ORBITAL STABILITY IN A PROTON SYNCHROTRON

N. H. FRANK AND R. Q. TWISS

TECHNICAL REPORT NO. 58

February 9, 1948

RESEARCH LABORATORY OF ELECTRONICS

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

The research reported in this document was made possible through support extended the Massachusetts Institute of Technology, Research Laboratory of Electronics, jointly by the Army Signal Corps, the Navy Department (Office of Naval Research), and the Air Force (Air Materiel Command), under the Signal Corps Contract No. W-36-039 sc-32037.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Research Laboratory of Electronics

Technical Report No. 58 February 9, 1948

ORBITAL STABILITY IN A PROTON SYNCHROTRON

N. H. Frank* and R. Q. Twiss

Abstract

The theory of the electron synchrotron has been amply covered elsewhere, but the proton synchrotron exhibits a number of distinct but important differences particularly at low energies. In the present paper the theory for this latter accelerator is developed with particular reference to the problems of injection and stability in the initial acceleration period. This theory is used to set up criteria relating the dimensions of a proton synchrotron to the conditions for stability and to the beam intensities obtainable.

---

* Work performed as a consultant for the Brookhaven National Laboratory under the auspices of the Atomic Energy Commission.

ORBITAL STABILITY IN A PROTON SYNCHROTRON

1. Introduction

The theory of orbital stability in an electron synchrotron has been discussed by a number of writers.[1] Since the energies of the electrons which are accelerated by synchrotron action lie in the extreme relativistic range, attention has been focussed largely on the case where the electron speeds are very close to that of light. There then result a number of simplifications of the general theory; in particular, the frequency of the accelerating r-f voltage may be taken as constant. In the proton synchrotron, however, particles are to be accelerated from initial energies small compared to their rest energy and hence the frequency of the accelerating voltage must be increased monotonically and synchronously with an increasing magnetic field. It is just this last feature which chiefly distinguishes the proton from the electron synchrotron, and requires that particular attention be given to the effects on the particle motions of departures from the desired time dependence of radio frequency.

No attempt will be made here to describe the basic principles in detail as these have been amply covered in the papers on the synchrotron listed in Reference 1. However, an outline of the process by which the equations of motion are derived will be presented. In order to simplify reference to previous publications, the notation employed here will conform as far as practicable, to that employed by one of us[2] in an article on the electron synchrotron. An analysis of the proton synchrotron has been presented by a group[3] working at Dr. Oliphant's Laboratory at Birmingham, England.

2. The Equations of Motion

It has been shown[2] that the equations of motion of a charged particle in an axially symmetric magnetic field $\underset{\sim}{B} = (-B_r, 0, -B_z)$ are

$$\left.\begin{aligned} \frac{d}{dt}(m\dot{r}) &= mr\dot{\theta}^2 - er\dot{\theta}B_z \\ \frac{d}{dt}\left(mr^2\dot{\theta} - \frac{e\Phi}{2\pi}\right) &= \frac{eV}{2\pi}\sin\left(\theta - \int \omega dt\right) \\ \frac{d}{dt}(m\dot{z}) &= er\dot{\theta}B_r \end{aligned}\right\} \qquad (1)$$

where m is the relativistic mass, e the proton charge, $\Phi=2\pi\int_0^r rB_z dr$ is the total magnetic flux linking a circle of radius r, V and ω are the peak voltage and angular frequency, respectively, of the r-f field across an accelerating gap at $\theta=0$. Radiation effects are omitted since, in the proton case, they are negligible for energies less than 10^5 Mev.

Circular motion in the median plane z=0 is possible for the particles if the parameters in these equations vary with time according to certain definite laws. Particles moving in this circle, of radius r_o, will always arrive at the accelerating gap with the same phase ψ_o relative to that of the accelerating voltage. This quasi-steady motion is given by the equations

$$\left.\begin{aligned} m\theta &= eB_z \\ \frac{d}{dt}\left(r_o{}^2B_o - \frac{\Phi_o}{2\pi}\right) &= \frac{V}{2\pi}\sin\psi_o \\ \psi_o &= \theta - \int\omega dt \end{aligned}\right\} \qquad (2)$$

Here B_o and Φ_o are the magnetic field at, and the flux linking, the circular orbit of radius r_o.

From these equations there follow directly expressions for the kinetic energy $\mathcal{E}$ and the required angular radio frequency ω_o in terms of B_o and r_o. They are

$$\left.\begin{aligned} \frac{\mathcal{E}}{m_oc^2} &= \sqrt{1+\left(\frac{eB_or_o}{m_oc}\right)^2} - 1 \\ \frac{r_o\omega_o}{c} &= \frac{\dfrac{eB_or_o}{m_oc}}{\sqrt{1+\left(\dfrac{eB_or_o}{m_oc}\right)^2}} \end{aligned}\right\} \qquad (3)$$

where, for protons, m_oc^2=938 Mev. These expressions are plotted as functions of eB_or_o/m_oc in Figs. 1 and 2. To provide an idea of the orders of magnitude involved, there are presented, in Table I, the design data for a proton synchrotron intended to yield particles of 10^4 Mev energy.

If the machine is to be run during any part of its operating cycle as a betatron with no r-f acceleration, the particles will remain in a stable circular orbit of radius r_o if

$$r_o{}^2B_o=\int_0^{r_o} B_zr\, dr, \qquad (4)$$

the well-known betatron condition.

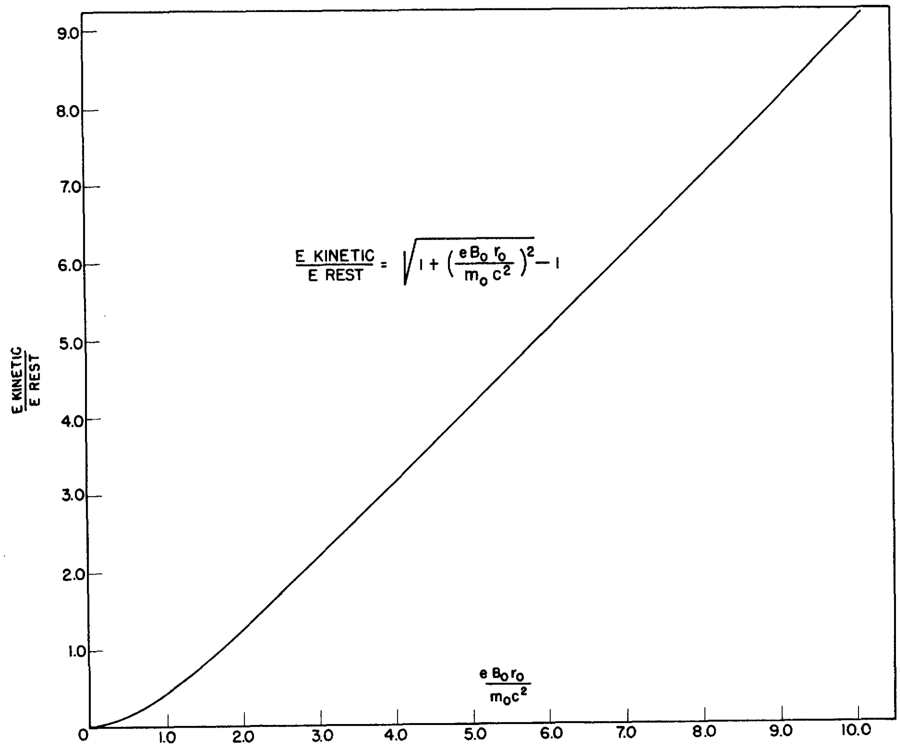

Figure 1. Kinetic energy of particle as a function of magnetic field.

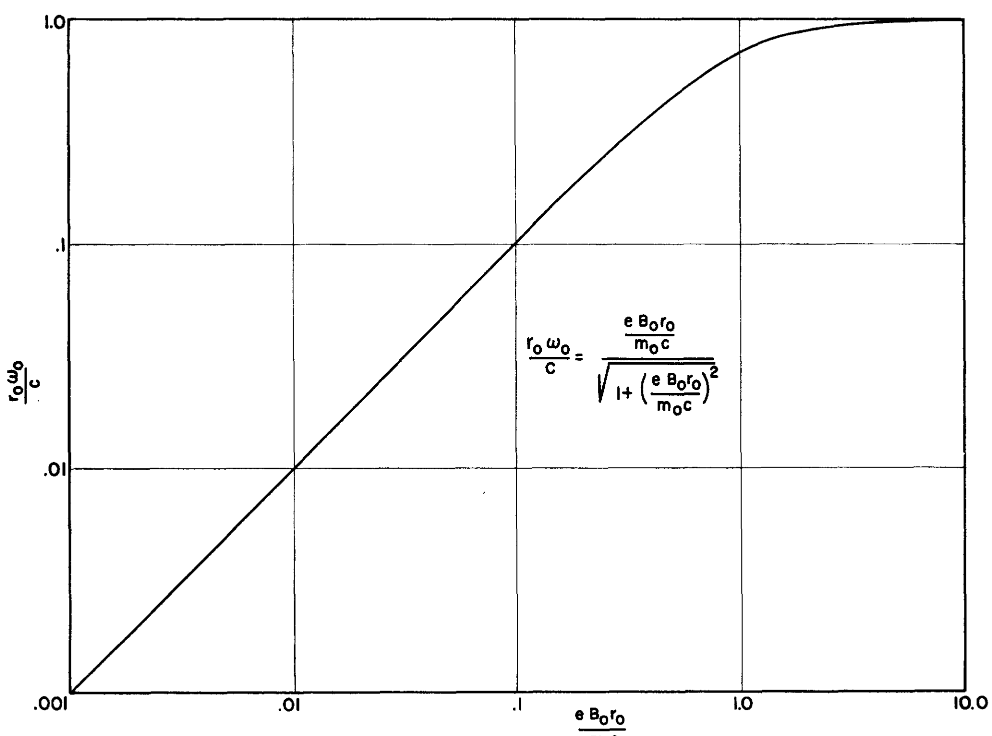

Figure 2. Angular velocity of particle as a function of magnetic field.

TABLE I. Preliminary Design Data for a 10-Bev Proton Synchrotron

Final energy of the protrons	10 Bev
Injection energy of the protons	4 Mev
Orbital radius	24.3 meters
Final magnetic field at orbit	15,000 gauss
Magnetic field at orbit at injection	110 gauss
Final radio frequency	1.96 Mc/sec
Radio frequency at injection	183 kc/sec
Frequency range	10.65 to 1
Acceleration period	1 second
Average energy gain per revolution	5500 ev

To find the oscillations about the stable orbit and phase in the general case one sets

$$\left.\begin{aligned} r &= r_o+x \\ \omega &= \omega_o+\omega_1 \\ \dot{\psi} &= \dot{\theta}-\omega = \dot{\varphi}-\omega_1 \end{aligned}\right\} \tag{5}$$

where ω_1 is the departure of the angular radio frequency from its correct value ω_o, and $\dot{\varphi}$ is the perturbation in the particle angular velocity caused by the correct r-f field. We take x, ω_1, and $\dot{\varphi}$ as first-order small quantities, the squares and products of which will be neglected.

If the vertical magnetic field B_z in the median plane z=0 decreases with radius in the neighborhood of r_o according to the law

$$B_z = B_o\left(\frac{r}{r_o}\right)^{-n} \tag{6}$$

then <u>n</u> must satisfy the inequality

$$0 < n < 1$$

if the orbit r_o is to be stable for both vertical and radial oscillations. In the following it will be assumed that <u>n</u> is independent of <u>r</u>.

Substituting from (5) and (6) into (1), for the case when the particle is moving in the plane z=0, we obtain, after some algebraic manipulation, the following equations describing the phase and radial oscillations:

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{B_o\dot{x}}{\omega_o}\right) &= r_o\omega_o B_o\left\{\left(\frac{eB_o}{m_o\omega_o}\right)^2-(1-n)\right\}\left\{\frac{x}{r_o}+\eta\frac{\dot{\varphi}}{\omega_o}\right\} \\ \frac{d}{dt}\left[\frac{r_o}{\omega_o}\frac{d}{dt}\left(\frac{B_o\dot{x}}{\omega_o}\right)+(1-n)r_oB_ox\right]+\frac{V\sin\psi_o}{2\pi} &= \frac{V}{2\pi}\sin\psi \end{aligned}\right\} \qquad (7)$$

where

$$\eta = \frac{\left(\frac{eB_o}{m_o\omega_o}\right)^2}{\left(\frac{eB_o}{m_o\omega_o}\right)^2-(1-n)} = \frac{\left(\frac{m}{m_o}\right)^2}{\left(\frac{m}{m_o}\right)^2-(1-n)} \qquad (8)$$

with m the relativistic and m_o the rest mass of the particle, respectively.

From Eq. (8), it is clear that η is a very slowly varying function of time, decreasing monotonically from $1/n$ to 1 as the particle kinetic energy increases from zero to infinity. In the non-relativistic range of energies, it may be taken as constant and equal to $1/n$. In Fig. 3 η is plotted as a function of eB_or_o/m_oc.

In the important case where the inequality

$$\frac{4V}{2\pi r_o^2B_o\omega_o\eta(1-n)^2} << 1 \qquad (9)$$

is satisfied, the equations of motion (7) can be regarded as describing two independent oscillations; the so-called betatron oscillations, which are independent of V, satisfy the pair of equations

$$\left.\begin{aligned} \frac{x}{r_o}+\frac{\dot{\varphi}}{\omega_o} &= 0 \\ \frac{d}{dt}\left(\frac{B_o\dot{x}}{\omega_o}\right)+(1-n)\omega_oB_ox &= 0 \end{aligned}\right\} \qquad (10)$$

for those particles, the angular momenta of which are such that they can perform stable betatron motion in the circle of radius r_o. The much lower frequency forced oscillations known as the synchrotron oscillations satisfy the following pair of equations:

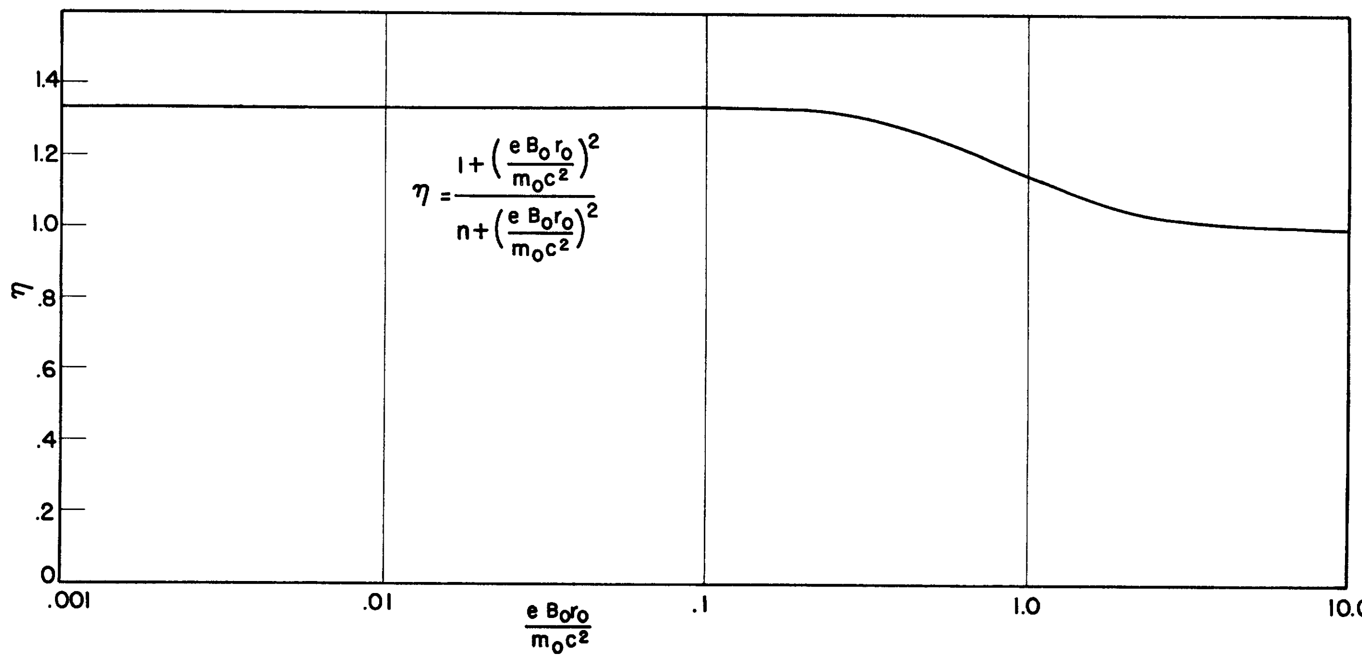

Figure 3. η as a function of magnetic field.

$$\left.\begin{aligned} \frac{x}{r_o} + \eta\frac{\dot{\varphi}}{\omega_o} &= 0 \\ (1-n)r_o^2\frac{d}{dt}\left(\eta\frac{B_o}{\omega_o}\dot{\varphi}\right) + \frac{V}{2\pi}\sin\psi &= \frac{V}{2\pi}\sin\psi_o \\ \dot{\varphi} &= \dot{\psi}+\omega_1 \end{aligned}\right\} \qquad (11)$$

When the inequality (9) is not fully satisfied, the two oscillations are coupled, but the effect of the small coupling term thus obtained is also small and does not modify appreciably the physical behavior of the protons in the accelerating chamber.

Equations (10) are the well-known equations of oscillation for betatron particles with both B_o and ω_o slowly varying functions of the time. The radial oscillations are essentially sinusoidal oscillations of slowly varying frequency $\sqrt{1-n}\,\omega_o$ and amplitude which is damped proportional to $B_o^{-\frac{1}{2}}$. Thus, if x_1 and $\dot{x}_1$ are the initial values of x and $\dot{x}$, respectively,*

$$x(t) = \left(\frac{B_1}{B_o}\right)^{\frac{1}{2}}\left[x_1\cos\left(\sqrt{1-n}\int\omega_o dt\right) + \frac{\dot{x}_1}{\sqrt{1-n}\,\omega_1}\sin\left(\sqrt{1-n}\int\omega_o dt\right)\right] \qquad (12)$$

* x_1 is measured from that orbit radius which corresponds exactly to the initial kinetic energy.

where ω_1 and B_1 are initial values. In view of the fact that the betatron oscillations have been fully analyzed elsewhere, (see Ref. 1) no further discussion of them will be given here and we shall concern ourselves with Eq. (11).

The equation that determines the phase oscillations may be written in the form:

$$(1-n)r_o^2\frac{d}{dt}\left(\eta\frac{B_o}{\omega_o}\dot{\psi}\right) + \frac{V}{2\pi}\sin\psi = \frac{V}{2\pi}\sin\psi_o - (1-n)r_o^2\frac{d}{dt}\left(\eta\frac{B_o\omega_1}{\omega_o}\right). \quad (13)$$

In general, B_o, ω_o, η, V, and ψ_o will be slowly varying functions of the time. We exclude for the present the exceptional case of transition from betatron to synchrotron operation, during which V and ψ_o may vary in a manner which cannot be considered slow. Following McMillan, one may interpret Eq. (13) as the equation of motion of a circular pendulum of slowly varying mass and length, acted on by a unidirectional tangential force together with an irregular forcing term depending on $d\omega_1/dt$. The motion of such a pendulum will be oscillatory if ψ never exceeds the critical angle $\pi-\psi_o$. If it does, the motion will be rotatory in an anti-clockwise direction with increasing angular speed under the action of the forcing term $(V/2\pi)\sin\psi_o$. Physically, the particle slips in phase relative to the accelerating voltage and moves continuously from an accelerating to a decelerating phase and back to an accelerating phase, repeating this process. It thus gains no energy from the r-f voltage on the average. The magnetic field, however, is increasing continuously and hence the particle orbit shrinks inward until the particle is lost to the walls, according to the first of Eqs. (11).

3. The Nature of the Synchrotron Oscillations

3.1. Conditions of Acceptance of a Particle into the Acceleration Cycle. Since a necessary condition for continuous acceleration of a particle is that its phase may not exceed $\pi-\psi_o$, restrictions on the initial phase and phase velocity with which a particle is injected into the synchrotron accelerating cycle follow therefrom. An exact calculation would require knowledge of the time variations of η, B_o, ω_1, and V. However, sufficiently accurate information can be obtained by taking $\omega_1=0$ and neglecting the time dependence of η, B_o, ω_o, and V. (This is particularly the case during the initial non-relativistic stages of operation if the magnetic field varies linearly with the time when $\eta B_o/\omega_o$ is constant).

A first integral of (13) can now be obtained as

$$\dot{\psi}^2 = \frac{2\Omega_o^2}{\cos\psi_o}\left[\cos\psi + \psi\sin\psi_o - \cos\psi_1 - \psi_1\sin\psi_o\right] + \dot{\psi}_1^2 \qquad (14)$$

where

$$\Omega_o^2 = \frac{V\omega_o\cos\psi_o}{2\pi r_o^2(1-n)\eta B_o} \qquad (15)$$

and ψ_1 and $\dot{\psi}_1$ are initial values. Ω_o is the angular frequency of small synchrotron phase oscillations about the equilibrium phase ψ_o. The equivalent pendulum will oscillate between two limiting values of ψ providing the representative point in phase space lies within the boundary determined by the equation:

$$\cos\psi_1 + \psi_1\sin\psi_o - \frac{\dot{\psi}_1^2\cos\psi_o}{2\Omega_o^2} = (\pi-\psi_o)\sin\psi_o - \cos\psi_o\,. \qquad (16)$$

Figure 4 shows the boundary curves for a number of values of ψ_o. The extreme phase angle limits of the oscillatory motion are given by

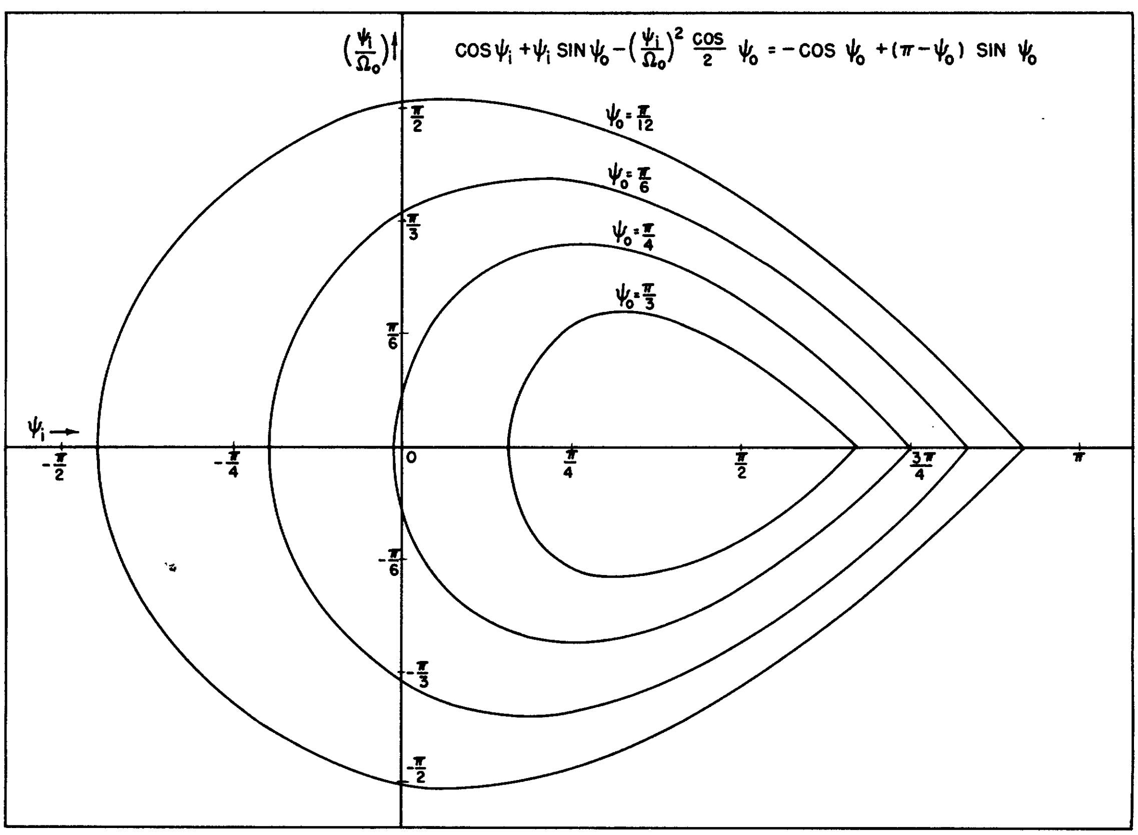

Figure 4. The acceptance domain in phase space at injection for various values of stable phase ψ_o.

$\psi_1 \leq \psi \leq \pi - \psi_0$, where ψ_1 is that solution of

$$\cos\psi_1 + \psi_1 \sin\psi_0 \quad = \quad (\pi - \psi_0)\sin\psi_0 - \cos\psi_0$$

which satisfies the inequality $-\pi \leq \psi_1 \leq \psi_0$. In Fig. 5 the total acceptance angle $(\pi - \psi_0 - \psi_1)/2\pi$ is shown plotted as a function of ψ_0.

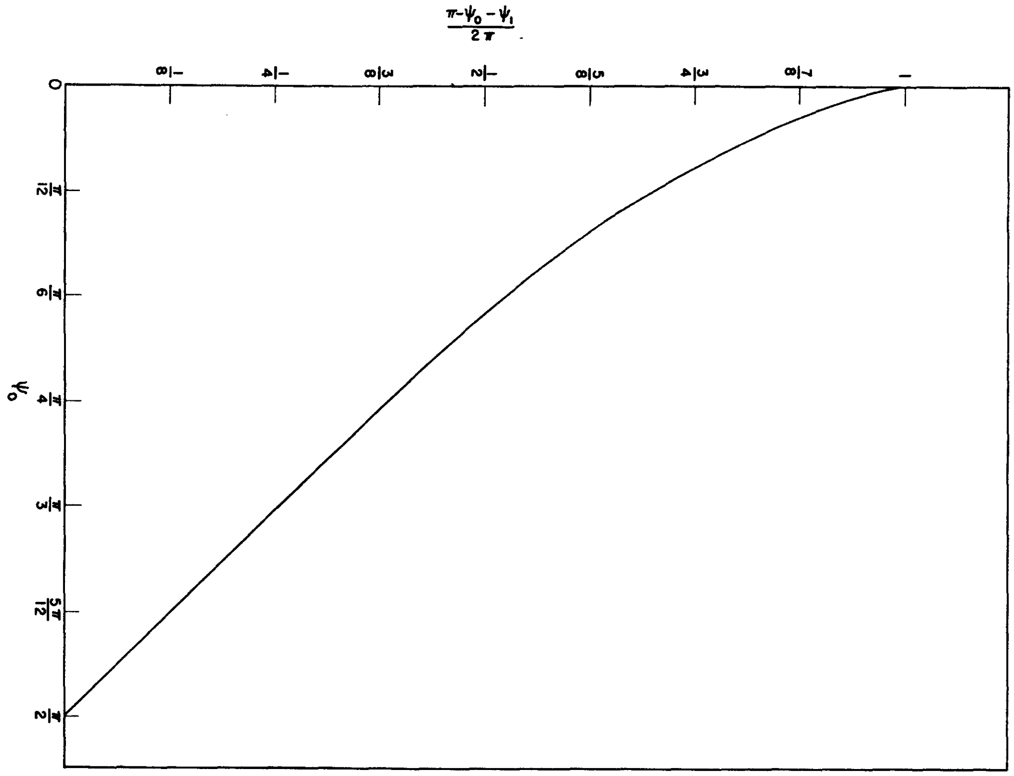

Figure 5. The normalized injection acceptance angle as a function of the stable phase ψ_0.

3.2. Oscillation Periods. Equation (14) cannot be integrated directly except in the special case that $\psi_0=0$. However, it can be integrated numerically for given values of ψ_0, ψ_1 and $\dot{\psi}$. In Fig. 6 the normalized oscillation period (the ratio of the period T_0 to $2\pi/\Omega_0$) is plotted as a function of $\psi_1 - \psi_0$ for various values of ψ_0 with $\dot{\psi}_1=0$. It will be noted that the period increases monotonically with ψ_1 and tends to infinity as ψ_1 approaches $\pi - \psi_0$. To get the order of magnitude of the periods involved, we use the values,

V = 5500 volts
ω_0 = $4\pi \times 10^5$/sec
ψ_0 = $\pi/4$
n = 3/4
r_0 = 24.3 m
B_0 = 110 gauss = 1.1×10^{-2} webers/m^2

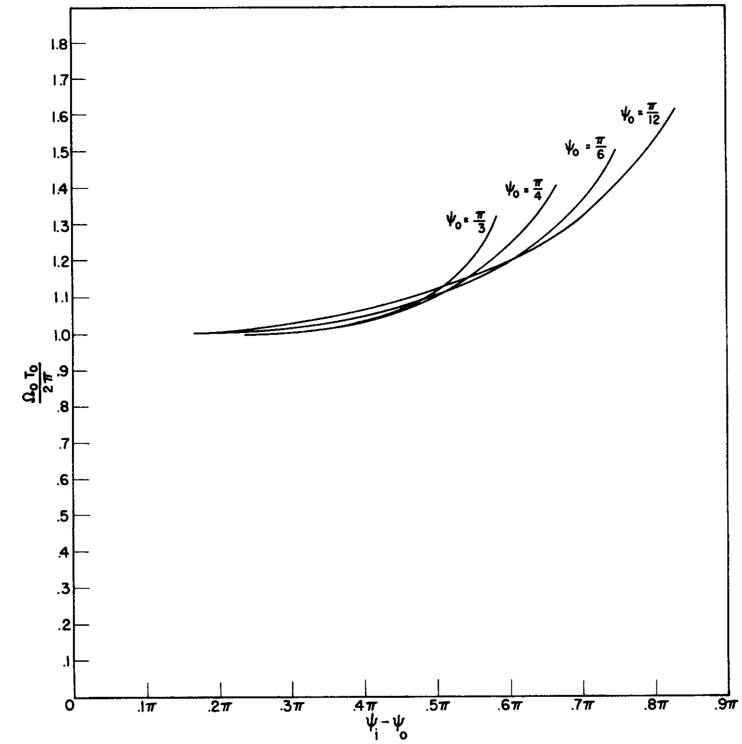

Figure 6. The period of a synchrotron oscillation as a function of the injection phase.

This gives $\Omega_o \cong 2\mathrm{x}10^{-2}\omega_o = 8\pi\mathrm{x}10^3$/sec during the initial stages of operation; i.e., one phase oscillation period occurs in about 50 revolutions of the particle.

3.3. The Stability of the Phase Oscillations. First consider small oscillations about the equilibrium phase, setting

$$\psi = \psi_o + \vartheta \qquad \vartheta << 1 .$$

The equation of undisturbed phase motion becomes

$$\frac{d}{dt}\left(\eta \frac{B_o}{\omega_o}\dot{\vartheta}\right) + \frac{V\cos\psi_o}{2\pi r_o^2(1-n)} \vartheta = 0$$

and the W.K.B. approximation to the solution of this equation yields

$$\psi = \left(\frac{V\cos\psi_o \eta B_o}{\omega_o}\right)_i^{\frac{1}{4}} \left(\frac{V\cos\psi_o \eta B_o}{\omega_o}\right)^{-\frac{1}{4}} \left\{\vartheta_i \cos\int\Omega_o dt + \frac{\dot{\vartheta}_i}{\Omega_i}\sin\int\Omega_o dt\right\} \quad (17)$$

with $\vartheta_i = \psi_i - \psi_o$ and $\dot{\vartheta}_i = -\omega_o x_i / r_o \eta$; x_i is the initial radial distance from the stable synchrotron orbit and ψ_i the initial phase.

Thus the amplitude A of the phase oscillations varies as

$$A \sim \left(\frac{V\cos\psi_o \eta B_o}{\omega_o}\right)^{-\frac{1}{4}} \tag{18}$$

where $V\sin\psi_o = \frac{d}{dt}(r_o^2 B_o - \frac{\Phi_o}{2\pi})$. (19)

Since $\dot{\Phi}_o/\dot{B}_o$ will be essentially constant, independent of time, one may write alternatively,

$$A \sim \left(\eta\frac{\dot{B}_o B_o}{\omega_o}\cot\psi_o\right)^{-\frac{1}{4}}. \tag{20}$$

The two limiting cases of the non-relativistic motion with $m=m_o$ and the extreme relativistic case with the particle speed practically equal to that of light are of particular interest.

The non-relativistic case. In this case $\eta B_o/\omega_o$ is constant and positive amplitude damping will occur only if $V\cos\psi_o$ increases with the time. In practice, B_o will increase with time, the initial curvature being due to eddy currents in the magnet core. Injection may occur at the start of this cycle or at any subsequent time. Since $V\sin\psi_o$ is proportional to dB_o/dt, some positive damping can be attained in the early stages of the acceleration cycle even if ψ_o is kept constant. However, this effect is small and to obtain appreciable damping, it is necessary that V be increased more rapidly than $\dot{B}_o$. The effect of the change in ψ_o is to alter the damping only slightly if $\cos\psi_o$ is near unity.

If V is kept constant, there will be a small increase in oscillation amplitude, proportional to $(\cos\psi_o)^{-\frac{1}{4}}$, so that it is desirable to increase V during the initial non-relativistic stages of acceleration. (A very small additional damping is obtained from the slowly increasing mass during these early stages if $n > 2/3$.)

An additional reason for the desirability of positive damping is that of radio-frequency error. Such an error leads to a forcing term $-r_o^2(1-n)\frac{d}{dt}(\eta B_o\omega_1/\omega_o)$. The effects of this term will be more or less random, sometimes increasing and sometimes decreasing the oscillation amplitude. As time goes on, there will be an ever increasing probability that the fluctuating amplitude will exceed the critical limits of Eq. (16) and then the particle will be lost. This probability is of course enormously reduced if steady positive damping is present.

The extreme relativistic case. In this case, ω_o is constant and the phase oscillation amplitude varies as $(V\cos\psi_o B_o)^{-\frac{1}{4}}$. The increasing magnetic field provides sufficient damping so that V may be kept constant. As a numerical example of the magnitude of the damping for the whole cycle, we take

ψ_o = constant

V increasing from 7700 to 10,000 volts

B_o increasing from 110 to 15,000 gauss

$\omega_o/2\pi$ increasing from 180 kc/sec to 2.00 Mc/sec

n = 3/4

and η decreasing from 4/3 to unity.

Then, according to Eq. (18), the ratio of initial to final amplitude is about 1.9 to 1.

For large amplitudes, the damping may be obtained with the help of the adiabatic theorem. This states that the $\oint p_\psi d\psi$ taken over a complete libration period of ψ is an invariant for slow variations of the parameters in the Hamiltonian of the system, with p_ψ the generalized momentum associated with the coordinate ψ. Now the equation of phase oscillation can be derived from a Lagrangian function

$$L = \frac{1}{2}\left[\frac{r_o^2(1-n)\eta B_o}{\omega_o}\dot{\psi}^2\right] + \frac{V}{2\pi}(\cos\psi + \psi\sin\psi_o) \qquad (21)$$

if effects of frequency error are neglected. Hence

$$p_\psi = \frac{\partial L}{\partial\dot{\psi}} = \frac{r_o^2(1-n)\eta B_o\dot{\psi}}{\omega_o} \qquad (22)$$

and one has

$$\oint \frac{\eta B_o}{\omega_o}\dot{\psi}\, d\psi = \text{constant}.$$

If now, at any time, ψ oscillates between ψ_1 and ψ_2, these limiting angles are determined by

$$\int_{\psi_1}^{\psi_2}\sqrt{\frac{\eta B_o V}{\omega_o\cos\psi_o}}\left[\cos\psi + \psi\sin\psi_o - \cos\psi_2 - \psi_2\sin\psi_o\right]^{\frac{1}{2}} d\psi = \text{constant} \qquad (23)\ .$$

where ψ_1 and ψ_2 are related by

$$\cos\psi_2 + \psi_2\sin\psi_o - \cos\psi_1 - \psi_1\sin\psi_o = 0\ ; \quad -\pi < \psi_1 < \psi_o < \psi_2 < \pi - \psi_o\ .$$

In the special case $\psi_0=0$, Eq. (23) can be evaluated in terms of complete elliptic functions, and gives

$$h\left[\sin\left(\frac{\psi_m}{2}\right)\right] = \text{constant}\left(\frac{\omega_0}{\eta B_0 V}\right)^{\frac{1}{2}}$$

with $h(k)=E(k)-(1-k^2)K(k)$, K and E being complete elliptic functions of the first and second kind. Figure 7 is a plot of $h(\sin \psi_m/2)$ against ψ_m, the maximum amplitude, and shows that the damping for finite amplitudes is in general less than for small amplitudes in this case.

3.4. <u>The Stability of the Radial Oscillations</u>. From the solution of the equation for phase oscillations, one obtains the radial oscillations immediately from the first of Eqs. (11). For small oscillations, one has

$$\frac{x}{r_0} = \frac{B_1}{B_0}\,\frac{(\eta V\cos\psi_0 B/\omega_0)^{\frac{1}{4}}}{(\eta V\cos\psi_0 B/\omega_0)_1^{\frac{1}{4}}}\left\{\frac{x_1}{r_0}\cos\int\Omega_0 dt + \frac{\dot{x}_1}{\Omega_0 r_0}\sin\int\Omega_0 dt\right\}$$

so that the amplitude A is damped according to the law

$$A \sim \left(\eta\frac{V\cos\psi_0 B_0}{\omega_0}\right)^{\frac{1}{4}} \cdot \frac{1}{B_0} \cdot \qquad (24)$$

<u>The non-relativistic case.</u> In this case $\eta B_0/\omega_0$ is constant and the radial oscillation amplitude varies as $(V\cos\psi_0)^{\frac{1}{4}}/B_0$. The increase of $V\cos\psi_0$ with time needed to obtain a small phase damping must be less than the increase of $B_0{}^4$ with time to insure positive radial damping. This restriction is purely academic, however, because of the impracticality of attaining such a rapid rate of voltage increase.

<u>The extreme relativistic case.</u> Here ω_0 is constant and the radial oscillation amplitude varies as $(V\cos\psi_0)^{\frac{1}{4}}/B_0^{\frac{3}{4}}$. Considering the same numerical example as in the case of phase oscillations, one finds the ratio of initial to final amplitude of radial oscillation to be 48 to 1.

3.5. <u>The Effects of Frequency Error</u>. From the outset, it should be noted that frequency variations which are rapid compared to the synchrotron oscillations have a negligible effect and will be ignored in the following. Departures in frequency from the correct law affect both the radial and phase motions of the particles. The average position of the particle is displaced according to the formula

$$\frac{x}{r_0} = -\eta\left(\frac{\dot{\psi}+\omega_1}{\omega_0}\right) \quad \text{or} \quad \frac{\bar{x}}{r_0} = -\frac{\omega_1\eta}{\omega_0} \cdot \qquad (25)$$

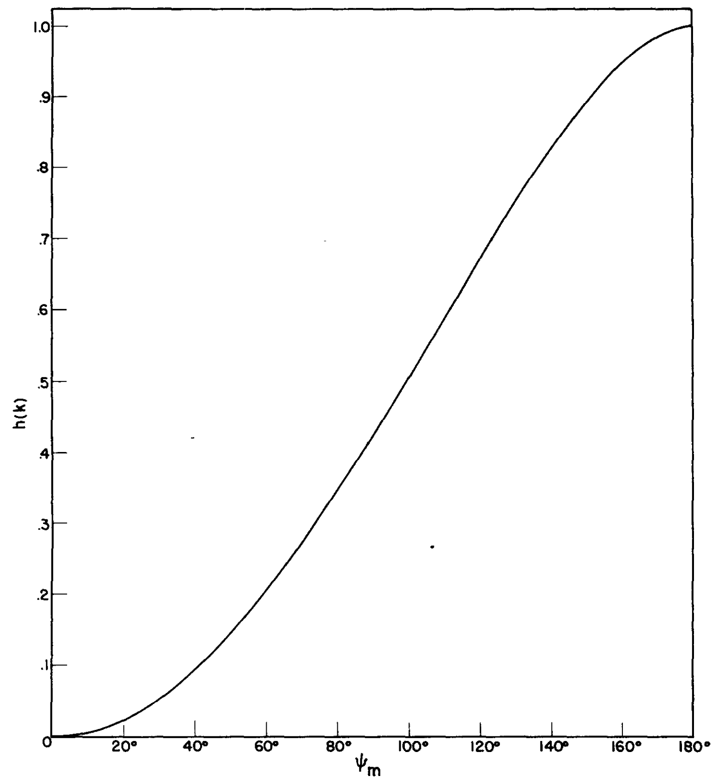

Figure 7. h(k) as a function of injection angle ψ_m.

If this displacement is too great the particle will be lost to the walls and hence a definite limit exists of the maximum allowed absolute value of the frequency error. With A the <u>total</u> oscillation amplitude and 2b the width of the accelerating chamber, the fractional frequency error must satisfy the inequality

$$\frac{\omega_1}{\omega_0} < \left(\frac{b-A}{r_0}\right)\frac{1}{\eta} \cong \left(\frac{b-A}{r_0}\right)n \qquad \text{at low energies .} \qquad (26)$$

This requirement is most severe at injection, as then A is a maximum and $1/\eta = n$, a minimum. Thus <u>n</u> should be as large as possible in the proton synchrotron if frequency errors are to be minimized. This is opposite to the betatron case where, to minimize errors in satisfying the betatron flux condition, (1-n) should be large and hence n small.

The general analysis of the effect of frequency error on the phase oscillations is extremely complicated because of the non-linear character of the oscillations. However, the very important practical case of a linear variation with time of the frequency error ω_1 allows a solution. If $\omega_1 = \alpha t$, the equation of phase motion becomes

$$r_0^2(1-n)\frac{d}{dt}\left(\eta\frac{B_0\dot{\psi}}{\omega_0}\right) + \frac{V}{2\pi}\sin\psi = \frac{V}{2\pi}\sin\psi_0 - r_0^2(1-n)\frac{d}{dt}\left(\frac{\alpha\eta B_0 t}{\omega_0}\right). \qquad (27)$$

In the low energy range, $\eta B_o/\omega_o$ is constant and (27) takes the form

$$r_o^2(1-n)\eta \frac{B_o}{\omega_o}\ddot{\psi} + \frac{V}{2\pi}\sin\psi = \frac{V}{2\pi}\sin\psi_o' \qquad (28)$$

with

$$\frac{V}{2\pi}\sin\psi_o' = \frac{V}{2\pi}\sin\psi_o - \frac{r_o^2(1-n)\eta\alpha B_o}{\omega_o} \,. \qquad (29)$$

Thus the effect of a frequency error is to produce a change in the stable phase angle. This will change the amplitude of oscillation and also alter the limits of stable oscillation from

$$\psi_1 < \psi < \pi - \psi_o \quad \text{to} \quad \psi_1' < \psi < \pi - \psi_o'$$

where

$$\left.\begin{aligned} \cos\psi_1 + \psi_1\sin\psi_o &= (\pi-\psi_o)\sin\psi_o - \cos\psi_o \\ \cos\psi_1' + \psi_1'\sin\psi_o' &= (\pi-\psi_o')\sin\psi_o' - \cos\psi_o' \,. \end{aligned}\right\} \qquad (30)$$

The range of ψ for stable phase oscillations will increase if $|\psi_o'| < \psi_o$, hence if

$$o < \alpha < \frac{\omega_o V \sin\psi_o}{\pi r_o^2(1-n)\eta B_o} \,.$$

Thus an increase in radio frequency from the proper value is less serious than a corresponding decrease. Increasing frequency implies a smaller orbital radius and hence a smaller r-f energy gain per revolution is required to maintain this orbit than an expanded orbit. Particles will be lost if their altered amplitudes exceed the limits of stable phase oscillation corresponding to the new stable phase; i.e., if the representative points in phase space fall outside the appropriate boundary of Fig. 4.

In general the oscillation amplitude will decrease if $|\psi_o'| < \psi_o$, but the amount of this decrease depends on the exact phase when the frequency begins to depart from the correct law. Finally, the speed with which the average radial position follows the frequency change is limited by the period of synchrotron oscillations. Thus essentially no change in radius will occur until an elapsed time equal to an appreciable fraction of the synchrotron period.

As pointed out, a negative frequency error increases the stable phase angle and decreases the stable phase range with a resulting loss of particles.

The principal conclusions may be summarized as follows:

i. The synchrotron oscillation frequency is small compared to the accelerating radio frequency, from 50-200 times smaller in the case under discussion.

ii. The radial oscillations are heavily damped;

$$A \sim \left(\frac{V\cos\psi_o \eta B_o}{\omega_o}\right)^{\frac{1}{4}} \cdot \frac{1}{B_o} \cong (B_o)^{-1}.$$

iii. The phase oscillations are less heavily damped;

$$A \sim \left(\frac{V\cos\psi_o \eta B_o}{\omega_o}\right)^{-\frac{1}{4}}.$$

In the non-relativistic region there is no damping unless V increases with time.

iv. A number of constraints exist on the allowable frequency error. The most important is that the long term error satisfy the inequality

$$\frac{\omega_1}{\omega_o} < \frac{1}{\eta}\left(\frac{b-A}{r_o}\right) \cong n\left(\frac{b-A}{r_o}\right) \quad \text{for the non-relativistic case.}$$

This constraint becomes less severe as time increases because A decreases as $1/B_o$.

v. It is desirable to make n as near unity as possible from the standpoint of synchrotron oscillations.

4. Initial Betatron Acceleration

In the majority of existing electron-synchrotrons the machine is operated initially as a betatron in order to obviate the necessity for frequency modulation of the r-f accelerating voltage. It has been suggested that a stage of betatron acceleration be utilized in the proton synchrotron not, of course, to remove the necessity for frequency modulation, but to reduce the range. This will be particularly valuable in a machine that employs a resonant tuned accelerating cavity and yet requires a large accelerating voltage over the whole frequency range. It is correspondingly less useful if an untuned accelerating system is used.

The maximum particle energy attainable in the betatron phase will be determined by economic considerations; the minimum initial energy will depend upon the injection system and will be limited by the necessity of avoiding excessive gas scattering.

There are, however, additional arguments in favor of a betatron acceleration phase, and as some of them are peculiar to the proton synchrotron, they will be discussed in greater detail. It should be emphasized from the outset that we are now only concerned with the case where the betatron phase carries the particles up to energies of a few Mev.

It has been stated above that the amplitude of the horizontal oscillation in a synchrotron is the sum of the high-frequency betatron and of the low-frequency synchrotron oscillations. If particles are injected into the betatron phase the low-frequency oscillations will not be present, and we can use this fact either to lower the tolerance on the injection energy or to increase the number of particles accepted into the acceleration cycle.

The use of a betatron phase also enables us to get increased phase damping in the early stages. If the r-f voltage be turned on while the betatron condition is satisfied, synchrotron oscillations will take place about a stable phase angle of zero. Under these conditions the stable acceptance angle is equal to 2π. If now the r-f voltage increases linearly with the time to a limiting value V in time T_o, then the particles will be bunched about zero phase.

If $$\Omega_o^{\,2} = \frac{V\omega_o \cos\psi_o}{2\pi r_o^{\,2}(1-n)\eta B_o} ,$$

then the phase of a particle at time T is given by

$$\psi = \frac{\psi_1 \Omega_{oi}^{\frac{1}{3}} \Gamma(2/3)}{3^{\frac{1}{3}} T_o^{\frac{1}{6}}} \sqrt{t}\, J_{-\frac{1}{3}}\left(\int_0^t \sqrt{\frac{\Omega_o^{\,2}\tau}{T_o}}\, d\tau\right)$$

where ψ_1 is the injection phase, and we have made the approximate assumption that ψ_1 is small. If $T_o\Omega_o$ is >> 1 an asymptotic expansion may be used to yield

$$\frac{\psi_F}{\psi_1} = \frac{1}{\sqrt{\pi}}\left(\frac{\Omega_{oi}}{\Omega_{oF}}\right)^{\frac{1}{3}} \frac{3^{\frac{1}{6}}\cdot\Gamma(2/3)}{(\Omega_{oF}T_o)^{\frac{1}{6}}} \left(\frac{T_o}{t}\right)^{\frac{1}{4}} = \frac{1}{\sqrt{\pi}}\left(\frac{\Omega_{oi}}{\Omega_{oF}}\right)^{\frac{1}{3}} \frac{3^{\frac{1}{6}}\Gamma(2/3)}{(\Omega_{oF}T_o)^{\frac{1}{6}}} \quad \text{if} \quad t = T_o \qquad (31)$$

where subscript i refers to initial values and subscript F refers to final values.

In the numerical case quoted above where V is taken to be 10,000 volts and T_o is the time required to raise the energy of the particle to 8 Mev,

$$\frac{\psi_F}{\psi_1} = \frac{1}{2.3} ,$$

a larger damping than that obtained during the entire synchrotron acceleration period. The damping for large phase angles will be less than this, but even so, by far the great majority of the particles injected will be accepted into the synchrotron acceleration cycle if this prebunching is employed. In addition a much smaller percentage will be lost because of random frequency errors.

It will be noted, from Eq. (31), that the amount of damping obtained depends very little upon the amplitude of the r-f voltage; in fact, $\psi_F \sim V^{-\frac{1}{12}}$. Thus the bunching can be obtained with an r-f voltage only a small fraction of that which would be needed in the synchrotron phase. This may be important if the r-f accelerating system is a resonant one.

To conclude this section we shall give a brief discussion of the effect of an error in the betatron flux condition. In the so-called flux-forcing scheme, prescribed by D. W. Kerst, the betatron coil is placed partly in series, partly in parallel with the field coils the whole system being powered from the same generator. Under these conditions it can be shown that there is a self compensation mechanism which automatically preserves the correct field-flux ratio. However a complicated switching problem arises if this scheme is to be incorporated into the proton synchrotron, where the current in the betatron coils has to be turned off at some point in the acceleration cycle. Accordingly, it may be necessary to power the betatron flux coils from a separate generator, and then the possibility arises that the flux is out of phase with the field in the accelerating chamber. One way of avoiding this trouble is to provide an additional field-compensating coil, the current in which is automatically adjusted to give the correct flux-field ratio. Alternatively, the correct orbit can be maintained by a small r-f voltage.

Let it be supposed that the betatron flux departs by a small amount $\Delta\Phi_o$ from the correct value and that the r-f frequency error is ω_1, then the particles in the acceleration chamber will oscillate about a stable phase angle ψ_B given by

$$V\sin\psi_B = \frac{d}{dt}\left[\Delta\Phi_o - 2\pi r_o^2 B_o(1-n)\omega_1\eta/\omega_o\right] \tag{32}$$

and ψ_B can be held small, even with small V, if $\Delta\Phi_o$ and ω_1 are small. If prebunching is used, it may thus be unnecessary to employ any automatic flux compensation even if

$$\frac{\Delta\Phi_o}{\Phi_o} > \frac{(1-n)(b-A)}{r_o} \tag{33}$$

where A is the amplitude of the betatron oscillations and 2b the width of the acceleration chamber. The inequality (33) gives the condition that the particle be lost to the walls in the absence of an r-f voltage.

5. Injection Methods

A number of injection methods have been proposed for the proton synchrotron. For injection directly into the synchrotron stage of operation, it is convenient to classify these methods into one or more of the four following categories. One attempts to avoid loss of injected particles on subsequent revolutions to the gun by

(1) Adiabatic damping of the oscillations before the particles return to the gun,

(2) Rapid decrease of the free-oscillation amplitude by the use of auxiliary transient fields,

(3) Rapid decrease of synchrotron phase and radial oscillations by rapid alteration of the stable phase angle,

(4) Adiabatic shift of the stable orbit by frequency modulation of the r-f accelerating voltage. With a preliminary betatron stage methods (1) and (2) are pertinent in addition to the possibility of an adiabatic shift of the betatron orbit.

Most likely injection will take place with energies in excess of 1 Mev, into an initial magnetic field greater than 50 gauss, from an auxiliary accelerator by means of a suitable gun. In practice intensities will be so low that the effects of space charge, even during injection, can be neglected. We shall not concern ourselves with the nature of this primary accelerator. When injecting directly into the synchrotron phase, it is desirable to minimize the horizontal betatron oscillations by making the injection momentum correspond to the value of $B\rho$ at the injection orbit. If particles are injected over a number of r-f cycles while the magnetic field is increasing, one should increase the injection energy in step with the magnetic field. This is not too difficult if the initial accelerator is a low-energy transformer but it is much harder if a Van de Graaff accelerator is used and probably impossible if a cyclotron or linear accelerator is employed.

We shall now discuss four alternative injection proposals, chosen to illustrate the four basic methods stated above. The first one is discussed in somewhat more detail than its importance warrants because it serves to clarify the difficulties inherent in injection. The numerical estimates for the number of particles accepted into the acceleration cycle are derived for the machine described by the data in Table I.

5.1. Adiabatic Damping of Oscillations before Particles Return to the Neighborhood of the Gun. In this system the gun is located vertically above the central orbit and particles are injected over several r-f cycles. It is highly desirable that the particle energy on injection just equal the energy appropriate to the central orbit, so that for maximum intensity the injection gun's energy must be modulated. Let us start by supposing that the injection energy has just the correct value and consider the effects of departure from this energy later. Furthermore let us for the moment neglect the fact that the injected beam has a certain angular spread.

Consider a particle arriving at the acceleration gap with a phase ψ. The initial value of the radial velocity $\dot{x}_1$ is given approximately by

$$\frac{\dot{x}_1}{r_o} = -\frac{\ddot{\psi}}{\omega_o n} = \frac{V(\sin\psi-\sin\psi_o)}{2\pi r_o^2(1-n)B_o}$$

so that during a single revolution the particle moves radially a distance x_1 given by

$$x_1 = \frac{V(\sin\psi-\sin\psi_o)}{(1-n)r_oB_o\omega_o} = \frac{eV(\sin\psi-\sin\psi_o)r_o}{(1-n)2E_i}$$

with E_i the injection energy.

If the accelerating gap is located near the gun, x_1 is the radial distance the particle traverses before returning to the neighborhood of the gun. If $x_1 > s$, the half-width of the gun, the particle will certainly not be lost in the second revolution. Another circumstance which allows the particle to miss the gun is that the frequency of the vertical betatron oscillations is $\sqrt{n}\, f_o$, where f_o is the particle rotation frequency. If the particle is injected at a height $\underline{h}$ above the median plane, it returns to the neighborhood of the gun at a height $h\cos(2\pi\sqrt{n})$. Thus if $h(1-\cos 2\pi\sqrt{n}) > t$, the half-thickness of the gun, the particle will not be lost even if $|x_1| < s$. In practice one can choose n so that the particle first returns to the vicinity of the gun after about five revolutions, although ideally it is possible to increase this to $2\pi/\cos^{-1}(1-t/h)$ revolutions if h is increased suitably to allow for the length of the gun and the finite spread of the input beam. This assumes that all but a few particles miss the gun on the first few revolutions.

The particles escaping the gun during the first few revolutions will not return to the gun for a half period of a synchrotron oscillation. In this time the amplitude of the vertical oscillation will be damped to

$$h\left(\frac{B_1}{B_1 + \frac{1}{2\Omega_o}\frac{dB}{dt}}\right)^{\frac{1}{2}} = h\left[1 - \frac{f_o V \sin\psi_o}{4\Omega_o E_1}\right]$$

and if $(hV\sin\psi_o/4E_1)(f_o/\Omega_o) < t$, the particles will not be lost to the gun on subsequent revolutions. In practice, this is impossible to achieve and a certain proportion of the particles will be lost, equal very roughly to $(1/\pi)\left[\cos^{-1}(1-t/h)\right]$.

The oscillation period of a particle depends on its injection phase as well as upon B_o, ω_o, and V, all of which change continuously with time. Since the synchrotron period is much larger than a period of vertical betatron oscillations, we can regard their relative phase as a random quantity. With these assumptions, the total number of particles accepted into the machine of Table I is

$$N = 2.5 \times 10^7 SI\left(\frac{4}{E_o}\right)\left(2 - \frac{\psi_o}{\frac{\pi}{4}}\right)\left[1 - \frac{1}{\pi}\cos^{-1}\left(1 - \frac{t}{h}\right)\right]^{\ell}$$

where

$$\ell\,\frac{hV\sin\psi_o}{4E_1}\left(\frac{f_o}{\Omega_o}\right) = t\,,$$

S is the total number of cycles over which injection takes place, E_1 the injection energy in Mev, and I the injection current in microamperes.

If particles are not injected with the correct energy they will have, superimposed on the horizontal synchrotron oscillations, horizontal betatron oscillations of frequency $\sqrt{1-n}\, f_o$. This means that the particles will reappear in the vicinity of the gun at intervals during the time required for the radial synchrotron oscillation to move a radial distance equal to the sum of the gun thickness and twice the amplitude of the horizontal betatron oscillations. Thus the possibility of losing particles to the gun is very greatly increased.

5.2. Single-Shot Injection. In this system an auxiliary electric field is maintained for a time interval less than the rotation period of the particles to remove completely the vertical betatron oscillations. The gun is located exactly as in case (1) and a uniform vertical electric field is established over an arc of the acceleration chamber. The field strength and arc length are to be so chosen that the injected particles emerge into the field-free region with zero vertical velocity, and by tilting the gun the arc length may be shortened. These particles will remain in the median plane until they return to the gun. If by this time the vertical electric field has been removed, the particles will continue to move in the median plane and none will be lost to the gun.

When the finite spread of the injected beam is taken into account, one can show that, as far as subsequent oscillations are concerned, the electric field serves to move the gun into the median plane. In this system particles can be injected only over a single cycle and the resulting loss of intensity may require compensation by the use of an exploded ion source. An important advantage is that initially the particles are essentially all grouped along the central line of the acceleration chamber, while pulsing the source will ensure that no wasted particles are introduced into the chamber. This arrangement is very desirable from the point of view of an automatic frequency control system, which is particularly necessary during the initial stages of operation, and it may well become the determining factor in the choice of an injection system.

If upon injection the particle energy is E_1 Mev, one can show that the total number accepted into the acceleration cycle will be equal approximately to

$$2.50 \times 10^7 \; I\sqrt{\frac{4}{E_1}}\left(1 - \frac{2\psi_0}{\pi}\right) \qquad (34)$$

with I the injection current in microamperes and ψ_0 the stable phase angle.

<u>5.3. Rapid Change of the Stable Phase</u>. If the stable phase angle for synchrotron oscillations be suddenly changed, the particles will execute new oscillations with decreased or increased amplitudes depending on their phase at the instant of change of the stable phase. Thus some of the particles will have their maximum phase velocity and hence their maximum radial displacement from the stable orbit decreased, and one has a mechanism for capturing particles into synchrotron acceleration without hitting the injecting gun.

Since the sine of the stable phase angle is proportional to the time rate of change of magnetic field and inversely proportional to the accelerating voltage, a sudden change in either of these quantities will provide the necessary sudden change in stable phase. Particles are injected from a gun placed on the inside of the accelerating chamber in the median plane (to keep the vertical aperture a minimum).

During injection the r-f voltage is maintained at a constant frequency with a constant magnetic field, so chosen as to keep the stable orbit centered in the accelerating chamber. The machine thus acts as a d-c cyclotron or as a synchrotron with $\psi_0=0$. The sudden jump in ψ_0 occurs when the a-c magnetic field is turned on. A particle arriving at the accelerating gap during the injection period with a positive phase will pick up energy and move inwards into the chamber. If the radial displacement in a single revolution exceeds half the gun width, the particle will not hit the gun on the second revolution. In a practical case this radial shift is so large that practically all the particles arriving at the gap with positive phase will miss the gun and all those arriving with negative phase will be lost to the inner walls or to the gun.

During injection the phase motion of the protons is that of the free oscillations of a pendulum with arbitrary initial phase and initial phase velocity given by $\dot{\psi}_i = - n\omega_0 d/r_0$, with $\underline{d}$ the distance of the gun from the stable orbit of radius r_0. Thus the phase angle ψ is an elliptic function of the time. In particular, if the a-c magnetic field is turned on at a definite instant of time, the phase ψ_F and phase velocity $\dot{\psi}_F$ at this instant can be written in terms of ψ_i and $\dot{\psi}_i$. If now ψ_0 is the final stable phase angle, it follows readily that the particle will be accepted into the acceleration cycle if the following inequalities are satisfied:

$$\left.\begin{aligned} &\psi_i > 0 \\ &\left(\frac{d}{r_0}\right)^2 \geq \left(\frac{\dot{\psi}_F}{n\omega_0}\right)^2 + 2\left(\frac{\Omega_0}{n\omega_0}\right)^2 \left[(\cos\psi_0 - \cos\psi_F) - (\psi_F - \psi_0)\sin\psi_0\right] \\ &\cos\psi_0 - \cos\psi_F - (\psi_F - \psi_0)\sin\psi_0 + \tfrac{1}{2}\left(\frac{\dot{\psi}_F}{\Omega_0}\right)^2 \leq \cos 2\psi_0 - (\pi - 2\psi_0)\sin\psi_0 \end{aligned}\right\} \quad (35)$$

with $\Omega_0 = n\omega_0 V / \left[2\pi r_0^{\,2}(1-n)B_0\right]$, the angular frequency of small free oscillations during injection. The first inequality states that the particle does not hit the gun or inner walls on the first revolution. The second is the condition the amplitude of the new radial oscillations be less than the distance of the gun from the stable orbit, and the third condition ensures that the particle does not move out of the stable phase range for oscillations on subsequent revolutions.

Inserting the expressions for ψ_F and $\dot{\psi}_F$ into the conditions (35) then gives the acceptable injection phase angle as a function of the time of injection of the particle relative to the instant when the magnetic field is turned on. Summing over all injection times for which the acceptable injection phase angle is positive, one then can obtain the total range of acceptable phase angles Θ for all the particles which are injected. This process will not be carried out here in detail but an approximate expression for Θ, obtained by replacing the elliptic functions by trigonometric functions, may be written as

$$\Theta = \frac{f_o}{\Omega_o} \int_0^{\pi} \left[\psi_o \sin y \left(\frac{2dn\omega_o}{r_o\Omega_o} - \psi_o \sin y \right) \right]^{\frac{1}{2}} dy \; . \qquad (36)$$

which is valid when $2dn\omega_o/r_o\Omega_o > \psi_o$. ($f_o$ is the frequency of rotation of the particles.)

If numerical values taken from Table I are inserted, it can be shown that the total number of particles accepted into the stable phase range is given by

$$N \cong 6.3 \times 10^7 I \sqrt{\frac{4}{E_1}}$$

with I the injection current in microamperes, E_1 the injection energy in Mev; ψ_o has been taken as $\pi/4$, and it may be noted that N varies but slowly with ψ_o.

The chief advantages of this scheme are the following:

(a) No auxiliary equipment is required.

(b) It is not necessary to energy-modulate the gun.

(c) There are no critical timing circuits.

(d) The attainable intensity is at least as great as that for any of the proposed alternatives.

5.4. Adiabatic Shift of the Stable Synchrotron Orbit by FM Techniques. The scheme described here is essentially that proposed by D. M. Dennison for the electron synchrotron at the University of Michigan. The very different conditions at injection in a proton synchrotron, however, result in considerable difference in detail.

Consider particles injected from a gun at the outside of the accelerating chamber. Initially the radio frequency is chosen so that the stable orbit passes through the gun and this frequency is so modulated that the stable orbit moves in toward the center at a steady predetermined rate. Particles are injected in bursts over a time interval encompassing a number of revolutions so that energy modulation of the injector is necessary to attain maximum beam intensity.

Particles will miss the gun during the next few revolutions provided they arrive at the accelerating gap with negative phase. Furthermore, if the time required for the stable orbit to reach the center of the chamber is not much longer than the synchrotron oscillation period, particles will not be lost to the gun. If they are lost at all, they will be lost to the inner wall.

The phase acceptance angle ψ_1 on the first revolution must lie within the limits $\psi_o' < \psi_1 < \pi - \psi_o'$, where ψ_o' is the stable phase angle during injection. As the stable orbit moves inward the phase acceptance angle will decrease, for a given amplitude of synchrotron oscillation, vanishing when the stable orbit reaches the center of the chamber since the maximum amplitude of radial synchrotron oscillation equals the distance d of the gun from the central orbit.

If the frequency error ω_1 has the form

$$\omega_1 = \omega_o\left(-\frac{nd}{r_o} + \alpha t\right) \tag{37}$$

the first of Eqs. (11) yields

$$\frac{x}{r_o} = \frac{d}{r_o} - \left(\frac{\dot{\psi} + \omega_o \alpha t}{n\omega_o}\right) \tag{38}$$

and the second, in the non-relativistic range which applies at injection, yields for the equation of phase oscillation:

$$\frac{(1-n)r_o^2 B_o}{n\omega_o}\ddot{\psi} + \frac{V}{2\pi}\sin\psi = \frac{V}{2\pi}\sin\psi_o' \tag{39}$$

with

$$\sin\psi_o' = \sin\psi_o - \frac{2\pi r_o^2(1-n)B_o\alpha\omega_o}{n\omega_o V} . \tag{39a}$$

From Eq. (38), one sees that the stable orbit, which initially lies at a distance d from the center of the chamber, moves toward the center at a constant rate, so that the frequency law (37) accomplishes the desired result. If the stable orbit is to reach the center in a time less than one synchrotron period, α must satisfy the inequality

$$\alpha > \frac{nd}{r_o}\frac{\Omega_o'}{2\pi} \quad \text{with} \quad \Omega_o'^2 = \frac{\omega_o n V}{2\pi r_o^2(1-n)B_o} .$$

If furthermore $|\psi_o'|$ is not to exceed $|\psi_o|$, then $\alpha < (\Omega_o'/\omega_o)2\Omega_o'\sin\psi_o$, so that we have, as an additional requirement,

$$\frac{2\Omega_o'\sin\psi_o}{\omega_o} > \frac{nd}{2\pi r_o} .$$

For the 10-Bev machine this inequality is certainly fulfilled and allows a decrease of V during injection with a consequent decrease of Ω_o' and increase of the number of particles accepted into the stable phase range.

We now consider the particles to be injected in bursts with a recurrence frequency f_o equal to the rotation frequency of the particles. If the phase of a particle injected in the s^{th} burst is ψ_1 and the energy is appropriate to the injection orbit, then the particle will not be lost to the gun on subsequent revolutions if

$$\left[\left(\frac{\alpha s}{f_o}\right)^2 + \frac{\{\cos\psi_o' - \cos\psi_1 + (\psi_o' - \psi_1)\sin\psi_o'\}\, nV}{2\pi r_o^2(1-n)B_o\omega_o}\right]^{\frac{1}{2}} < \frac{nd}{r_o} \; .$$

From this one sees that there is no point in injecting particles after the orbit for which

$$s = \frac{f_o}{\alpha}\,\frac{nd}{r_o} \; .$$

The total number of particles accepted is given approximately by

$$N \cong 10^7 I \sqrt{\frac{4}{E_1}}\left(1 - \frac{2\psi_o}{\pi}\right) \frac{nd}{r_o} \cdot \frac{f_o}{\alpha}$$

with E_1 the average injection energy in Mev, ψ_o the stable phase, and I the beam current in microamperes.

Since the above expression is inversely proportional to α, the maximum number of particles will be obtained when $\alpha = nd/r_o \cdot \Omega_o'/2\pi$. In this case

$$N \cong 10^7 I \sqrt{\frac{4}{E_1}}\left(1 - \frac{2\psi_o}{\pi}\right) \frac{\omega_o}{\Omega_o'} = 4 \times 10^8 I \sqrt{\frac{4}{E_1}}\left(1 - \frac{2\psi_o}{\pi}\right) .$$

It is virtually impossible to attain this value in practice and a much more realistic value is that for which $\alpha = (\Omega_o'^2 \sin\psi_o)/\omega_o$, i.e. with $\psi_o' = 0$. In this case

$$N \cong 10^8 I \sqrt{\frac{4}{E_o}}\left(1 - \frac{2\psi_o}{\pi}\right) .$$

If the gun is not energy-modulated, injection will start when the energy required in the central orbit equals the injection energy, and will cease when the energy appropriate to the injection orbit equals the injection energy. The accepted number of particles will then be very much reduced and is probably no greater than that obtained by single-shot injection.

We have now given one example of each of the four main alternative injection schemes. Of these, if intensity is not the primary consideration, the single-shot scheme is most attractive. In contrast to the other methods, the injected ions are caught into the acceleration cycle and lie initially in the central orbit. This is highly desirable in view of the proposed frequency-control system which, with this injection system, can operate right from the beginning when control requirements are most critical. In the other schemes, a large number of particles not caught into the accelerating phase will drift to the inner wall. Until all these have been removed, the frequency-control system cannot begin to function properly.

If maximum intensity is the primary factor, probably scheme (3) is next best because of its simplicity. No energy modulation of the gun, no auxiliary electric fields or special frequency-modulation techniques are required, and the timing of the various steps in the injection process are not critical.

It must be emphasized once again, however, that a large number of alternative injection schemes of greater or lesser complexity can be evolved, each with advantages peculiar to itself. In a practical case a choice can only be made when all the relevent factors are taken into account.

REFERENCES

1. E. M. McMillan, Phys. Rev. 68, 143 (1945;
   V. Veksler, J. Phys. U.S.S.R. 9, 153 (1945);
   D. M. Dennison and T. H. Berlin, Phys. Rev. 70,58 (1946);
   D. S. Saxon and J. Schwinger, Phys. Rev. 69,702(A) (1946);
   N. H. Frank, Phys. Rev. 70, 177 (1946);
   D. Bohm and L. Foldy, Phys, Rev. 70, 249 (1946).
2. N. H. Frank, Phys. Rev, 70, 177 (1946).
3. J. S. Gooden, H. H. Jensen, and J. L. Symonds , Proc. Phys. Soc. London, 59, 677 (1947); see also H. L. Oliphant, J. S. Gooden, and G. S. Hide, Proc.Phys. Soc. London, 59, 660 (1947).