Pacific Journal of Mathematics

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Volume 208 No. 1 January 2003

PACIFIC JOURNAL OF MATHEMATICS
Vol. 208, No. 1, 2003

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We identify the twisted sectors of a compact simplicial toric variety. We do the same for a generic nondegenerate Calabi-Yau hypersurface of an n-dimensional simplicial Fano toric variety and then explicitly compute $h^{1,1}_{\mathrm{orb}}$ and $h^{n-2,1}_{\mathrm{orb}}$ for the hypersurface. We give applications to the orbifold string theory conjecture and orbifold mirror symmetry.

1. Introduction.

The *K-Orbifold string theory conjecture* states that there is a natural isomorphism between the *Orbifold K-theory* of a Gorenstein orbifold and the ordinary K-theory of its crepant resolution (see [**AR**], [**Ru**]). To construct a natural isomorphism as the conjecture demands, is a very hard problem. But weaker versions of the conjecture that compare Euler numbers, Hodge numbers, etc. have been studied extensively in the literature in the case of orbifolds that are global-quotients. Batyrev [**B2**], and Batyrev and Dais [**BD**] proved, in particular, the equality of orbifold Hodge numbers and the Hodge numbers of smooth crepant resolutions for Gorenstein global-quotient orbifolds. But there were no results for nonglobal-quotient orbifolds.

In this paper, we show that the orbifold Hodge numbers of a generic Calabi-Yau hypersurface in a complex 4-dimensional simplicial Fano toric variety coincide with the Hodge numbers of its smooth crepant resolution. Besides being the first nonglobal-quotient example, this is also an important example in mirror symmetry. An immediate corollary of this is the pairing of orbifold Hodge numbers of Calabi-Yau 3-folds and their Batyrev mirrors.

While this paper was being refereed, extensive generalisations and related results were reported in [**BoM**], [**P**] and [**Y**]. [**BoM**] and [**P**] use the characterisation of twisted sectors presented here. [**Y**] uses the theory of algebraic stacks and achieves a deep result. The survey article [**Re**] nicely explains the heart of the matter.

Now we briefly describe how this article is organised. In Sections 2 and 3 we review relevant facts from orbifold cohomology and toric geometry respectively. In Section 4 we find characterisations for the twisted sectors of complete simplicial toric varieties and nondegenerate Calabi-Yau hypersurfaces of simplicial Fano toric varieties. In Section 5 we compute formulas for some orbifold Hodge numbers of these hypersurfaces, state some corollaries and then give an example.

2. Orbifolds.

2.1. Orbifold structure. Let U be a connected topological space, V a connected n-dimensional smooth manifold and G a finite group acting smoothly on V. Then an n-dimensional *uniformising system* of U is a triple (V, G, π), where $\pi : V \to U$ is a continuous map inducing a homeomorphism between the quotient space V/G and U. Two uniformising systems $(V_i, G_i, \pi_i), i = 1, 2$, are *isomorphic* if there is a diffeomorphism $\phi : V_1 \to V_2$ and an isomorphism $\lambda : G_1 \to G_2$ such that ϕ is λ-equivariant, and $\pi_2 \circ \phi = \pi_1$.

If (ϕ, λ) is an automorphism of (V, G, π), then there is a $g \in G$ such that $\phi(x) = g.x$ and $\lambda(a) = gag^{-1}$, for any $x \in V$ and $a \in G$.

Let $i : U' \to U$ be a connected open subset of U. An uniformising system (V', G', π') of U' is said to be induced from (V, G, π) if there is a monomorphism $\lambda : G' \to G$ and a λ-equivariant open embedding $\phi : V' \to V$ such that $i \circ \pi' = \pi \circ \phi$. The pair $(\phi, \lambda) : (V', G', \pi') \to (V, G, \pi)$ is called an *injection*. Two uniformising systems (V_1, G_1, π_1) and (V_2, G_2, π_2) of neighbourhoods U_1 and U_2 of a point p are *equivalent* at p if they induce isomorphic uniformising systems for a neighbourhood U_3 of p.

Let X be a Hausdorff, second countable topological space. An n-dimensional *orbifold structure* on X is given by the following data: For every point $p \in X$, there is an assigned neighbourhood U_p of p and an n-dimensional uniformising system (V_p, G_p, π_p) of U_p. The assignment satisfies the condition that for any point $q \in U_p$, (V_p, G_p, π_p) and (V_q, G_q, π_q) are equivalent at q.

Two orbifold structures $\{(V_p, G_p, \pi_p) : p \in X\}$ and $\{(V'_p, G'_p, \pi'_p) : p \in X\}$ are equivalent if for any $p \in X$, (V_p, G_p, π_p) and (V'_p, G'_p, π'_p) are equivalent at p. With a given equivalence class of orbifold structures on it, X is called an *orbifold.*

We call each U_p a uniformised neighbourhood of p, and (V_p, G_p, π_p) a chart at p. In fact we choose U_p to be small enough that G_p has the minimum possible order; that is, every element of G_p fixes the preimage of p in V_p. In what follows, this choice is assumed. Then a point p is called *smooth* if G_p is trivial; otherwise, it is called *singular.* X is called a *global-quotient* orbifold if X itself is an uniformised open set.

An orbifold X is called *reduced* if G_p acts effectively on V_p. Furthermore if a group element acts nontrivially, we require the fixed-point set to be of at least (real) codimension two, so that the complement is locally connected. We will deal with reduced orbifolds only. Note that even a reduced non-smooth orbifold can have a smooth underlying variety because of examples with complex reflections. *Gorenstein* orbifolds do not present this problem as they do not admit such complex reflections.

2.2. Orbifold (Chen-Ruan) cohomology. First we will describe the so-called twisted sectors. Consider the set of pairs:

$$\widetilde{X} = \{(p,(g)_{G_p}) \mid p \in X, g \in G_p\},$$

where $(g)_{G_p}$ denotes the conjugacy class of g in G_p. Then Kawasaki showed (see [**CR**]) that $\widetilde{X}$ has a natural orbifold structure. We will describe the connected components of $\widetilde{X}$. Recall that each point p has a local chart (V_p, G_p, π_p) which gives a local uniformised neighbourhood $U_p = \pi_p(V_p)$. If $q \in U_p$, up to conjugation, there is an injective homomorphism $G_q \to G_p$. For $g \in G_q$, the conjugacy class $(g)_{G_p}$ is well-defined. We define an equivalence relation $(g)_{G_q} \cong (g)_{G_p}$. Let T denote the set of equivalence classes. By an abuse of notation, we use (g) to denote the equivalence class to which $(g)_{G_q}$ belongs. $\widetilde{X}$ is decomposed as a disjoint union of connected components

$$\widetilde{X} = \bigsqcup_{(g)\in T} X_{(g)},$$

where

$$X_{(g)} = \{(p,(g')_{G_p}) \mid g' \in G_p, (g')_{G_p} \in (g)\}.$$

Definition 1. $X_{(g)}$ is called a *twisted sector* if $g \neq 1$. We call $X_{(1)} = X$ the *nontwisted* sector.

Assume that X is an almost complex orbifold with an almost complex structure J (see [**CR**]). Then for a singular point p, J gives rise to an effective representation $\rho_p : G_p \to GL(n, \mathbb{C})$. For any $g \in G_p$ we write $\rho_p(g)$, up to conjugation, as a diagonal matrix $\mathrm{diag}\left(e^{2\pi i \frac{m_{1,g}}{m_g}}, \dots, e^{2\pi i \frac{m_{n,g}}{m_g}}\right)$, where m_g is the order of g in G_p, and $0 \le m_{i,g} < m_g$. Define a function $\iota : \widetilde{X} \to \mathbb{Q}$ by

$$\iota(p,(g)_{G_p}) = \sum_{i=1}^{n} \frac{m_{i,g}}{m_g}.$$

This function $\iota : \widetilde{X} \to \mathbb{Q}$ is locally constant. Denote its value on $X_{(g)}$ by $\iota_{(g)}$. $\iota_{(g)}$ is called the *degree shifting number* of $X_{(g)}$. It has the following properties:

(1) $\iota_{(g)}$ is integral iff $\rho_p(g) \in SL(n, \mathbb{C})$.
(2) $\iota_{(g)} + \iota_{(g^{-1})} = \mathrm{rank}\,(\rho_p(g) - Id) = n - \dim(X_{(g)})$.

Definition 2. An almost complex orbifold is called *Gorenstein* if $\iota_{(g)}$ is integral for all (g).

Remark. An almost complex, complex or Kähler structure on X induces a corresponding similar structure on each $X_{(g)}$.

Definition 3. Let $\mathbb{F}$ be any field containing $\mathbb{Q}$ as a subfield. We define the *orbifold chomology groups* of X with coefficients in $\mathbb{F}$ by

$$H^d_{\mathrm{orb}}(X;\mathbb{F}) = \oplus_{(g)\in T} H^{d-2\iota_{(g)}}(X_{(g)};\mathbb{F}).$$

Definition 4. Let X be a closed complex orbifold. We define, for $0 \le p, q \le \dim_{\mathbb{C}} X$, *orbifold Dolbeault cohomology groups*

$$H^{p,q}_{\mathrm{orb}}(X;\mathbb{C}) = \oplus_{(g)} H^{p-\iota_{(g)},q-\iota_{(g)}}(X_{(g)};\mathbb{C}).$$

Remark. When X is a closed Kähler orbifold (so is each $X_{(g)}$), these Dolbeault groups are related to the singular cohomology groups of X and $X_{(g)}$ as in the manifold case, and the Hodge decomposition theorem holds for these cohomology groups.

Definition 5. We define *orbifold Hodge numbers* by

$$h^{p,q}_{\mathrm{orb}}(X) = \dim H^{p,q}_{\mathrm{orb}}(X;\mathbb{C}).$$

3. Facts from toric geometry.

3.1. Orbits, divisors and polytopes. A complex n-dimensional toric variety X_Ξ is constructed from an n-dimensional lattice N and a fan Ξ in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. We will write X for X_Ξ when there is no confusion. Let $M = \mathrm{Hom}\,(N,\mathbb{Z})$ denote the dual lattice, with dual pairing denoted by $\langle , \rangle$. If σ is a cone in N, the dual cone $\check{\sigma}$ in $M_{\mathbb{R}}$ determines a finitely generated commutative semigroup $R_\sigma = \check{\sigma} \cap M$. $\mathbb{C}[R_\sigma]$ is the $\mathbb{C}$-algebra with generators χ^m for each $m \in R_\sigma$ and relations $\chi^m\chi^{m'} = \chi^{m+m'}$. It gives an open affine subset $U_\sigma := \mathrm{spec}(\mathbb{C}[R_\sigma])$ of X. A face τ of σ gives an inclusion $U_\tau \to U_\sigma$.

$\Xi(d)$ denotes the set of d-dimensional cones of Ξ. We reserve the letter η to denote elements of $\Xi(1)$. For each η, let v_η denote the unique generator of the semigroup $\eta \cap N$. The $v_\eta \in \sigma$ are the *generators* of σ. If $r = |\Xi(1)|$ is the number of 1-dimensional cones, we sometimes write the v_η's as $v_1, \ldots, v_r$.

X is nonsingular iff for every cone in Ξ, its generators are part of a $\mathbb{Z}$-basis of N. Such a fan is called *smooth.* X is an orbifold iff the generators for every cone in Ξ are linearly independent over $\mathbb{R}$; and we say X and Ξ are *simplicial.*

The action of the torus $T_N = U_{\{0\}} = N \otimes \mathbb{C}$ on X has exactly one orbit O_τ corresponding to each cone $\tau \in \Xi$. Each $\overline{O_\eta}$ is an irreducible T_N-invariant Weil divisor denoted D_η. If X is complete, these generate the Chow group $A_{n-1}(X)$. Two T_N-invariant Weil divisors are linearly equivalent iff they differ by $\mathrm{div}\,(\chi^m) = \sum_\eta \langle m, v_\eta\rangle D_\eta$ for some $m \in M$. A Weil divisor $D = \sum_\eta a_\eta D_\eta$ is Cartier iff for each $\sigma \in \Xi$, there is $m_\sigma \in M$ such that

$\langle m_\sigma, v_\eta \rangle = -a_\eta$ whenever $\eta \subset \sigma$. A Cartier divisor D is ample iff $\langle m_\sigma, v_\eta \rangle > -a_\eta$ whenever η is not in σ and σ is n-dimensional.

If X is complete and $D = \sum_\eta a_\eta D_\eta$ is Cartier, then $\Delta_D = \{m \in M_\mathbb{R} : \langle m, v_\eta \rangle \geq -a_\eta \forall \eta\}$ is a polytope. A polytope is called *integral* if its vertices are integral. Δ_D is integral if D is ample. Conversely, given any n-dimensional integral polytope Δ one can canonically associate a projective toric variety $\mathbb{P}_\Delta$ to it. See [**CK**], Section 3.2.2 for details. It comes with a specific choice of ample divisor D_Δ such that $\Delta_{D_\Delta} = \Delta$. The T_N orbit closures of $\mathbb{P}_\Delta$ are in one-to-one correspondence with the nonempty faces F of Δ. There is a natural inclusion of toric varieties $\mathbb{P}_F \hookrightarrow \mathbb{P}_\Delta$.

Choose a basis for M. This corresponds to picking coordinates $t_1, \ldots, t_n$ for the torus T_N. Then, if $m \in M$ is written $m = (a_1, \ldots, a_n)$, we have $\chi^m = \prod_{i=1}^n t_i^{a_i}$, so we can write t^m instead of χ^m. For any $k \geq 0$, we have the space of Laurent polynomials $L(k\Delta) = \{f : f = \sum_{m \in k\Delta \cap M} \lambda_m t^m, \lambda_m \in \mathbb{C}\}$. Each $f \in L(k\Delta)$ gives the affine hypersurface $Z_f \subset T_N$ defined by $f = 0$. There is a T_N-equivariant map $H^0(X, \mathcal{O}(D)) \simeq \bigoplus_{m \in \Delta_D \cap M} \mathbb{C}\chi^m$. So $L(k\Delta) \simeq H^0(\mathbb{P}_\Delta, \mathcal{O}_{\mathbb{P}_\Delta}(kD_\Delta))$. Under this isomorphism, f corresponds to an effective divisor $\overline{Z_f} \subset \mathbb{P}_\Delta$. $\overline{Z_f}$ is a hypersurface. It is a compactification of Z_f for generic f.

We will use the following notation:

(a) $l(k\Delta) = |k\Delta \cap M| = \dim(L(k\Delta))$,

(b) $l^*(k\Delta) = |\{m \in k\Delta \cap M : m$ is not in any facet of $k\Delta \cap M\}|$.

3.2. Homogeneous coordinate ring. We introduce a variable x_η for each $\eta \in \Xi(1)$ and consider the polynomial ring $S = \mathbb{C}[x_\eta : \eta \in \Xi(1)]$. A monomial in S is written $x^D = \prod_\eta x_\eta^{a_\eta}$, where $D = \sum_\eta a_\eta D_\eta$ is an effective torus-invariant divisor on X. We say that x^D has degree $deg(x^D) = [D] \in A_{n-1}(X)$. Thus, S is graded by $A_{n-1}(X)$. Given a divisor class $\alpha \in A_{n-1}(X)$, S_α denotes the graded piece of S of degree α. We often write the variables as $x_1, \ldots, x_r$, where x_i corresponds to the cone generator v_i and $r = |\Xi(1)|$. The ring S, together with the grading defined above is called the *homogeneous coordinate ring* of X. See [**CK**], Chapter 3.2 for more details.

If τ is any cone of Ξ then the orbit closure $\overline{O_\tau}$ is given by the ideal $(x_i : v_i$ is a generator of $\tau)$ of S. Also the graded pieces of S have nice cohomological interpretation. We noted that $L(\Delta) \simeq H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(D_\Delta))$. Now the map sending the Laurent monomial t^m to $\prod_\eta x_\eta^{\langle m, v_\eta \rangle + a_\eta}$ induces an isomorphism $H^0(X, \mathcal{O}_X(D)) \simeq S_\alpha$, where $\alpha = [D] \in A_{n-1}(X)$.

3.3. Fano toric varieties. For any toric variety X, the anticanonical divisor $-K_X = \sum_\eta D_\eta$. A complete toric variety X is called *Fano* if $-K_X$ is Cartier and ample.

The anticanonical divisor of a Fano toric variety X detemines a *reflexive* polytope Δ. An integral polytope is called reflexive if:

(a) All facets Γ of Δ are supported by an affine hyperplane of the form $\{m \in M_{\mathbb{R}} : \langle m, v_\Gamma \rangle = -1\}$ for some $v_\Gamma \in N$.
(b) $\mathrm{Int}(\Delta) \cap M = \{0\}$.

The *polar polytope* Δ° of the reflexive polytope Δ is obtained by $\Delta^\circ = \{v \in N_{\mathbb{R}} : \langle m, v \rangle \geq -1 \text{ for all } m \in \Delta\} \subset N_{\mathbb{R}}$. The fan Ξ of X can be retrieved by coning over the proper faces of Δ°. This fan is called the *normal fan* of Δ and $X = \mathbb{P}_\Delta$. Δ° is also reflexive and $(\Delta^\circ)^\circ = \Delta$. The Fano toric variety constructed from the normal fan of Δ° is denoted by $\mathbb{P}_{\Delta^\circ}$.

We shall use F and F° to denote a face of Δ and Δ° respectively. There exists an inclusion reversing duality between the faces of Δ and Δ°. For instance, the face of Δ dual to the face F° of Δ° is defined as $\widehat{F^\circ} := \{m \in \Delta : \langle n, m \rangle = -1 \forall n \in F^\circ\}$. If τ is the cone in the normal fan of Δ associated to the face F°, then the orbit closure $\overline{O_\tau} = \mathbb{P}_{\widehat{F^\circ}}$.

Generic anticanonical hypersurfaces V in $\mathbb{P}_\Delta$ and V° in $\mathbb{P}_{\Delta^\circ}$ constitute two families of Calabi-Yau varieties, which are conjectured to be mirror families in the sense of Conformal Field Theory and called *Batyrev mirrors* in the literature. These varieties are orbifolds if the corresponding ambient toric variety is simplicial. Let $\widehat{V}$ and $\widehat{V^\circ}$ denote the MPCP resolutions (see [**B1**] or [**CK**]) of V and V° respectively. These are again Calabi-Yau. These are smooth if $n = 4$.

Remark. A simplicial Fano toric variety or its Calabi-Yau hypersurfaces are Gorenstein orbifolds, the orbifold structures arising naturally from the algebraic structures. In particular, all the degree shifting numbers are integers and the singular locus is of at least complex codimension two.

4. Twisted sectors.

We claim that the twisted sectors of a toric variety or a Calabi-Yau hypersurface, up to reduction of orbifold structure, can be identified with subvarieties. Note that in general a twisted sector could be a multiple cover of the corresponding singular locus even if the group actions are all Abelian.

4.1. Twisted sectors in simplicial toric variety. Let Ξ be any complete simplicial fan. Then the orbifold structure of the toric variety X_Ξ can be described as follows. Let σ be any n-dimensional cone of Ξ. Let $v_1, \ldots, v_n$ be the generators of σ. These are linearly independent in $N_{\mathbb{R}}$. Let N_σ be the sublattice of N generated by $v_1, \ldots, v_n$. Let $G_\sigma := N/N_\sigma$ be the quotient group. G_σ is finite and abelian.

Let σ' be the cone σ regarded in N_σ. Let $\check{\sigma'}$ be the dual cone of σ' in M_σ, the dual lattice of N_σ. $U_{\sigma'}$ =spec($\mathbb{C}[\check{\sigma'} \cap M_\sigma]$). Note that σ' is a smooth cone in N_σ. So $U_{\sigma'} \cong \mathbb{C}^n$.

There is a canonical dual pairing $M_\sigma/M \times N/N_\sigma \to \mathbb{Q}/\mathbb{Z} \to \mathbb{C}^*$, the first map by the pairing $\langle , \rangle$ and the second by $q \mapsto \exp(2\pi i q)$. Now G_σ acts on

$\mathbb{C}[M_\sigma]$, the group ring of M_σ, by $v(\chi^u) = \exp(2\pi i\langle u, v\rangle)\chi^u$, for $v \in N$ and $u \in M_\sigma$. Note that

$$(4.1.1) \qquad (\mathbb{C}[M_\sigma])^{G_\sigma} = \mathbb{C}[M].$$

Thus G_σ acts on $U_{\sigma'}$. Let π_σ be the quotient map. Then $U_\sigma = U_{\sigma'}/G_\sigma$. So U_σ is uniformised by $(U_{\sigma'}, G_\sigma, \pi_\sigma)$. For any $\tau < \sigma$, the orbifold structure on U_τ is same as the one induced from the uniformising system on U_σ. Then in the absence of complex reflections, toric gluing implies that $\{(U_{\sigma'}, G_\sigma, \pi_\sigma) : \sigma \in \Xi(n)\}$ defines a reduced orbifold structure on X. We show this in the general case.

Let $\mathcal{B}$ be the nonsingular matrix with generators $v_1, \dots, v_n$ of σ as rows. Then $\check{\sigma}'$ is generated in M_σ by the the column vectors $v^1, \dots, v^n$ of the matrix $\mathcal{B}^{-1}$. So $\chi^{v^1}, \dots, \chi^{v^n}$ are the coordinates of $U_{\sigma'}$. For any $\kappa = (k_1, \dots, k_n) \in N$, the corresponding coset $[\kappa] \in G_\sigma$ acts on $U_{\sigma'}$ in these coordinates as a diagonal matrix: $\mathrm{diag}(e^{2\pi i c_1}, \dots, e^{2\pi i c_n})$ where $c_i = \langle \kappa, v^i\rangle$. Such a matrix is uniquely represented by an n-tuple $a = (a_1, \dots, a_n)$ where $a_i \in [0, 1)$ and $c_i = a_i + b_i, b_i \in \mathbb{Z}$. In matrix notation, $\kappa\mathcal{B}^{-1} = a + b \iff \kappa = a\mathcal{B} + b\mathcal{B}$. We denote the integral vector $a\mathcal{B}$ in N by κ_a and the diagonal matrix corresponding to a by g_a. $\kappa_a \leftrightarrow g_a$ gives a one to one correspondence between the elements of G_σ and the integral vectors in N that are linear combinations of the generators of σ with coefficients in $[0, 1)$.

Now let us examine the orbifold chart induced by $(U_{\sigma'}, G_\sigma, \pi_\sigma)$ at any point $x \in U_\sigma$. By the orbit decomposition of U_σ, there is a unique face τ of σ such that $x \in O_\tau$. Without loss of generality assume that τ is generated by $v_1, \dots, v_j, j \le n$. Then any preimage of x with respect to π_σ has coordinates $\chi^{v^i} = 0$ iff $i \le j$. Let $z = (0, \dots, 0, z_{j+1}, \dots, z_n)$ be one such preimage. Let $G_\tau := \{g_a \in G_\sigma : a_i = 0 \text{ if } j + 1 \le i \le n\}$. We can find a small neighbourhood $W \subset (\mathbb{C}^*)^{n-j}$ of $(z_{j+1}, \dots, z_n)$ such that the inclusions $\mathbb{C}^j \times W \hookrightarrow U_{\sigma'}$ and $G_\tau \hookrightarrow G_\sigma$ induces an injection of uniformising systems $(\mathbb{C}^j \times W, G_\tau, \pi) \hookrightarrow (U_{\sigma'}, G_\sigma, \pi_\sigma)$ on some small open neighbourhood U_x of x. So we have $G_x = G_\tau$ and an orbifold chart $(\mathbb{C}^j \times W, G_\tau, \pi)$. Note that G_τ can be constructed from the set $\{\kappa_a = \sum_{i=1}^{j} a_i v_i : \kappa_a \in N, a_i \in [0, 1)\}$ which is completely determined by τ and hence is independent of σ.

Now we determine the twisted sectors. Take any $x \in X$. x belongs to a unique O_τ. Assume the generators of τ are $v_1, \dots, v_j$. Consider any n-dimensional $\sigma > \tau$. Assume $v_1, \dots, v_n$ generate σ. First suppose there is a g_a in G_x such that $a_i \ne 0, \forall i \le j$ i.e., κ_a lies in the interior of τ. We want to find the twisted sector $X_{(g_a)}$. Consider g_a as an element of G_σ. It is clear that g_a fixes $z \in U_{\sigma'}$ iff $z_1 = \cdots = z_{j+s} = 0$, for some $s \ge 0$. Hence $\pi_\sigma(z) \in O_\tau$ or $\pi_\sigma(z) \in O_\delta$ for some $\delta > \tau$. So $X_{(g_a)} \cap U_\sigma = \overline{O_\tau} \cap U_\sigma$. Since a twisted sector is connected, $X_{(g_a)} = \overline{O_\tau}$. If $g_a \in G_x$ is such that (without loss of generality) only $a_1, \dots, a_k \ne 0, k < j$, then $g_a \in G_\delta$ where δ is the cone generated by $v_1, \ldots, v_k$ and by the above argument $X_{(g_a)} = \overline{O_\delta}$. Thus we have proved the following theorem:

Theorem 1. *A twisted sector of any complete simplicial toric variety X_Ξ is isomorphic to a subvariety $\overline{O_\tau}$ of X_Ξ for some cone $\tau \in \Xi$. Moreover, there is a one-to-one correspondence between the set of twisted sectors of the type $\overline{O_\tau}$ and the set of integral vectors in the interior of τ which are linear combinations of the* 1*-dimensional generators of τ with coeffients in* $(0, 1)$.

Note that the degree shifting number $\iota_{(g_a)} = \sum a_i$. Now if X_Ξ is Fano, i.e., Ξ is obtained by coning over the faces of a reflexive polytope Δ°, then the twisted sectors with $\iota = 1$ are in one to one correspondence with the integral interior points of faces of Δ°.

4.2. Twisted sectors of a hypersurface of a Fano variety. We identify the twisted sectors of a generic nondegenerate anticanonical (Calabi-Yau) hypersurface V of a simplicial Fano toric variety $X = \mathbb{P}_\Delta$. *Nondegenerate* means that $V \cap O_\tau$ is either empty or a smooth subvariety of codimension one in O_τ, for each torus orbit O_τ in X. Then V turns out to be a suborbifold of X. Also nondegeneracy is a generic condition. We show that $V = \overline{Z_f}$, for a generic $f \in L(\Delta)$, is nondegenerate and a suborbifold of X. For a different treatment of this, see [**BC**].

Consider any n-dimensional cone σ with generators $v_1, \ldots, v_n$. For notational simplicity set $\chi^{v^i} = z_i$. Then $z_1, \ldots, z_n$ are the coordinates of $U_{\sigma'}$. Let Y be the preimage of $V \cap U_\sigma$ in $U_{\sigma'}$ with respect to π_σ. Then Y is defined by the equation $\sum_{m \in \Delta \cap M} \lambda_m \prod_{i=1}^n z_i{}^{\langle m, v_i \rangle + 1} = 0$. This is because, $t^m = \prod_{i=1}^n z_i{}^{q_i} \iff m = \sum q_i v^i = \mathcal{B}^{-1} q \iff q = m\mathcal{B} \iff q_i = \langle m, v_i \rangle$. The one is added to ensure that V is anticanonical. Note that by definition of Δ, $\langle m, v_i \rangle + 1 \geq 0$. If $\lambda_{m_\sigma} \neq 0$ then Y does not pass through the origin. It can be checked from this description using Bertini's theorem that for generic values of the coefficients λ_m, Y is a smooth submanifold of $U_{\sigma'}$ that intersects the coordinate planes $z_{i_1} = \cdots = z_{i_j} = 0$ transversely.

Y is G_σ-stable by (4.1.1). When Y is smooth, all singularities of $V \cap U_\sigma$ are quotient singularities induced by action of G_σ on Y. Since there are only finitely many n-dimensional cones, V is nondegenerate and a suborbifold of X. $(Y, G_\sigma, \pi_\sigma)$ is an uniformising system for $V \cap U_\sigma$. Let τ be the face of σ obtained by coning over the face F° of Δ°. Without loss of generality let $v_1, \ldots, v_j : j \geq 2$ be the generators of τ. (We remarked in Section 3.3 that there is no codimension one singularity.) We want to find a chart for any point $x \in V \cap O\tau$. By our earlier remark that Y misses the origin of $U_{\sigma'}$, $V \cap O_\sigma$ is empty. So we need only consider proper faces τ of σ. First assume that F° has codimension 2. This means that $\overline{O_\tau}$ has dimension 1. Then the corollary on page 112 of [**F**] implies that the number of points in $V \cap \overline{O_\tau}$ is the normalised volume of $\widehat{F^\circ}$, which equals $l^*(\widehat{F^\circ}) + 1$ since $\widehat{F^\circ}$

has dimension 1. Since the only other points in $\overline{O_\tau}$ in this case are O_σ for n-dimensional cones $\sigma > \tau$, all the intersection points actually lie in O_τ. If codimension F° is bigger than 2, then $V \cap \overline{O_\tau}$ is irreducible by Bertini.

Following the same notation as before, x has a small neighbourhood $U_x \cap Y$ such that $((\mathbb{C}^j \times W) \cap Y, G_\tau, \pi)$ is a chart for V at x. $\mathbb{C}^j \times W$ is , as before, a suitable neighbourhood of some preimage z of x in $U_{\sigma'}$. The tangent space TY_z is a G_τ-stable subspace of $T\mathbb{C}^n_z$. Any $g_a \in G_\tau$ acts trivially on TW_z =span$\{\partial/\partial z_i, i = j+1, \ldots, n\}$. By transversality, we can choose basis $\{\xi_1, \ldots, \xi_n\}$ of $T\mathbb{C}^n_z$ such that $\xi_i \in TY_z$ $\forall i \leq n-1$ and $\xi_n \in TW_z$. g_a acts trivially on ξ_n. This implies that the degree shifting number of $g_a|_{TY_z}$ is still $\sum_{i=1}^j a_i$.

From the description of the charts, it is clear that twisted sectors of V are isomorphic to $V \cap \overline{O_\tau}$ where $2 \leq \dim(\tau) \leq n-1$. Recall that $\overline{O_\tau} = \mathbb{P}_{\widehat{F^\circ}}$ where $\widehat{F^\circ}$ is the face of Δ dual to F°. In particular we have the following theorem:

Theorem 2. *Let V be a generic nondegenerate anticanonical hypersurface of an n-dimensional simplicial Fano toric variety $\mathbb{P}_\Delta$. Then the twisted sectors of V are isomorphic to $V \cap \mathbb{P}_{\widehat{F^\circ}}$ for some face F° of Δ° such that $1 \leq \dim(F^\circ) \leq n-2$. There is exactly one twisted sector of this type having $\iota = 1$, corresponding to each integral interior point of F° if $\dim(F^\circ) < n-2$. If $\dim(F^\circ) = n-2$, then there are exactly $l^*(\widehat{F^\circ}) + 1$ twisted sectors of this type having $\iota = 1$, corresponding to each integral interior point of F°.*

5. Orbifold Hodge numbers.

5.1. $h^{1,1}_{\mathrm{orb}}(V)$. Let $V_{(g)}$ denote a twisted sector of the hypersurface V and $\iota_{(g)}$ its degree shifting number. $h^{1,1}_{\mathrm{orb}}(V) = h^{1,1}(V) + \sum_{\iota_{(g)}=1} h^{0,0}(V_{(g)})$. Since $h^{0,0}(V_{(g)}) = 1$ for each twisted sector, by Theorem 2 we obtain

$$\sum_{\iota_{(g)}=1} h^{0,0}(V_{(g)}) = \sum_{1 \leq \dim(F^\circ) \leq n-2} l^*(F^\circ) + \sum_{\dim(F^\circ)=n-2} l^*(F^\circ) l^*(\widehat{F^\circ})$$

$$= l(\Delta^\circ) - r - 1 - \sum_{\dim(F^\circ)=n-1} l^*(F^\circ) + \sum_{\dim(F^\circ)=n-2} l^*(F^\circ) l^*(\widehat{F^\circ}).$$

To compute $h^{1,1}(V)$ we invoke the following Lefschetz hyperplane theorem ([**BC**, Proposition 10.8]):

Lemma 1. *Let V be a nondegenerate ample hypersurface of an n-dimensional complete simplicial toric variety X. Then the natural map induced by inclusion $j^* : H^i(X) \to H^i(V)$, is an isomorphism for $i < n-1$ and an injection for $i = n-1$.*

In our case V is anticanonical, and since the anticanonical divisor of a Fano variety is ample, V is ample. Also it is well-known ([**F**, Section 5.1]) that for any simplicial toric variety X, $H^2(X,\mathbb{R}) = H^{1,1}(X,\mathbb{R}) = A_{n-1}(X) \otimes \mathbb{R}$. So for $n \geq 4$, $h^{1,1}(V) = h^{1,1}(\mathbb{P}_\Delta) = \mathrm{rank} A_{n-1}(\mathbb{P}_\Delta) = r - n$. Thus we have the following theorem:

Theorem 3. *For any generic nondegenerate anticanonical hypersurface V of an n-dimensional simplicial Fano toric variety $\mathbb{P}_\Delta$, $n \geq 4$,*

$$h^{1,1}_{\mathrm{orb}}(V) = l(\Delta^\circ) - n - 1 - \sum_{\dim(F^\circ)=n-1} l^*(F^\circ) + \sum_{\dim(F^\circ)=n-2} l^*(F^\circ) l^*(\widehat{F^\circ}).$$

5.2. $h^{n-2,1}(V)$. Next we compute $h^{n-2,1}(V)$ for $n \geq 4$. For this we use the homogeneous coordinate ring S of $X = \mathbb{P}_\Delta$. Let $v_1, \ldots, v_r$ be the one-dimensional cones of the normal fan of Δ. Let $x_1, \ldots, x_r$ be the corresponding homogeneous coordinates. Let $\beta_0 = [-K_X] = [\sum_{i=1}^r D_i] \in A_{n-1}(X)$. Then $S_{\beta_0} \simeq L(\Delta)$. And the divisor $f \in L(\Delta)$ corresponding to V can be written in the homogeneous coordinates as $f = \sum_{m \in M \cap \Delta} \lambda_m \prod_{i=1}^r {x_i}^{\langle m, v_i \rangle + 1}$. For notational simplicity, we will denote $\prod_{i=1}^r {x_i}^{\langle m, v_i \rangle + 1}$ by $\mathbf{x}^m$ for any $m \in M$. Define the Jacobian ideal of f to be $J(f) = \langle \partial f/\partial x_1, \ldots, \partial f/\partial x_r \rangle$.

First we quote the following theorem ([**BC**, Theorem 10.13]):

Lemma 2. *Let X be an d-dimensional complete simplicial toric variety and $V \subset X$ be a quasi-smooth (i.e., suborbifold) ample hypersurface defined by $f \in S_\beta$. Then for $k \neq (d/2) + 1$, there exists a canonical isomorphism*

$$(S/J(f))_{k\beta - \beta_0} \simeq PH^{d-k,k-1}(V).$$

Remark. The primitive cohomolgy $PH^{d-1}(V) := H^{d-1}(V)/(\mathrm{im}\, H^{d-1}(X))$. This coincides with the usual cohomology if d is even.

For our application of Lemma 2, set $d = n$, $k = 2$ and $\beta = \beta_0$. With $n \geq 4$ these choices imply that $k \neq (d/2)+1$ in the lemma. Also, $H^{n-2,1}(X) = 0$ if $n \geq 4$, so that $PH^{n-2,1}(V) = H^{n-2,1}(V)$. Thus $h^{n-2,1}(V) = \mathrm{rank}\, (S/J(f))_{\beta_0}$.

Lemma 3. *$x_i \partial f/\partial x_i \in J(f)_{\beta_0}, i = 1, \ldots, r$, and the space of complex linear relations among these has dimension $r - (n+1)$.*

Proof.

$$x_i \frac{\partial f}{\partial x_i} = \sum_m \lambda_m (\langle m, v_i \rangle + 1) \mathbf{x}^m,$$

$$\sum_i c_i x_i \frac{\partial f}{\partial x_i} = \sum_m \lambda_m \left(\left\langle m, \sum_i c_i v_i \right\rangle + \sum_i c_i \right) \mathbf{x}^m.$$

For a generic f we can assume that $\lambda_m \neq 0$ for each $m \in \Delta \cap M$. Hence $\sum_i c_i x_i \partial f/\partial x_i \equiv 0 \iff \langle m, \sum_i c_i v_i \rangle + \sum_i c_i = 0 \; \forall m \in \Delta \cap M$.

In particular, taking $m=0$ we get $\sum_i c_i = 0$. Therefore $\langle m, \sum_i c_i v_i \rangle = 0$ $\forall m \in \Delta \cap M$ and since Δ is n-dimensional we have $\sum_i c_i v_i = 0$.

Thus $\sum_i c_i x_i \partial f/\partial x_i \equiv 0 \iff \sum_i c_i v_i = 0$ and $\sum_i c_i = 0$.

Now let $\widetilde{v_i} = (v_i, 1) \in \mathbb{R}^{n+1} \subset \mathbb{C}^{n+1} = \mathbb{R}^{n+1} \otimes \mathbb{C}$.

Since the v_i's are vertices of Δ°, the $\widetilde{v_i}$'s are generators of the $(n+1)$-dimensional cone $\{q \in \mathbb{R}^{n+1} : qt \in \Delta^\circ \times \{1\}$ for some $t \in \mathbb{R}_{>0}\}$. So the $\widetilde{v_i}$'s span $\mathbb{R}^{n+1}$ over $\mathbb{R}$, and hence they span $\mathbb{R}^{n+1} \otimes \mathbb{C}$ over $\mathbb{C}$. Note that $(\sum c_i v_i, \sum c_i) = \sum c_i \widetilde{v_i}$. Hence the lemma follows. □

Without loss of generality assume that the $x_k \partial f/\partial x_k, k = 1, \ldots, n+1$, are linearly independent. In other words $\widetilde{v_k}, k = 1, \ldots, n+1$ are linearly independent. We will consider monomials $\prod_{j \neq i} {x_j}^{p_j}$ that have same degree in S as x_i. So we want $m^* \in M$ such that $\langle m^*, v_j \rangle = p_j \geq 0 > -1$ if $j \neq i$ and $\langle m^*, v_i \rangle = -1$. Such m^* is given by interior lattice points of F_i, the $(n-1)$-dimensional face of Δ that is dual to the 0-dimensional face $\{v_i\}$ of Δ°. Then for each $m^* \in \mathrm{Int}(F_i) \cap M$, $\prod_{j \neq i} {x_j}^{\langle m^*, v_j \rangle} \partial f/\partial x_i$ belongs to $J(f)_{\beta_0}$. Together with the $x_i \partial f/\partial x_i$, these generate $J(f)_{\beta_0}$ as we vary over all i.

In the following computation, we denote the characteristic function of a set by $I(\cdot)$. For instance, $I(m' \in \Delta)$ is 1 when $m' \in \Delta$ and 0 otherwise. Also, recall that $f = \sum_{m' \in M \cap \Delta} \lambda_{m'} \mathbf{x}^{m'}$.

$$
\begin{aligned}
&\left(\prod_{j \neq i} {x_j}^{\langle m^*, v_j \rangle} \right) \partial f/\partial x_i \\
&= \left(\prod_{j=1}^{r} {x_j}^{\langle m^*, v_j \rangle} \right) x_i \partial f/\partial x_i \\
&= \sum_{m' \in \Delta \cap M} \lambda_{m'} (\langle m', v_i \rangle + 1) \mathbf{x}^{m'+m^*} \\
&= \sum_{m'+m^* \in \Delta \cap M} \lambda_{m'} (\langle m', v_i \rangle + 1) I(m' \in \Delta) \mathbf{x}^{m'+m^*} \\
&= \sum_{m \in \Delta \cap M} \lambda_{m-m^*} (\langle m - m^*, v_i \rangle + 1) I(m - m^* \in \Delta) \mathbf{x}^{m}.
\end{aligned}
$$

To justify the fourth line in the above calculation, note that given $m' \in \Delta \cap M$, either $m' + m^* \in \Delta \cap M$ or $\langle m', v_i \rangle + 1 = 0$. Then setting $m = m' + m^*$ leads to the last line.

Let $\mathrm{Int}(F_i) \cap M = \{m_{i,i_s} : 1 \leq s \leq t_i; t_i \geq 0\}$.

Then $J(f)_{\beta_0} = \mathrm{span}\{x_k \partial f/\partial x_k, \prod_{j=1}^{r} {x_j}^{\langle m_{i_s}, v_j \rangle} x_i \partial f/\partial x_i : 1 \leq k \leq n+1, 1 \leq i_s \leq t_i, i = 1, \ldots, r\}$. We want to find the dimension of this complex vector space. So we study the space of linear relations:

$$\sum_k c_k x_k \partial f/\partial x_k + \sum_{i,i_s} d_{i,i_s} \prod_{j=1}^{r} x_j{}^{\langle m_{i_s}, v_j\rangle} x_i \partial f/\partial x_i \equiv 0$$

$$\Longleftrightarrow$$

$$\sum_m \Big\{ \sum_k c_k \lambda_m(\langle v_k, m\rangle + 1) + \sum_{i,i_s} d_{i,i_s} \lambda_{m-m_{i,i_s}} I(m - m_{i,i_s} \in \Delta)(\langle m - m_{i,i_s}, v_i\rangle + 1) \Big\} \mathbf{x}^m \equiv 0$$

$$\Longleftrightarrow$$

$$\sum_k c_k \lambda_m(\langle v_k, m\rangle + 1) + \sum d_{i,i_s} \lambda_{m-m_{i,i_s}} I(m - m_{i,i_s} \in \Delta)\langle m, v_i\rangle \equiv 0$$

for each $m \in \Delta \cap M$, [note: $\langle m_{i,i_s}, v_i\rangle = -1$].

This is a system of $l(\Delta)$ number of linear equations in $\gamma = n + 1 + \sum_{i=1}^{r} l^*(F_i)$ variables namely c_k, d_{i,i_s}. Note that $l(\Delta) \geq \gamma$. We shall find a nonsingular subsystem of rank γ.

To do so pick n linearly independent vertices $m_1, \ldots, m_n$ of Δ and let $m_{n+1} = 0$, the origin. Then from the above system we pick the equations corresponding to $m = m_1, \ldots, m_{n+1}$ and $m = m_{i,i_s} : i = 1, \ldots, r; 0 \leq i_s \leq t_i$. Denote this $\gamma \times \gamma$ system by $(**)$. It can be written as:

$$\begin{bmatrix} \mathbf{P} & \mathbf{A} \\ \mathbf{B} & \mathbf{Q} \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where

$$\mathbf{P} = \begin{bmatrix} \lambda_{m_1}(\langle m_1, v_1\rangle + 1) & \ldots & \lambda_{m_1}(\langle m_1, v_{n+1}\rangle + 1) \\ \ldots & \ldots & \ldots \\ \lambda_{m_n}(\langle m_n, v_1\rangle + 1) & \ldots & \lambda_{m_n}(\langle m_n, v_{n+1}\rangle + 1) \\ \lambda_0 & \ldots & \lambda_0 \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} \lambda_{m_{1,1}-m_{1,1}} I(.)\,(\langle m_{1,1} - m_{1,1}, v_1\rangle + 1) & \ldots & \lambda_{m_{1,1}-m_{r,t_r}} I(.)\,(\langle m_{1,1} - m_{r,t_r}, v_r\rangle + 1) \\ \ldots & \ldots & \ldots \\ \lambda_{m_{r,t_r}-m_{1,1}} I(.)\,(\langle m_{r,t_r} - m_{1,1}, v_1\rangle + 1) & \ldots & \lambda_{m_{r,t_r}-m_{r,t_r}} I(.)\,(\langle m_{r,t_r} - m_{r,t_r}, v_r\rangle + 1) \end{bmatrix}.$$

Observe that all the diagonal entries of $\mathbf{Q}$ are λ_0, and none of its off-diagonal entries has λ_0. Also any entry of $\mathbf{A}$ is of the form $\lambda_{m_k - m_{i,i_s}} I(.)$ and hence does not involve λ_0. Similarly an entry of $\mathbf{B}$ is of the form $\lambda_{m_{i,i_s}}(\langle m_{i,i_s}, v_k\rangle + 1)$ and so does not have λ_0.

Consider the determinant of the coefficient matrix $\begin{bmatrix} \mathbf{P} & \mathbf{A} \\ \mathbf{B} & \mathbf{Q} \end{bmatrix}$ as a polynomial in the λ's. Then the term of this determinant having the highest power of λ_0 is $(\lambda_0)^{\sum l^*(F_i)} \det \mathbf{P}$. We will show below that $\det \mathbf{P}$ = nonzero constant times $\lambda_{m_1} \dots \lambda_{m_n} \lambda_0$. Thus the determinant of the coeffcient matrix of the system (**) is a nontrivial polynomial in the λ's and is therefore nonzero for generic choice of the λ's. Hence $J(f)_{\beta_0}$ has rank γ as a complex vector space, for a generic $f \in L(\Delta)$. Since $S_{\beta_0} \simeq L(\Delta)$, so $(S/J(f))_{\beta_0}$ has rank $l(\Delta) - \gamma$, for a generic f.

Lemma 4. *The* $(n+1) \times (n+1)$ *matrix* $\mathbf{P} = ((\, P_{i,j} = \lambda_{m_i}(\langle m_i, v_j \rangle + 1)\,))$ *is nonsingular for generic chioce of* λ*'s.*

Proof. Let $\mathbf{E}$ be the $(n+1) \times (n+1)$ matrix $((\, E_{i,j} = (\langle m_i, v_j \rangle + 1)\,))$. Then $\det \mathbf{P} = \lambda_{m_1} \dots \lambda_{m_{n+1}} \det \mathbf{E}$. We claim that $\mathbf{E}$ is nonsingular. Otherwise there exists a nontrivial vector $(c_1, \dots, c_{n+1})$ such that $\sum_{k=1}^{n+1} c_k(\langle m_i, v_k \rangle + 1) = 0$ for all $i = 1, \dots, n+1$. In particular, for $i = n+1$ we get $\sum_{k=1}^{n+1} c_k = 0$. This implies $\sum_{k=1}^{n+1} c_k \langle m_i, v_k \rangle = 0$ for all $i = 1, \dots, n$. Since $m_1, \dots, m_n$ are linearly independent, this would imply that $\sum_{k=1}^{n+1} c_k \langle m, v_k \rangle = 0$ for all $m \in \Delta$. Therefore $\sum_{k=1}^{n+1} c_k v_k = 0$. this combined with $\sum c_k = 0$ implies that $\sum_{k=1}^{n+1} c_k \widetilde{v_k} = 0$ which contradicts the linear independence of $\widetilde{v_1}, \dots, \widetilde{v_{n+1}}$. Thus the lemma holds. □

So we have the following theorem:

Theorem 4. *For any generic nondegenerate anticanonical hypersurface* V *of an* n*-dimensional simplicial Fano toric variety* $\mathbb{P}_\Delta$*, and* $n \geq 4$*,*

$$h^{n-2,1}(V) = l(\Delta) - n - 1 - \sum_{\dim(F)=n-1} l^*(F).$$

5.3. Cohomology of the twisted sectors of V. We now want to compute $h^{n-3,0}(V_{(g)})$ for any twisted sector $V_{(g)} \cong V \cap \mathbb{P}_{\widehat{F^\circ}}$. This is obviously zero if $\dim(F^\circ) > 1$, since $\dim(\mathbb{P}_{\widehat{F^\circ}}) = n - 1 - \dim(F^\circ)$. So we will only consider the case $\dim(F^\circ) = 1$. Let τ be the 2-dimensional cone obtained by coning over F°. As noted earlier $\overline{O_\tau} = \mathbb{P}_{\widehat{F^\circ}}$. The restriction of V to $\overline{O_\tau}$ gives a quasi-smooth ample hypersurface of $\overline{O_\tau}$, which we shall identify with $V_{(g)}$. So we are again in a situation where we can invoke Lemma 2.

For this we need to understand the homogeneous coordinate ring S' of $\overline{O_\tau}$. According to Fulton [**F**], Section 3.1, a fan for $\overline{O_\tau}$ can be constucted from the fan Ξ of X as follows.

Let N_τ be the sublattice of N generated by the primitive one dimensional generators of τ. Let $N(\tau) = N/N_\tau$. The dual lattice of $N(\tau)$ is given by $M(\tau) = \tau^\perp \cap M$. The *star* of a cone τ can be defined abstractly as the set of cones σ in Ξ that contain τ as a face. Such cones σ are determined by their images in $N(\tau)$ i.e., by $\bar{\sigma} = (\sigma + (N_\tau)_\mathbb{R})/(N_\tau)_\mathbb{R} \subset N_\mathbb{R}/(N_\tau)_\mathbb{R} = N(\tau)_\mathbb{R}$. These cones $\{\bar{\sigma} : \tau < \sigma\}$ form a fan $\mathrm{Star}(\tau)$ in $N(\tau)$. $\overline{O_\tau}$ is the toric variety corresponding to this fan. Without loss of generality, let v_1, v_2 be the generators of τ. The corresponding Weil divisors in X are D_1 and D_2.

Assume that $v_j, j = 3, \dots, l$ are the 1-dimensional cones of Ξ such that $\{v_1, v_2, v_j\}$ generate a 3-dimensional cone of Ξ. In other words, $\overline{v_j}, j = 3, \dots, l$ are the 1-dimensional cones of $\mathrm{Star}(\tau)$. Let $\widetilde{D_j} := D_1 D_2 D_j$ for $j = 3, \dots, r$. Note that $\widetilde{D_j} = 0$ if $j > l$. The divisor of $\overline{O_\tau}$ corresponding to $\overline{v_j}$ is $\widetilde{D_j}$ for $j = 3, \dots, l$. So the homogeneous coordinate ring S' is generated by variables y_j corresponding to $\widetilde{D_j}$ for $j = 3, \dots, l$. Denote by α_0 the anticanonical class $\sum_{j=3}^{l} \widetilde{D_j}$ in S'.

On the other hand the divisor V restricts to $-K_X D_1 D_2 = (D_1 + \cdots + D_r) D_1 D_2 = \sum_{j=3}^{l} \widetilde{D_j} + (D_1 + D_2) D_1 D_2$. To see what $(D_1 + D_2) D_1 D_2$ is in terms of the $\widetilde{D_j}$s , we can pick a point $m \in \widehat{F^\circ} \cap M$ and let $b_i := \langle m, v_i \rangle, 1 \leq i \leq r$. Then $\sum_{i=1}^{r} b_i D_i$ is linearly equivalent to zero. Note that $b_1 = b_2 = -1$. Hence we have $D_1 + D_2 = \sum_{i=3}^{r} b_i D_i$. So, $(D_1 + D_2) D_1 D_2 = \sum_{i=3}^{l} b_i \widetilde{D_i}$. Let α be the class in S' representing the effective ample divisor $-K_X D_1 D_2$. Let f' be the associated homogeneous polynomial in the y_js. Now we can apply Lemma 2 to the $(n-2)$-dimensional variety $\overline{O_\tau}$ and the ample hypersurface $V_{(g)}$. Choose $k = 1$ in the lemma to get

$$(S'/J(f'))_{\alpha - \alpha_0} \simeq PH^{n-3,0}(V_{(g)}) = H^{n-3,0}(V_{(g)})$$

since $H^{n-3,0}(\overline{O_\tau}) = 0$.

Now $\alpha - \alpha_0 = [\alpha - \sum_{j=3}^{l} \widetilde{D_j}]$. A typical generator $\partial f' / \partial y_i \in J(f')$ has degree $[\alpha - \widetilde{D_i}]$. There are no nonconstant regular functions on the projective variety $\overline{O_\tau}$. So any nontrivial effective divisor, and in particular $\widetilde{D_i}$, $\sum_{j=3,\dots,l; j \neq i} \widetilde{D_j}$ and $\sum_{j=3,\dots,l} \widetilde{D_j}$ are not linearly equivalent to zero. This implies that $J(f')_{\alpha - \alpha_0} = 0$. Hence we obtain that

$$(S')_{\alpha - \alpha_0} \simeq H^{n-3,0}(V_{(g)}).$$

Now $\alpha - \alpha_0 = [\sum_{j=3}^{l} b_j \widetilde{D_j}]$. We want to identify the effective divisors in this class. So we want $m_* \in M(\tau)$ such that $\sum_{j=3}^{l} (b_j + \langle m_*, \overline{v_j} \rangle) \widetilde{D_j}$ is effective. This is if and only if $(b_j + \langle m_*, \overline{v_j} \rangle) \geq 0$ for all $j = 3, \dots, l$ $\iff$ $(\langle m + m_*, v_j \rangle) \geq 0 > -1$ for $j = 3, \dots, l$ $\iff$ $m + m_* \in \mathrm{Int}(\widehat{F^\circ}) \cap M$. To justify the last step note that $\langle m + m_*, v_i \rangle = -1$ for $i = 1, 2$.

Since m is fixed, the required effective divisors are in one-to-one correspondence with the interior lattice points of $\widehat{F^\circ}$. Hence $h^{n-3,0}(V_{(g)}) = l^*(\widehat{F^\circ})$. Since there are $l^*(F^\circ)$ twisted sectors isomorphic to $V \cap \mathbb{P}_{\widehat{F^\circ}}$ we have the following:

$$h^{n-2,1}_{\mathrm{orb}}(V)$$

$$= h^{n-2,1}(V) + \sum_{\iota(g)=1} h^{n-3,0}(V_{(g)})$$

$$= l(\Delta) - n - 1 - \sum_{\dim(F)=n-1} l^*(F) + \sum_{\dim(F^\circ)=1} l^*(F^\circ) l^*(\widehat{F^\circ})$$

$$= l(\Delta) - n - 1 - \sum_{\dim(F)=n-1} l^*(F) + \sum_{\dim(F)=n-2} l^*(F) l^*(\widehat{F}).$$

For the last step we used the one-to-one correspondence between faces of Δ and Δ°.

5.4. Main results. Combining the above formula with Theorem 4 we have the following theorem:

Theorem 5. *For any generic nondegenerate anticanonical hypersurface V of an n-dimensional simplicial Fano toric variety $\mathbb{P}_\Delta$, $n \geq 4$,*

$$h^{n-2,1}_{\mathrm{orb}}(V) = l(\Delta) - n - 1 - \sum_{\dim(F)=n-1} l^*(F) + \sum_{\dim(F)=n-2} l^*(F) l^*(\widehat{F}).$$

Corollary 1. *If $\widehat{V}$ is an MPCP desingularisation of any generic nondegenerate anticanonical hypersurface V of an n-dimensional simplicial Fano toric variety $\mathbb{P}_\Delta$, $n \geq 4$, then $h^{p,1}_{\mathrm{orb}}(V) = h^{p,1}(\widehat{V})$ for $p = 1$ and $p = n-2$.*

Proof. The formulas for $h^{p,1}(\widehat{V})$ for $p = 1, n-2$ computed in [**B1**] by Batyrev match the orbifold Hodge numbers for V obtained in Theorem 3 and Theorem 5. ☐

Corollary 2. *In the case $n = 4$, $h^{p,q}_{\mathrm{orb}}(V) = h^{p,q}(\widehat{V})$ for any p and q.*

Proof. We need only consider $p, q \leq 3$. Also $h^{p,0}_{\mathrm{orb}} \equiv h^{p,0}$ by definition since ι is nonnegative. So, by Serre duality for ordinary and orbifold cohomologies, it is enough to consider just the cases $p = 1, q = 1$ and $p = 2, q = 1$. These are addressed by Corollary 1. (We should remark here that in this case $\widehat{V}$ is actually smooth.) ☐

Corollary 3. *If $\mathbb{P}_{\Delta^\circ}$ is also simplicial, and V° is a generic nondegenerate anticanonical hypersurface of $\mathbb{P}_{\Delta^\circ}$, then $h^{1,1}_{\mathrm{orb}}(V) = h^{n-2,1}_{\mathrm{orb}}(V^\circ)$ and vice versa.*

Proof. Follows from interchanging the roles of Δ and Δ° in the formulas. ☐

Remark. In particular, for the $n = 4$ case, we have $h^{p,q}_{\mathrm{orb}}(V) = h^{3-p,q}_{\mathrm{orb}}(V^\circ)$. This is an example of 'mirror symmetry' of orbifold hodge numbers.

5.5. An example. This example first appeared in the Greene-Plesser mirror construction [**GP**] and was also studied in [**COFKM**] in the context of mirror symmetry.

Consider the complex 4-dimensional weighted projective space $X = \mathbb{P}(1,1,2,2,2)$. It is a simplicial Fano toric variety. Its fan Ξ has the following

1-dimensional cones in $N \cong \mathbb{Z}^4$: $v_1 = (-1,-2,-2,-2)$, $v_2 = (1,0,0,0)$, $v_3 = (0,1,0,0)$, $v_4 = (0,0,1,0)$, $v_5 = (0,0,0,1)$. Ξ has five 4-dimensional cones, obtained by dropping one of the v_i's at a time and taking the cone generated by the remaining four.

Let D_i denote the torus-invariant divisor given by the orbit closure $\overline{O_{v_i}}$. It is easy to check that in $A_3(X)$, $[D_2] = [D_1]$ and $[D_i] = 2[D_1]$ for $i \geq 3$. Construct the homogeneous coordinate ring of X by introducing variables x_i corresponding to v_i. Then $\deg(x_1) = \deg(x_2) = 1$ $(= [D_1])$ and $\deg(x_i) = 2$ for $i \geq 3$.

This leads to the more familiar description of $\mathbb{P}(1,1,2,2,2)$ as $(\mathbb{C}^5 - \{0\})/\mathbb{C}^*$. The action of any $\alpha \in \mathbb{C}^*$ on $\mathbb{C}^5 - \{0\}$ is as follows:

$$\alpha.[x_1 : x_2 : x_3 : x_4 : x_5] = [\alpha x_1 : \alpha x_2 : \alpha^2 x_3 : \alpha^2 x_4 : \alpha^2 x_5].$$

In this description, D_i corresponds to the hyperplane $\{x_i = 0\}$ and the 4-dimensional cones of Ξ correspond to the open sets $\{x_i \neq 0\}$. It is also easily seen that the singular locus of X is precisely the surface $\{x_1 = x_2 = 0\}$. In fact, this represents the only twisted sector of X. The v_i's are the vertices of a reflexive polytope Δ°. The faces of Δ° have only one interior lattice point: $(0,-1,-1,-1) = \frac{1}{2}(v_1 + v_2)$. This lattice point corresponds to the twisted sector and the local isotropy group is $\mathbb{Z}_2$.

The dual reflexive polytope Δ in $M_{\mathbb{R}}$ has the following vertices:

$$w_1 = (-1,-1,-1,-1), \quad w_2 = (7,-1,-1,-1), \quad w_3 = (-1,3,-1,-1),$$
$$w_4 = (-1,-1,3,-1), \quad w_5 = (-1,-1,-1,3).$$

Δ is the polytope corresponding to the anticanonical divisor $-K_X = \sum_{i=1}^{5} D_i$ of X, and $X = \mathbb{P}_\Delta$. If V is a generic nondegenerate Calabi-Yau (anticanonical) hypersurface of X, then V has just one twisted sector namely $C = V \cap \{x_1 = x_2 = 0\}$. One can directly compute the genus of this curve C by using the Riemann-Hurwitz formula. It turns out to be 3. V has the Hodge numbers: $h^{1,0} = h^{2,0} = 0$, $h^{3,0} = 1$, $h^{1,1} = 1$, $h^{2,1} = 83$. Since the degree shifting number of the twisted sector C is 1, we compute $h^{1,1}_{\mathrm{orb}}(V) = h^{1,1}(V) + h^{0,0}(C) = 1+1 = 2$, and $h^{2,1}_{\mathrm{orb}}(V) = h^{2,1}(V) + h^{1,0}(C) = 83+3 = 86$.

The dual Fano veriety $\mathbb{P}_{\Delta^\circ}$ is also simplicial. This is easily checked since its fan is is obtained by coning over the faces of Δ. In fact, $\mathbb{P}_{\Delta^\circ} = \mathbb{P}_\Delta/\mathbb{Z}_4^3$. This is also shown easily. First, observe that $w_1 = -w_2 - 2w_3 - 2w_4 - 2w_5$. Secondly, if $\overline{M}$ is the sublattice of M generated by w_2, w_3, w_4, w_5, then $M/\overline{M} = \mathbb{Z}_4^3$.

Acknowledgments. This work was done under the supervision of my doctoral advisor at UW-Madison, Yongbin Ruan. I am indebted to him for his guidance. My special thanks are also due to Lev Borisov, David Cox and Igor Dolgachev for answering my questions. I thank Prof. Cox for suggesting many improvements, and Anvar Mavlyutov for his comments.

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Received October 18, 2001 and revised May 6, 2002.

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