4,0} = -\frac{7}{41990400} + \frac{\chi(X)}{87091200}, N_{5,0} = \frac{3161}{77598259200} - \frac{\chi(X)}{2554675200}, \ldots$$

[4]We would like to thank A. Klemm for the computation of the invariants with $g \geq 3$.

where $\chi(X)$ is the "Euler number" of $X = \mathcal{O}(-3) \to \mathbb{P}^2$. Although $\chi(X)$ is not really well defined mathematically since X is noncompact, its natural value can be found by noting that any vector bundle retracts to its zero section; hence, since $X = \mathcal{O}(-3) \to \mathbb{P}^2$, we obtain that $\chi(X) = \chi(\mathbb{P}^2) = 3$. Note that the numbers $N_{2,0}$ and $N_{3,0}$ have now been computed independently by myself and R. Cavalieri in [8], by evaluating directly the Hodge integrals entering in the definition of orbifold Gromov-Witten invariants.

3.5.4 *Symplectic transformation.* If we wanted to implement the second strategy to compute the invariants, as in [1], we would need the symplectic transformation between the basis of periods $(t_D, t, 1)$ at q_0 and the basis of periods $(\sigma_2, \sigma_1, 1)$ at q_{orb} canonically chosen by the mirror maps. The transformation is easily found by analytic continuation of the periods; see for example [20]. Define

$$c_1 = -\frac{1}{2\pi i}\frac{\Gamma(1/3)}{\Gamma(2/3)^2}, \qquad c_2 = \frac{1}{2\pi i}\frac{\Gamma(2/3)}{\Gamma(1/3)^2}, \qquad \beta = \frac{1}{(2\pi i)^3}, \qquad \omega = \mathrm{e}^{2\pi i/3}. \tag{3.25}$$

We get the transformation

$$\begin{pmatrix} t_D \\ t \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\beta\omega^2}{c_1} & \frac{\beta\omega}{c_2} & \frac{1}{3} \\ -c_2 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_2 \\ \sigma_1 \\ 1 \end{pmatrix}. \tag{3.26}$$

Note that this transformation is not quite symplectic, since its determinant is $-\beta$; that is, it changes the scale of the symplectic form. However, this can be taken into account by renormalizing the string coupling constant; see [1] for the details of this procedure.

4 Open mirror symmetry

So far we considered A- and B-model closed topological string theory, and used mirror symmetry and the holomorphic anomaly equations to compute orbifold Gromov-Witten invariants. In this section we would like to extend this line of thought to open topological string theory, following the recent work of [9].

The general idea is the same: we want to use mirror symmetry and B-model topological string theory to compute the A-model amplitudes at the orbifold point, which are related to Gromov-Witten theory of the orbifold $\overline{X}$. However, we now consider open topological string theory, which is a theory of maps with boundaries ending on submanifolds (usually called *branes*) of the target space. At large radius, A-model open topological strings compute the so-called open Gromov-Witten invariants of X [24]. On the contrary, to the best of our knowledge it is not clear what A-model open topological string theory computes in orbifold enumerative geometry; it should be some sort of generalization of open Gromov-Witten invariants to the orbifold setting. Keeping this in mind, we use the mirror symmetric physical approach to compute some generating functions for such as yet undefined "open orbifold Gromov-Witten invariants".

The main differences in considering open topological string theory are:

- the mirror map now comprises an open sector, which maps the moduli associated to the branes on the A- and the B-model sides;
- the holomorphic anomaly equations are no longer sufficient to compute the open B-model amplitudes. Hence, we need a new formalism to compute the amplitudes at the orbifold point.

We (partially) overcame these two obstacles in [9] for simple orbifolds such as $\mathbb{C}^3/\mathbb{Z}_3$. Let us first review general features of topological open string theory and the new B-model formalism of [9], which is based on the ideas of [33]. We then study how the open/closed mirror map can be fixed at the orbifold point, and compute some "open orbifold Gromov-Witten invariants" of $\mathbb{C}^3/\mathbb{Z}_3$.

4.1 Open A-model on X. A-model topological open string theory is a theory of holomorphic maps $f : \Sigma_{g,h} \to X$, where $\Sigma_{g,h}$ is a genus g Riemann surface with h holes, and X is a Calabi-Yau threefold, such that f maps the boundaries of $\Sigma_{g,h}$ to a special Lagrangian submanifold $L \subset X$, which is called a *brane*. Near a large radius point $p_0 \in \mathcal{KM}(X)$, the A-model produces the generating functionals

$$A_h^{(g)}(X_1, \ldots, X_h) = \sum_{w_i \in \mathbb{Z}} F_{g,w} X_1^{w_1} \cdots X_h^{w_h}, \tag{4.1}$$

where the X_i are the open moduli associated to the brane. The $F_{g,w}$ here are generating functions for open Gromov-Witten invariants of (L, X), counting maps at genus g, indexed by their relative homology class $\beta \in H_2(X, L)$, and with winding numbers w_i, $i = 1, \ldots, h$, specifying how many times the i-th boundary wraps around the one-cycle in L:

$$F_{g,w} = \sum_{\beta \in H_2(X,L)} N_{g,w,\beta}\, Q^\beta. \tag{4.2}$$

To define the open A-model amplitudes we needed to fix the brane L. When X is a toric threefold, the usual branes considered are noncompact and have topology $\mathbb{C} \times S^1$; we will call these branes *toric branes.*[5]

4.2 Open B-model on Y. In the B-model, the Lagrangian branes are mapped to holomorphic submanifolds of Y. More precisely, the toric brane L introduced above maps to a one-complex dimensional holomorphic submanifold of Y, given by

$$G(x, y; q_1, \ldots, q_r) = 0 = w'. \tag{4.3}$$

Hence, it is parameterized by w, and its moduli space corresponds to the mirror curve

$$\Sigma = \{G(x, y; q_1, \ldots, q_r) = 0\}. \tag{4.4}$$

The open string moduli X_i thus become variables on the mirror curve Σ.

We proposed in [9], following the ideas of [33], a complete recursive formalism to generate the B-model open and closed amplitudes. The formalism is based on the recursion relations of Eynard and Orantin which solve the loop equations of matrix models [22]. We refer the reader to [9] for the details of the formalism. To be concise, let us simply say that the open amplitudes are encoded by some meromorphic differentials living on the mirror curve Σ, while the closed amplitudes are functions on Σ. The recursive process can be understood as some sort of gluing process for the open amplitudes, whereby the gluing procedure consists in taking residues of some meromorphic differentials at the ramification points of the projection map $\Sigma \to \mathbb{C}^*$. It is important to note that, in contrast to the holomorphic anomaly equations for closed amplitudes (or their generalization to open amplitudes [36]), the recursive formalism proposed in [9] is complete, in the sense that there is

[5] See [9] for an extensive discussion of these branes and the different phases in the open moduli space.

no ambiguity at each genus, and no additional data is needed to fix the topological string amplitudes.

The input of the recursive process is rather simple. Using the notation above, to generate all the open and closed amplitudes near a given point in $\mathcal{M}(Y)$, say a maximally unipotent boundary point $q_0 \in \mathcal{M}(Y)$, one only needs to know the disk amplitude $A_1^{(0)}$ and the annulus amplitude $A_2^{(0)}$ near q_0.

4.2.1 *Disk amplitude.* Following the work of [4, 5], we know that the disk amplitude is simply given by the Abel-Jacobi map of the curve $\Sigma \subset \mathbb{C}^* \times \mathbb{C}^*$:

$$A_1^{(0)}(x) = \int \lambda = \int \log y(x) \frac{\mathrm{d}x}{x}, \tag{4.5}$$

where $y(x)$ is obtained by solving the defining equation of Σ for y; x here is the B-model open string modulus. Note that $A_1^{(0)}$ is globally defined all over the moduli space $\mathcal{M}(Y)$, and can be expanded both at maximally unipotent boundary points and the point of finite monodromy.

4.2.2 *Annulus amplitude.* The annulus amplitude is slightly more complicated. Generically, it is given by

$$A_2^{(0)}(x_1, x_2) = \int \left(B(x_1, x_2) - \frac{\mathrm{d}x_1 \mathrm{d}x_2}{(x_1 - x_2)^2} \right), \tag{4.6}$$

where $B(x_1, x_2)$ is the Bergman kernel of the Riemann surface Σ. This is defined to be the unique meromorphic differential on Σ with a double pole at $x_1 = x_2$ with no residue, and no other pole, and normalized such that

$$\oint_{A_I} B(x_1, x_2) = 0, \tag{4.7}$$

where (A_I, B^I) is a canonical basis of one-cycles on Σ. Note that by definition the Bergman kernel depends on a choice of canonical basis of cycles, or periods, on Σ.

In fact, we can be more precise on how the Bergman kernel depends on the choice of cycles. Under an $\mathrm{Sp}(2g, \mathbb{Z})$ transformation of the periods, the Bergman kernel transforms with a shift, as follows:

$$B(x_1, x_2) \mapsto B(x_1, x_2) - 2\pi i \omega(x_1)(C\tau + D)^{-1} C \omega(x_2), \tag{4.8}$$

with

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2g, \mathbb{Z}), \tag{4.9}$$

and τ is the period matrix. Here, $\omega(p)$ is the holomorphic differentials put in vector form. Hence, $B(x_1, x_2)$ provides an "open analog" – since it is a differential on Σ – to the second Eisenstein series $E_2(\tau)$, which also transforms with a shift. For more on the modular properties of the Bergman kernel, see [21, 22].

Therefore, once one knows the disk and the annulus amplitude at a given point in $\mathcal{M}(Y)$, one can generate unambiguously all other open and closed amplitudes at this point, using the recursive process of [9]. However, to map the results to the more interesting A-model where the amplitudes are related to Gromov-Witten theory, one must also know the open/closed mirror map near this point, which we now turn to.

4.3 The open/closed mirror map. In the context of open topological string theory, the usual mirror map must be supplemented with an open sector, which relates the A-model open modulus X to the B-model open modulus x. Consider first a large radius point $p_0 \in \mathcal{KM}(X)$ and its corresponding maximally unipotent boundary point $q_0 \in \mathcal{M}(Y)$. The open mirror map is defined such that, in conjunction with the closed mirror map, it lifts to an isomorphism of the open and closed A- and B-model amplitudes near p_0 and q_0.

Since x and X are $\mathbb{C}^*$ coordinates, as is customary in the physics literature we write $x = \mathrm{e}^u$, $X = \mathrm{e}^U$. While the closed mirror map is given by the integrals of the one-form λ over one-cycles on the mirror curve Σ (the "periods" of λ), the open mirror map, as argued in [4], is given by a chain integral of the one-form λ:

$$U = \int_{\alpha_u} \lambda, \tag{4.10}$$

where α_u is a given chain on Σ. In analogy with the flat coordinates on $\mathcal{M}(Y)$ — which we now call "closed flat coordinates" to avoid confusion — we will call this integral the *open flat coordinate* on the open/closed moduli space. As explained in [9], once evaluated, the chain integral always takes the form

$$U = u + 2\pi i \sum_{i=1}^{r} r_i^u \left(\frac{1}{2\pi i} \log q_i - \omega^i \right), \tag{4.11}$$

where the q_i and ω^i were introduced in section 2.2; the ω^i are the logarithmic periods (the closed flat coordinates) at $q_0 \in \mathcal{M}(Y)$, while the q_i are the usual coordinates on $\mathcal{M}(Y)$ centered at q_0. The coefficients r_i^u are rational numbers. In physics language, what this statement means is that the open string coordinates only receive corrections coming from closed string instantons.

What we are now interested in is determining the open mirror map at the point of finite monodromy $q_{orb} \in \mathcal{M}(Y)$ mirror dual to the orbifold point $p_{orb} \in \mathcal{KM}(X)$. By definition, we know that the open flat coordinate at q_{orb}, which we denote by U_{orb}, will always be given by a linear combination of the period integrals of λ and the chain integral (4.10). We claim that, as in the closed case, for simple orbifolds such as $\mathbb{C}^3/\mathbb{Z}_3$, the good linear combination can be uniquely determined, up to scale, by requiring that the disk amplitude be invariant under finite monodromy around q_{orb}.

Let us now put these ideas into practice for our orbifold friend $\mathbb{C}^3/\mathbb{Z}_3$.

4.4 Example: $\mathbb{C}^3/\mathbb{Z}_3$. Consider again the crepant resolution $X = \mathcal{O}(-3) \to \mathbb{P}^2$ of $\mathbb{C}^3/\mathbb{Z}_3$. We now fix a toric brane, that is, a special Lagrangian submanifold of X with topology $\mathbb{C} \times S^1$; for example, we consider an "outer brane" with zero framing, in the terminology of [9]. The mirror curve reads

$$\Sigma = \{ y^2 + y(1+x) + qx^3 = 0 \}. \tag{4.12}$$

The maximally unipotent boundary point is at $q = 0$. Recall that the only logarithmic period is given by

$$\omega(q) = \frac{1}{2\pi i} \left(\log q + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{(3n)!}{(n!)^3} q^n \right). \tag{4.13}$$

The chain integral (4.10) giving the open mirror map at large radius was performed in [4], and the result is

$$U = u + \frac{2\pi i}{3}\left(\frac{1}{2\pi i}\log q - \omega(q)\right) = u - \frac{1}{3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{(3n)!}{(n!)^3}q^n. \tag{4.14}$$

To compute the disk amplitude we need to integrate the one-form λ. Solving (4.12) for y, we get

$$y(x) = -\frac{1+x}{2} - \frac{1}{2}\sqrt{(1+x)^2 - qx^3}, \tag{4.15}$$

where we kept the branch of the square root which is relevant for the disk amplitude. The B-model disk amplitude at q_0 is then given by expanding

$$A_1^{(0)} = \int \log y(x)\frac{\mathrm{d}x}{x} \tag{4.16}$$

around $q = 0$. To get the A-model amplitude at large radius, we simply plug in the open and closed mirror maps.

4.4.1 *Orbifold mirror map.* We now want to compute the amplitude at the orbifold point $p_{orb} \in \mathcal{KM}(X)$. We first need to compute the B-model amplitude at the point of finite monodromy $q_{orb} \in \mathcal{M}(Y)$. Recall that q_{orb} is located at $\psi = 0$, with

$$\psi = q^{-1/3}. \tag{4.17}$$

The disk amplitude at q_{orb} is simply given by writing (4.15) in terms of ψ, and then expanding (4.16) around $\psi = 0$. However, to obtain the A-model amplitude at the orbifold point we need to plug in the open and closed mirror maps near the orbifold point. We already found the closed mirror map in (3.19), which is given by

$$(\sigma_1, \sigma_2) = (B_1(\psi), B_2(\psi)). \tag{4.18}$$

We now need to find the open mirror map.

Well, the open flat parameter must be given by a linear combination of the periods of λ and the chain integral (4.10). It turns out that the only combination, up to scale, that gives a disk amplitude invariant under the $\mathbb{Z}_3$-monodromy is

$$U_{orb} = U + \frac{2\pi i}{3}\omega(q) = u + \frac{1}{3}\log q. \tag{4.19}$$

In terms of exponentiated parameters, $X_{orb} = \mathrm{e}^{U_{orb}}$ and $x = \mathrm{e}^u$, we get

$$X_{orb} = xq^{1/3} = \frac{x}{\psi}. \tag{4.20}$$

This gives the open mirror map at the orbifold point, where X_{orb} is the A-model open modulus and x is the B-model open modulus.

4.4.2 *Orbifold disk amplitude.* Now, expanding the disk amplitude around $\psi = 0$ and plugging in the open and closed orbifold mirror maps (4.18) and (4.20) we obtain

$$
\begin{aligned}
A_1^{(0)} = &\left(\sigma_1 + \frac{\sigma_1^4}{648} - \frac{29\,\sigma_1^7}{3674160} + \frac{6607}{71425670400}\sigma_1^{10} + \dots\right) X_{orb} \\
&+ \left(-\frac{\sigma_1^2}{4} - \frac{\sigma_1^5}{1296} + \frac{197\,\sigma_1^8}{58786560} - \frac{5737}{142851340800}\sigma_1^{11} + \dots\right) X_{orb}^2 \\
&+ \left(-\frac{1}{3} + \frac{\sigma_1^3}{9} + \frac{\sigma_1^6}{1944} - \frac{\sigma_1^9}{544320} + \dots\right) X_{orb}^3 + \mathcal{O}(X_{orb}^4).
\end{aligned}
\tag{4.21}
$$

Up to the scale of X_{orb}, which we could not fix in the mirror map (4.20), we predict that this amplitude should generate "open orbifold Gromov-Witten invariants" of $\mathbb{C}^3/\mathbb{Z}_3$ at genus 0 with one hole; it would be fascinating to understand what these invariants really are. Finally, note that (4.21) is indeed invariant under the $\mathbb{Z}_3$-monodromy given by $\psi \mapsto e^{2\pi i/3}\psi$, as it should be.

A little note of care may be added here. In the B-model, the open amplitudes explicitly depend on a choice of parameterization for the mirror curve Σ. At large radius, this corresponds on the mirror A-model side to a choice of framing and phase for the Lagrangian brane [9]. However, for the A-model amplitudes at the orbifold point it is unclear what this freedom means. In particular, we should be more precise and say that, in the terminology of [9], the orbifold disk amplitude presented above should be for an "outer brane with zero framing" in the orbifold $\mathbb{C}^3/\mathbb{Z}_3$, even though we do not really understand what this means yet.

4.4.3 *Orbifold annulus amplitude.* To compute the remaining amplitudes at the orbifold point, we first need to obtain the B-model annulus amplitude at the point of finite monodromy. In turn, this involves computing the Bergman kernel at this point. The strategy goes as follows; we will use our knowledge of the Bergman kernel at q_0 to compute it at q_{orb}.

In terms of closed periods, moving from q_0 to q_{orb} involves a symplectic transformation of the periods; that is, the periods $(t_D, t, 1)$ at q_0 and the periods $(\sigma_2, \sigma_1, 1)$ at q_{orb} are related by the transformation (3.26). However, we have seen that the Bergman kernel is *not* modular under symplectic transformations of the periods; instead, it transforms with a shift, see (4.8). Hence, we can use this transformation formula, in conjunction with (3.26), to extract the Bergman kernel at q_{orb} from the Bergman kernel at q_0. This also involves analytically continuing the large radius Bergman kernel to the point $q_{orb} : \{\psi = 0\}$, which can be done with standard methods. In fact, a consistency check is that the analytic continuation of the large radius annulus amplitude, when expanded in terms of σ_1 after plugging in the open/closed orbifold mirror map, does not have rational coefficients; however, after shifting it using (4.8), the result should have a rational expansion.

We are presently in the process of computing the annulus amplitude at q_{orb} [10]. When this is done, we can generate all open and closed amplitudes at the point of finite monodromy q_{orb} unambiguously, and then plug in the open and closed orbifold mirror maps found above to compute the open and closed A-model amplitudes at the orbifold point. We hope to report on that in the near future.

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