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Orbits on affine symmetric spaces under the action of parabolic subgroups

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Introduction

Let G be a connected Lie group, σ an involutive automorphism of G and H a subgroup of G satisfying $(G_\sigma)_0 \subset H \subset G_\sigma$ where $G_\sigma = \{x \in G \mid \sigma(x) = x\}$ and $(G_\sigma)_0$ is the identity component of G_σ. Then the triple (G, H, σ) is called an affine symmetric space. We assume that G is real semisimple throughout this paper.

Let P be a minimal parabolic subgroup of G. Then the double coset decomposition $H\backslash G/P$ is studied in [3] and [4]. Let P' be an arbitrary parabolic subgroup of G containing P. Then we have a canonical surjection

$$f: H\backslash G/P \longrightarrow H\backslash G/P'.$$

The purpose of this paper is to determine $f^{-1}(\mathcal{O})$ for an arbitrary double coset $\mathcal{O}$ in $H\backslash G/P'$.

When G is a complex semisimple Lie group and H is a real form of G, the double coset decomposition $H\backslash G/P$ is studied in [1] and [7] and structures of H-orbits on G/P' are studied in [7].

When G is a complex semisimple Lie group, H is a complex subgroup of G and P' is a parabolic subgroup of G corresponding to a simple root, the structure of $f^{-1}(\mathcal{O})$ is determined for an arbitrary double coset $\mathcal{O}$ in $H\backslash G/P'$ in [5], p. 29, Lemma 5.2.

The results of this paper are as follows. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively, and the automorphism σ of $\mathfrak{g}$ be the one induced from the automorphism σ of G. Let θ be a Cartan involution of $\mathfrak{g}$ such that $\sigma\theta = \theta\sigma$. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ (resp. $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$) be the decomposition of $\mathfrak{g}$ into the $+1$ and -1 eigenspaces for σ (resp. θ).

Let P^0 be a minimal parabolic subgroup of G. Then the factor space G/P^0 is identified with the set of minimal parabolic subalgebras of $\mathfrak{g}$. By Theorem 1 of [3], every H-conjugacy class of minimal parabolic subalgebras of $\mathfrak{g}$ contains a minimal parabolic subalgebra of the form $\mathfrak{P} = \mathfrak{P}(\mathfrak{a}, \Sigma(\mathfrak{a})^+)$ where $\mathfrak{a}$ is a σ-stable maximal abelian subspace of $\mathfrak{p}$, $\Sigma(\mathfrak{a})^+$ is a positive system of the root system $\Sigma(\mathfrak{a})$ of the pair $(\mathfrak{g}, \mathfrak{a})$ and $\mathfrak{P}(\mathfrak{a}, \Sigma(\mathfrak{a})^+) = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ is the corresponding minimal parabolic subalgebra of $\mathfrak{g}$.

Thus the problem is reduced to the following. Fix a σ-stable maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and a minimal parabolic subalgebra $\mathfrak{P}=\mathfrak{P}(\mathfrak{a}, \Sigma(\mathfrak{a})^+)$. Let $\mathfrak{P}'$ be an arbitrary parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{P}$ and P' the corresponding parabolic subgroup of G. Then we have only to determine the double coset decomposition

$$H\backslash HP'/P.$$

Since there is a canonical bijection $H\cap P'\backslash P'/P \simeq H\backslash HP'/P$ and since the factor space P'/P is identified with the set of minimal parabolic subalgebras of $\mathfrak{g}$ contained in $\mathfrak{P}'$, we have only to consider $H\cap P'$-conjugacy classes of minimal parabolic subalgebras of $\mathfrak{g}$ contained in $\mathfrak{P}'$. Let $\mathfrak{P}'=\mathfrak{m}'+\mathfrak{a}'+\mathfrak{n}'$ be the Langlands decomposition of $\mathfrak{P}'$ such that $\mathfrak{a}'\subset\mathfrak{a}$. A subset $\mathfrak{a}'_+$ of $\mathfrak{a}'$ is defined by $\mathfrak{a}'_+=\{Y\in\mathfrak{a}' \mid \alpha(Y)>0$ for all $\alpha\in\Sigma(\mathfrak{a})$ satisfying $\mathfrak{g}(\mathfrak{a};\alpha)\subset\mathfrak{n}'\}$ ($\mathfrak{g}(\mathfrak{a};\alpha)=\{X\in\mathfrak{g} \mid [Y,X]=\alpha(Y)X$ for all $Y\in\mathfrak{a}\}$). Now we can state the main result of this paper as follows.

THEOREM. *Every minimal parabolic subalgebra of* $\mathfrak{g}$ *contained in* $\mathfrak{P}'$ *is* $H\cap P'$*-conjugate to a minimal parabolic subalgebra* $\mathfrak{P}_1$ *of* $\mathfrak{g}$ *of the form*

$$\mathfrak{P}_1 = \mathfrak{P}(\mathfrak{a}_1, \Sigma(\mathfrak{a}_1)^+)$$

where $\mathfrak{a}_1$ *is a* σ*-stable maximal abelian subspace of* $\mathfrak{p}$ *such that* $\mathfrak{a}_1\supset\mathfrak{a}'$ *and* $\Sigma(\mathfrak{a}_1)^+$ *satisfies* $\langle\Sigma(\mathfrak{a}_1)^+, \mathfrak{a}'_+\rangle\subset\boldsymbol{R}_+$ $(=\{t\in\boldsymbol{R} \mid t\geq 0\})$.

Let $\mathfrak{Z}_{\mathfrak{g}}(\mathfrak{a}'+\sigma\mathfrak{a}')$ denote the centralizer of $\mathfrak{a}'+\sigma\mathfrak{a}'$ and $\mathfrak{Z}$ the center of $\mathfrak{Z}_{\mathfrak{g}}(\mathfrak{a}'+\sigma\mathfrak{a}')$. Define a subalgebra $\mathfrak{m}''$ of $\mathfrak{Z}_{\mathfrak{g}}(\mathfrak{a}'+\sigma\mathfrak{a}')$ by $\mathfrak{m}''=\{X\in\mathfrak{Z}_{\mathfrak{g}}(\mathfrak{a}'+\sigma\mathfrak{a}') \mid B(X, \mathfrak{Z}\cap\mathfrak{a})=\{0\}\}$ where $B(\ ,\ )$ is the Killing form of $\mathfrak{g}$. Then a subspace $\mathfrak{a}_1$ of $\mathfrak{p}$ satisfying the condition of Theorem contains $\mathfrak{Z}\cap\mathfrak{a}$. For such a subspace $\mathfrak{a}_1$ of $\mathfrak{p}$, define subsets $\Sigma(\mathfrak{a}_1)_{\mathfrak{m}'}$ and $\Sigma(\mathfrak{a}_1)_{\mathfrak{m}''}$ of $\Sigma(\mathfrak{a}_1)$ by

$$\Sigma(\mathfrak{a}_1)_{\mathfrak{m}'} = \{\alpha\in\Sigma(\mathfrak{a}_1) \mid \langle\alpha, \mathfrak{a}'\rangle = \{0\}\}$$

and

$$\Sigma(\mathfrak{a}_1)_{\mathfrak{m}''} = \{\alpha\in\Sigma(\mathfrak{a}_1) \mid \langle\alpha, \mathfrak{a}'+\sigma\mathfrak{a}'\rangle = \{0\}\}.$$

We consider closed H-orbits and open H-orbits on HP'/P with respect to the topology of HP'/P.

COROLLARY 1. (a) *A minimal parabolic subalgebra* $\mathfrak{P}_1=\mathfrak{P}(\mathfrak{a}_1, \Sigma(\mathfrak{a}_1)^+)$ *satisfying the conditions of Theorem is contained in a closed H-orbit on* HP'/P *(here we identified* $\mathfrak{P}_1$ *with a point in* P'/P*) if and only if the following three conditions are satisfied*:

(i) $\langle\Sigma(\mathfrak{a}_1)^+_{\mathfrak{m}'}, \sigma\mathfrak{a}'_+\rangle\subset\boldsymbol{R}_+$ *where* $\Sigma(\mathfrak{a}_1)^+_{\mathfrak{m}'}=\Sigma(\mathfrak{a}_1)_{\mathfrak{m}'}\cap\Sigma(\mathfrak{a}_1)^+$,

(ii) $\Sigma(\mathfrak{a}_1)^+_{\mathfrak{m}''}$ *is* σ*-compatible (i.e.* $\alpha\in\Sigma(\mathfrak{a}_1)^+_{\mathfrak{m}''}$, $\alpha|_{\mathfrak{m}''\cap\mathfrak{a}_1\cap\mathfrak{q}}\neq 0\Rightarrow\sigma\alpha\in\Sigma(\mathfrak{a}_1)^+_{\mathfrak{m}''}$) *where* $\Sigma(\mathfrak{a}_1)^+_{\mathfrak{m}''}=\Sigma(\mathfrak{a}_1)_{\mathfrak{m}''}\cap\Sigma(\mathfrak{a}_1)^+$,

(iii) $\mathfrak{m}'' \cap \mathfrak{a}_1 \cap \mathfrak{h}$ *is maximal abelian in* $\mathfrak{m}'' \cap \mathfrak{p} \cap \mathfrak{h}$.

(b) *A minimal parabolic subalgebra* $\mathfrak{P}_1 = \mathfrak{P}(\mathfrak{a}_1, \Sigma(\mathfrak{a}_1)^+)$ *satisfying the conditions of Theorem is contained in an open H-orbit on* HP'/P *if and only if the following three conditions are satisfied*:

(i) $\langle \Sigma(\mathfrak{a}_1)^+_{\mathfrak{m}'}, \sigma\theta\mathfrak{a}'_+\rangle \subset \boldsymbol{R}_+$,

(ii) $\Sigma(\mathfrak{a}_1)^+_{\mathfrak{m}'}$ *is* $\sigma\theta$*-compatible* (*i.e.* $\alpha \in \Sigma(\mathfrak{a}_1)^+_{\mathfrak{m}''}, \alpha|_{\mathfrak{m}'' \cap \mathfrak{a}_1 \cap \mathfrak{h}} \neq 0 \Rightarrow \sigma\theta\alpha \in \Sigma(\mathfrak{a}_1)^+_{\mathfrak{m}''}$),

(iii) $\mathfrak{m}'' \cap \mathfrak{a}_1 \cap \mathfrak{q}$ *is maximal abelian in* $\mathfrak{m}'' \cap \mathfrak{p} \cap \mathfrak{q}$.

For an affine symmetric space (G, H, σ), the associated affine symmetric space $(G, H', \sigma\theta)$ is defined by $H' = (K \cap H)\exp(\mathfrak{p} \cap \mathfrak{q})$. Then there exists a one-to-one correspondence between the double coset decompositions $H\backslash G/P$ and $H'\backslash G/P$. If $\mathfrak{a}$ is a σ-stable maximal abelian subspace of $\mathfrak{p}$, then the H-orbit containing $\mathfrak{P}(\mathfrak{a}, \Sigma(\mathfrak{a})^+)$ corresponds to the H'-orbit containing the same $\mathfrak{P}(\mathfrak{a}, \Sigma(\mathfrak{a})^+)$ ([3], Corollary 2 of Theorem 1).

COROLLARY 2. (a) *In the above correspondence between* $H\backslash G/P$ *and* $H'\backslash G/P$, $H\backslash HP'/P$ *corresponds to* $H'\backslash H'P'/P$. *Moreover closed H-orbits on* HP'/P *correspond to open H'-orbits on* $H'P'/P$ *and open ones to closed ones.*

(b) *Let* P'' *be a parabolic subgroup of G containing* P'. *Then there is a one-to-one correspondence between* $H\backslash HP''/P'$ *and* $H'\backslash H'P''/P'$. *In this correspondence closed H-orbits on* HP''/P' *correspond to open H'-orbits on* $H'P''/P'$ *and open ones to closed ones.*

Lastly we state an explicit formula for the decomposition $H\backslash HP'/P$ applying the method used in §2 of [3]. Let $\mathfrak{a}_0$ be a σ-stable maximal abelian subspace of $\mathfrak{p}$ such that $\mathfrak{a}_0 \subset \mathfrak{a}'$ and that $\mathfrak{m}'' \cap \mathfrak{a}_0 \cap \mathfrak{h}$ is maximal abelian in $\mathfrak{m}'' \cap \mathfrak{p} \cap \mathfrak{h}$. Fix a positive system $\Sigma(\mathfrak{a}_0)^+$ of $\Sigma(\mathfrak{a}_0)$ such that $\langle \Sigma(\mathfrak{a}_0)^+, \mathfrak{a}'_+\rangle \subset \boldsymbol{R}_+$. Then $\mathfrak{P}_{(0)} = \mathfrak{P}(\mathfrak{a}_0, \Sigma(\mathfrak{a}_0)^+)$ is contained in $\mathfrak{P}'$. Let $P_{(0)}$ be the corresponding minimal parabolic subgroup of G.

Let $\bar{\mathfrak{a}}$ be a σ-stable maximal abelian subspace of $\mathfrak{p}$ such that $\bar{\mathfrak{a}} \cap \mathfrak{h}$ is maximal abelian in $\mathfrak{p} \cap \mathfrak{h}$, $\bar{\mathfrak{a}} \cap \mathfrak{h} \supset \mathfrak{a}_0 \cap \mathfrak{h}$ and $\bar{\mathfrak{a}} \cap \mathfrak{q} \subset \mathfrak{a}_0 \cap \mathfrak{q}$. Put $\mathfrak{r} = \{Y \in \bar{\mathfrak{a}} \cap \mathfrak{h} | B(Y, \mathfrak{a}_0 \cap \mathfrak{h}) = \{0\}\}$. Put $\Sigma_{\mathfrak{h}}(\mathfrak{a}_0)_{\mathfrak{m}''} = \{\alpha \in \Sigma(\mathfrak{a}_0)_{\mathfrak{m}''} \mid H_\alpha \in \mathfrak{m}'' \cap \mathfrak{a}_0 \cap \mathfrak{h}\}$ where $H_\alpha \in \mathfrak{a}_0$ is defined by $B(H_\alpha, Y) = \alpha(Y)$ for $Y \in \mathfrak{a}_0$. Then a set of root vectors $Q = \{X_{\alpha_1}, \dots, X_{\alpha_k}\}$ is said to be a $\mathfrak{q}$-orthogonal system of $\Sigma_{\mathfrak{h}}(\mathfrak{a}_0)_{\mathfrak{m}''}$ if the following two conditions are satisfied:

(i) $\alpha_i \in \Sigma_{\mathfrak{h}}(\mathfrak{a}_0)_{\mathfrak{m}''}$ and $X_{\alpha_i} \in \mathfrak{g}(\mathfrak{a}_0; \alpha_i) \cap \mathfrak{q} - \{0\}$ for $i = 1, \dots, k$,

(ii) $[X_{\alpha_i}, X_{\alpha_j}] = [X_{\alpha_i}, \theta X_{\alpha_j}] = 0$ for $i \neq j$.

We normalize X_{α_i}, $i = 1, \dots, k$ so that $2\alpha_i(H_{\alpha_i})B(X_{\alpha_i}, \theta X_{\alpha_i}) = -1$. Define an element $c(Q)$ of M''_0 by

$$c(Q) = \exp(\pi/2)(X_{\alpha_1} + \theta X_{\alpha_1}) \cdots \exp(\pi/2)(X_{\alpha_k} + \theta X_{\alpha_k}).$$

Then $\mathfrak{a}^1 = \mathrm{Ad}\,(c(Q))\mathfrak{a}_0$ is a σ-stable maximal abelian subspace of $\mathfrak{p}$ such that $\mathfrak{a}' \subset \mathfrak{a}^1$.

Let $\{Q_0, \dots, Q_n\}$ $(Q_0 = \emptyset)$ be a complete set of representatives of $\mathfrak{q}$-orthogonal systems of $\Sigma_{\mathfrak{h}}(\mathfrak{a}_0)_{\mathfrak{m}''}$ with respect to the following equivalence relation $\sim$. For two $\mathfrak{q}$-orthogonal systems $Q = \{X_{\alpha_1}, \dots, X_{\alpha_k}\}$ and $Q' = \{X_{\beta_1}, \dots, X_{\beta_{k'}}\}$ of $\Sigma_{\mathfrak{h}}(\mathfrak{a}_0)_{\mathfrak{m}''}$, $Q \sim Q'$ if and only if there exists a $w \in W_{K\cap H}(\bar{\mathfrak{a}}) (= N_{K\cap H}(\bar{\mathfrak{a}})/Z_{K\cap H}(\bar{\mathfrak{a}}))$ such that

$$w(\mathfrak{r} + \textstyle\sum_{j=1}^{k} H_{\alpha_j}) = \mathfrak{r} + \sum_{j=1}^{k'} H_{\beta_j}.$$

Put $\mathfrak{a}_i = \mathrm{Ad}(c(Q_i))\mathfrak{a}_0$, $i = 1, \dots, n$. Then we have the following corollary.

COROLLARY 3. *$HP' = \bigcup_{i=0}^{n} \bigcup_{j=1}^{m(i)} Hw_j^i c(Q_i) P_{(0)}$ (disjoint union) where $\{w_1^i, \dots, w_{m(i)}^i\}$ is a complete set of representatives of $W_{K\cap H}(\mathfrak{a}_i) \cap W(\mathfrak{a}_i)_{\mathfrak{m}'} \backslash W(\mathfrak{a}_i)_{\mathfrak{m}'}$ in $N_{K\cap M'}(\mathfrak{a}_i)$ $(W(\mathfrak{a}_i)_{\mathfrak{m}'} = N_{K\cap M'}(\mathfrak{a}_i)/Z_{K\cap M'}(\mathfrak{a}_i))$. Moreover we have*

$$H'P' = \bigcup_{i=0}^{n} \bigcup_{j=1}^{m(i)} H' w_j^i c(Q_i) P_{(0)} \quad (\textit{disjoint union}).$$

§ 1. Notations and preliminaries

Let $\boldsymbol{R}$ denote the set of real numbers and $\boldsymbol{R}_+$ the subset of $\boldsymbol{R}$ defined by $\boldsymbol{R}_+ = \{t \in \boldsymbol{R} \mid t \geq 0\}$. Let G be a Lie group with Lie algebra $\mathfrak{g}$. For subsets $\mathfrak{s}$ and $\mathfrak{t}$ in $\mathfrak{g}$ and a subset S in G, $\mathfrak{z}_{\mathfrak{s}}(\mathfrak{t})$, $Z_S(\mathfrak{t})$ and $N_S(\mathfrak{t})$ are the subsets of $\mathfrak{g}$, G and G defined by

$$\mathfrak{z}_{\mathfrak{s}}(\mathfrak{t}) = \{X \in \mathfrak{s} \mid [X, Y] = 0 \quad \text{for all} \quad Y \in \mathfrak{t}\},$$

$$Z_S(\mathfrak{t}) = \{x \in S \mid \mathrm{Ad}\,(x)Y = Y \quad \text{for all} \quad Y \in \mathfrak{t}\}$$

and

$$N_S(\mathfrak{t}) = \{x \in S \mid \mathrm{Ad}\,(x)\mathfrak{t} = \mathfrak{t}\},$$

respectively.

Let G be a connected real semisimple Lie group, σ an involutive automorphism of G (i.e. $\sigma^2 = \text{identity}$) and H a subgroup of G satisfying $(G_\sigma)_0 \subset H \subset G_\sigma$ where $G_\sigma = \{x \in G \mid \sigma(x) = x\}$ and $(G_\sigma)_0$ is the identity component of G_σ. Then the triple (G, H, σ) is an affine symmetric space such that G is real semisimple.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively, and the automorphism σ of $\mathfrak{g}$ be the one induced from the automorphism σ of G. There exists a Cartan involution θ of $\mathfrak{g}$ such that $\sigma\theta = \theta\sigma$ ([2], cf. Lemmas 3 and 4 in [3]). Fix such a Cartan involution θ of $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ (resp. $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$) be the decomposition of $\mathfrak{g}$ into the $+1$ and -1 eigenspaces for σ (resp. θ). Then we have the following direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} \cap \mathfrak{h} + \mathfrak{k} \cap \mathfrak{q} + \mathfrak{p} \cap \mathfrak{h} + \mathfrak{p} \cap \mathfrak{q}$$

of $\mathfrak{g}$. Let K denote the analytic subgroup of G for $\mathfrak{k}$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Then the space of real linear forms on $\mathfrak{a}$ is denoted by $\mathfrak{a}^*$. For an $\alpha \in \mathfrak{a}^*$, let $\mathfrak{g}(\mathfrak{a}; \alpha)$ denote the subspace of $\mathfrak{g}$ defined by

$$\mathfrak{g}(\mathfrak{a}; \alpha) = \{X \in \mathfrak{g} \mid [Y, X] = \alpha(Y)X \quad \text{for all} \quad Y \in \mathfrak{a}\}.$$

Then the root system $\Sigma(\mathfrak{a})$ of the pair $(\mathfrak{g}, \mathfrak{a})$ is the finite subset of $\mathfrak{a}^*$ defined by

$$\Sigma(\mathfrak{a}) = \{\alpha \in \mathfrak{a}^* - \{0\} \mid \mathfrak{g}(\mathfrak{a}; \alpha) \neq \{0\}\}.$$

Let $\Sigma(\mathfrak{a})^+$ be a positive system of $\Sigma(\mathfrak{a})$. Then we can define a minimal parabolic subalgebra $\mathfrak{P}(\mathfrak{a}, \Sigma(\mathfrak{a})^+)$ of $\mathfrak{g}$ and a minimal parabolic subgroup $P(\mathfrak{a}, \Sigma(\mathfrak{a})^+)$ of G by

$$\mathfrak{P}(\mathfrak{a}, \Sigma(\mathfrak{a})^+) = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$$

and

$$P(\mathfrak{a}, \Sigma(\mathfrak{a})^+) = MAN,$$

respectively, where $\mathfrak{m} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$, $M = Z_K(\mathfrak{a})$, $A = \exp \mathfrak{a}$, $\mathfrak{n} = \sum_{\alpha \in \Sigma(\mathfrak{a})^+} \mathfrak{g}(\mathfrak{a}, \alpha)$ and $N = \exp \mathfrak{n}$.

Let $\mathfrak{P}'$ be an arbitrary parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{P}(\mathfrak{a}, \Sigma(\mathfrak{a})^+)$ and P' the corresponding parabolic subgroup of G. Then there is a unique Langlands decomposition

$$\mathfrak{P}' = \mathfrak{m}' + \mathfrak{a}' + \mathfrak{n}'$$

of $\mathfrak{P}'$ such that $\mathfrak{a}' \subset \mathfrak{a}$. Let $\mathfrak{a}'_+$ denote the subset of $\mathfrak{a}$ defined by

$$\mathfrak{a}'_+ = \{Y \in \mathfrak{a}' \mid \alpha(Y) > 0 \quad \text{for all} \quad \alpha \in \Sigma(\mathfrak{a}) \quad \text{such that} \quad \mathfrak{g}(\mathfrak{a}; \alpha) \subset \mathfrak{n}'\}.$$

The corresponding Langlands decomposition of P' is denoted by $P' = M'A'N'$.

Let P^0 be a minimal parabolic subgroup of G and $\mathfrak{P}^0$ the corresponding minimal parabolic subalgebra of $\mathfrak{g}$. Then the factor space G/P^0 is identified with the set of minimal parabolic subalgebras of $\mathfrak{g}$ by the correspondence $xP^0 \mapsto \mathrm{Ad}(x)\mathfrak{P}^0$, $x \in G$. Thus the H-orbits on G/P^0 are identified with the H-conjugacy classes of minimal parabolic subalgebras of $\mathfrak{g}$.

Here we review a main result of [3]. Let $\{\mathfrak{a}_i \mid i \in I\}$ be a complete set of representatives of the $K \cap H$-conjugacy classes of σ-stable maximal abelian subspace of $\mathfrak{p}$. Let $W(\mathfrak{a}_i) = N_K(\mathfrak{a}_i)/Z_K(\mathfrak{a}_i)$ be the Weyl group of $\Sigma(\mathfrak{a}_i)$ and $W_{K \cap H}(\mathfrak{a}_i)$ the subgroup of $W(\mathfrak{a}_i)$ defined by

$$W_{K \cap H}(\mathfrak{a}_i) = N_{K \cap H}(\mathfrak{a}_i)/Z_{K \cap H}(\mathfrak{a}_i).$$

PROPOSITION (Corollary 1 of Theorem 1 in [3]). *There is a one-to-one correspondence between the set of H-conjugacy classes of minimal parabolic subalgebras of* $\mathfrak{g}$ *and the set* $\bigcup_{i \in I} W_{K \cap H}(\mathfrak{a}_i) \backslash W(\mathfrak{a}_i)$ *(disjoint union). Fix a positive* *system $\Sigma(\mathfrak{a}_i)^+$ of $\Sigma(\mathfrak{a}_i)$ for each $i \in I$. Then $W_{K \cap H}(\mathfrak{a}_i)w \in W_{K \cap H}(\mathfrak{a}_i) \backslash W(\mathfrak{a}_i)$ corresponds to the H-conjugacy class of minimal parabolic subalgebras of $\mathfrak{g}$ containing $\mathfrak{P}(\mathfrak{a}_i, w\Sigma(\mathfrak{a}_i)^+)$.*

§2. Theorem and its corollaries

Let $\mathfrak{P}^{0\prime}$ be an arbitrary parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{P}^0$ and $P^{0\prime}$ the corresponding parabolic subgroup of G. Then we have a canonical surjection

$$f: H \backslash G/P^0 \longrightarrow H \backslash G/P^{0\prime}.$$

For every double coset $\mathcal{O} = HxP^{0\prime} \in H \backslash G/P^{0\prime}$ $(x \in G)$, we want to study $f^{-1}(\mathcal{O}) = H \backslash HxP^{0\prime}/P^0$. It follows from Proposition in §1 that there exist an $h \in H$, a σ-stable maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and a positive system $\Sigma(\mathfrak{a})^+$ of $\Sigma(\mathfrak{a})$ such that $\mathrm{Ad}(hx)\mathfrak{P}^0 = \mathfrak{P}(\mathfrak{a}, \Sigma(\mathfrak{a})^+)$. Thus we have only to study the double coset decomposition $H \backslash HP'/P$ for such a minimal parabolic subalgebra $\mathfrak{P} = \mathfrak{P}(\mathfrak{a}, \Sigma(\mathfrak{a})^+)$ where P is the minimal parabolic subgroup corresponding to $\mathfrak{P}$ and $P' = hxP^{0\prime}x^{-1}h^{-1}$.

Therefore we fix a σ-stable maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ and a positive system $\Sigma(\mathfrak{a})^+$ of $\Sigma(\mathfrak{a})$. Put $\mathfrak{P} = \mathfrak{P}(\mathfrak{a}, \Sigma(\mathfrak{a})^+)$ and let $\mathfrak{P}'$ be the parabolic subalgebra of $\mathfrak{g}$ which is conjugate to $\mathfrak{P}^0$ and contains $\mathfrak{P}$. Notations $\mathfrak{P} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$, $P = MAN$, $\mathfrak{P}' = \mathfrak{m}' + \mathfrak{a}' + \mathfrak{n}'$, $P' = M'A'N'$ and $\mathfrak{a}'_+$ are the same as in §1.

Since $H \backslash HP'$ is isomorphic to $H \cap P' \backslash P'$, there is a canonical bijection

$$H \cap P' \backslash P'/P \xrightarrow{\sim} H \backslash HP'/P. \tag{2.1}$$

Then the following theorem gives standard representatives for $H \cap P' \backslash P'/P$ since P'/P is identified with the set of minimal parabolic subalgebras of $\mathfrak{g}$ contained in $\mathfrak{P}'$.

THEOREM. *Every minimal parabolic subalgebra of $\mathfrak{g}$ contained in $\mathfrak{P}'$ is $H \cap P'$-conjugate to a minimal parabolic subalgebra $\mathfrak{P}_1$ of $\mathfrak{g}$ of the form*

$$\mathfrak{P}_1 = \mathfrak{P}(\mathfrak{a}_1, \Sigma(\mathfrak{a}_1)^+)$$

where $\mathfrak{a}_1$ is a σ-stable maximal abelian subspace of $\mathfrak{p}$ such that $\mathfrak{a}_1 \supset \mathfrak{a}'$ and $\Sigma(\mathfrak{a}_1)^+$ is a positive system of $\Sigma(\mathfrak{a}_1)$ such that

$$\langle \Sigma(\mathfrak{a}_1)^+, \mathfrak{a}'_+ \rangle \subset \boldsymbol{R}_+.$$

REMARK. Conversely if $\mathfrak{a}_1$ and $\Sigma(\mathfrak{a}_1)^+$ satisfy the conditions in Theorem, then $\mathfrak{P}_1 = \mathfrak{P}(\mathfrak{a}_1, \Sigma(\mathfrak{a}_1)^+)$ is contained in $\mathfrak{P}'$. In fact, write $\mathfrak{P}_1 = \mathfrak{m}_1 + \mathfrak{a}_1 + \mathfrak{n}_1$ where $\mathfrak{m}_1 = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}_1)$ and $\mathfrak{n}_1 = \sum_{\alpha \in \Sigma(\mathfrak{a}_1)^+} \mathfrak{g}(\mathfrak{a}_1; \alpha)$. Note that

$$\mathfrak{P}' = \textstyle\sum_\alpha \mathfrak{g}(\mathfrak{a}'; \alpha) \text{ (the sum is taken over all } \alpha \in (\mathfrak{a}')^* \text{ such that } \langle \alpha, \mathfrak{a}'_+ \rangle \supset \boldsymbol{R}_+)$$

where $(\mathfrak{a}')^*$ is the space of real linear forms on $\mathfrak{a}'$ and $\mathfrak{g}(\mathfrak{a}';\alpha)=\{X\in\mathfrak{g}\mid[Y,X]=\alpha(Y)X\}$. Then it follows from the condition for $\mathfrak{a}_1$ that $\mathfrak{m}_1+\mathfrak{a}_1\subset\mathfrak{g}(\mathfrak{a}';0)$. On the other hand it follows from the condition for $\Sigma(\mathfrak{a}_1)^+$ that $\mathfrak{g}(\mathfrak{a}_1;\alpha)\subset\mathfrak{g}(\mathfrak{a}';\alpha|_{\mathfrak{a}'})\subset\mathfrak{P}'$ for $\alpha\in\Sigma(\mathfrak{a}_1)^+$. Thus we have $\mathfrak{P}_1\subset\mathfrak{P}'$.

We use the following method of Lusztig and Vogan ([5], p. 29, Lemma 5.2). Let $\pi: P'\to M'$ be the projection with respect to the Langlands decomposition $P'=M'A'N'$. Then π is a group homomorphism and induces an isomorphism of P'/P onto $M'/M'\cap P$. Put $J=\pi(H\cap P')$. Then there is a canonical bijection

$$H\cap P'\backslash P'/P \xrightarrow{\sim} J\backslash M'/M'\cap P. \tag{2.2}$$

(In [5], G and H are complex groups and P' is a parabolic subgroup of G corresponding to a simple root of $\Sigma(\mathfrak{a})^+$.)

Let J_0 and M'_0 be the identity components of J and M' respectively. Since $M'\cap P\supset M$, every connected component of M' has a non-trivial intersection with $M'\cap P$. Thus $M'/M'\cap P$ is isomorphic to $M'_0/M'_0\cap P$ and we have a canonical surjection

$$J_0\backslash M'_0/M'_0\cap P\longrightarrow J\backslash M'/M'\cap P. \tag{2.3}$$

It is clear that the subalgebras $\mathfrak{m}'\cap\mathfrak{P}$ and $\mathfrak{m}'\cap\sigma\mathfrak{P}'$ are a minimal parabolic subalgebra and a parabolic subalgebra of $\mathfrak{m}'$ respectively. Let $\Sigma(\mathfrak{a})_{\mathfrak{m}'}$ and $\Sigma(\mathfrak{a})_{\mathfrak{n}'}$ be the subsets of $\Sigma(\mathfrak{a})$ defined by $\Sigma(\mathfrak{a})_{\mathfrak{m}'}=\{\alpha\in\Sigma(\mathfrak{a})\mid\langle\alpha,\mathfrak{a}'\rangle=\{0\}\}$ and $\Sigma(\mathfrak{a})_{\mathfrak{n}'}=\{\alpha\in\Sigma(\mathfrak{a})\mid\langle\alpha,\mathfrak{a}'_+\rangle\subset\boldsymbol{R}_+-\{0\}\}$ respectively. Then

$$\mathfrak{m}'+\mathfrak{a}'=\mathfrak{m}+\mathfrak{a}+\textstyle\sum_{\alpha\in\Sigma(\mathfrak{a})_{\mathfrak{m}'}}\mathfrak{g}(\mathfrak{a};\alpha)$$

and

$$\mathfrak{n}'=\textstyle\sum_{\alpha\in\Sigma(\mathfrak{a})_{\mathfrak{n}'}}\mathfrak{g}(\mathfrak{a};\alpha).$$

Let

$$\mathfrak{m}'\cap\sigma\mathfrak{P}'=\mathfrak{m}''+\mathfrak{a}''+\mathfrak{n}''$$

be the Langlands decomposition of $\mathfrak{m}'\cap\sigma\mathfrak{P}'$ such that $\mathfrak{a}''\subset\mathfrak{a}$. Let $\Sigma(\mathfrak{a})_{\mathfrak{m}''}$ and $\Sigma(\mathfrak{a})_{\mathfrak{n}''}$ be the subsets of $\Sigma(\mathfrak{a})_{\mathfrak{m}'}$ defined by $\Sigma(\mathfrak{a})_{\mathfrak{m}''}=\{\alpha\in\Sigma(\mathfrak{a})\mid\langle\alpha,\mathfrak{a}'+\sigma\mathfrak{a}'\rangle=\{0\}\}$ and $\Sigma(\mathfrak{a})_{\mathfrak{n}''}=\{\alpha\in\Sigma(\mathfrak{a})_{\mathfrak{m}'}\mid\langle\alpha,\sigma\mathfrak{a}'_+\rangle\subset\boldsymbol{R}_+-\{0\}\}$ respectively. Then we have

$$\mathfrak{m}''+\mathfrak{a}''+\mathfrak{a}'=\mathfrak{m}+\mathfrak{a}+\textstyle\sum_{\alpha\in\Sigma(\mathfrak{a})_{\mathfrak{m}''}}\mathfrak{g}(\mathfrak{a};\alpha)$$

and

$$\mathfrak{n}''=\textstyle\sum_{\alpha\in\Sigma(\mathfrak{a})_{\mathfrak{n}''}}\mathfrak{g}(\mathfrak{a};\alpha).$$

LEMMA. *Let $\mathfrak{j}$ be the Lie algebra of J and $\mathfrak{a}''_{\mathfrak{j}}$ be the subspace of $\mathfrak{a}''$ given by $\mathfrak{a}''_{\mathfrak{j}}=\pi((\mathfrak{a}'+\mathfrak{a}'')\cap\mathfrak{h})$. Then*

$$\mathfrak{j} = \mathfrak{m}' \cap \mathfrak{h} + \mathfrak{a}''_{\mathfrak{j}} + \mathfrak{n}''.$$

PROOF. Put $A_1 = \Sigma(\mathfrak{a})_{\mathfrak{m}'} \cap \sigma\Sigma(\mathfrak{a})_{\mathfrak{m}'} = \Sigma(\mathfrak{a})_{\mathfrak{m}''}$, $A_2 = \Sigma(\mathfrak{a})_{\mathfrak{m}'} \cap \sigma\Sigma(\mathfrak{a})_{\mathfrak{n}'} = \Sigma(\mathfrak{a})_{\mathfrak{n}''}$ and $A_3 = \Sigma(\mathfrak{a})_{\mathfrak{n}'} \cap \sigma\Sigma(\mathfrak{a})_{\mathfrak{n}'}$, and set

$$\mathfrak{A}_i = \sum_{\alpha \in A_i} (\mathfrak{g}(\mathfrak{a}; \alpha) + \mathfrak{g}(\mathfrak{a}; \sigma\alpha)) \cap \mathfrak{h} \qquad (i = 1, 2, 3).$$

Then

$$\mathfrak{P}' \cap \mathfrak{h} = \mathfrak{P}' \cap \sigma\mathfrak{P}' \cap \mathfrak{h} = \mathfrak{m} \cap \mathfrak{h} + \mathfrak{a} \cap \mathfrak{h} + \mathfrak{A}_1 + \mathfrak{A}_2 + \mathfrak{A}_3.$$

Since $\pi \colon \mathfrak{P}' \to \mathfrak{m}'$ is the projection with respect to the decomposition $\mathfrak{P}' = \mathfrak{m}' + \mathfrak{a}' + \mathfrak{n}'$, we have

$$\begin{aligned}\mathfrak{j} = \pi(\mathfrak{P}' \cap \mathfrak{h}) &= \mathfrak{m} \cap \mathfrak{h} + \pi(\mathfrak{a} \cap \mathfrak{h}) + \mathfrak{A}_1 + \textstyle\sum_{\alpha \in A_2} \mathfrak{g}(\mathfrak{a}; \alpha) \\ &= \mathfrak{m} \cap \mathfrak{h} + \mathfrak{m}'' \cap \mathfrak{a} \cap \mathfrak{h} + \mathfrak{a}''_{\mathfrak{j}} + \mathfrak{A}_1 + \mathfrak{n}'' = \mathfrak{m}'' \cap \mathfrak{h} + \mathfrak{a}''_{\mathfrak{j}} + \mathfrak{n}''.\end{aligned}$$

q. e. d.

Let $W(\mathfrak{a})_{\mathfrak{m}'}$ and $W(\mathfrak{a})_{\mathfrak{m}''}$ denote the subgroups of $W(\mathfrak{a})$ generated by the reflections with respect to the roots of $\Sigma(\mathfrak{a})_{\mathfrak{m}'}$ and $\Sigma(\mathfrak{a})_{\mathfrak{m}''}$ respectively.

PROOF OF THEOREM. We have only to find a set of standard representatives $S \subset M'_0$ of $J_0 \backslash M'_0 / M'_0 \cap P$ since the set S becomes a set of representatives of $H \backslash HP'/P$ in view of the above arguments.

$M'_0 \cap P$ is a minimal parabolic subgroup of M'_0 since $\mathfrak{m}' \cap \mathfrak{P}$ is a minimal parabolic subalgebra of $\mathfrak{m}'$ and since $Z_{K \cap M'_0}(\mathfrak{a}) = M'_0 \cap M$ is contained in $M'_0 \cap P$. In the same way $M'_0 \cap \sigma P'$ is proved to be a parabolic subgroup of M'_0. Thus we have the Bruhat decomposition

$$M'_0 = \bigcup_{w \in W_1} (M'_0 \cap \sigma P') w (M'_0 \cap P)$$

where W_1 is a complete set of representatives of $W(\mathfrak{a})_{\mathfrak{m}''} \backslash W(\mathfrak{a})_{\mathfrak{m}'}$ in $N_{K \cap M'_0}(\mathfrak{a})$.

Let $M'_0 \cap \sigma P' = M''A''N''$ be the Langlands decomposition of $M'_0 \cap \sigma P'$ corresponding to $\mathfrak{m}' \cap \sigma\mathfrak{P}' = \mathfrak{m}'' + \mathfrak{a}'' + \mathfrak{n}'$. Then it follows from Lemma that

$$(M'_0 \cap \sigma P') w (M'_0 \cap P) = J_0 M'' A'' w (M'_0 \cap P)$$

for every $w \in W_1$. Therefore we have only to study the decomposition

$$J_0 \cap M''A'' \backslash M''A'' / wPw^{-1} \cap M''A''.$$

Since $M''A''/wPw^{-1} \cap M''A''$ is isomorphic to $M''_0/wPw^{-1} \cap M''_0$ (M''_0 is the identity component of M'') and since $J_0 \cap M''A'' = (M'' \cap H)_0 \exp \mathfrak{a}''_{\mathfrak{j}}$ (Lemma), there is a canonical bijection

$$(2.4) \qquad (M'' \cap H)_0 \backslash M''_0 / wPw^{-1} \cap M''_0 \xrightarrow{\sim} J_0 \cap M''A'' \backslash M''A'' / wPw^{-1} \cap M''A''.$$

Here we note that M_0'' is σ-stable. Thus the triple $(M_0'', (M'' \cap H)_0, \sigma)$ is an affine symmetric space such that M_0'' is a connected real reductive Lie group. Moreover $wPw^{-1} \cap M_0''$ is a minimal parabolic subgroup of M_0''. Therefore the result of [3] can be applied to the left hand side of (2.4). For every $x \in M_0''$ there is a $y \in (M'' \cap H)_0 x(wPw^{-1} \cap M_0'')$ such that $\mathfrak{a}_1'' = \mathrm{Ad}(y)(\mathfrak{a} \cap \mathfrak{m}'')$ is a σ-stable maximal abelian subspace of $\mathfrak{m}'' \cap \mathfrak{p}$ (Proposition in § 1).

Thus we have proved the following. For an arbitrary $x \in HP'$ there exists a $w \in W_1$ and a $y \in M_0''$ such that $\mathfrak{a}_1 = \mathrm{Ad}(y)\mathfrak{a}$ is σ-stable and that $yw \in HxP$. Then it is clear that $\mathfrak{a}_1$ and $\mathfrak{P}_1 = \mathrm{Ad}(yw)\mathfrak{P} = \mathfrak{P}(\mathfrak{a}_1, \Sigma(\mathfrak{a}_1)^+)$ satisfy the conditions of the theorem. Hence the theorem is proved. q.e.d.

For a σ-stable maximal abelian subspace $\mathfrak{a}_1$ of $\mathfrak{p}$ satisfying $\mathfrak{a}_1 \supset \mathfrak{a}'$, we can define subsets $\Sigma(\mathfrak{a}_1)_{\mathfrak{m}'}$ and $\Sigma(\mathfrak{a}_1)_{\mathfrak{m}''}$ of $\Sigma(\mathfrak{a}_1)$ in the same manner as $\Sigma(\mathfrak{a})_{\mathfrak{m}'}$ and $\Sigma(\mathfrak{a})_{\mathfrak{m}''}$. If $\Sigma(\mathfrak{a}_1)^+$ is a positive system of $\Sigma(\mathfrak{a}_1)$, then $\Sigma(\mathfrak{a}_1)_{\mathfrak{m}'}^+$ and $\Sigma(\mathfrak{a}_1)_{\mathfrak{m}''}^+$ are defined by $\Sigma(\mathfrak{a}_1)_{\mathfrak{m}'}^+ = \Sigma(\mathfrak{a}_1)_{\mathfrak{m}'} \cap \Sigma(\mathfrak{a}_1)^+$ and $\Sigma(\mathfrak{a}_1)_{\mathfrak{m}''}^+ = \Sigma(\mathfrak{a}_1)_{\mathfrak{m}''} \cap \Sigma(\mathfrak{a}_1)^+$ respectively.

Now we consider closed H-orbits and open H-orbits on HP'/P with respect to the topology of HP'/P.

COROLLARY 1. *Retain the notations in Theorem.*

(a) *A minimal parabolic subalgebra $\mathfrak{P}_1 = \mathfrak{P}(\mathfrak{a}_1, \Sigma(\mathfrak{a}_1)^+)$ satisfying the conditions of Theorem is contained in a closed H-orbit on HP'/P ($\mathfrak{P}_1$ is identified with a point in P'/P) if and only if the following three conditions are satisfied:*

(i) $\langle \Sigma(\mathfrak{a}_1)_{\mathfrak{m}'}^+, \sigma\mathfrak{a}_+' \rangle \subset \boldsymbol{R}_+$,

(ii) $\Sigma(\mathfrak{a}_1)_{\mathfrak{m}''}^+$ *is σ-compatible (i.e. $\alpha \in \Sigma(\mathfrak{a}_1)_{\mathfrak{m}''}^+$, $\alpha|_{\mathfrak{m}'' \cap \mathfrak{a}_1 \cap \mathfrak{q}} \neq 0 \Rightarrow \sigma\alpha \in \Sigma(\mathfrak{a}_1)_{\mathfrak{m}''}^+$),*

(iii) $\mathfrak{m}'' \cap \mathfrak{a}_1 \cap \mathfrak{h}$ *is maximal abelian in* $\mathfrak{m}'' \cap \mathfrak{p} \cap \mathfrak{h}$.

(b) *A minimal parabolic subalgebra $\mathfrak{P}_1 = \mathfrak{P}(\mathfrak{a}_1, \Sigma(\mathfrak{a}_1)^+)$ satisfying the conditions of Theorem is contained in an open H-orbit on HP'/P if and only if the following three conditions are satisfied:*

(i) $\langle \Sigma(\mathfrak{a}_1)_{\mathfrak{m}'}^+, \sigma\theta\mathfrak{a}_+' \rangle \subset \boldsymbol{R}_+$,

(ii) $\Sigma(\mathfrak{a}_1)_{\mathfrak{m}''}^+$ *is $\sigma\theta$-compatible (i.e. $\alpha \in \Sigma(\mathfrak{a}_1)_{\mathfrak{m}''}^+$, $\alpha|_{\mathfrak{m}'' \cap \mathfrak{a}_1 \cap \mathfrak{h}} \neq 0 \Rightarrow \sigma\theta\alpha \in \Sigma(\mathfrak{a}_1)_{\mathfrak{m}''}^+$),*

(iii) $\mathfrak{m}'' \cap \mathfrak{a}_1 \cap \mathfrak{q}$ *is maximal abelian in* $\mathfrak{m}'' \cap \mathfrak{p} \cap \mathfrak{q}$.

PROOF. Since the bijections (2.1) and (2.2) come from the topological isomorphisms $H \cap P' \backslash P' \simeq H \backslash HP'$ and $P'/P \simeq M'/M' \cap P$ respectively, we have only to consider closed double cosets and open double cosets in the decomposition

$$J \backslash M'/M' \cap P.$$

For $x \in M'$ and $y \in J$, we have $J_0 yx(M' \cap P) = yJ_0 x(M' \cap P)$. Hence $Jx(M' \cap P)$ is closed (resp. open) in M' if and only if $J_0 x(M' \cap P)$ is closed (resp. open) in M' and therefore we have only to consider closed double cosets and open double cosets in the decomposition

$$J_0\backslash M_0'/M_0' \cap P.$$

Consider the decomposition

$$M_0' = \cup_{w \in W_1} J_0 M''A''w(M_0' \cap P).$$

Then open double cosets in $J_0\backslash M_0'/M_0' \cap P$ are contained in

$$J_0 M''A''w_2(M_0' \cap P) = (M_0' \cap \sigma P')w_2(M_0' \cap P)$$

where w_2 is the unique element in W_1 satisfying

$$(\mathfrak{m}' \cap \sigma\mathfrak{P}') + \mathrm{Ad}(w_2)(\mathfrak{m}' \cap \mathfrak{P}) = \mathfrak{m}'. \tag{2.5}$$

On the other hand closed double cosets in $J_0\backslash M_0'/M_0' \cap P$ are contained in

$$J_0 M''A''w_1(M_0' \cap P)$$

where w_1 is the unique element in W_1 satisfying

$$\mathrm{Ad}(w_1)(\mathfrak{m}' \cap \mathfrak{P}) \supset \mathfrak{n}''. \tag{2.6}$$

This is proved as follows. Let $g: J_0 \to M''A'' \cap J_0$ be the projection with respect to the decomposition $J_0 = (M''A'' \cap J_0)N''$. For $x \in M''A''$ and $w \in W_1$, we have

$$J_0 xw(M_0' \cap P)/M_0' \cap P \cong J_0/J_0 \cap xw(M_0' \cap P)w^{-1}x^{-1}.$$

Then the map g induces a projection

$$J_0/J_0 \cap xw(M_0' \cap P)w^{-1}x^{-1} \longrightarrow (M''A'' \cap J_0)/g(J_0 \cap xw(M_0' \cap P)w^{-1}x^{-1})$$

with fibres isomorphic to $F = N''/N'' \cap xw(M_0' \cap P)w^{-1}x^{-1}$. Since $x^{-1}N''x = N''$, we have $F \cong N''/N'' \cap w(M_0' \cap P)w^{-1}$. If we apply Lemma 1.1.4.1 in [6] to $\mathfrak{n}''$ and $\mathfrak{n}'' \cap \mathrm{Ad}(w)(\mathfrak{m}' \cap \mathfrak{P})$, it follows easily that F is topologically isomorphic to $\boldsymbol{R}^k$ where $k = \dim \mathfrak{n}'' - \dim(\mathfrak{n}'' \cap \mathrm{Ad}(w)(\mathfrak{m}' \cap \mathfrak{P}))$. If the double coset $J_0 xw(M_0' \cap P)$ is closed in M_0', then $J_0 xw(M_0' \cap P)/(M_0' \cap P)$ is compact and therefore $k=0$. Hence $\mathrm{Ad}(w)(\mathfrak{m}' \cap \mathfrak{P}) \supset \mathfrak{n}''$ and $w = w_1$.

The assertion (a) is proved as follows. Since the canonical map

$$M_0''/w_1 P w_1^{-1} \cap M_0'' \longrightarrow M''A''/w_1 P w_1^{-1} \cap M''A''$$

is a topological isomorphism and since (2.5) is a bijection, we have only to consider closed double cosets in

$$(M'' \cap H)_0\backslash M_0''/w_1 P w_1^{-1} \cap M_0''. \tag{2.7}$$

For each double coset in (2.7), take a representative $x \in M_0''$ so that $\mathrm{Ad}(x)(\mathfrak{m}'' \cap \mathfrak{a}) = \mathfrak{a}_1''$ is σ-stable. Then x is contained in a closed double coset in (2.7) if and only if $\mathfrak{a}_1'' \cap \mathfrak{h}$ is maximal abelian in $\mathfrak{m}'' \cap \mathfrak{p} \cap \mathfrak{h}$ and the positive system $\Sigma(\mathfrak{a}_1'')^+$ of $\Sigma(\mathfrak{a}_1'')$ corresponding to $xw_1Pw_1^{-1}x^{-1} \cap M_0''$ is σ-compatible ([3], § 3, Proposition 2). Put $\mathfrak{a}_1 = \mathrm{Ad}(x)\mathfrak{a}$ and $\mathfrak{P}_1 = \mathrm{Ad}(xw_1)\mathfrak{P} = \mathfrak{P}(\mathfrak{a}_1, \Sigma(\mathfrak{a}_1)^+)$. Then it is clear that (2.6) is equivalent to the condition (i) in (a) and that the above conditions for $\mathfrak{a}_1''$ $(=\mathfrak{a}_1 \cap \mathfrak{m}'')$ and $\Sigma(\mathfrak{a}_1'')^+$ are equivalent to the conditions (ii) and (iii) in (a). Hence the assertion (a) is proved.

The assertion (b) is proved by a similar argument using (2.5) and Proposition 1 in [3]. q.e.d.

For an affine symmetric space (G, H, σ) such that G is semisimple, the associated affine symmetric space $(G, H', \sigma\theta)$ is defined by $H' = (K \cap H)\exp(\mathfrak{p} \cap \mathfrak{q})$. Then there exists a one-to-one correspondence between the double coset decompositions $H\backslash G/P$ and $H'\backslash G/P$. If $\mathfrak{a}$ is a σ-stable maximal abelian subspace of $\mathfrak{p}$, an H-orbit containing $\mathfrak{P}(\mathfrak{a}, \Sigma(\mathfrak{a})^+)$ corresponds to the H'-orbit containing the same $\mathfrak{P}(\mathfrak{a}, \Sigma(\mathfrak{a})^+)$ ([3], Corollary 2 of Theorem 1).

COROLLARY 2. (a) *In this correspondence,* $H\backslash HP'/P$ *corresponds to* $H'\backslash H'P'/P$. *Moreover closed H-orbits on* HP'/P *correspond to open H'-orbits on* $H'P'/P$ *and open ones to closed ones.*

(b) *Let* P'' *be a parabolic subgroup of G containing* P'. *Then there is a one-to-one correspondence between* $H\backslash HP''/P'$ *and* $H'\backslash H'P''/P'$ *which is compatible with the canonical surjections* $f: H\backslash HP''/P \to H\backslash HP''/P'$ *and* $f': H'\backslash H'P''/P \to H'\backslash H'P''/P'$ *and with the correspondence* $H\backslash HP''/P \simeq H'\backslash H'P''/P$. *In this correspondence closed H-orbits on* HP''/P' *correspond to open H'-obrits on* $H'P''/P'$ *and open ones to closed ones.*

PROOF. The first assertion in (a) is clear from Theorem. The second assertion in (a) is clear from Corollary 1. Since a double coset HxP' in HP'' is closed (resp. open) in HP'' if and only if HxP' contains a closed (resp. open) double coset HyP in HP'', and since the same holds for H', the assertions in (b) are clear from (a). q.e.d.

REMARK. Let $\mathfrak{a}^o$ be a σ-stable maximal abelian subspace of $\mathfrak{p}$ such that $\mathfrak{a}^o \cap \mathfrak{q}$ is maximal abelian in $\mathfrak{p} \cap \mathfrak{q}$ and let $\Sigma(\mathfrak{a}^o)^+$ be a $\sigma\theta$-compatible positive system of $\Sigma(\mathfrak{a}^o)$. Then $\mathfrak{P}^o = \mathfrak{P}(\mathfrak{a}^o, \Sigma(\mathfrak{a}^o)^+)$ is contained in an open H-orbit on G/P. Let $\mathfrak{P}'^o$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{P}^o$ and $W^o_{\mathfrak{P}'}$ the subgroup of $W(\mathfrak{a}^o)$ corresponding to $\mathfrak{P}'^o$. Then it follows easily from Theorem and [3], Proposition 1 that there is a one-to-one correspondence between the set of open double cosets in $H\backslash G/P'^o$ and

$$W_{K \cap H}(\mathfrak{a}^o)\backslash W_\sigma(\mathfrak{a}^o)/W_\sigma(\mathfrak{a}^o) \cap W^o_{\mathfrak{P}'}$$

where $W_\sigma(\mathfrak{a}^o)=\{w\in W(\mathfrak{a}^o)\mid w\sigma=\sigma w\}$. This fact is also proved in [4], Corollary 16.

Let $\mathfrak{a}^c$ be a σ-stable maximal abelian subspace of $\mathfrak{p}$ such that $\mathfrak{a}^c\cap\mathfrak{h}$ is maximal abelian in $\mathfrak{p}\cap\mathfrak{h}$ and let $\Sigma(\mathfrak{a}^c)^+$ be a σ-compatible positive system of $\Sigma(\mathfrak{a}^c)$. Let $\mathfrak{P}'^c$ be a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{P}^c=\mathfrak{P}(\mathfrak{a}^c, \Sigma(\mathfrak{a}^c)^+)$ and $W^c_{\mathfrak{P}'}$ the subgroup of $W(\mathfrak{a}^c)$ corresponding to $\mathfrak{P}'^c$. Then there is a one-to-one correspondence between the set of closed double cosets in $H\backslash G/P'^c$ and

$$W_{K\cap H}(\mathfrak{a}^c)\backslash W_\sigma(\mathfrak{a}^c)/W_\sigma(\mathfrak{a}^c)\cap W^c_{\mathfrak{P}'}$$

where $W_\sigma(\mathfrak{a}^c)=\{w\in W(\mathfrak{a}^c)\mid w\sigma=\sigma w\}$ (Theorem and [3], Proposition 2).

In the following we shall give an explicit formula for the decomposition $H\backslash HP'/P$ applying the method used in §2 of [3]. Let $\mathfrak{a}_0$ be a σ-stable maximal abelian subspace of $\mathfrak{p}$ such that $\mathfrak{a}_0\supset\mathfrak{a}'$ and that $\mathfrak{m}''\cap\mathfrak{a}_0\cap\mathfrak{h}$ is maximal abelian in $\mathfrak{m}''\cap\mathfrak{p}\cap\mathfrak{h}$. Such a subspace $\mathfrak{a}_0$ of $\mathfrak{p}$ is constructed as follows. Let $\mathfrak{a}''_{0+}$ be a maximal abelian subspace of $\mathfrak{m}''\cap\mathfrak{p}\cap\mathfrak{h}$ and $\mathfrak{a}''_0$ a maximal abelian subspace of $\mathfrak{m}''\cap\mathfrak{p}$ containing $\mathfrak{a}''_{0+}$. Then $\mathfrak{a}_0=\mathfrak{a}''_0+\mathfrak{a}''+\mathfrak{a}'$ is a desired one. By [3], p. 341, Lemma 7, all the maximal abelian subspace $\mathfrak{a}''$ of $\mathfrak{m}''\cap\mathfrak{p}$ such that $\mathfrak{a}''\cap\mathfrak{h}$ is maximal abelian in $\mathfrak{m}''\cap\mathfrak{p}\cap\mathfrak{h}$ are mutually $(M''\cap H)_0$-conjugate. Thus the choice of $\mathfrak{a}_0$ is unique up to $(M''\cap H)_0$-conjugacy. Fix a positive system $\Sigma(\mathfrak{a}_0)^+$ of $\Sigma(\mathfrak{a}_0)$ such that $\langle\Sigma(\mathfrak{a}_0)^+, \mathfrak{a}'_+\rangle\subset\boldsymbol{R}_+$. Then $\mathfrak{P}_{(0)}=\mathfrak{P}(\mathfrak{a}_0, \Sigma(\mathfrak{a}_0)^+)$ is contained in $\mathfrak{P}'$. Let $P_{(0)}$ be the corresponding minimal parabolic subgroup of G.

Let $\bar{\mathfrak{a}}$ be a σ-stable maximal abelian subspace of $\mathfrak{p}$ such that $\bar{\mathfrak{a}}\cap\mathfrak{h}$ is maximal abelian in $\mathfrak{p}\cap\mathfrak{h}$, $\bar{\mathfrak{a}}\cap\mathfrak{h}\supset\mathfrak{a}_0\cap\mathfrak{h}$ and $\bar{\mathfrak{a}}\cap\mathfrak{q}\subset\mathfrak{a}_0\cap\mathfrak{q}$. The existence of such a subspace $\bar{\mathfrak{a}}$ of $\mathfrak{p}$ is an easy consequence of [3], p. 342, Lemma 8. Put $\mathfrak{r}=\{Y\in\bar{\mathfrak{a}}\cap\mathfrak{h}\mid B(Y, \mathfrak{a}_0\cap\mathfrak{h})=\{0\}\}$. Then $\bar{\mathfrak{a}}\cap\mathfrak{h}=\mathfrak{a}_0\cap\mathfrak{h}+\mathfrak{r}$ (direct sum).

Put $\Sigma_{\mathfrak{h}}(\mathfrak{a}_0)_{\mathfrak{m}''}=\{\alpha\in\Sigma(\mathfrak{a}_0)_{\mathfrak{m}''}\mid H_\alpha\in\mathfrak{m}''\cap\mathfrak{a}_0\cap\mathfrak{h}\}$ where $H_\alpha\in\mathfrak{a}_0$ is defined by $B(H_\alpha, Y)=\alpha(Y)$ for all $Y\in\mathfrak{a}_0$. Then a set of root vectors $Q=\{Y_{\alpha_1},\dots, X_{\alpha_k}\}$ is said to be a $\mathfrak{q}$-orthogonal system of $\Sigma_{\mathfrak{h}}(\mathfrak{a}_0)_{\mathfrak{m}''}$ if the following two conditions are satisfied:

(i) $\alpha_i\in\Sigma_{\mathfrak{h}}(\mathfrak{a}_0)_{\mathfrak{m}''}$ and $X_{\alpha_i}\in\mathfrak{g}(\mathfrak{a}_0;\alpha_i)\cap\mathfrak{q}-\{0\}$ for $i=1,\dots, k$,

(ii) $[X_{\alpha_i}, X_{\alpha_j}]=[X_{\alpha_i}, \theta X_{\alpha_j}]=0$ for $i\neq j$.

We normalize X_{α_i}, $i=1,\dots, k$ so that $2\alpha_i(H_{\alpha_i})B(X_{\alpha_i}, \theta X_{\alpha_i})=-1$. Define an element $c(Q)$ of M''_0 by

$$c(Q)=\exp(\pi/2)(X_{\alpha_1}+\theta X_{\alpha_1})\cdots\exp(\pi/2)(X_{\alpha_k}+\theta X_{\alpha_k}).$$

Then $\mathfrak{a}^1=\mathrm{Ad}(c(Q))\mathfrak{a}_0$ is a σ-stable maximal abelian subspace of $\mathfrak{p}$ such that $\mathfrak{a}^1\supset\mathfrak{a}'$.

Let $\{Q_0,\dots, Q_n\}$ $(Q_0=\phi)$ be a complete set of representatives of $\mathfrak{q}$-orthogonal systems of $\Sigma_{\mathfrak{h}}(\mathfrak{a}_0)_{\mathfrak{m}''}$ with respect to the following equivalence relation $\sim$. For two $\mathfrak{q}$-orthogonal systems $Q=\{X_{\alpha_1},\dots, X_{\alpha_k}\}$ and $Q'=\{X_{\beta_1},\dots, X_{\beta_{k'}}\}$ of $\Sigma_{\mathfrak{h}}(\mathfrak{a}_0)_{\mathfrak{m}''}$, $Q\sim Q'$ if and only if there exists a $w\in W_{K\cap H}(\bar{\mathfrak{a}})(=N_{K\cap H}(\bar{\mathfrak{a}})/Z_{K\cap H}(\bar{\mathfrak{a}}))$ such that

$$w(\mathfrak{r}+\textstyle\sum_{j=1}^{k} H_{\alpha_j}) = \mathfrak{r} + \textstyle\sum_{j=1}^{k'} H_{\beta_j}.$$

Put $\mathfrak{a}_i=\mathrm{Ad}(c(Q_i))\mathfrak{a}_0$, $i=1,\dots, n$. Then the following is a trivial consequence of Theorem in this paper, Corollary 1 of Theorem 1 in [3] (Proposition in § 1) and Theorem 2 in [3].

COROLLARY 3. *$HP' = \cup_{i=0}^{n} \cup_{j=1}^{m(i)} Hw_j^i c(Q_i)P_{(0)}$ (disjoint union) where $\{w_1^i,\dots, w_{m(i)}^i\}$ is a complete set of representatives of $W_{K\cap H}(\mathfrak{a}_i)\cap W(\mathfrak{a}_i)_{\mathfrak{m}'}\backslash W(\mathfrak{a}_i)_{\mathfrak{m}'}$ in $N_{K\cap M'}(\mathfrak{a}_i)$. Moreover we have*

$$H'P' = \cup_{i=0}^{n} \cup_{j=1}^{m(i)} H'w_j^i c(Q_i)P_{(0)}\ (\textit{disjoint union}).$$

EXAMPLE 1. Suppose that $G=G_1\times G_1$ where G_1 is a connected real semisimple Lie group with Lie algebra $\mathfrak{g}_1$ and that $H=\Delta G_1=\{(x, x)\in G \mid x\in G_1\}$. Let $\mathfrak{g}_1=\mathfrak{k}_1+\mathfrak{p}_1$ be a Cartan decomposition of $\mathfrak{g}_1$ and put $\mathfrak{k}=\mathfrak{k}_1+\mathfrak{k}_1$ and $\mathfrak{p}=\mathfrak{p}_1+\mathfrak{p}_1$. Then a σ-stable maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ is of the form $\mathfrak{a}=\mathfrak{a}_1+\mathfrak{a}_1$ where $\mathfrak{a}_1$ is a maximal abelian subspace of $\mathfrak{p}_1$. Let $\mathfrak{P}^0$ be a minimal parabolic subalgebra of $\mathfrak{g}$ of the form $\mathfrak{P}^0=\mathfrak{P}_1+\mathfrak{P}_1$ where $\mathfrak{P}_1=\mathfrak{P}(\mathfrak{a}_1, \Sigma(\mathfrak{a}_1)^+)$ for some positive system $\Sigma(\mathfrak{a}_1)^+$ of $\Sigma(\mathfrak{a}_1)$. Then there is a one-to-one correspondence

$$\Delta W(\mathfrak{a}_1)\backslash W(\mathfrak{a}_1)\times W(\mathfrak{a}_1) \xrightarrow{\sim} H\backslash G/P^0$$

which is induced by the map $(w_1, w_2)\mapsto \mathrm{Ad}(w_1)\mathfrak{P}_1+\mathrm{Ad}(w_2)\mathfrak{P}_1$ $(w_1, w_2\in W(\mathfrak{a}_1))$ where $\Delta W(\mathfrak{a}_1)=\{(w, w)\in W(\mathfrak{a}_1)\times W(\mathfrak{a}_1)\mid w\in W(\mathfrak{a}_1)\}$. If we identify $H\backslash G$ with G_1 by the map $(x, y)\mapsto x^{-1}y$ $(x, y\in G_1)$, the decomposition $H\backslash G/P^0$ is equivalent to the Bruhat decomposition

$$P_1\backslash G_1/P_1 \cong W(\mathfrak{a}_1).$$

Fix $(w_1, w_2)\in W(\mathfrak{a})(=W(\mathfrak{a}_1)\times W(\mathfrak{a}_1))$ and put $\mathfrak{P}=\mathrm{Ad}(w_1)\mathfrak{P}_1+\mathrm{Ad}(w_2)\mathfrak{P}_1$. Let $\mathfrak{P}^{0\prime}=\mathfrak{P}_1'+\mathfrak{P}_1''$ be an arbitrary parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{P}^0$ and let $W_{\mathfrak{P}_1'}$ and $W_{\mathfrak{P}_1''}$ be the subgroups of $W(\mathfrak{a}_1)$ corresponding to $\mathfrak{P}_1'$ and $\mathfrak{P}_1''$ respectively. The parabolic subalgebra $\mathfrak{P}'=\mathrm{Ad}(w_1)\mathfrak{P}_1'+\mathrm{Ad}(w_2)\mathfrak{P}_1''$ contains $\mathfrak{P}$ and then $W(\mathfrak{a})_{\mathfrak{m}'}=w_1W_{\mathfrak{P}_1'}w_1^{-1}\times w_2W_{\mathfrak{P}_1''}w_2^{-1}$. Thus the minimal parabolic subalgebras of $\mathfrak{g}$ given in Theorem are of the form $\mathrm{Ad}(w_1w_1')\mathfrak{P}_1+\mathrm{Ad}(w_2w_2')\mathfrak{P}_1$ $(w_1'\in W_{\mathfrak{P}_1'}$, $w_2'\in W_{\mathfrak{P}_1''})$. Hence there is a bijection

$$\Delta W(\mathfrak{a}_1)\backslash W(\mathfrak{a}_1)\times W(\mathfrak{a}_1)/W_{\mathfrak{P}_1'}\times W_{\mathfrak{P}_1''} \xrightarrow{\sim} H\backslash G/P^{0\prime}.$$

If we identify $H\backslash G$ with G_1, the above decomposition $H\backslash G/P^{0\prime}$ is equivalent to the well-known decomposition

$$P_1'\backslash G_1/P_1'' \cong W_{\mathfrak{P}_1'}\backslash W(\mathfrak{a}_1)/W_{\mathfrak{P}_1''}.$$

EXAMPLE 2 ([5], p. 29, Lemma 5.2). Let G be a connected complex semisimple Lie group and σ a complex linear involution of G. Then H is a complex subgroup of G. A Cartan involution θ is a conjugation of $\mathfrak{g}$ with respect to a compact real form $\mathfrak{k}$ of $\mathfrak{g}$ and $\mathfrak{p}=(-1)^{1/2}\mathfrak{k}$. Let $\mathfrak{a}$ be a σ-stable maximal abelian subspace of $\mathfrak{p}$ and $\Sigma(\mathfrak{a})^+$ a positive system of $\Sigma(\mathfrak{a})$. Then $\mathfrak{P}=\mathfrak{P}(\mathfrak{a}, \Sigma(\mathfrak{a})^+)$ is a Borel subalgebra of $\mathfrak{g}$. Let $\mathfrak{P}'$ be a parabolic subalgebra of $\mathfrak{g}$ corresponding to a simple root α of $\Sigma(\mathfrak{a})^+$. Then the simple root α is called (i) compact imaginary if $\mathfrak{g}(\mathfrak{a}; \alpha)\subset\mathfrak{h}$, (ii) non-compact imaginary if $\mathfrak{g}(\mathfrak{a}; \alpha)\subset\mathfrak{q}$, (iii) real if $\sigma\alpha=-\alpha$ and (iv) complex if $\sigma\alpha\neq\pm\alpha$. In [5], $H\backslash HP'/P\subset H\backslash G/P$ is determined in each case (i)~(iv). Therefore $f^{-1}(f(\mathcal{O}))$ is determined for an arbitrary $\mathcal{O}\in H\backslash G/P$ if P' is a parabolic subgroup of G corresponding to a simple root.

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