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Order Function and Macroscopic Mutual Entrainment in Uniformly Coupled Limit-Cycle Oscillators

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A concept of *order function* is proposed to develop a general self-consistent theory of mutual entrainment in large populations of limit-cycle oscillators such that each element is uniformly coupled to every other. The onset of entrainment is revealed to be a bifurcation of the order function in functional space. Numerical evidence for the theory is also presented.

Large populations of coupled limit-cycle oscillators have been useful models in studies of collective temporal rhythmicity observed in a variety of far from equilibrium systems such as chemical reactors, engineering circuits, biological populations and diverse physiological organisms.[1] As the coupling strength becomes large enough to compensate the desynchronizing effect due to the dispersion of natural frequencies, there appears a macroscopic cluster of mutually entrained oscillators with a common frequency, thus global oscillations of the population switched on.

In order to investigate such a phenomenon, so-called phase models are most frequently used because of their simple forms, which are derived by means of averaging from underlying equations when the dispersion of natural frequencies as well as coupling is weak.[2)~4)] Among them, a class of models with uniform couplings expressed as

$$d\theta_j/dt=\Omega_j+\epsilon N^{-1}\sum_{i=1}^{N}h(\theta_i-\theta_j) \tag{1}$$

for $j=1,\cdots,N$ (the range of j will be omitted hereafter) appears to be particularly simple, tempting one to attempt analytic investigations, where θ_j is the phase variable of the jth oscillator (normalized to unity), Ω_j its natural frequency which is assumed to be distributed over the population with a density denoted hereafter by $f(\Omega)$, $\epsilon\geq 0$ the coupling strength and $h(\theta)$ the coupling function with period one. In fact, the simplest case $h(\theta)=\sin 2\pi\theta$[2] and its phase-shifted version[5] are known to allow rigorous analytic treatments based on a self-consistent equation of an order parameter (Z_1 defined below) in the thermodynamic limit, $N\to\infty$, therefore having played an invaluable role in the study of macroscopic mutual entrainment (MME). Indeed, some insights have been obtained into the nature of MME through investigations of that case, e.g., close analogies to[2),6)] and clear distinctions from[6] phase transitions in conventional cooperative systems.

It is clear, however, that one needs to explore the case of $h(\theta)$ more generally because otherwise one cannot know how universal the results obtained for the trigonometric coupling functions are. More importantly, the real nature of MME would never be clarified if one sticks to the convenient form of $h(\theta)$. It should also be pointed out that coupling functions would be more or less modulated by higher

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harmonics in any real coupled-oscillator systems.

In the following a self-consistent theory of MME will be developed for the phase model (1) in the infinite-size limit without specifying the form of $h(\theta)$, though at least a certain property needs to be assumed if one wishes to keep self-consistency up to ϵ infinitely large (see a remark below (13)). The central idea is to introduce a new concept of *order function* , $H(\theta)$, which will be shown in a self-consistent way to obey an explicit functional equation involving essentially ϵ, $f(\Omega)$ and $h(\theta)$ alone. Below the threshold of MME, where every oscillator runs with its intrinsic frequency, $H(\theta)$ is identically zero, while in the supercritical regime it attains a nontrivial shape, governing the asymptotic dynamics of the system. It turns out that the onset of MME can be viewed as a bifurcation of a nontrivial order function in functional space.

Entrainment discussed here is phase-locking with a common frequency, say, Ω_e, so that it is convenient to change variables by $\psi_j=\theta_j-\Omega_e t$. Moreover, one may expand $h(\theta)$ as

$$h(\theta)=\sum_{k=1}^{\infty}\{h_k{}^{(s)}\sin 2\pi k\theta+h_k{}^{(c)}\cos 2\pi k\theta\}\,, \tag{2}$$

where $h_0{}^{(c)}$ is set to be zero since it can be removed by redefining Ω_j. Then, it is possible to rewrite (1) as follows:

$$d\psi_j/dt=\Delta_j-\epsilon H(\psi_j,\,t)\,, \tag{3}$$

where $\Delta_j\equiv\Omega_j-\Omega_e$, and

$$H(\psi,\,t)\equiv\sum_{k=1}^{\infty}\{(h_k{}^{(s)}X_k(t)-h_k{}^{(c)}Y_k(t))\sin 2\pi k\psi-(h_k{}^{(c)}X_k(t)+h_k{}^{(s)}Y_k(t))\cos 2\pi k\psi\}\,, \tag{4}$$

in which X_k and Y_k are the real and the imaginary part of the kth complex order parameter, $Z_k\equiv N^{-1}\sum_{j=1}^{N}\exp(2\pi i' k\psi_j)$ $(i'\equiv\sqrt{-1})$, respectively. The basic hypothesis adopted here is the existence of $\lim_{t\to\infty}H(\psi,\,t)\equiv H(\psi)$ in the infinite-size system, which allows one to replace (3) by

$$d\psi_j/dt=\Delta_j-\epsilon H(\psi_j)\equiv F(\psi_j) \tag{5}$$

for t asymptotically large. The function, H, will be called the *order function* (OF) for convenience because it identically vanishes in the nonentrained regime where $\lim_{t\to\infty}Z_k(t)=0$ for all $k\geq 1$, while it begins to be nontrivially shaped above the threshold of entrainment, thus embodying the order created.

Assume now that $H(\psi)$ exhibits only one minimum and only one maximum within a unit interval as illustrated in Fig. 1. Then, the whole population is divided into two parts with qualitatively different behaviors: One is a group of mutually phase-locked oscillators satisfying $\epsilon H_{\min}<\Delta_j<\epsilon H_{\max}$ with ψ_j converging to $H^{-1}(\Delta_j/\epsilon)\equiv\psi_j^*$ for $t\to\infty$, where $\psi^{(1)}<\psi_j^*<\psi^{(2)}$ by the stability condition, $F'(\psi_j^*)<0$ $(F'(\psi)\equiv dF/d\psi)$. The other is the collection of remaining oscillators whose ψ_j periodically rotates on the unit circle with the period $T_j=s_jC(\Delta_j)^{-1}$ as well as the invariant probability density $C(\Delta_j)(\Delta_j-\epsilon H(\psi))^{-1}$, where

$$C(\Delta)\equiv 1\Big/\int_0^1 d\psi/\{\Delta-\epsilon H(\psi)\} \tag{6}$$

and $s_j=1$ if $\Delta_j>\epsilon H_{\max}$, $=-1$ otherwise. Then, it follows that $P(\psi)$, the distribution function of ψ_j over the population, is expressed as

$$P(\psi)=P_e(\psi)+P_{ne}(\psi), \quad (7)$$

where P_e and P_{ne} respectively are contributions from the subpopulations of entrained and nonentrained oscillators:

$$P_e(\psi)=\epsilon\tilde{f}(\epsilon H(\psi))H'(\psi) \quad (\psi^{(1)}<\psi<\psi^{(2)}),$$
$$=0 \text{ (otherwise)},$$
$$P_{ne}(\psi)=\int_{\Delta>\epsilon H_{\max},\ \Delta<\epsilon H_{\min}} \times d\Delta\tilde{f}(\Delta)\frac{C(\Delta)}{\Delta-\epsilon H(\psi)},$$

where $\tilde{f}(\Delta)\equiv f(\Omega_e+\Delta)$. The asymptotic value of Z_k for $t\to\infty$ is then expressed by

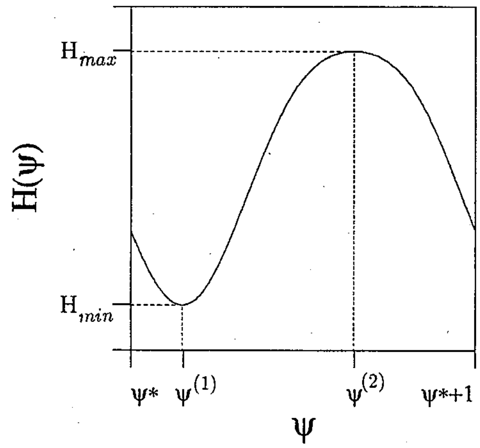

Fig. 1. Schematic form assumed for H.

$$Z_k=\int_0^1 d\psi e^{2\pi i' k\psi}P(\psi). \quad (8)$$

Combining (8), (7), (4) (for $t\to\infty$) and (2), one is able to obtain the following equation of the OF :

$$H(\theta)=-\epsilon\int_{\psi^{(1)}}^{\psi^{(2)}} d\psi\tilde{f}(\epsilon H(\psi))H'(\psi)h(\psi-\theta)-\int_{\Delta>\epsilon H_{\max},\Delta<\epsilon H_{\min}} d\Delta\tilde{f}(\Delta)C(\Delta)\int_0^1 d\psi\frac{h(\psi-\theta)}{\Delta-\epsilon H(\psi)}$$
$$=-\int_0^1 d\psi P(\psi)h(\psi-\theta). \quad (9)$$

For any ϵ, this nonlinear functional equation has the trivial solution, $H(\theta)\equiv 0$, which corresponds to the disordered state where no entrainment occurs. The onset of MME is nothing but the bifurcation of a nontrivial solution from it. In general such a solution may be found together with the threshold value of ϵ, ϵ_c, and the frequency of entrainment, Ω_e, by numerically solving (9), e.g., through its conversion to simultaneous equations for Fourier coefficients of $H(\theta)$, where it is useful to note that by definition, only the harmonics contained in $h(\theta)$ are relevant. Recall that at the beginning, $H(\theta)$ was assumed to be of the form illustrated in Fig. 1. Therefore, consistency requires that solutions to (9) shaped otherwise, if any, be discarded.

Since the OF governs the asymptotic dynamics of the population, all the information of the asymptotic state ought to be extracted from it. For example, the ratio of entrained oscillators, R, is evaluated with

$$R=\int_{\epsilon H_{\min}}^{\epsilon H_{\max}} d\Delta\tilde{f}(\Delta), \quad (10)$$

and the distribution of average frequencies, $\omega_j\equiv\lim_{t\to\infty}\{\theta_j(t)-\theta_j(0)\}/t$, is described by

$$g(\omega)=R\delta(\omega-\Omega_e)+\{\tilde{f}(C^{-1}(\omega-\Omega_e))\}/\{C'(C^{-1}(\omega-\Omega_e))\}, \quad (11)$$

where $\delta(\omega)$ is the Dirac function. Z_k as well as $P(\psi)$ also falls within one's reach by (7) and (8) once the OF is known.

In previous studies of the model with $h(\theta)=\sin 2\pi\theta$,[2),5),6)] Z_1 has been the most popular as an order parameter of MME mainly because it obeys a self-consistent equation, thus easily analyzable. The present theory suggests, however, that the most appropriate one is any norm of the OF, e.g.,

$$\|H\|\equiv\sqrt{\int_0^1 d\theta H(\theta)^2}\,, \tag{12}$$

which *never* loses effect by definition of H, while no reason seems to exist denying the possibility that Z_1 accidentally vanishes despite the presence of order.

A couple of remarks are now in order: (i) If $H(\theta)$ is a solution of (9), so is $H(\theta+c)$ for arbitrary constant c because of the translational invariance of (1). (ii) For $\epsilon\to\infty$, all the oscillators get entrained with a common value of ψ_j^*, implying that by (9)

$$\lim_{\epsilon\to\infty} H(\theta)=-h(-\theta) \tag{13}$$

apart from the arbitrary phase constant. By this result, it follows that $h(\theta)$ itself needs to be subject to the condition of Fig. 1 in order for theory to maintain consistency up to ϵ arbitrarily large.

For brevity, suppose that $h_k^{(c)}=h_{2k}^{(s)}=0$ for all $k\geq 1$ as well as $lh_l^{(s)}\neq l'h_{l'}^{(s)}$ for any different odd l and l' (the superscript, "$^{(s)}$", will be left out below), and also that $f(\Omega)$ is symmetric, meaning that $f(\Omega^*+\Delta)=f(\Omega^*-\Delta)$ for a certain Ω^*. Noting $h(-\theta)=-h(\theta)$ as well as the symmetry of f, one may put $\Omega_e=\Omega^*$. It is also derived from the definition of H that $\psi^{(2)}=-\psi^{(1)}=1/4$ and $H_{\min}=-H_{\max}$. Moreover, $C(\Delta)$ is easily proved to be odd. One can then show that the second term on the r.h.s. of (9) vanishes, obtaining

$$H(\theta)=\epsilon\int_{-1/4}^{1/4} d\psi \tilde{f}(\epsilon H(\psi))H'(\psi)h(\psi+\theta)\,. \tag{14}$$

The dominant part of H in the limit $\epsilon\to\epsilon_c+0$, $\hat{H}$, is now found to satisfy

$$\lambda\hat{H}(\theta)=\int_{-1/4}^{1/4} d\psi\hat{H}'(\psi)h(\psi+\theta)\equiv\mathcal{L}\{\hat{H}\}(\theta) \tag{15}$$

with $\lambda=\{\epsilon_c f(\Omega^*)\}^{-1}$. This linear eigenvalue problem is readily solved in relevant functional space: The spectrum of $\mathcal{L}$ and associated eigenfunctions are $\lambda_k=(\pi/2)(2k-1)h_{2k-1}$ and $\hat{H}_k(\theta)=\sin\{2\pi(2k-1)\theta\}$ for $k=1, 2, \cdots$. One then chooses λ_1 as λ because otherwise broken is the premise posed by Fig. 1, thus finding

$$\epsilon_c=2/\{\pi f(\Omega^*)h_1\}\,, \tag{16}$$

which is consistent with a previous result derived for the case $h(\theta)=\sin 2\pi\theta$.[2] Remarkably, ϵ_c *does not depend on higher harmonics of* $h(\theta)$.

It is now easy to develop a bifurcation theory of the OF, which is now outlined for the particular class of h. First, it is shown that the bifurcation is supercritical as long as $f''(\Omega^*)<0$ and $h_1>0$. Under these conditions, the standard technique of bifurcation theory[7] yields from (14)

$$H(\theta)=\delta A\sin 2\pi\theta+\delta^3(B\sin 2\pi\theta+C\sin 6\pi\theta)+\cdots\,, \tag{17}$$

where $\delta\equiv\sqrt{\epsilon-\epsilon_c}$ and

$$A=\{(\pi h_1)^{3/2}f(\Omega^*)^2\}/\sqrt{-f''(\Omega^*)}\,,$$

$$B=\frac{1}{2}C+\frac{(\pi h_1)^{5/2}f(\Omega^*)^3}{4(-f''(\Omega^*))^{3/2}}\left\{3f''(\Omega^*)-\frac{1}{3}\frac{f(\Omega^*)f^{(4)}(\Omega^*)}{f''(\Omega^*)}\right\},$$

$$C=\{(\pi h_1)^{5/2}f(\Omega^*)^3 h_3\}/\{2(h_1-3h_3)\sqrt{-f''(\Omega^*)}\}\,.$$

It is important to note that $H(\theta)$ obtained this way certainly meets the condition of Fig. 1 for δ small, guaranteeing self-consistency of the theory near the onset of MME.

Figure 2 displays a piecewise-linear coupling function, which can be expressed as

$$h(\theta)=\frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k-1)^2}\sin\{2\pi(2k-1)\theta\}\,,$$

hence (14)~(17) being applicable. Numerical simulations have been carried out by the Euler scheme with the time increment of $\Delta t=0.01$ for $N=100$ and for $\tilde{f}(\Delta)=(\gamma/\pi)\times(\Delta^2+\gamma^2)^{-1}$ with $\gamma=0.4$. The initial condition was set up by a uniform random number generator of the interval (0, 1). Figures 3 and 4 demonstrate excellent agreement between simulations and theory except the vicinity of the threshold where finite-size effects come out inevitably large.[6] All the theoretical results were obtained by solving (14). Note that $H(\theta)$ displayed in Fig. 4 indeed keeps the form of Fig. 1, assuring self-consistency. It is also evident in the same figure that the OF tends to approach $-h(-\theta)=h(\theta)$ (depicted in Fig. 2) for increasing ϵ, in accordance with (13). (See captions for details.)

As will be fully discussed elsewhere, it is possible to relax the constraints on $h(\theta)$ and $f(\Omega)$ introduced above for brevity. By assuming that $h(\theta)$ contains only odd harmonics in (2), hence not necessarily symmetric with respect to the origin, one can show for example that ϵ_c and $\Omega_c\equiv\lim_{\epsilon\to\epsilon_c}\Omega_e$ are determined from

$$\epsilon_c=2h_1^{(s)}/[\pi f(\Omega_c)\{(h_1^{(s)})^2+(h_1^{(c)})^2\}]\,, \tag{18}$$

$$f(\Omega_c)^{-1}\int_0^{\infty}d\Delta\{f(\Omega_c+\Delta)-f(\Omega_c-\Delta)\}/\Delta=-\pi h_1^{(c)}/h_1^{(s)}\,, \tag{19}$$

where no symmetry is assumed for $f(\Omega)$. Again, ϵ_c *is not affected by higher harmonics of* $h(\theta)$. These formulae are consistent with a previous result derived for the special case $h(\theta)=\sin 2\pi(\theta+\alpha)$.[5]

In summary, by introducing the order function, a way has been opened to a general theory of macroscopic mutual entrainment in uniformly coupled limit-cycle oscillators as modeled by (1). The functional equation of the OF, (9), has been derived in a self-consistent way, which enables one not only to investigate the properties of the asymptotic state of the system with its, in general, numerical solutions, but also to develop analytic theories to describe the onset of MME for some types of $h(\theta)$. The present approach has provided a new picture for the onset of MME in uniformly coupled oscillators: It is a bifurcation of the OF in *functional* space. In the light of the general theory, the case $h(\theta)=\sin 2\pi\theta$ or its relative, to which previous studies have all been restricted, turns out to be

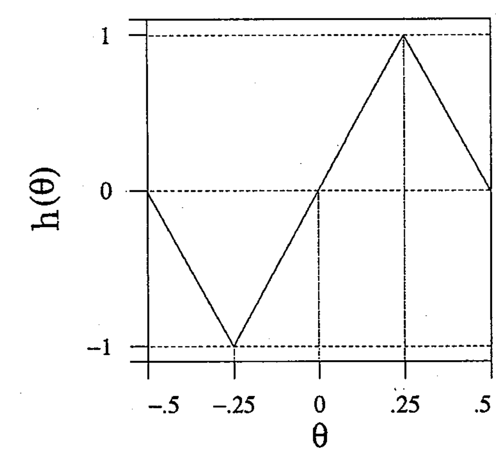

Fig. 2. Piecewise-linear coupling function.

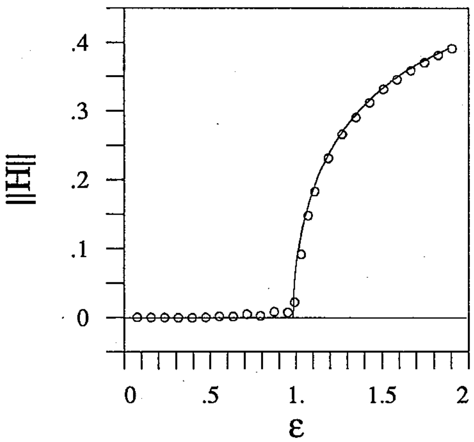

Fig. 3. Norm of the OF, (12), for $h(\theta)$ displayed in Fig. 2. Open circles represent data obtained by simulations over $60 \leq t \leq 240$ with the earlier period neglected. The theoretical line is based on numerical solutions of (14) converted to equations of the first three harmonics (others were found negligibly small in the regime shown), except a close neighborhood of ϵ_c(= 0.98696⋯by (16)): $\epsilon_c < \epsilon < 1.014$, where (17) is invoked because of slowing-down in convergence of the solutions.

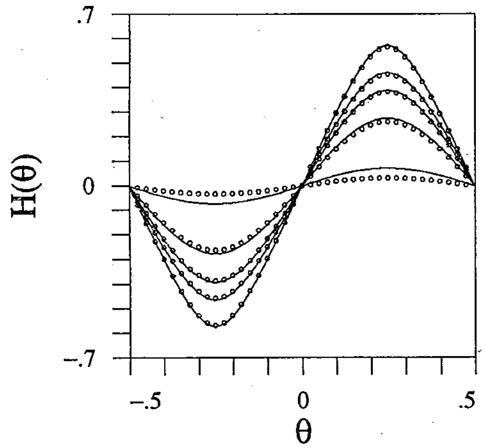

Fig. 4. Growth of the OF for ϵ=0.995, 1.114, 1.273, 1.432, 1.830 with the same details as in Fig. 3.

such a special case that the OF rigidly remains trigonometric for all ϵ, which is the reason why just an equation for the single complex number is sufficient for that case.[2),5)] Extensions of the theory are now planned to such cases that the r.h.s of (1) is accompanied by some additional terms, e.g., $-a \sin 2\pi\theta_j$ and noise. Finally, two remaining subjects should be mentioned: One is to find the necessary and sufficient condition for self-consistency of the theory to be kept for *all* ϵ. Speculation sounds plausible that it would be $h(\theta)$'s being in the shape of Fig. 1, but can one raise this up to a theorem? The other is to rigorously establish the stability of the OF via the analysis of the time-dependent OF, $H(\theta, t)$.

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