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ORDER AND EQUIVALENCE OF RATIONAL NUMBERS: A CLINICAL TEACHING EXPERIMENT

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Fourth-grade students' understanding of the order and equivalence of rational numbers was investigated in 11 interviews with each of 12 children during an 18-week teaching experiment. Six children were instructed individually and as a group at each of two sites. The instruction relied heavily on the use of manipulative aids. Children's explanations of their responses to interview tasks were used to identify strategies for comparing fraction pairs of three types: same numerators, same denominators, and different numerators and denominators. After extensive instruction, most children were successful but some continued to demonstrate inadequate understanding. Previous knowledge relating to whole numbers sometimes interfered with learning about rational numbers.

Rational number concepts are among the most complex and most important mathematical ideas that children encounter before they reach secondary school. The increased attention being given to research on children's acquisition of such concepts reflects their importance. Recent results from national assessments have shown that children have significant difficulty learning and applying concepts related to rational numbers.

In a recent national assessment, 30% of the nation's 13-year-olds added the numerators and the denominators to find the sum of 1/2 and 1/3 (Post, 1981), even though a bit of reflection would have suggested that the sum of two positive quantities should not be less than one of them. Only 24% of the 13-year-olds were able to estimate 12/13 + 7/8 by selecting the correct response, 2, from [1, 2, 19, 21, I don't know]. The most recent Minnesota State Assessment in Mathematics identified fractions as the topic most in need of attention (Minnesota Department of Education & Minnesota Council of Teachers of Mathematics, 1976), a finding that is not unique to Minnesota.

State and national assessments suggest that students have often failed to internalize a workable concept of rational number. Students often do not consider numerators and denominators in relation to one another but, rather,

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handle them as separate entities to be operated on independently. For example, 28% of the 13-year-olds in the national assessment sample selected 19 as the estimate for 12/13 + 7/8, and another 27% selected 21. Many students appear to lack a quantitative notion of rational number. What is such a quantitative notion? How do we want students to think about the rational numbers they encounter?

A recent review and synthesis of the research on rational number learning (Behr, Lesh, Post, & Silver, 1983) indicates that researchers have focused on various aspects of the topic. Numerous researchers (e.g., E. F. Karplus, R. Karplus, & Wollman, 1974; R. Karplus, Adi, & Lawson, 1980; Kieren, 1976, 1980, 1981; Noelting, 1980a, 1980b; Owens, 1980) have conducted status studies of children's cognitive structures for performing rational number tasks. These researchers have investigated, through clinical research procedures, children's thinking about the rational number subconstructs (Kieren, 1976) in ordinary school settings.

Other researchers (Harrison, Brindley, & Bye, 1981; Payne, 1976; Streefland, 1981) have studied children's learning in the context of experimental teaching materials. These studies have emphasized children's understanding of rational number concepts and their ability to perform operations on rational numbers. Behr, Wachsmuth, and Post (1983) recently observed that children's ability to acquire a quantitative notion of rational number is crucial to the development of other rational number concepts. What meaning, for example, do 2/3 × 5/6 or 2/3 + 5/8 have for children who lack a well-internalized concept of the "bigness" of rational numbers?

One measure of children's quantitative notion of rational number is their ability to perceive the relative size of the rational numbers in a pair or a larger set; that is, their ability to determine which of the relations, *is equal to, is less than,* or *is greater than,* holds for a given pair of rational numbers. Prior research (e.g., E. F. Karplus et al., 1974; R. Karplus et al., 1980; Noelting, 1980a, 1980b) has given considerable attention to questions of the equivalence of fractions by means of tasks requiring proportional reasoning. Little attention appears to have been given to children's ability to deal with the relations *is less than* and *is greater than.* The work reported in this paper, in addition to a consideration of equivalence, provides information about children's thinking in tasks involving the ordering of unequal fractions.

This paper is a report from the Rational Number Project, a multiuniversity research effort funded by the National Science Foundation from 1979 through 1983. The project developed instructional and assessment materials concerned with the learning of the rational number subconcepts of *part-whole, measure, quotient, decimal,* and *ratio.* The materials were written to reflect cognitive psychological principles as suggested by Piaget (1960), Bruner (1966), Dienes (1967), and Gagné and White (1978). Of particular concern was the question of the impact of manipulative materials on the learning of the subconcepts.

A major focus of the project was the role of physical models in facilitating the acquisition and use of mathematical concepts as the learner's understanding progresses from concrete to abstract. Psychological analyses show that physical aids are just one component in the development of representational systems, and that other modes of representation—verbal, pictorial, and symbolic—also play a role in the acquisition and use of concepts (Lesh, Hamilton, & Landau, 1981). A major hypothesis of the project was that the ability to make translations among and within modes of representation is what makes ideas meaningful to learners.

METHOD

An 18-week teaching experiment was conducted between October 1980 and March 1981 in schools located in St. Paul, Minnesota, and DeKalb, Illinois. The subjects were 12 fourth graders, 6 at each site. The instructional program consisted of 13 lessons, from 3 to 8 days each. The children worked on some activities individually and others in a group. The instructional and assessment materials were identical at both sites. While the teaching experiment was in progress, the children did not receive any other formal instruction on rational numbers.

The children were introduced to the part-whole interpretation of rational number by means of circular and rectangular pieces of laminated colored construction paper. Each unit fraction $1/n$, for n from 1 to 10, 12, and 15, was represented by a different colored paper. The fraction m/n was represented by juxtaposing m pieces of the appropriate color. Each child had his or her own set of materials. Later in the instruction, the materials were expanded to include Cuisenaire rods, paper folding, poker chips, and number lines.

The instruction covered five topics: naming fractions, equivalent fractions, comparing fractions, adding fractions with the same denominators, and multiplying fractions. The children were expected to model inherent and related ideas using materials, pictures, symbols, and words. They were also expected to translate within and among the various modes of representation. An outline of the instructional sequence is given in Table 1.

INTERVIEWS

Each child was interviewed individually on 11 separate occasions. The interviews were conducted approximately every 8 days during the 18-week instructional period. Each interview was audiotaped or videotaped and later transcribed.

The interviews contained items dealing with the major strands considered by the Rational Number Project: order and equivalence, translations within and between modes of representation, the concept of unit, the quantitative notion of rational number, and the ability to apply rational number concepts in problem situations. This paper concerns only those items dealing with order and equivalence. Such items were contained in most of the interviews.

Table 1
Design of Instructional Materials

Lesson	Embodiment[a]	Activity
1	Color-coded circular pieces	Name pieces Compare sizes Observe that as size decreases, number to make whole increases Observe equivalence
2	Color-coded rectangular pieces	Name pieces Compare sizes Observe that as size decreases, number to make whole increases Observe equivalence
3	Color-coded circular and rectangular pieces	Observe similarities and differences between circular and rectangular pieces Translate between circular and rectangular pieces
4	Color-coded circular and rectangular pieces	Attach unit fraction names (one fourth, one fifth) to parts of whole Work with equivalent sets of fractions
5	Paper folding with circles and rectangles	Attach fraction names to shaded parts of folded regions Learn names for unit and proper fractions Associate names with color-coded parts and with shaded parts of folded regions Note similarities and differences
6	Cuisenaire rods	Attach fraction names to display of rods Note fractions as sums of unit fractions Compare display with colored pieces and paper folding Investigate real-world problems
7	All materials from Lessons 1 to 6	Review, using all four embodiments Identify proper fractions orally, in written form, symbolically, and pictorially Translate from one mode to another
8	Chips	Review division as partitioning (18 ÷ 3 = 6 represents 18 chips, 3 groups, 6 in each group) Represent fractions by covering equal-sized groups of chips Associate fractions with amount covered (3 of 4 groups covered is 3/4)
9	All materials from Lessons 1 to 7	Translate to any mode, using any embodiment, a fraction represented with physical objects, orally, or with a written symbol
10	Color-coded pieces Paper folding Chips	Represent improper fractions using pieces Translate between improper and mixed number notation orally and in writing Translate representation to paper folding Translate representation to chips
11	Number line	Associate whole numbers, fractions, and mixed numbers with points on number line Convert improper fractions to whole or mixed numbers Determine equivalence Add fractions with same denominators
12	All materials from Lessons 1 to 11	Using pieces, chips, rods, or pictures of parts of a unit, construct the unit
13	Chips Paper folding	Represent a model for multiplication of fractions Generalize to algorithm with product of numerators and product of denominators

[a]All embodiments are continuous until Lesson 8.

An ordering item typically asked the child to decide which of two (or three) fractions was the lesser (or the least) and then to explain the reasons for the decision. An item on equivalence asked the child to decide whether two fractions were equivalent (termed a *comparison* item) or to generate a fraction equivalent to a given one by supplying a missing numerator or denominator (termed a *missing-value* item). (R. Karplus, Pulos, & Stage, 1983a, 1983b, have made a similar distinction between types of proportion items.) For each item, the interviewer solicited a verbal explanation of the child's response or a demonstration of how it had been obtained. As the instruction progressed, the fractions increased in complexity: from unit fractions to proper fractions to improper fractions and whole numbers, and from small-sized to larger sized numerators and denominators.

The children's responses were coded in a matrix prepared for each interview. The matrices were examined within and between children and sites for patterns in the children's thinking. The strategies discussed in this paper emerged from that examination.

RESULTS

Similarities and differences were observed in the children's approaches to questions of order and equivalence. Although the children's level of success was of interest, the major concern of the study was the description of the thought processes, or solution strategies, that the children brought to bear on the interview items. The strategies were inferred from the explanations that the children gave for their responses. The results were analyzed, and are reported, according to the three classes of fractions used in the items: fractions with the same numerators (whether unit fractions or nonunit fractions), fractions with the same denominators, and fractions with different numerators and denominators. The last class of fractions was the only one to give rise to items on the equivalence of fractions because equivalence of fractions with the same numerators or the same denominators occurs only in the trivial case in which the fractions are identically equal. In each analysis, several distinct strategies were identified together with a category labeled *other*.

Fractions With the Same Numerators

The analysis suggested five distinct strategies. The first four are valid in the sense that unless the child made a "mental slip," the strategy should have led to a correct response. The fifth strategy, however, is invalid.

Numerator and denominator. This strategy is evidenced by explanations in which the child referred to both the numerators and the denominators, indicating that the same number of parts was present (the numerators) but that the fraction with the larger (or largest) denominator had the smaller (or smallest) sized parts.

- Two fifths is less than two thirds because "there are two pieces in each, but the pieces in two fifths are smaller. So a smaller amount of the unit is covered for two fifths."
- One twentieth is less than one seventeenth because "if we took two equal units and cut one into 20 pieces and one into 17 pieces, the one with the 20 has more. So they'd have to be smaller. And there's one of each, so one twentieth is less."

Denominator only. The explanations associated with this strategy referred only to the denominators of the fractions.

- One ninth is less than one fifth because "the bigger the number is, the smaller the pieces get. One ninth would be less because it has smaller pieces."
- Two fifths is less than two thirds because "you look at the bottom numbers. . . . In numbers the bigger one is more [referring to 2/5]. Then you look at the other one [2/3]. In numbers that one is small, but in fractions that makes it big . . . less pieces, so they're quite big."

Reference point. In using this strategy, the child compared the given fractions to a third number. In some cases, the explanation indicated that an amount or number of pieces was needed to complete a whole or attain a reference point; in other cases, an amount or number of pieces in excess of a whole or beyond a reference point was specified.

- Three ninths is less than three sixths because ". . . three ninths doesn't cover one half . . . less than one half. And three sixths is one half, so three ninths is less than three sixths."
- "Three ninths is less than three fifths is less than three fourths because three fourths has one fourth left over, three fifths has two fifths left over, and three ninths has six ninths left over."

Manipulative. The child explained his or her response using pictures or manipulative materials.

- One fifth is less than one fourth, and one fourth is less than one third because [after drawing and referring to a picture] "five pieces to make one whole, so [one fifth] must be smaller."

Whole number dominance. The child's explanation suggested the application of a rule that centered exclusively on the values of the denominators. The rule gave an ordering consistent with whole number arithmetic but failed to incorporate the inverse relation between numerator and denominator.

- One third is less than one fourth "because three is less than four."
- One seventeenth is less than one twentieth because "20 pieces and 17 pieces. It takes 20 pieces [1/20] to cover; it takes 17 pieces to cover. . . . One seventeenth [is less]."

Other. The child gave an uninterpretable explanation or said, "I don't know."

For the analysis of the items with the same numerators, the interviews were categorized as *early* (Interviews 1 and 2) or *late* (Interviews 3 through 11). The late interviews took place after Lesson 5 (see Table 1). The distribution of explanations given by the children to the items on fractions with the same numerators is shown in Table 2. The explanations are classified by strategy, site, fraction type (unit vs. nonunit), and time of interview (early vs. late).

Table 2
Explanations of Responses to Items on Fractions With the Same Numerators Classified by Strategy, Site, Fraction Type, and Time of Interview

	Unit fractions		Nonunit fractions	
Strategy	Early	Late	Early	Late
		DeKalb site		
Numerator & denominator	0	7	0	12
Denominator only	4	34	0	23
Reference point	0	0	0	2
Manipulative	0	0	0	0
Whole number dominance	17	2	0	4
Other	2	2	0	2
		Minneapolis site		
Numerator & denominator	0	4	0	5
Denominator only	4	28	0	14
Reference point	0	0	0	11
Manipulative	0	3	0	5
Whole number dominance	2	0	0	0
Other	2	6	0	4

Note. Early = Interviews 1–2; Late = Interviews 3–11.

In the early interviews, the items involved unit fractions. The invalid whole-number-dominance strategy was especially frequent at the DeKalb site, and the only valid strategy employed at either site was denominator only. In the late interviews, the denominator-only strategy predominated at both sites and for both unit and nonunit fractions. When the data from the late interviews are pooled across sites and fraction types, of the 166 explanations, 89% are associated with valid strategies (60% with the denominator-only strategy), 2% are associated with the whole-number-dominance strategy, and 8% with other strategies.

Fractions With the Same Denominators

The analysis suggested five distinct strategies. Again, the first four are valid, and the fifth is invalid.

Numerator and denominator. The child referred to both the numerators and the denominators, indicating that the size of the parts was the same (the denominators) but that there were more parts (the numerators).

- Seventeen thirtieths is less than twenty thirtieths because "if you have 30 pieces, and 17 are covered, 17 is less than 20; 20 is a bigger number, so there would be more thirtieths."

Reference point. The strategy is the same as that for fractions with the same numerators.

- Eleven thirds is greater than one and one third because "eleven thirds is greater than three wholes. . . ."
- Given three fractions to order from smallest to largest, the child orders them 3/7, 8/7, 10/7, explaining that it "takes one (whole) unit for this [3/7], two units for this [8/7], and two units for this [10/7]."

Manipulative. The strategy is the same as that for fractions with the same numerators.

- The child draws pictures to represent 8/7, 10/7, and 3/7 and then correctly orders them from smallest to largest, saying, "The size of the piece is the same. The number of pieces is important."

Whole number consistent. The explanations associated with this strategy suggested that the child ordered the fractions according to the sizes of the numerators. The child may or may not have referred to the denominators being the same and did not make reference to the size of any parts shown in pictures or materials.

- Four thirteenths is less than nine thirteenths because "four is less than nine."
- "Seventeen thirtieths is smaller [than twenty thirtieths] because both are the same [thirtieths], but one has 20 and the other only 17."

Incorrect numerator and denominator. The child made an incorrect comparison of the sizes of the parts, inverting the relation between numerator and denominator.

- Nine thirteenths is less than four thirteenths because "four pieces are so big, nine pieces would have to be smaller to fit the whole."

Other. The child gave an uninterpretable explanation or said, "I don't know."

Table 3 contains a classification of the explanations for the responses to items on fractions with the same denominators. Such items were not given in the early interviews; the data in the table are for the late interviews only. The data are classified by strategy and site.

The numerator-and-denominator strategy predominated at both sites. When the data are pooled across sites, the strategies associated with the 77 explanations are as follows, in decreasing order of frequency: numerator and denominator (45%), whole number consistent (17%), reference point (13%), manipulative (9%), and incorrect numerator and denominator (6%). Other strategies account for the remaining 9%.

Table 3
Explanations of Responses to Items on Fractions With the Same Denominators Classified by Strategy and Site

	Site	
Strategy	DeKalb	Minneapolis
Numerator & denominator	26	9
Reference point	4	6
Manipulative	0	7
Whole number consistent	6	7
Incorrect numerator & denominator	4	1
Other	4	3

Fractions With Different Numerators and Denominators

The analysis suggested six strategies. The first three are valid; the last three are invalid.

Application of ratios. The child used ratios to determine the equivalence of the fractions.

- Three fifths is not equal to six eighths because "if they were equal, three goes into six, but five doesn't go into eight."
- Asked to solve $3/4 = 9/\square$, after writing 12 in the box, the child says, "Three goes into 9 three times, and 4 goes into 12 three times."

Reference point. The strategy is the same as that for fractions with the same numerators and for fractions with the same denominators.

- Three elevenths is less than eleven thirds because ". . . three elevenths is much less. Here we have eleven thirds, and eleven thirds is three and two thirds. But three elevenths isn't equal to one unit."
- Asked to compare 3/5 and 6/8, the child responds, "Three fifths, you have two fifths left. Six eighths, you have two eighths left. Two fifths is greater than two eighths, so three fifths is less than six eighths."
- Asked whether $11/3 = 3$, the child says, "Nine thirds equals three. Eleven thirds is more than nine thirds. . . ."

Manipulative. The strategy is the same as that for fractions with the same numerators and for fractions with the same denominators.

- Six eighths equals three fourths because "I started with four parts. Then I didn't have to change the size of the paper at all. I just folded it, and then I got eight."

Addition. The child compared fractions by adding to a numerator and a denominator.

- Three fourths equals seven eighths because "three plus four equals seven, and four plus four equals eight."

Incomplete proportion. The child's explanation made use of one ratio in the proportion but did not apply it correctly.

- Asked about 3/5 and 6/10, the child says, "They are not equivalent because 6 divided by 3 equals 2, 5 plus 2 equals 7, and 7 is not equal to 10."
- Asked to solve 6/4 = □/8, the child writes 3 in the box because "three into six twice; two times four equals eight."

Whole number dominance. The child's explanation suggested a strategy of making separate comparisons of the numerators and the denominators using the ordering of whole numbers.

- Three fifths is less than six tenths because "3 is less than 6, and 5 is less than 10."

Other. The child gave an uninterpretable explanation or said, "I don't know."

The distribution of explanations for the items on fractions with different numerators and denominators is shown in Table 4. The explanations are classified by strategy, site, and item type (comparison vs. missing value).

Table 4
Explanations of Responses to Items on Fractions With Different Numerators and Denominators Classified by Strategy, Site, and Item Type

	DeKalb		Minneapolis	
Strategy	Comparison	Missing value	Comparison	Missing value
Application of ratios	21	9	2	5
Reference point	13	4	18	8
Manipulative	10	0	29	4
Addition	0	1	4	1
Incomplete proportion	1	1	0	0
Whole number dominance	4	0	0	0
Other	10	4	9	2

The application-of-ratios strategy was the most common strategy at the DeKalb site for both comparison and missing-value items. The manipulative strategy was relatively more common at the Minneapolis site than at the DeKalb site; it was the most common strategy used for comparison items at the Minneapolis site. Valid strategies were much more common than invalid strategies, but a relatively large number of explanations could not be classified. When the data are pooled across sites and item types, the strategies associated with the 160 explanations are as follows, in decreasing order of frequency: reference point and manipulative (tied at 27% each), application of ratios (23%), addition (3%), whole number dominance (2%), and incomplete proportion (1%). Other strategies account for the remaining 16%.

DISCUSSION

When Fractions Have the Same Numerators

Early in the instruction (after the first 4 or 5 days), the children were given items in which unit fractions were to be compared. At that point the children were midway through the fourth grade. A large number of the explanations they gave for their responses were of the type "one third is less than one fourth because three is less than four" (the whole-number-dominance strategy—see Table 2). The denominators were sufficiently small so that they would have had direct experience in ordering the fractions using manipulative aids. Furthermore, even much younger children display considerable knowledge about the fractions 1/2, 1/3, and 1/4 (Gunderson & Gunderson, 1957). Nevertheless, the performance of the children in the sample seemed to be dominated, or overpowered, by their knowledge of the ordering of whole numbers. The results suggest that (a) children's schemas for ordering whole numbers are very strong and, at least during initial instruction in fractions, are overgeneralized; and (b) by the middle of fourth grade, or during initial instruction in fractions, children have not developed a quantitative notion of fractions that is strong enough to deal with questions of their order—even in the case of unit fractions with small denominators.

Data obtained later, however, suggest that this dominance by whole numbers diminishes in the face of instruction. In the late interviews, 89% of the children's explanations of their responses to items on fractions with the same numerators indicated that they were using a valid strategy—that is, a strategy that reflected appropriate thinking, that (except for a possible mental slip) led to a correct answer, and whose application represented understanding. However, the instruction was not universally successful. In the late interviews, two explanations (from the same child) for the ordering of unit fractions indicated use of the whole-number-dominance strategy, and four others (also from that same child) for the ordering of nonunit fractions indicated use of the same strategy. It is worth noting that the items were not of the application type. Instruction may be less successful when children are required to apply their knowledge in new situations. Work in progress suggests that even late into instruction a substantial number of children "back slide" into a whole-number-dominance strategy when confronted with problem-solving situations where they must apply their knowledge of the order and equivalence of fractions (Wachsmuth, Behr, & Post, 1983).

In the late interviews, more than three fourths of the children's explanations of their responses to items on fractions with the same numerators indicated that they were using either the numerator-and-denominator strategy or the denominator-only strategy. The use of these strategies suggests an understanding of the inverse relationship between the number of parts into which a whole is partitioned and the size of each part. This observation should be tempered by the fact that the denominator-only strategy accounted for three fifths of the explanations. Most of the children did not give overt evidence of being aware that both the numerator and the denominator must be considered when judging the order or equivalence of two fractions. This lack of awareness may not lead to errors when children are dealing consistently with problems in which the fractions have the same numerators, but it may cause difficulty when the children encounter other types of problems. When the fractions in the problems change from one class to another, the strategies for solving the problems must also change. If children are not aware of the need to consider both the numerator and the denominator of each fraction, confusion may result. Work in progress is dealing more substantially with this issue.

When Fractions Have the Same Denominators

Items on fractions with the same denominators were not included in the early interviews because the early lessons dealt mainly with order situations in which the numerators were the same (and therefore the denominators were different). The restriction to such order situations was designed to (a) facilitate (and permit investigation of) the children's acquisition of the inverse relation between the size of the parts and their number when a unit is partitioned equally, and (b) avoid directing the children's thinking alternately to same-denominator and same-numerator situations early in the instruction. In the first few lessons, many children gave evidence that they were still struggling to separate their thinking about fractions from their schemas for whole numbers.

The data indicate that almost all of the children in the sample successfully learned to order fractions with the same denominators during the course of the teaching experiment (see Table 3). Success was not universal, however; almost one sixth of the explanations indicated the use of the incorrect-numerator-and-denominator strategy or some other unspecified strategy.

The explanations given by the many children who used the numerator-and-denominator strategy suggest that they based their thinking on a mental image of their experience with manipulative aids. Even without a manipulative aid present, their thinking could refer to the compensating relation between the number of equal parts of a whole and their size. This strategy seems to reflect a desirable type of thinking. It has a concrete basis yet is free from a reference to direct action with a physical embodiment of the fraction concept. It represents thought that has advanced from embodiment dependence to embodiment independence. Such thought seems to be both generalized and abstract: *generalized,* in that the numerator-and-denominator strategy appears in response to both same-numerator and same-denominator items—thereby accommodating to some extent the principle of mathematical variability (Dienes, 1967)—and *abstract,* in that the reference to the embodiments does not depend on direct and observable physical actions on them. The abstract character of the thought can also be seen in the references to the general concept of "pieces" rather than to the more specific concept of "pieces of x," where x is some embodiment for representing fractions.

The thought represented by the manipulative strategy is more embodiment dependent, and therefore less abstract, than the thought represented by the numerator-and-denominator strategy. The children who spontaneously used a manipulative aid, or a drawing of one, to facilitate their thinking were directing their thinking toward their own actions on the aid.

The reference-point strategy was interesting because unlike the numerator-and-denominator and manipulative strategies it had not been specifically taught in the experimental lessons. The use of a third number as a reference point in ordering fractions reflects thought that is generalized and abstract. This use of a reference point is similar to the way a reference point is used in making an estimate (Trafton, 1978). There appears to be a positive relation between thinking based on a reference point and a quantitative understanding of rational numbers (Behr et al., 1983).

The thinking represented by the whole-number-consistent strategy appears to be consistent with the thinking children sometimes use in situations requiring them to order whole numbers, as, for example, when they use counting strings (Resnick, 1983). The strategy may have developed out of instructional activities that emphasized the formation of nonunit fractions by iterating the associated unit fraction (e.g., 3/8 is 1/8 and 1/8 and 1/8). For example, the thinking represented by the statement "seventeen thirtieths is smaller [than twenty thirtieths] because both are the same [thirtieths], but one has 20 and the other only 17," seems consistent with the argument that 17 one thirtieths is less than 20 one thirtieths because 17 is less than 20. In this sense, the basis of the strategy is a process similar to the ordering of whole numbers based on counting. It should again be noted that thinking strategies based on counting or enumerating are appropriate only in situations where the denominators of the fractions are the same.

Use of the invalid incorrect-numerator-and-denominator strategy clearly indicates that the child does not understand the inverse, or compensating, relation between number and size of parts when a unit is partitioned equally. The five explanations associated with this strategy (see Table 3) came from only two children, one at each site. We conclude that most of the children had a consistent understanding of the inverse relation in partitioning. Again, one should note that the interview items did not deal with applications. We are continuing to investigate the persistence of the incorrect-numerator-and-denominator strategy in some children and the effect of variation in tasks on children's regression to such primitive strategies.

When Fractions Have Different Numerators and Denominators

Items concerning the general case of fraction comparison—when both fractions have different numerators and different denominators—were not included in the early interviews for the same reason that items with the same denominators were not included. The general case of fraction comparison permits the investigation of equivalence, nonequivalence, and the emerging ratio concept. Items concerning the general case assess a knowledge of rational number concepts that Post, Behr, and Lesh (1983) argued is prerequisite to proportional reasoning in various situations. If the argument is valid, then the degree to which children employ appropriate strategies in dealing with this class of fractions might be an indicator of their readiness to deal with problems involving proportional reasoning.

The application-of-ratios strategy for tasks involving fraction equivalence represents what E. F. Karplus et al. (1974) describe as the highest level of thinking related to proportional reasoning. Therefore, children who consistently use an application-of-ratios strategy should have at least some prerequisite knowledge for understanding proportional reasoning and using it in applications. However, the data we report in Table 4 came from items that did not involve applications. R. Karplus et al. (1983a) found that the relative frequency with which various strategies were used was greatly affected by the context of the task, its numerical content, and the tasks immediately preceding it. The application-of-ratios strategy is likely to be one of several thinking strategies necessary for proportional reasoning, but it is not sufficient.

E. F. Karplus et al. (1974) describe the application-of-ratios strategy as "using all the data." In the context of fraction equivalence, that phrase suggests that reference was made to the numerator and the denominator of each fraction. Thus the application-of-ratios strategy in the general case of fraction comparison is similar to the numerator-and-denominator strategies for ordering fractions when their numerators or their denominators are the same.

The reference-point strategy when applied to items on fractions with different numerators and denominators is the same strategy as when the numerators or the denominators were the same. Although the reference-point strategy was not specifically taught to the children, it was as common as the manipulative strategy in dealing with questions of the equivalence of fractions (Table 4). The prevalence of the manipulative strategy suggests that much of the children's thought was dependent on the use of manipulatives. For some children, this manipulative dependence continued through the final interview.

The invalid addition and incomplete-proportion strategies represent thought similar to that identified by R. Karplus et al. (1980) in the context of proportional reasoning. Very few explanations were associated with these strategies, which suggests that the strength of the manipulative-based instruction was sufficient to overcome the tendency to use additivity that is apparently not uncommon among children up through the seventh grade (R. Karplus, E. Karplus, Formisano, & Paulsen, 1979).

Although only one child gave all four of the very few explanations that reflected the whole-number-dominance strategy in dealing with items on fraction equivalence (Table 4), the strategy is still important to consider. The child gave the responses late in the teaching experiment, which illustrates again how children whose knowledge of rational numbers is insecure can regress to more primitive strategies in the face of cognitive disequilibrium. The whole-number-dominance strategy is similar to the strategy used in the context of fraction addition when children erroneously write such sums as $2/4 + 3/7 = 5/11$. The presence of the number pairs 2,3 and 4,7, together with the plus sign, can activate the very powerful *primary-grade undifferentiated binary-operation frame* (Davis, 1980). This schema leads the child to perform two additions. Similarly, one can hypothesize an order schema for whole numbers that a child might use with the pair of numerators and the pair of denominators. A child who does not have a good concept of the size of a rational number and who lacks an easily retrieved strategy for ordering rational numbers may exhibit a "back sliding" phenomenon by interpreting the comparison of 3/5 and 4/7, for example, as two comparisons of whole numbers: 3 versus 4, and 5 versus 7. The observation that $3 < 4$ and $5 < 7$ leads the child to conclude that $3/5 < 4/7$.

Commonalities Across Classes of Fractions

A comparison of Tables 2, 3, and 4 indicates considerable commonality in the thinking strategies used with items on fractions from the three classes. We see four common features across the fraction classes in the strategies used:

1. Thinking that demonstrates attention to both the numerator and the denominator of each fraction (the numerator-denominator strategy for same-numerator and same-denominator items; the application-of-ratios strategy when numerators and denominators differ)
2. Thinking that depends on manipulative aids (manipulative strategy)
3. Thinking that compares the fractions in question to a third fraction or whole number as a reference (reference-point strategy)
4. Thinking influenced by one's knowledge of whole numbers (a debilitating influence when numerators are the same or when both numerators and denominators differ; a facilitating influence when denominators are the same)

IMPLICATIONS

The results indicate that with adequate instruction over an extended period of time, most children by late in the fourth grade are able to develop adequate thinking to deal with questions of the order and equivalence of fractions. This observation is restricted, however, to questions that do not involve the application of fraction knowledge to a new situation. Even after extensive instruction, the performance of a significant number of fourth graders demonstrates a substantial lack of understanding.

Understanding the order and equivalence of fractions requires an understanding of the compensatory relation between the size and number of equal parts in a partitioned unit. A small percentage of children are able to exhibit an understanding of this relation after only brief instruction. Other children grasp it after additional lessons. For still others, the relation remains elusive even after they have had ample opportunities to learn and practice. The variation in children's ability to attain this understanding has implications for instruction and is an important concern for research.

Instruction aimed at developing an understanding of the compensatory relation will require more instructional time than has been given in most curricula, in addition to a careful spiraling of the concept through several grade levels. We recommend that fractions be introduced in the third grade. The introduction should be limited to establishing elementary meanings for fractions, with a heavy emphasis on unit fractions. As the compensatory relation is being learned, its application to the problem of ordering unit fractions can begin. Such experience would provide a good foundation for establishing a quantitative concept of rational number. At the end of the third grade or at the beginning of the fourth, instruction would incorporate the concept of nonunit fractions, which would be developed through the iteration of unit fractions. The concept of order would be extended to fractions with the same numerators and then to fractions with different numerators and denominators.

Our observations suggest that children whose rational number concepts are insecure tend to have a continuing interference from their knowledge of whole numbers. This interference needs careful consideration by researchers, curriculum developers, and teachers. It would clearly be inadequate simply to inform children when the schemata they have developed for dealing with whole numbers are appropriate and when they are not; children need to learn how to make such determinations on their own.

Some children invent the strategy of using a reference point to determine the order and the equivalence of fractions. The strategy may be related to estimation skill. Further investigation is needed of children's ability to learn the reference-point strategy and how that might affect their ability to perform other tasks involving rational numbers.

Some children's thinking remains dependent on the use of manipulative aids over an extended period of instruction. Such a dependence is not necessarily inappropriate, but there is an obvious advantage for the learner whose thought is based on mental images of relations expressed through objects rather than on direct actions with the objects. A question for research is whether children's thought can be advanced from manipulative dependence to manipulative independence, and if so, how and when. In instruction, manipulative-dependent children will probably need extended experience with aids, whereas manipulative-independent children probably should not be required to use them—at least within a specific content domain. A change in the content domain may necessitate the use of manipulative aids. Manipulative independence is likely to be domain specific; the question of its transfer awaits further research.

LIMITATIONS

The data reported and the results obtained and discussed in this paper were from 132 individual interviews (11 interviews × 6 children per site × 2 sites) conducted during an 18-week teaching experiment. Many decisions about the content of the instruction, and thus of the interviews, were made on the basis of day-to-day observations of the children's responses to instruction. The teaching experiment was intended to yield rather broad insights into children's thinking about rational number concepts. As the experiment progressed, data were collected along several strands of which order and equivalence was just one. Consequently, the data on concepts of order and equivalence are more sparse than if the experiment had had a single focus. This paper reports on the thinking strategies that the children used in dealing with order and equivalence of fractions; questions relating to the transition from one strategy to another are not considered. A companion report (Post, Wachsmuth, Lesh, & Behr, in press) treats the issue of transition among strategies and deals more directly with the comments that the children made during the interviews and the observational data obtained during the lessons.

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