Orchestrating Mathematics Lessons: Beyond the Use of a Single Rich Task

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Teachers have several challenges when designing and implementing mathematically-rich tasks, and hence, these tasks are not prevalent in many mathematics classrooms. Instead, teachers often use *typical problems*, such as standard textbook tasks and examination questions, to develop students' procedural fluency. This begs the question of whether, and if so, how teachers can think about, and use these typical problems differently to develop conceptual understanding. In this paper, we report findings drawn from a two-year design-based research project and highlight two teaching vignettes to illustrate how typical problems were used to orchestrate instructional activities. Our findings suggest three important principles for teachers to consider when using typical problems.

Orchestrating discussions around mathematically-rich tasks (Grootenboer, 2009) can potentially enhance students' learning experiences by providing opportunities for students to reason about, and communicate their mathematical ideas (Smith & Stein, 2011). To support teachers in developing their competencies in this high-leverage teaching practice, Smith and Stein (2011) proposed five inter-dependent practices—anticipating, monitoring, selecting, sequencing, and connecting—which hinge on a single high cognitive-demand task. However, the use of high cognitive-demand tasks in the classrooms remains relatively infrequent, and often problematic. For example, Kaur (2010) suggested that teachers in Singapore may prefer to use standard examination-type questions, or what we termed as *typical problems* (Choy & Dindyal, 2017) during day-to-day teaching. This is not surprising because teachers in an examination-driven education system, such as Singapore, may believe that it is "important to prepare students to do well in tests than to implement problem-solving lessons" (Foong, 2009, p. 279). Furthermore, mathematically-rich tasks have a high entry-point for students, and teachers have to provide additional support or prompts for students (Sullivan et al., 2014). These factors limit the use of rich tasks as teachers may find these tasks time-consuming and pedagogically challenging to implement. While acknowledging the importance of mathematically-rich tasks in mathematics lessons, we also wonder whether typical problems have a role to play to develop conceptual understanding, and if so, how can these problems be used to orchestrate instructional activities? In this paper, we draw data from a bigger study to describe two teaching vignettes of an experienced teacher, Alice (pseudonym), and highlight how she had used typical problems differently to develop relational understanding (Skemp, 1978).

Orchestrating Instructional Activities

Instructional activities refer to the ways in which "teacher, content, and diverse students would interact within work on authentic problems, how materials of instruction would be used, how the space would be arranged, and how the teacher would move around the room" (Lampert & Graziani, 2009, p. 493). As argued by Lampert, Beasley, Ghousseini, Kazemi, and Franke (2010), these interactions form part of the core high-leverage teaching practices

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needed to enact *ambitious teaching*, which focus on responding to and building on students' answers as they work on problem-solving tasks. The idea is to *co-construct* instructional explanations and dialogues with students (Lampert et al., 2010). Co-constructing mathematical conversations with students places tremendous demands on teachers, and teachers would need support to do this work. A key strategy to break down the complexity of this classroom practice is then to use *routines* to make some of this work "automatic", as highlighted by Leinhardt and Steele (2005, p. 142).

Lampert et al. (2010) see these routines, particularly the exchange routines (Leinhardt & Steele, 2005), as the foundation of ambitious teaching. They highlight three exchange routines—*call-on, revise, and clarification*—to be one of the means that make the work of orchestrating discussions learnable by novice teachers (Lampert et al., 2010; Leinhardt & Steele, 2005). First, the *call-on* routine seeks to invite students to respond to a problem, and is followed by an exchange involving analysis of, justifications for, and critiques of ideas by the other students. Next, the *revise* routine facilitates students to rethink ideas put forth by their peers before they explain their new ideas. Finally, the clarification routine seeks to understand the source of any confusion regarding the ideas discussed.

Another important aspect of preparing teachers to orchestrate discussion is recognise that certain aspects of this teaching expertise can be planned. To this end, Smith and Stein (2011) suggested an instructional sequence, where teachers use five practices around a single rich task in which students attempt, present, and discuss the mathematics embedded in the task. The crux of orchestrating discussions is to purposefully select students' work so that both important mathematical ideas as well as common misconceptions are addressed. This provides opportunities for the teacher to share useful alternative strategies that were not presented by the students. The motivation for carefully selecting and sequencing responses is to lay the groundwork for the teacher to connect these different responses to important mathematical ideas. By directing students' attention to the connections between different strategies, and by shifting their focus from solutions to mathematical ideas, teachers can begin to support students' efforts in understanding the concepts targeted in the lesson (Smith & Stein, 2011).

The idea of focusing on connections reflect a *connectionist* orientation (Askew, Rhodes, Brown, Wiliam, & Johnson, 1997), which emphasises connections within mathematics when teaching numeracy. According to Askew et al. (1997), a connectionist numeracy teacher strikes a balance between a transmission orientation and a discovery orientation, and is more likely to be more effective in the classrooms. Although these beliefs pertain to the teaching of numeracy, we have found these beliefs useful in explaining the teaching practices of the experienced teachers in our study, who had exploited the use of typical problems to orchestrate discussions.

In our earlier paper (Choy & Dindyal, 2017), we described how Alice, an experienced teacher, orchestrated a mathematically productive discussion (Smith & Stein, 2011) by carefully attending to students' answers to a typical problem before she asked for volunteers during the whole class discussion. Alice's orchestration of instructional activities differed from the five practices in two important ways. First, Alice used a selection of four contextual questions on matrix multiplication, taken from past-year examination papers. This stands in contrast to Smith and Stein's idea of using a single rich task for the lesson. Second, although Alice's way of orchestrating discussion seems to reflect the five practices, Alice interjected to explain the connections in between the different solutions, instead of connecting the solutions at the end of the presentation. This provided opportunities for her to emphasise the *connections* between matrix multiplication and arithmetic to provide meaning to matrix operations in between students' presentations. Hence, Alice kept the concept in focus and ensured coherence in the discussion by *co-constructing the explanations* for the different approaches with her students (Lampert et al., 2010). In this paper, we build on our previous paper to describe two more vignettes, taken from different lessons conducted by Alice, to build up a more complete picture of how typical problems can be used to develop conceptual understanding.

Methodology

The data reported in this paper came from a larger study on orchestrating learning experiences in a secondary school mathematics classroom in Singapore. We used a design-based research approach to develop a toolkit for our teachers as a means of supporting their orchestration of learning experiences, as well as to develop a theory about teachers' productive noticing (Choy, Thomas, & Yoon, 2017) in the context of orchestrating learning experiences. Details on how we worked with the teachers can be found in Choy and Dindyal (2017). Besides audio-taping the pre-lesson and post-lesson discussions, we also made video recordings of the lesson and collected lesson artefacts used during the lessons. The recordings were transcribed, and segments related to the major divisions of the lessons were identified and analysed. The findings were developed through identifying themes related to the five practices, as envisioned by Smith and Stein (2011), and the notion of routines used in orchestrating instructional dialogue (Leinhardt & Steele, 2005). In this paper, we examine the instructional activities of Alice, a Senior Teacher at Coventry Secondary School (pseudonym), which is a government-funded school. As a Senior Teacher, which is an official appointment in the school, she has demonstrated strong content knowledge, familiarity with the national curriculum, and strong pedagogical content knowledge. For each of the vignettes, we will describe briefly the context of the lesson, and highlight how Alice's orchestration of instructional activities using typical problems reflect a connectionist orientation towards teaching.

Two Teaching Vignettes

Techniques of Differentiation

This lesson for Secondary Four (Grade 10) students focused on developing procedural fluency in differentiating real-valued functions of one variable using one, or more of the following formula: (a) the "basic" rule ($\frac{d}{dx}x^n = nx^{n-1}$); (b) Chain rule; (c) Product rule; and (d) Quotient rule. On the surface, Alice's lesson appeared to focus solely on developing skills, but closer examination reveals that she was deliberate in the selection of the functions for differentiation. In particular, Alice wanted her students to develop the reasoning skills to determine which of the rules is most efficient for a given function. As each of the functions lend themselves to be differentiated using a variety of methods, her choice of functions afforded opportunities for students to focus on the structure of the given function. In the following exchange, we see Alice's discussion with the class after students had worked through a series of questions together in groups.

294	Alice:	So, I want to look at a few questions, like this one. (Proceeded to write the following on the board) $y = \frac{(2-3x^2)^2}{\sqrt{2-3x^2}}$

Ok what comes to your mind? I asked that group (pointing to a group of students) just now, what comes to you mind when you first see this? Ok so initially, they told me it's u/v, right? u/v, ok? So, if you use quotient rule, right? Ok, it's correct la I never say it's wrong, you can still do it. However, you can do it in a simpler way if you represent it in another manner.
(Pointed to Student S5)
So, [Student S5], come and write down. How would you represent it? Just write down quickly. How you re-write the y function?

295	S5:	Ok. (Walked to the whiteboard and wrote the following: $y=\left(2-3x^2\right)^{3/2}$
296	Alice:	Yes, ok, so he has represented this function like that. Ok, so he has represented it like this. Once he has represented it like this, you see that it is what rule? Which rule can apply? (Inaudible answers by students) Ok, then you can see that you can apply chain rule easily, instead of the quotient rule, ok alright?

Alice then followed up with another function $y=\frac{1}{\sqrt{2x}}$ and called on Student S6 to write a representation for the class.

297	Alice:	So, another question, I see some interesting representations of this. [Student S6], how do you simplify this initial y function? What do you re-write it as first, before you differentiate?
298	S6:	(Walked to the whiteboard and wrote the following) $y=\frac{1}{\sqrt{2x}}=\left(2^{1/2}x\right)^{-1}$
299	Alice:	Ok, then you intend to use?
300	S6:	Chain rule.
301	Alice:	Chain rule, ok, alright, thank you. [Explained why Student S6's expression is right.] … Is there another group that wrote it differently? Anybody wrote it differently? … (after a while) Ok, alright, [Student S7] has written it like that: (wrote the following) $y=\frac{1}{\sqrt{2x}}=\frac{1}{\sqrt{2}}x^{-1}$ If a function is written like that, what rule would you use?
302	Students:	Normal rule.
303	Alice:	Basic rule, correct or not? This is you're a (referring to the constant), this is your x to the power of minus 1, so straight away you can use this, which is a very simple, basic rule. … Can? [Student S8], [Student S9], can you see this? Ok, this is different from this ah. We're not saying that the chain rule is incorrect, we're saying that if you can represent it like that, it becomes the basic rule. Let's do another one. (Wrote the following expression) $y=x\sqrt{x}$ …

In the preceding vignette, Alice invited a few students to share their ideas with the class with the purpose of directing students' attention to the structure of the given function. For example, in the case of $y=\frac{1}{\sqrt{2x}}$, we see that Alice was aware of the different ways students might have perceived the function ("I see some interesting representations of this."). We can infer that Alice was purposeful in the selection of Student S6 and Student S7 to drive home the point that it is important to consider different possible "representations" of the same function. In many ways, Alice's teaching reflected a connectionist orientation, in which "more efficient methods are offered" and "discussed the sort of contexts where different representations would be used" (Askew et al., 1997, p. 30). Furthermore, there are instances of the five practices, such as monitoring (Line 297), selecting, sequencing (Lines 298 to 301), and connecting (Line 303) in this short exchange.

Standard Deviation Lesson

In this lesson for Secondary Four students, Alice used a sequence of examination-type items to get students think more deeply about the concept of standard deviation. In particular, she adapted an EXCEL worksheet for students can manipulate a data set to explore statistical diagrams (such as histograms). Students had about 40 minutes to work through the nine items while Alice circulated the various groups to offer prompts and assistance when requested. The vignette, which follows, centres around the discussion on Question 3 as shown in Figure 1.

3 Which of the two histograms below represents the data distribution with the greater standard deviation? Explain your choice.

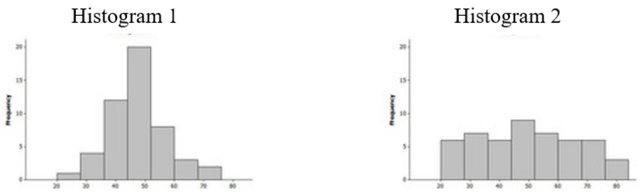

Figure 1. Question on Histogram and Standard Deviation.

248	Alice	Ok maybe uh [S8] can tell us, which histogram you chose for question 3.
249	Student S8	We chose histogram 1.
250	Alice	Yeah, can you tell us why you chose histogram 1? Why 1?
251	Student S8	Greater spread of data.
252	Alice	Why is there greater spread?
253	Student S8	There's a lot of variation.
254	Alice	There's a lot of variation, ok?
255	Student S9	Because histogram 2 everything is almost the same…
256	Alice	Ok, what does standard deviation measure?
257	Students	Spread
258	Alice	It measures spread. So, for this ah, where do you think the mean is? Somewhere... Where's the mean? Because standard deviation, we are measuring the deviation from the mean right? So where do you expect the mean to be, roughly?
259	Students	In the middle.
260	Alice	Centre ah, ok so let's say you have it here, ok, alright? So how does that give you more deviation from this centre? [S10], what do you think?
261	Student S10	[inaudible]
262	Alice	Huh? [S11] what do you think? Why is there more deviation from the mean as compared to this?
263	Student S11	The data is clustered around the middle.
264	Alice	Which one?
265	Student S11	[inaudible]
266	Alice	The data is not clustered. Which one is not clustered? Which diagram are you talking about? This one? (Pointed to Histogram 1) But you compare to this, you see ah, ok? Standard deviation measures spread ah, you look at this, this one, you got high frequencies of data at the 2 ends, correct or not? ...

Alice demonstrated the clarification routine (Lampert et al., 2010) to understand students' erroneous thinking about the notion of spread (Lines 249 to 265). Although she could have revealed the answer earlier, Alice withheld her explanation until she got a sense of what students were confused about. By listening to her students, Alice realised that students did not pay attention to the idea that the heights of the histogram bars refer to frequencies and tried to emphasise that idea in her explanation (Line 266). After her explanation, she called on another student, S12, to give his reasoning because she had a brief discussion with him during the seatwork. By engaging Student S12 to give his comments, Alice provided a platform for her students to understand the justification for the correct answer—Histogram 2. To make the explanation more accessible to students, Alice decided to use the EXCEL worksheet to show how the standard deviation relates to the shape of the histogram. Hence, we can say that Alice use of discussion routines and spreadsheet around the typical problem to make connections between explanations, concepts, and representations reflects her connectionist orientation.

Orchestrating Instructional Activities Using Typical Problems

In many ways, Alice's use of typical problems to orchestrate instructional activities for a day-to-day lesson is not unique but is commonly practised by the experienced teachers in our study. Her discussion moves around typical problems to develop relational understanding—knowing how and why (Skemp, 1978)—suggest new affordances for typical problems, beyond its use to develop procedural skills. Notwithstanding the limitations of a case study, we think that Alice's case provides an existential proof for how typical problems can be used differently to develop a relational understanding of mathematics. Furthermore, we argue that teachers can better tap the affordances of typical problems when they adopt a connectionist approach to teaching.

Affordances of Typical Problems

Alice's use of typical problems, and other teachers in our study, highlight new affordances of typical problems in developing a relational understanding of mathematics. As we have highlighted earlier in this paper, typical problems are widely used in mathematics classrooms to develop procedural fluency. Often, these problems are used with a transmission approach to teaching (Askew et al., 1997), or what some may termed as "drill and practice". However, Alice use of typical problems suggests a more balanced view of what these problems can afford beyond developing an instrumental understanding of mathematics. As many of the typical problems are narrowly focused on one or two instructional outcomes, they provide an excellent avenue to direct learners' attention to specific features of the target concept. For example, in the vignette on *Techniques of Differentiation*, Alice used a sequence of questions to highlight the importance of examining the structure of the functions given before deciding on the appropriate technique for differentiation. In the *Standard Deviation* lesson, we see how Alice used a single typical problem to highlight the relationship between the standard deviation and the statistical diagram representation of a given data set. Furthermore, typical problems also lend themselves to be modified slightly to open up its solution space, so that teachers can discuss different solutions and the connections between these solutions (Choy & Dindyal, 2017). In all these cases, Alice could have simply used the problems in a transmission approach by highlighting the solutions and the procedures needed to solve the problem. Instead, we see how she had noticed productively (Choy et al., 2017) about the affordances of typical problems, and had orchestrated discussions around these problems by making explicit the connections between the problems and the concepts taught.

A Connectionist Approach to Teach Mathematics

A key aspect of Alice's use of typical problems lies in the connections she made when orchestrating instructional activities in class. As we have already seen in Choy and Dindyal (2017), Alice tried to connect different students' solutions by highlighting the connection between arithmetic and the method of matrix multiplication, and the connection between the solution methods in relation to the use of matrices to represent information in a systematic manner. In this way, Alice tried to highlight the "links between different aspects of mathematics", which reflect a strong connectionist orientation (Askew et al., 1997, p. 32). Similarly, we see that Alice made connections between representations of mathematics as she attempted to clarify students' thinking about standard deviation in the *Standard Deviation* vignette. In addition, referring to the vignette on differentiation techniques, Alice used a series of focused discussions to support students in making sense of the efficiency of different differentiation techniques by considering the structure of the given functions. Hence, we see Alice's use of typical problems to make connections within mathematics as an extension of the connectionist orientation to teach mathematics. We believe that it is the connectionist mindset adopted by Alice, and other teachers in the study, which made it possible for teachers to exploit the affordances of typical problems to teach mathematics in a more relational way (Skemp, 1978).

Partial or Rapid Cycles of Five Practices

Another aspect of Alice's orchestration of instructional activities, which brought forth the affordances of typical problems, is how she directed the discussions during her lessons. As discussed in each of the vignettes, we see some elements of Smith's and Stein's (2011) five practices in the way Alice orchestrated the discussions. In the *Differentiation Technique* vignette, we see Alice monitored students' answers, selected, sequenced, and connected their responses to highlight the thinking behind the choice of differentiation rules to apply. Here, Alice's lesson differed, in terms of structure, from that envisioned by Smith and Stein (2011) in the plurality of tasks within the same lesson, punctuated by several more rapid successions of the same discussion moves: monitoring, selecting, sequencing, and connecting. This structure was made feasible by Alice's choice to use typical problems, which generally take a shorter time to complete. Moreover, there were times when Alice did not use all the practices. Instead, she employed rapid but partial cycles of the five practices to discuss a modified typical problem, as in the case of the *Standard Deviation* vignette. Smith and Stein (2011) highlight the five practices as inter-dependent moves that hinge on the use of a single high cognitive demand task for the lesson. However, our data suggest that partial or rapid cycles of the five practices can be used effectively with typical problems to emphasise the connections between mathematical ideas and representations.

Concluding Remarks

This paper explores the possibility of using typical problems for developing conceptual understanding and highlights how Alice orchestrates discussions around typical problems. More specifically, Alice recognised the affordances of typical problems and exploited them effectively through partial or rapid cycles of the five practices, to help connect students' thinking to the mathematical ideas embedded in the typical problems. Given the time constraints in Singapore and other examination-oriented systems, typical problems offer a way to strike the balance between a transmission orientation and a discovery orientation for teaching mathematics. Like Alice, connectionist teachers can better initiate and sustain productive mathematics discussions, which can potentially support students in developing a relational understanding of mathematics. The question now is not whether typical problems can be used, but rather, how teachers can be supported to notice more productively the affordances of typical problems and orchestrate instructional activities around them.

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