Tôhoku Math. J.
43 (1991), 289–299

ORBITS AND THEIR ACCUMULATION POINTS OF CYCLIC SUBGROUPS OF MODULAR GROUPS

Dedicated to Professor Tatuo Fuji'i'e on his sixtieth birthday

HARUMI TANIGAWA

(Received May 10, 1990, revised October 15, 1990)

Introduction. As is well known, the Teichmüller space $T(S)$ of a Riemann surface S of finite analytic type (p, n) with $3p-3+n>0$ is a complex manifold of dimension $3p-3+n$, and is complete with respect to the Teichmüller metric. Bers [B2] gave a classification of modular transformations in terms of the translation lengths, and showed that the types of modular transformations are characterized by the self-mappings of S inducing them. By definition and facts shown in [B2], hyperbolic modular transformations are expected to have properties similar to that of hyperbolic Möbius transformations. For example Bers [B2] showed that a non-periodic modular transformation is hyperbolic if and only if it has an invariant Teichmüller line, and gave a remark (without proof) that for each hyperbolic modular transformation the invariant line is unique by Thurston's theory. He also posed a problem to prove the uniqueness of the invariant line using quasiconformal mappings. In this paper we show (§2) this by combining the theory of quasiconformal mappings and the result of Bowen and Marcus [BM]. Using this fact we give a simple proof of the theorem of McCarthy about the centralizer and normalizer of a hyperbolic cyclic subgroup of the modular group (Theorem 2.4).

It is also well-known that the Teichmüller space $T(S)$ is identified with a bounded domain of C^{3p-3+n}, via the embedding introduced by Bers, and each point of the boundary corresponds to a Kleinian group. From the discontinuity of the action of the modular group, for every non-periodic modular transformation $[f]_*$, induced by a self-mapping $f: S\to S$, and for a point $\tau\in T(S)$ the accumulation points set of the sequence $\{[f]_*^m(\tau)\}_{m=1}^{\infty}$ is contained in the boundary of $T(S)$. Interesting investigations about relations between the type of the modular transformation $[f]_*$ and the type of the Kleinian groups corresponding to accumulating points of $\{[f]_*^m(\tau)\}_{m=1}^{\infty}$ are seen in [B3], [S], and [H]. It is natural to expect that the Kleinian group corresponding to an accumulation point of the sequence $\{[f]_*^m(\tau)\}_{m=1}^{\infty}$ inherits the property of f, if the mapping f has some symmetric property. This line of thought is developed in §3. The argument there yields a different way of approach to necessary conditions, studied by Birman, Lubotsky and McCarthy [BLM], for two non-hyperbolic modular transformations to commute.

The author is grateful to Professor H. Shiga for his helpful suggestions. She is also

grateful to Professor S. Morita for useful information about mapping class groups.

1. Notation. In this section we fix our notation and recall some known results.

Let G be a finitely generated torsion free Fuchsian group of the first kind acting on the upper half plane U. The Riemann surface $S=U/G$ has a finite genus $p\geq 0$ and is compact except for $n\geq 0$ punctures with $3p-3+n>0$. First we describe the Teichmüller space $T(S)$ of S. Later we recall the relation between $T(S)$ and the Teichmüller space $T(G)$ of G. See Lehto [L] for details. Two quasiconformal homeomorphisms $f_1: S\rightarrow S_1$ and $f_2: S\rightarrow S_2$, where S_1 and S_2 are Riemann surfaces of type (p, n), are said to be equivalent if there is a conformal mapping $h: S_1\rightarrow S_2$ such that $f_2^{-1}\circ h\circ f_1: S\rightarrow S$ is homotopic to the identity. The *Teichmüller space* $T(S)$ of S is the set of all such equivalence classes. The Teichmüller space $T(S)$ is a complex manifold of dimension $3p-3+n$. The *Teichmüller metric* d on $T(S)$ is defined by

$$d([f], [g])=\inf\{2^{-1}\log K(f'\circ(g')^{-1}); [f']=[f], [g']=[g]\}$$

where $[f], [g]\in T(S), f'$ and g' are quasiconformal mappings, and $K(f'\circ(g')^{-1})$ is the maximal dilatation of $f'\circ(g')^{-1}$. A *Teichmüller disc* is the image of the unit disc $\Delta=\{|z|<1\}$ by a holomorphic isometry of Δ into $T(S)$ with respect to the non-Euclidean metric on Δ and the Teichmüller metric on $T(S)$. The image of a non-Euclidean geodesic line in Δ by such a mapping is called a *Teichmüller line.*

A quasiconformal homeomorphism f: $S\rightarrow S$ induces an automorphism $[f]_*$ of $T(S)$ defined by $[g]\rightarrow[g\circ f^{-1}]$, where $[g]$ denotes the equivalence class of a quasiconformal homeomorphism $g: S\rightarrow g(S)$. The automorphism $[f]_*$ depends only on the homotopy class of f. Such an automorphism is called a *modular transformation*, and the set of all modular transformations of $T(S)$ is called the *modular group* of $T(S)$ and is denoted by $\mathrm{Mod}(S)$. Every modular transformation is a holomorphic isometry.

Now we recall the classification of modular transformations. See Bers [B2]. For $\chi\in\mathrm{Mod}(S)$, let $a(\chi)$ denote the infimum of $d(\tau, \chi(\tau))$ for $\tau\in T(S)$. The modular transformation χ is said to be *elliptic* if it has a fixed point in $T(S)$, *parabolic* if $a(\chi)=0$ and there exists no fixed point in $T(S)$, *hyperbolic* if $a(\chi)>0$ and there is a point $\tau\in T(S)$ with $a(\chi)=d(\tau, \chi(\tau))$, *pseudo-hyperbolic* if $a(\chi)>0$ and $a(\chi)<d(\tau, \chi(\tau))$ for every point $\tau\in T(S)$. A non-periodic modular transformation χ is hyperbolic if and only if there exists an invariant Teichmüller line l, namely, $\chi(l)=l$. For a hyperbolic modular transformation χ, a point $\tau\in T(S)$ satisfies $d(\tau, \chi(\tau))=a(\chi)$ if and only if it is on an invariant line of χ. The type of a modular transformation is characterized by topological property of a self-mapping of S by which the modular transformation is induced, as follows (Bers [B2]): a system of disjoint Jordan curves $\Sigma=\{C_1, \cdots, C_s\}$ on S is said to be *admissible* if no C_j is homotopic to a point, a boundary component of S, or C_i with $i\neq j$. A self-mapping f: $S\rightarrow S$ is said to be reduced by a system of curves Σ if Σ is admissible and $f(\Sigma)=\Sigma$. A self-mapping f of S is said to be *reducible* if it is isotopic to a reduced mapping, *irreducible* otherwise. When f is reduced by Σ, f is said to be

completely reduced by Σ if for each component S_1 of $S-\Sigma$ and for the smallest positive integer j with $f^j(S)=S_1$, the map f^j is irreducible. Every reducible mapping is isotopic to a completely reduced mapping. Hence every modular transformation is induced either by a completely reduced mapping or by an irreducible mapping. If there is an integer k such that f^k is homotopic to the identity, we shall say that f is periodic. A modular transformation $[f]_*$ is elliptic if and only if it is induced by a periodic mapping, hyperbolic if and only if it is induced by a non-periodic irreducible mapping. If $[f]_*$ is induced by a non-periodic mapping f completely reduced by an admissible system of curves Σ, $[f]_*$ is parabolic if and only if for every component S_1 of $S-\Sigma$ and for the smallest integer with $f^m(S_1)=S_1$ the mapping $f^m|S_1$ is periodic, pseudo-hyperbolic if and only if there exists a component S_1 of $S-\Sigma$ such that for the smallest integer m with $f^m(S_1)=S_1$ the mapping $f^m|S_1$ is non-periodic irreducible.

Next we describe Teichmüller mappings. Let ϕ be an integrable holomorphic quadratic differential on S, namely, holomorphic quadratic differential with $\iint_S|\phi|<\infty$. For each point $p\in S$, there exists a local parameter ζ defined in a neighborhood of p, such that

$$\phi=\left(\frac{r+2}{2}\right)^2\zeta^r d\zeta^2,$$

where r is the order of ϕ at p. The parameter ζ is called a *natural parameter* (see Lehto [L]). A quasiconformal homeomorphism $f\colon S\to S'$ is called a *Teichmüller mapping* if its Beltrami differential is of the form $k\bar{\phi}/|\phi|$ where $0<k<1$ and ϕ is an integrable holomorphic quadratic differential on S which is not identically zero. The quadratic differential ϕ is called the *initial differential* of f. For the Teichmüller mapping f, there exists an integrable holomorphic differential ψ on the target Riemann surface S' with the following properties:

(a) For each point $p\in S$ and a natural parameter ζ of ϕ at p and a natural parameter ζ' of ψ at $f(p)$, the mapping f has the representation

$$\zeta'\circ f=\left(\frac{\zeta^{(j+2)/2}+k(\bar{\zeta})^{(j+2)/2}}{1-k}\right)^{2/(j+2)}$$

where $0<k<1$ and j is the order of ϕ at p.

(b) The order of ϕ at p equals the order of ψ at $f(p)$.

(c) $\iint_S|\phi|=\iint_{S'}|\psi|$.

The differential ψ is called the *terminal differential* of f.

Now we define the Teichmüller space of the Fuchsian group G. (Recall that G denotes the Fuchsian group such that $U/G=S$.) Let W_1 and W_2 be quasiconformal homeomorphisms of $\hat{\mathbf{C}}$ such that $W_j|L$ is conformal and $W_j(-i)=0$, $W_j(-2i)=1$, $W_j(-3i)=\infty$, and $W_j\circ g\circ(W_j)^{-1}$ is a Möbius transformation for each $g\in G$ $(i=1,2)$. Here L denotes the lower half plane. The quasiconformal homeomorphisms W_1 and W_2 are said to be equivalent if $W_1|L=W_2|L$. The set of all such equivalence classes is denoted by $T(G)$ and is called the *Teichmüller space* of G. The spaces $T(S)$ and $T(G)$ are identified in the following way: let f: $S \to S'$ be a quasiconformal homeomorphism. The mapping f is lifted to a quasiconformal self-mapping w of U via the canonical projection $U \to S$. Let W_w be the quasiconformal mapping such that

$$(1) \qquad W_w | U \circ w^{-1} \quad \text{and} \quad W_w | L \text{ are conformal}$$

and

$$(2) \qquad W_w(-i) = 0 , \qquad W_w(-2i) = 1 , \qquad W_w(-3i) = \infty .$$

The assignment $f \mapsto W_w$ induces a bijection between $T(S)$ and $T(G)$. Now let $B_2(L, G)$ denote the space of bounded holomorphic quadratic differentials on L for G, namely,

$$B_2(L, G) = \{\varphi\,;\ \text{holomorphic on } L,\ \sup |\{\operatorname{Im} z\}^2 \varphi(z)| < \infty,\ (\varphi \circ g)(g')^2 = \varphi \text{ for } \forall g \in G\} .$$

The space $B_2(L, G)$ is a $(3p-3+n)$-dimensional Banach space, where (p, n) is the signature of S. The *Bers embedding* (see Bers [B1]) Ψ: $T(G) \to B_2(L, G)$ is defined as follows: For each point $[W]$ of $T(G)$, $\Psi([W])$ is the Schwarzian derivative of $W|L$. Then the image $\Psi(T(G))$ is a bounded domain of $B_2(L, G)$. We identify the spaces $T(S)$, $T(G)$ and $\Psi(T(G))$, so we call the boundary of $\Psi(T(G))$ in $B_2(L, G)$ the boundary of $T(S)$ or $T(G)$, and denote it by $\partial T(S)$ or $\partial T(G)$. For a point $\varphi \in T(G) \cup \partial T(G)$, let W_φ denote the (necessarily univalent) meromorphic function on L with the Schwarzian derivative φ normalized by the formula (1). Then for each $g \in G$ there exists a Möbius transformation $\chi_\varphi(g)$ such that $W_\varphi \circ g = \chi_\varphi(g) \circ W_\varphi$. The map $G \ni g \mapsto \chi_\varphi(g)$ is an isomorphism, and the group $G_\varphi = \chi_\varphi(G)$ is a Kleinian group with an invariant component $W_\varphi(L)$. If a hyperbolic element $g \in G$ is mapped to a parabolic element $\chi_\varphi(g) \in G_\varphi$, the element $\chi_\varphi(g)$ is called an *accidental parabolic element* of G_φ.

2. Uniqueness of invariant lines of hyperbolic modular transformations. In this section we show the uniqueness of the invariant Teichmüller line of each hyperbolic modular transformation and investigate modular transformations commutative with a hyperbolic modular transformation as a corollary.

THEOREM 2.1. *For each hyperbolic modular transformation, there exists exactly one invariant Teichmüller line.*

Before proving this theorem, we recall the definition of a pseudo-Anosov mapping. Let $\bar{S}$ denote the compact Riemann surface into which S is embedded. A *singular foliation* $\mathscr{F}$ on $\bar{S}$ with a finite set E of singularities is a decomposition of S into a disjoint union of leaves as follows (see [CB], [FLP]): for each point $x \in \bar{S}$ there exists a neighborhood V of x and a local C^∞-chart φ: $V \to \boldsymbol{C}$ with $\varphi(x) = 0$ such that

(i) the decomposition $\mathscr{F} | V$ into leaves are obtained by φ^{-1} (horizontal lines in $\boldsymbol{C}$), if $x \in \bar{S} - E$,

(ii) the decomposition $\mathscr{F} | V$ into leaves are obtained by φ^{-1} (horizontal trajectories of a quadratic differential $z^{p-2}dz^2$), for some integer $p \geq 3$ if $x \in S \cap E$, and for some integer $p \geq 3$ or $p = 1$ if $x \in (\bar{S} - S) \cap E$.

A *transverse measure* μ to a singular foliation $\mathscr{F}$ defines a Borel measure $\mu|\alpha$ on each arc α transverse to $\mathscr{F}$ with the following properties:

(a) For every subarc α' of α, $\mu|\alpha'$ is the restriction of $\mu|\alpha$.

(b) If two arcs α_1 and α_2 transverse to $\mathscr{F}$ are homotopic via a homotopy Φ: $[0, 1] \times [0, 1] \to \bar{S}$ such that $\Phi([0, 1] \times 0) = \alpha_1$, $\Phi([0, 1] \times 1) = \alpha_2$, and that $\Phi(t \times [0, 1])$ is contained in a leaf $\mathscr{F}$ for all $t \in [0, 1]$, then $\mu(\alpha_1) = \mu(\alpha_2)$.

A pair $(\mathscr{F}, \mu)$ of a singular foliation $\mathscr{F}$ and its transverse measure μ is called a measured foliation. Two measured foliations $(\mathscr{F}_1, \mu_1)$ and $(\mathscr{F}_2, \mu_2)$ are said to be transverse if they have the same set of singularities E, and transverse to each other in $\bar{S} - E$. A self-mapping f: $S \to S$ is called a *pseudo-Anosov diffeomorphism* if there are transverse measured foliations $(\mathscr{F}^s, \mu^s)$ and $(\mathscr{F}^u, \mu^u)$ and a positive number $\lambda > 1$ such that

$$(f\mathscr{F}^s, f_*\mu^s) = \left(\mathscr{F}^s, \frac{1}{\lambda}\mu^s\right), \qquad (f\mathscr{F}^u, f_*\mu^u) = (\mathscr{F}^u, \lambda\mu^u),$$

and if f is differentiable in $\bar{S} - E$. The foliations $\mathscr{F}^s$ and $\mathscr{F}^u$ are called the stable foliation and the unstable foliation of f, respectively. For relations of non-periodic irreducible mappings and pseudo-Anosov mappings, see Bers [B2, §9] or Gardiner [G, §11].

Proof of Theorem 2.1. Let γ be a hyperbolic modular transformation and let l_1 and l_2 be invariant lines for γ. Fix a point τ_i of l_i $(i = 1.2)$. We may assume that the point τ_1 is the base point of the Teichmüller space. Let S_i denote the corresponding Riemann surface $(i = 1, 2)$. Take a diffeomorphism φ: $S_1 \to S_2$ representing the point τ_2, i.e., $\tau_2 = [\varphi] \in T(S_1)$. Then the modular transformation γ is induced by a Teichmüller mapping w_i: $S_i \to S_i$ (i.e., $\gamma = [w_1]_* = [\varphi^{-1} \circ w_2 \circ \varphi]_*$) whose initial differential and terminal differential coincide $(i = 1, 2)$ (Bers [B2]). Let Φ_i denote the initial (= terminal) differential of w_i. For each $z \in \Delta = \{|z| < 1\}$ let f^z denote the quasiconformal homeomorphism of S_1 onto another Riemann surface with Beltrami differential $z\bar{\Phi}_1/|\Phi_1|$. Then the Teichmüller line l_1 is represented as

$$l_1 = \{[f^k];\ -1 < k < 1\}$$

(Bers [B2, §6]. We shall show that the Teichmüller line l_2 is contained in the Teichmüller line l_1.

The family $\mathscr{F}_i^u$ of horizontal trajectories of Φ_i and the family $\mathscr{F}_i^s$ of vertical trajectories of Φ_i are made into measured foliations on S_i with transverse measures $\mu_i^u = |\mathrm{Im}\sqrt{\Phi_i}|$ and $\mu_i^s = |\mathrm{Re}\sqrt{\Phi_i}|$, respectively. The mapping w_i is a pseudo-Anosov mapping with transverse measured foliations $(\mathscr{F}_i^s, \mu_i^s)$ and $(\mathscr{F}_i^u, \mu_i^u)$ (Bers [B2, §9]). It follows that the mapping $\varphi^{-1} \circ w_2 \circ \varphi$ is homotopic to w_1, and is a pseudo-Anosov mapping of S_1 with transverse measured foliations $(\varphi^{-1}\mathscr{F}_2^s, \varphi^*\mu_2^s)$ and $(\varphi^{-1}\mathscr{F}_2^u, \varphi^*\mu_2^u)$.

Here we note the following two lemmas:

LEMMA 2.2. *Two homotopic pseudo-Anosov diffeomorphisms* φ_1 *and* φ_2 *are conjugate by a diffeomorphism homotopic to the identity.*

A proof appears in [FLP Exposé 12, p. 238–241], which consists of two parts. In the first part it is shown that the sequence $\{\varphi_2^{-n} \circ \varphi_1^n\}$ converges uniformly. The limit mapping h of the sequence satisfies $h \circ \varphi_1 \circ h^{-1} = \varphi_2$. In the second part h is shown to be a diffeomorphism. The argument there first shows that h maps leaves of the stable (resp. unstable) foliation of φ_1 to those of the stable (resp. unstable) foliation of φ_2. Then the following lemma is used:

LEMMA 2.3 (unique ergodicity). *Let* μ *and* ν *be transverse measures for the stable foliation of a pseudo-Anosov mapping. Then there exists a positive number* λ *such that* $\nu = \lambda\mu$.

This is a particular case of Bowen-Marcus [BM], and a proof also appears in [FLP, Exposé 12].

We proceed to prove Theorem 2.1. By Lemma 2.2, there exists a diffeomorphism s of S such that $w_1 = s^{-1} \circ \varphi^{-1} \circ w_2 \circ \varphi \circ s$, and from the argument of the proof of Lemma 2.2, the map s satisfies $s(\mathscr{F}_1^s) = \varphi^{-1}(\mathscr{F}_2^s)$ and $s(\mathscr{F}_1^u) = \varphi^{-1}(\mathscr{F}_2^u)$. This means that the mapping $\varphi \circ s$ sends the vertical and horizontal trajectories of the quadratic differential Φ_1 to the vertical and horizontal trajectories of Φ_2, respectively. Since μ_1^s and $(\varphi \circ s)^*(\mu_2^s)$ are both transverse measures of the foliation $\mathscr{F}_1^s$, there exists a positive number λ' such that $\mu_1^s = (\lambda')^{-1}(\varphi \circ s)^*(\mu_2^s)$. In the same way we have a positive number λ such that $\mu_1^u = (\lambda)^{-1}(\varphi \circ s)^*(\mu_2^u)$. In other words, for each regular point x of the quadratic differential Φ_1 and natural parameters ζ around x and ζ' around $\varphi \circ s(x)$ of Φ_1, the mapping $\varphi \circ s$ is represented as $\operatorname{Re}\zeta + i \operatorname{Im}\zeta \mapsto \lambda \operatorname{Re}\zeta' + i\lambda' \operatorname{Im}\zeta'$, namely, $\zeta \mapsto ((\lambda' + \lambda)/2)\zeta + ((\lambda - \lambda')/2)\bar{\zeta}'$. Hence the complex dilatation of $\varphi \circ s$ coincides with $((\lambda - \lambda')/(\lambda + \lambda'))(\bar{\Phi}_1/|\Phi_1|)$ almost everywhere on S_1. It follows that every point $\tau \in l_2$ is represented by a Teichmüller mapping with complex dilatation $k\bar{\Phi}_1/|\Phi_1|$ for some $k \in (-1, 1)$. This means that the Teichmüller line l_2 is contained in the Teichmüller line l_1. Hence the line l_2 necessarily coincides with l_1. ∎

REMARK. In [FLP, Exposé 12], the singularities of foliations are supposed to be those of $\sqrt{z^{p-2}dz^2}$, $p \geq 3$. But the same proof is valid for a pseudo-Anosov mapping, as w_1 in the above proof of Theorem 2.1, whose transverse foliations are induced by an integrable holomorphic differential on the Riemann surface S of finite analytic type. To follow the proof of Lemma 2.3 in [FLP], one needs to see some (of course not all) of the results in previous sections. But as for such a pseudo-Anosov mapping we can make somewhat shorter course to get to Lemma 2.3. (For example, it is easy to show that transverse foliations of such a mapping has no leaves of finite length.)

Using the above theorem we give a simple proof of the theorem by McCarthy [Mc].

THEOREM 2.4. *Let γ_1 be a hyperbolic modular transformation and let γ_2 be a modular transformation such that $\gamma_2\langle\gamma_1\rangle\gamma_2^{-1}=\langle\gamma_1\rangle$, where $\langle\gamma_1\rangle$ denotes the cyclic subgroup of the modular group generated by γ_1. Then $\gamma_2\circ\gamma_1\circ\gamma_2^{-1}=\gamma_1$ or $\gamma_2\circ\gamma_1\circ\gamma_2^{-1}=\gamma_1^{-1}$. In the former case the modular transformation γ_2 is either an elliptic transformation which fixes the invariant Teichmüller line of γ_1 pointwise, or a hyperbolic transformation such that $\gamma_1^m=\gamma_2^k$ for some integers m and k. In the latter case γ_2 is an elliptic transformation such that γ_2^2 fixes the invariant Teichmüller line of γ_1 pointwise.*

PROOF. The first statement is immediate since $a(\gamma_1)=a(\gamma_2\circ\gamma_1\circ\gamma_2^{-1})$ *and* $a(\gamma_1^j)=|j|\,a(\gamma_1)$.

Assume that $\gamma_1\circ\gamma_2=\gamma_2\circ\gamma_1$. Let l denote the invariant Teichmüller line of γ_1. Then by assumption the Teichmüller line $\gamma_2(l)$ is also invariant under γ_1. Since the invariant line is unique, $\gamma_2(l)=l$. Let D be the Teichmüller disc containing the Teichmüller line l, and let $\psi: \Delta\to D$ be a holomorphic isometry. Since γ_1 and γ_2 leave l invariant, the mappings $\psi^{-1}\circ\gamma_1\circ\psi$ and $\psi^{-1}\circ\gamma_2\circ\psi$ are Möbius transformations acting on Δ with an invariant non-Euclidean geodesic line $\psi^{-1}(l)$ and are commutative. Hence the Möbius transformation $\psi^{-1}\circ\gamma_2\circ\psi$ is either the identity or a hyperbolic transformation with the same fixed points as those of $\psi^{-1}\circ\gamma_1\circ\psi$. In the former case, it follows that $\gamma_2|D=\mathrm{id}$. In the latter case, there exist integers i and j such that $\psi^{-1}\circ\gamma_1^i\circ\psi=\psi^{-1}\circ\gamma_2^j\circ\psi$, since the modular group acts properly discontinuously on the Teichmüller space. It follows that $\gamma_1^i|D=\gamma_2^j|D$, hence $\gamma_1^i\circ\gamma_2^{-j}$ is an elliptic modular transformation. Therefore there exists an integer h such that $(\gamma_1^i\circ\gamma_2^{-j})^h=\gamma_1^{ih}\circ\gamma_2^{-jh}=\mathrm{id}$.

Assume that $\gamma_1\circ\gamma_2=\gamma_2\circ\gamma_1^{-1}$. Then, since the modular transformations γ_1^{-1} and γ_1 have the same invariant line l, we can again consider the Möbius transformations $\psi^{-1}\circ\gamma_1\circ\psi$ and $\psi^{-1}\circ\gamma_2\circ\psi$. This time $\psi^{-1}\circ\gamma_2\circ\psi$ is an elliptic Möbius transformation of order 2 which permutes the fixed points of $\psi^{-1}\circ\gamma_1\circ\psi$. Hence we have $\gamma_2^2|D=\mathrm{id}$. ■

3. Iteration of a parabolic or pseudo-hyperbolic transformation. In this section we investigate non-periodic modular transformations induced by reducible self-mappings of $S=U/G$. We utilize the following two lemmas:

LEMMA 3.1 (Maskit [M]). *Let $\varphi\in\partial T(G)$ and assume that the Kleinian group G_φ corresponding to φ contains accidental parabolic elements. Then there exist pairwise non-commuting hyperbolic elements $g_1,\cdots,g_s$ of G such that every accidental parabolic element of G_φ is conjugate in G_φ to $\chi_\varphi(g_i)^m$ for some $i=1,\cdots,s$ and some integer $m\neq 0$ and that the axes of $g_1,\cdots,g_s$ are mapped by the canonical projection to an admissible system of curves $\{C_1,\cdots,C_s\}$ on S.*

LEMMA 3.2 (Shiga [S]). *Let f: $S\to S$ be a non-periodic reducible quasiconformal selfmapping. Assume that the sequence $\{[f]_*^{m_j}(\tau)\}$ converges to a boundary point $\varphi\in\partial T(S)$ for a sequence of integer $\{m_j\}$ and a point $\tau\in T(S)$. Then the group G_φ contains an* *accidental parabolic transformation.*

Under the hypotheses of Lemma 3.2, Bers [B3] showed that there exists a mapping f' homotopic to f such that $f(\{C_1, \cdots, C_s\})=\{C_1, \cdots, C_s\}$, where $\{C_1, \cdots, C_s\}$ is the system of curves accompanied to the limit point φ of $\{[f]_*^{m_j}(\tau)\}$ as in Lemma 3.1. A similar argument yields the following theorem:

THEOREM 3.3. *Under the hypotheses of Lemma 3.2, assume that $[f]_*$ commutes with a modular transformation $[g]_*$ induced by a quasiconformal homeomorphism $g: S\to S$. Then there exists a mapping g' in the same equivalence class as g such that the system of curves $\{C_1, \cdots, C_s\}$, accompanied to $\varphi\in\partial T(G)$ as in Lemma 3.1, is invariant under g'. Furthermore, let S_1 and S_2 be components of $S-\{C_1, \cdots, C_s\}$ with $g'(S_1)=S_2$, and let G_1 and G_2 be the component subgroups of G_φ corresponding to S_1 and S_2, respectively. Then the groups G_1 and G_2 are quasiconformally equivalent. Here, a component subgroup G_1 corresponding to S_1 is the group χ_φ (the stabilizer in G of a component of $\pi^{-1}(S_1)$), where $\pi: U\to S$ is the canonical projection.*

The proof is parallel to Bers [B3].

PROOF. The quasiconformal mappings g and f are lifted to quasiconformal homeomorphisms ω_1 and ω_2 of U, respectively, conjugating G onto itself. Extend ω_1 and ω_2 to quasiconformal homeomorphisms of $\hat{C}$ by symmetry $\omega_i(\bar{z})=\overline{\omega_i(z)}$. Assume that the point τ in Lemma 3.2 is represented by a quasiconformal mapping h as $\tau=[h: S\to S']$, and lift h to a quasiconformal self-mapping w of U. Set

$$\omega_2'=\omega_1\circ\omega_2\circ\omega_1^{-1}, \quad W_m=W_{w\circ\omega_2^{-m}}, \quad \text{and} \quad W_m'=W_{w\circ(\omega_2')^{-m}}, \quad m=1, 2, \cdots.$$

Here, the quasiconformal homeomorphisms $W_{w\circ\omega_2^{-m}}$ and $W_{w\circ(\omega_2')^{-m}}$ of $\hat{C}$ are defined by the formulae (1), (2) in §1. Then from the assumption $[f]_*[g]_*[f]_*^{-1}=[g]_*$, if follows that $[W_m]=[W_m']\in T(G)$, and that this point corresponds to the point $[f]_*^m(\tau)\in T(S)$.

Set

$$X_m=W_m'\circ\omega_1\circ W_m^{-1}, \qquad m=1, 2, \cdots.$$

We now show that there exists a subsequence of $\{X_m\}$ which converges to a quasiconformal homeomorphism of $\hat{C}$. By the normalization of W_m, the quasiconformal mapping X_m satisfies

$$X_m(0)=W_m'\circ\omega_1(-i), \quad X_m(1)=W_m'\circ\omega_1(-2i), \quad X_m(\infty)=W_m'\circ\omega_1(-3i).$$

By the assumption $\lim[f]_*^{m_j}(\tau)=\varphi$ the sequences $\{W_{m_j}|L\}$ and $\{W_{m_j}'|L\}$ converge to the univalent function W_φ locally uniformly on L. Hence we have

$$\lim X_{m_j}(0)=W_\varphi\circ\omega(-i), \quad \lim X_{m_j}(1)=W_\varphi\circ\omega(-2i), \quad \lim X_{m_j}(\infty)=W_\varphi\circ\omega(-3i).$$

These three points are distinct. Next we note that the maximal dilatations of $\{X_{m_j}\}$

are uniformly bounded. Set

$$h_m = W_m \circ (w \circ \omega_2^{-m})^{-1}, \quad h'_m = W'_m \circ (w \circ (\omega'_2)^{-m})^{-1} \qquad \text{on} \quad U.$$

Then we have

$$X_m | W_m(U) = h'_m \circ w \circ (\omega'_2)^{-m} \circ \omega_1 \circ \omega_2^m \circ w^{-1} \circ h_m^{-1} = h'_m \circ w \circ \omega_1 \circ w^{-1} \circ h_m^{-1}.$$

Since the mappings h_m and h'_m are conformal, the maximal dilatation of $X_m | W_m(U)$ equals that of $w \circ \omega_1 \circ w^{-1}$. Since the mapping $W_m | L$ ($= W'_m | L$) is conformal, the maximal dilatation of $X_m | W_m(L)$ equals that of ω_1. The complement of $W_m(U \cup L)$ in $\hat{C}$ has measure 0.

From these facts we may assume, taking a subsequence if necessary, that the sequence $\{X_{m_j}\}$ converges to a quasiconformal homeomorphism X of $\hat{C}$ uniformly with respect to the spherical metric. Therefore we have

$$X \circ W_\varphi = W_\varphi \circ \omega_1$$

in L. By this formula

$$X \circ \chi_\varphi(\gamma) \circ X^{-1} = \chi_\varphi(\omega_1 \circ \gamma \circ \omega_1^{-1})$$

for every element γ of G. Hence the component subgroups G_1 and G_2 are quasiconformally equivalent, and $\chi_\varphi(\gamma)$ is parabolic if and only if $\chi_\varphi(\omega_1 \circ \gamma \circ \omega_1^{-1})$ is parabolic. It follows that the set of the homotopy class of $\{g(C_1), \cdots, g(C_s)\}$ equals that of $\{C_1, \cdots, C_s\}$. By a topological theorem due to Epstein [E], g is homotopic to a mapping g' with $g'(\{C_1, \cdots, C_s\}) = \{C_1, \cdots, C_s\}$. ■

The above theorem yields another method of giving necessary conditions for non-hyperbolic modular transformations to be commutative (see [BLM]). First assume that $[f]_*$ be a parabolic transformation induced by a self-mapping f: $S \to S$. We may assume that there exists an integer m such that f^m is a product of Dehn twists about an admissible system of curves $\{C_1, \cdots, C_s\}$ (cf. [S]). Evidently, such a system is determined uniquely by $[f]$ up to homotopy. We say that this system is accompanied to f and denote the system by Σ_f. Hejhal [H] proved that for every sequence $\{[f^m]_*^{n_j}\}(\tau)$ converging to a boundary point $\phi \in \partial T(S)$ the system of curves accompanied to ϕ as in Lemma 3.1 coincides (up to homotopy) with the system Σ_f. Hence if a modular transformation $[g]_*$ induced by a self-mapping $g: S \to S$ commutes with $[f]_*$, then by Theorem 3.3 we may assume that g is reduced by $\{C_1, \cdots, C_s\}$. By the argument in Bers [B2, §7], there exists an admissible system of curves $\{C_1, \cdots, C_t\}$ containing $\{C_1, \cdots, C_s\}$ by which some mapping g' homotopic to g is completely reduced. Assume that $[g]$ is also parabolic. Then there exists an integer k such that $(g')^k$ is homotopic to a product of Dehn twists about an admissible system of curves Σ_g contained in $\{C_1, \cdots, C_t\}$ (cf. [S]). Thus we have:

COROLLARY 3.4 (cf. [BLM]). *If parabolic transformations $[f]_*$ and $[g]_*$ are* *commutative, then for the admissible system of curves* Σ_f *and* Σ_g *accompanied to* f *and* g*, respectively, the union* $\Sigma_f \cup \Sigma_g$ *is an admissible system.*

In the general case, we show the following:

COROLLARY 3.5 (cf. [BLM]). *Let* f *be a reducible homeomorphism. Assume that the modular transformation* $[f]_*$ *commutes with a modular transformation* $[g]_*$ *induced by a quasiconformal self-mapping* $g: S \to S$*. Then there exist an admissible system of curves* Σ *and quasiconformal self-mappings* f' *and* g' *homotopic to* f *and* g*, respectively, with the following property: there exists an integer* m *such that* $(f')^m(\Sigma) = \Sigma$, $(g')^m(\Sigma) = \Sigma$, *and for each component* S_1 *of* $S - \Sigma$ *the mappings* $(f')^m | S_1$ *and* $(g')^m | S_1$ *are irreducible or periodic self-mappings whose isotopy classes are commutative (as investigated in the previous section).*

PROOF. We show the corollary by induction on $k = 3p - 3 + n$, where (p, n) is the type of S.

Assume that $k = 1$. If both of f and g are periodic, nothing remains to be proved. If f is non-periodic, take a sequence of positive integers $\{m_j\}$ such that the sequence $[f]_*^{m_j}(\tau)$ converges to a boundary point ϕ for a point $\tau \in T(S)$. There exists a simple closed curve C accompanied to ϕ as in Lemma 3.1. By Theorem 3.3, there exist quasiconformal mappings f' and g' homotopic to f and g, respectively, with $f'(C) = C$ and $g'(C) = C$. Hence the assertion is immediate. When f is periodic and g is non-periodic, the statement is shown in the same way.

Assume that $k = 3p - 3 + n > 1$. For the same reason as above, we may assume that f is non-periodic. Again take a sequence of positive integers $\{m_j\}$ such that $[f]_*^{m_j}(\tau)$ converges to a boundary point $\phi \in \partial T(S)$, and let Γ be the non-empty system of curves accompanied to ϕ as in Lemma 3.1. We may assume that $f(\Gamma) = \Gamma$ and $g(\Gamma) = \Gamma$. Let l be a positive integer such that f^l and g^l preserve each component of $S - \Gamma$. Since the statement of the corollary is true for $f^l | S'$ and $g^l | S'$ for each component S' of $S - \Gamma$, the statement of the corollary for the Riemann surface S of type (p, n) is now immediate. ■

Note that we can show the following theorem by Birman, Lubotsky and McCarthy [BLM] by the same argument as above: *An abelian subgroup* F *of the modular subgroup* Mod(S) *has a torsion free subgroup of finite index with rank* $\leq 3p - 3 + n$.

REFERENCES

[B1] L. BERS, On boundaries of Teichmüller spaces and on Kleinian groups I, Ann. of Math. (2) 91 (1970), 570–600.

[B2] L. BERS, An extremal problem for quasiconformal mappings and a theorem by Thurston, Acta Math. 141 (1978), 73–98.

[B3] L. BERS, On iterates of hyperbolic transformations, Amer. J. of Math. 105 (1983), 1–11.

[BLM] J. BIRMAN, A. LUBOTSKY AND J. MCCARTHY, Abelian and solvable subgroups of the mapping class group, Duke Math. J. 50 (1983), 1107–1120.

[BM] R. Bowen and B. Marcus, Unique ergodicity for horocycle foliations, Israel J. Math. 26 (1977), 43–67.

[CB] A. Casson and S. Bleiler, Automorphisms of Surfaces after Nielsen and Thurston, Cambridge University Press, Cambridge (1988).

[E] D. B. A. Epstein, Curves on 2-manifolds and isotopies, Acta Math. 115 (1966), 83–107.

[FLP] A. Fathi, F. Laudenbach and V. Poénaru, Travaux de Thurston sur les surfaces, Astérisque 66–67 (1979).

[G] F. Gardiner, Teichmüller Theory and Quadratic Differentials, Wiley and Sons, New York (1988).

[H] D. Hejhal, Regular *b*-group and repeated Dehn twists, in Complex Analysis, Joensuu 1987, Lecture Notes in Math. 1351 Springer-Verlag, New York (1988).

[L] O. Lehto, Univalent Functions and Teichmüller Spaces, Springer-Verlag, New York (1985).

[M] B. Maskit, On boundaries of Teichmüller spaces and on Kleinian groups, II, Ann. of Math. (2) 91 (1971), 607–639.

[Mc] J. McCarthy, Normalizers and Centralizers of pseudo-Anosov mapping classes, available upon request.

[S] H. Shiga, On Teichmüller spaces and modular transformations, J. Math. Kyoto Univ. 25 (1985), 619–626.

Department of Matehmatics
Faculty of Science
Nagoya University
Nagoya 464
Japan