ORDER AND NORM CONVERGENCE IN BANACH LATTICES

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Let $(V, \leqq, \|\cdot\|)$ be a Banach lattice, and denote $V\backslash\{0\}$ by V'. For the definition of a Banach lattice and other undefined terms used below, see Vulikh [**4**]. Leader [**3**] shows that, if norm convergence is equivalent to order convergence for sequences in V, then the norm is equivalent to an M-norm. By assuming the equivalence for nets in V we can strengthen this result.

THEOREM. *Let $(V, \leqq, \|\cdot\|)$ be a Banach lattice; then the following statements are equivalent:*

(i) *Norm convergence is equivalent to order convergence, for nets in V.*
(ii) *V is finite-dimensional.*

Proof. (i) implies (ii). If $\alpha, \beta \in V'$, write $\alpha \preccurlyeq \beta$ to mean $\|\alpha\| \geqq \|\beta\|$. Then $(V', \preccurlyeq)$ is a preordered set directed to the right. Let $x_\alpha = \alpha$ for all $\alpha \in V'$; then $\|\cdot\|-\lim x_\alpha = 0$, and so 0-lim $x_\alpha = 0$. Hence $(V, \leqq)$ has a strong unit, e say. Define $\|\cdot\|_e$ by $\|x\|_e = \inf\{\lambda: |x| \leqq \lambda e\}$, for $x \in V$. By Birkhoff [**1**], $\|\cdot\|$ and $\|\cdot\|_e$ are equivalent norms. In fact $(V, \leqq, \|\cdot\|_e)$ is a Banach lattice with unity e and so an M-space, Birkhoff [**1**]. So $(V, \leqq, \|\cdot\|_e)$ is isomorphic with $(C(X), \leqq$, sup norm), X compact Hausdorff, by Kelley and Namioka [**2**].

Let $x_0 \in X$ and let g be the characteristic function for the point x_0. Define

$$F = \{f \in C(X): f \geqq 0 \quad \text{and} \quad f(x_0) = 1\};$$

then $(F, \geqq)$ is directed to the right. Let $f_\alpha = \alpha$ for all $\alpha \in F$. Then, by Urysohn's Lemma, $f_\alpha \downarrow g$ pointwise. If $g \in C(X)$, then 0-lim $f_\alpha = g$; otherwise 0-lim $f_\alpha = 0$. Now $\|\cdot\|_e-\lim f_\alpha = 0$ is impossible; so $g \in C(X)$. Hence $\{x_0\}$ is open; so X is discrete and hence finite.

(ii) implies (i). The proof of this is trivial.

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