Order-Isomorphisms in Affine Spaces (*) (**).

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Summary. – *See Introduction.*

1. – Introduction.

We are currently investigating natural metric structures possessed by certain cones, for instance the cone of all positive-definite symmetric bilinear forms on a vector space. This cone plays a part in the kinematics of continuous media, and its metric structure can serve to define physical material properties involving fading memory. The natural metric structure of such cones is closely related to their natural order structure, and we found that we first had to understand the order-isomorphisms before getting a grasp on the isometries. The results of the investigation of the metric structures will be published later.

Our work thus led us to consider order-isomorphisms between subsets of directed affine spaces. Great interest clearly attaches to conditions that ensure that such an order-isomorphism is actually a restriction of an affine isomorphism between the whole spaces. The earliest significant advance in this direction appears to be the result that in the Minkowskian space-time of Special Relativity the automorphisms of the causal-order structure are necessarily automorphisms of the affine structure (and their gradients are positive scalar multiples of orthochronous Lorentz transformations). This result is due to ALEKSANDROV and OVČINNIKOVA [2] and, apparently independently, to ZEEMAN [5].

The ALEKSANDROV-OVČINNIKOVA-ZEEMAN result has been generalized in many directions. Of the work relevant to our present purpose, we should mention the result of ALEKSANDROV [1]: In a directed finite-dimensional affine space, every order-automorphism of the whole space is an affine automorphism, provided that the closure of the direction-cone has no extreme ray that is not included in the linear span of the union of all the others. (We use the terminology of the present paper; we describe the last-mentioned condition by saying « all extreme rays are engaged ».) ROTHAUS [4] obtained a similar result for order-automorphisms of the interior of the direction cone and also for the whole space, provided the space is finite-dimensional and the

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closure of the direction-cone has no isolated extreme rays (a condition unnecessarily stronger than « engagedness »).

One of our results (Corollary A1) is a generalization of Aleksandrov's result to order-isomorphisms between suitable, possibly proper, subsets of directed affine spaces (possibly distinct, not necessarily finite-dimensional); the « engagedness » condition plays a central part. We are, however, also interested in the structure of order-isomorphisms when the « engagedness » condition fails to hold; Corollary B1 and Proposition 4 give a complete description of this structure; such order-isomorphisms are in general not affine.

We are actually able to treat a problem that is somewhat more general than the one about order-isomorphisms, by considering injective mappings that map half-lines with directions in a certain set $\mathfrak{R}$ to half-lines with directions in a set $\mathfrak{R}'$ that contains no three complanar rays. Such mappings must be affine if $\mathfrak{R}$ satisfies the « engagedness » condition (Theorem A), and have a well-defined structure in the general case (Theorem B). We have now found that this approach to the determination of conditions that make all order-isomorphisms affine resembles the method used in the remarkable paper by BORCHERS and HEGERFELDT [3]. It differs in that (a) our consideration of half-lines eliminates the field-theoretic complications encountered in [3], which deals with whole lines; (b) our imposition of a restriction on $\mathfrak{R}'$ considerably weakens the requirements on $\mathfrak{R}$; (c) finite-dimensionality is not required, and the domain of the mapping need not be the whole space.

After fixing our notation and terminology in Sections 2 and 3, we give a precise statement of our problem and some of our results in Section 4. Section 5 is devoted to a geometric proposition that is the key to the rest of the paper. Section 6 contains the proof of Theorem A and some of the preliminaries for the study of the general case. The general case is dealt with in Section 7 and 8.

We are indebted to a referee for calling some of the references to our attention.

2. – Notation and terminology.

We use $:=$ to indicate an equality in which the left-hand side is defined by the right-hand side. $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{P}$ the set of non-negative real numbers.

If $\varphi\colon D \to D'$ is a mapping and X a subset of D, we write $\varphi_>(X) := \{\varphi(x)|x \in X\}$ for the image of X under φ.

Let V be a (real) vector space. We use such notations as $-R := \{-u|u \in R\}$, $\mathbb{P}R := \{tu|u \in R, t \in \mathbb{P}\}$, $R - S := \{u - v|u \in R, v \in S\}$, when R and S are subsets of V. If $\mathfrak{R}$ is a collection of subsets of V, each containing $0 \in V$, we write $\sum_{R\in\mathfrak{R}} R := \left\{\sum_{R\in\mathcal{F}} u_R|\mathcal{F} \text{ is a finite subset of } \mathfrak{R} \text{ and } u_R \in R \text{ for all } R \in \mathcal{F}\right\}$, and call this the *sum of the sets in* $\mathfrak{R}$. The (*linear*) *span* of a subset R of V, i.e., the smallest subspace of V including R, is denoted by $\operatorname{Sp} R$. A subset of V of the form $\mathbb{P}u$ for some $u \neq 0$ is a *ray*; if R is a ray, then $R = \mathbb{P}u$ for each $u \in R\setminus\{0\}$.

An *affine space* E is a non-empty set (also called E) endowed with structure by the prescription of a (real) vector space V and an injective and transitive action of the additive group of V on E. The vector space V is the *translation space of* E. If $v \in V$ and $x \in E$, we write $x + v := v(x)$. If $x, y \in E$, we write $y - x$ for the unique element of V whose value at x is y, so that $x + (y - x) = y$. The *dimension of* E is defined by $\dim E := \dim V$.

An orbit in E under the action of a subspace U of V is called a *flat*: it is an affine space with translation space U. A one-dimensional flat is a *line* and a two-dimensional flat is a *plane*. The *affine span* of a non-empty subset X of E is the smallest flat that includes X. If $x \in E$ and R is a ray in V, then $x + R$ is called the *half-line* with *apex* x and *direction* R; its affine span is a line whose translation space is $\mathrm{Sp}\, R = R - R$.

Every vector space V has the natural structure of an affine space: this structure is obtained by letting V be its own translation space and by letting $v \mapsto (u \mapsto u + v)$ be the action of V on itself. Thus every concept and result concerning affine spaces applies, in particular, to vector spaces.

Let E and E' be affine spaces with translation spaces V and V', respectively. A mapping $\alpha: E \to E'$ is *affine* if there is a linear mapping $\lambda: V \to V'$ such that

$$\alpha(x + v) - \alpha(x) = \lambda(v) \qquad \text{for all } x \in X,\ v \in V.$$

The linear mapping λ, uniquely determined by α, is the *gradient of* α.

An *order* $\lhd$ on a set E is a reflexive, antisymmetric, and transitive relation on E. We read « $x \lhd y$ » as « *y follows x* ». We say that an order $\lhd$ is *total on* a subset L of E if, for all $x, y \in L$, either $x \lhd y$ or $y \lhd x$. Given any $x \in E$ we call $\{y \in E | x \lhd y\}$ the *follower-set of* x, and $\{y \in E | x \lhd y \text{ and } x \neq y\}$ the *strict-follower-set of* x. The *follower-set of* a subset D of E is the union of the follower-sets of all the elements of D. We say that D is *follower-saturated* if it includes (and hence coincides with) its own follower-set. Given $x, y \in E$ such that $x \lhd y$, the *order-interval from x to y* is defined to be the set $\{z \in E | x \lhd z \lhd y\}$. We say that the order $\lhd$ on E is *directing* if for all $x, y \in E$ there is $z \in E$ such that $x \lhd z$ and $y \lhd z$.

If D and D' are sets provided with orders $\lhd$ and $\lhd'$, respectively, a mapping $\varphi: D \to D'$ is an *order-isomorphism* if it is invertible and both it and its inverse are isotone; i.e., if φ is bijective and

$$x \lhd y \Leftrightarrow \varphi(x) \lhd' \varphi(y) \qquad \text{for all } x, y \in D.$$

3. – Ordered affine spaces.

Let E be an affine space with translation space V. We say that an order $\lhd$ on E is *translation-invariant* if

$$x \lhd y \Rightarrow x + v \lhd y + v \qquad \text{for all } x, y \in E,\ v \in V,$$

and that it is *connected* if, for all $x, y \in E$ with $x \lhd y$, the order-interval from x to y includes the line segment from x to y. If $\lhd$ is a translation-invariant and connected order, then $K := \{y - x | x, y \in E, x \lhd y\}$ satisfies (i): $K + K = K$, (ii): $\mathbb{P}K = K$, (iii): $K \cap (-K) = \{0\}$. Conversely, if a subset K of V satisfies (i), (ii), (iii), then

$$x \lhd y :\Leftrightarrow y - x \in K \qquad \text{for all } x, y \in E$$

defines a translation-invariant and connected order $\lhd$ on E. The subset K is called the *direction-cone of* $\lhd$. The follower-set of a subset D of E is given by $D + K$; thus D is follower-saturated if and only if $D + K = D$.

We say that E is an *ordered affine space* if it is endowed with additional structure by the prescription of a translation-invariant and connected order $\lhd$, or of its direction-cone K. We now assume that E is such a space.

We say that a ray $R \subset K$ is an *extreme ray* (*of* K) if $R \subset S + T$ is possible for rays $S, T \subset K$ only if $S = R$ or $T = R$. We say that a half-line in E is an *extreme half-line* if its direction is an extreme ray. The extreme half-lines have a purely order-theoretic characterization, as follows.

PROPOSITION 1. – *Assume that* $K \neq \{0\}$. *A subset of* E *is an extreme half-line with apex* $x \in E$ *if and only if it is maximal among the subsets* H *of the follower-set of* x *such that* $x \in H$ *and such that the order* $\lhd$ *is total on the order-interval from* x *to* y *for every* $y \in H$.

The proof is straightforward and is left to the reader. Observe that the order-interval from x to y is $(x + K) \cap (y - K)$.

We say that the order $\lhd$ of E is *closed* if, for all $x, y \in E$, the strict-follower-set of x includes the strict-follower-set of y only when $x \lhd y$; i.e., if

$$x \lhd y \Leftrightarrow y + (K \setminus \{0\}) \subset x + (K \setminus \{0\}) .$$

If E is finite-dimensional, it is easily seen that the order $\lhd$ is closed if and only if the direction-cone K is a closed subset of V.

The following result is an easy consequence of the well-known fact that every compact convex set in a finite-dimensional affine space is the convex hull of the set of its extreme points.

PROPOSITION 2. – *If* E *is finite-dimensional, then the order* $\lhd$ *is closed if and only if the direction-cone* K *is the sum of its extreme rays.*

REMARK 1. – If E is not finite-dimensional, it is still true that $\lhd$ is closed if K is the sum of its extreme rays; but $\lhd$ may be closed even when K has no extreme rays at all.

Let E be an affine space with a translation-invariant and connected order $<$ that is not necessarily closed, and let J be the direction-cone of $<$. We can then define a reflexive and transitive relation $\lhd$ on E by

$$x \lhd y :\Leftrightarrow y + (J \setminus \{0\}) \subset x + (J \setminus \{0\}) \quad \text{for all } x, y \in E.$$

In general, $\lhd$ is not antisymmetric; if it is, we say that the order $\prec$, and its direction-cone J, are *genuine*, and we call the order $\lhd$ the *closure of* $\prec$. Assume that $\prec$ is genuine; then its closure $\lhd$ is translation-invariant and connected, and the direction-cone of $\lhd$ is $K := \{u \in V | u + (J \setminus \{0\}) \subset J\}$; it is easily seen that the closure $\lhd$ is a closed order. If $\prec$ is itself closed, then it is necessarily genuine and equal to its own closure. If E is finite-dimensional, then K is the (topological) closure of J. It is not hard to see that the translation-invariant and connected order $\prec$ is genuine if and only if its direction-cone J includes no line; this is the case if and only if no follower-set $x + J$ of any point $x \in E$ includes a line.

We say that the ordered affine space E is a *directed affine space* if its order is directing. This is the case if and only if the direction-cone K spans the whole translation space V:

$$V = \operatorname{Sp} K = K - K\,. \tag{3.1}$$

REMARK 2. – It is clear that, if the order of E is not directing, E is partitioned into flats with translation space Sp K, and that the restriction of the order to each of these flats is directing, while elements in different flats of this partition are unrelated by the order.

REMARK 3. – The structure of space-time in the theory of Special Relativity is a four-dimensional directed affine space whose direction-cone K has the property that $K \cup (-K)$ is the set on which a certain non-degenerate quadratic form with Sylvester index 1 has its non-negative values. The order $\lhd$ of E is closed: it is usually called the *causal order* of space-time. The causal order is the closure of another genuine directing order $\prec$, usually called the *chronological order* of space-time. If J is the direction-cone of $\prec$, then $J \setminus \{0\}$ is the interior of K; it is the set of all vectors in K at which the quadratic form has a positive value. The extreme half-lines of E with respect to the causal order have a physical interpretation as *light rays*.

4. – Statement of the problem.

We assume that the following are given: directed affine spaces E and E', with respective translation spaces V and V'; and follower-saturated subsets D and D' of E and E', respectively. The problem we consider is: *What is the structure of an order-isomorphism* $\varphi: D \to D'$? In particular, *Must* φ *be the restriction of an affine mapping*? We are able to give a complete answer to this problem when the direction-cone K of E is the sum of its extreme rays.

Let $\mathfrak{R}$ and $\mathfrak{R}'$ be the sets of extreme rays of the direction-cones K and K' of E and E', respectively, and assume that $K = \sum_{R\in\mathfrak{R}} R$. Then the following conditions are satisfied (use (3.1) for the second one):

(R1): *If* $R \in \mathfrak{R}$, *then* $-R \notin \mathfrak{R}$;

(R2): $V = \mathrm{Sp} \cup \mathfrak{R} = \sum_{R\in\mathfrak{R}} (R - R)$;

(R3): *If* R', S', $T' \in \mathfrak{R}'$ *are distinct, then*

$$\dim \mathrm{Sp}\,(R' \cup S' \cup T') = 3\,;$$

(D): *For all* $R \in \mathfrak{R}$, $D + R = D$; *and for all* $R' \in \mathfrak{R}'$, $D' + R' = D'$.

If $\varphi\colon D \to D'$ is an order-isomorphism, it follows from Proposition 1 that φ has the following property:

(H): *For all* $x \in D$ *and* $R \in \mathfrak{R}$, *we have*

$$\varphi_{>}(x + R) - \varphi(x) \in \mathfrak{R}'\,;$$

so that the image under φ *of every half-line in* D *with direction in* $\mathfrak{R}$ *is a half-line in* D' *with direction in* $\mathfrak{R}'$.

It turns out that for the solution of our problem only the conditions (R1), (R2), (R3), (D) are significant, and that it is essentially sufficient to consider arbitrary injections $\varphi\colon D \to D'$ that satisfy (H).

The following concept will be crucial for the statement of our results. Let $\mathcal{S}$ be a collection of rays in a vector space. A ray $R \in \mathcal{S}$ is *engaged in* $\mathcal{S}$ if $R \subset \mathrm{Sp} \cup (\mathcal{S} \setminus \{R\})$. If $R \in \mathcal{S}$ is not engaged in $\mathcal{S}$, then R is *disengaged in* $\mathcal{S}$.

THEOREM A. – *Let* E *and* E' *be affine spaces with repective translation spaces* V *and* V' *and let* $\mathfrak{R}$ *and* $\mathfrak{R}'$ *be collections of rays in* V *and* V', *respectively. Assume that* $\mathfrak{R}$ *and* $\mathfrak{R}'$ *satisfy the conditions* (R1), (R2), (R3), *and that* D *and* D' *are non-empty subsets of* E *and* E', *respectively, that satisfy* (D). *Assume, moreover, that every ray of* $\mathfrak{R}$ *is engaged in* $\mathfrak{R}$. *Then every injection* $\varphi\colon D \to D'$ *with the property* (H) *is the restriction of an affine mapping from* E *to* E'.

REMARK 4. – This result can be cast as a necessary and (trivially) sufficient condition by replacing the last sentence of the statement by: *Then a mapping* $\varphi\colon D \to D'$ *is an injection with the property* (H) *if and only if it is the restriction of an injective affine mapping from* E *to* E' *whose gradient* λ *satisfies* $\lambda_{>>}(\mathfrak{R}) \subset \mathfrak{R}'$.

COROLLARY A1. – *Let* E *and* E' *be directed affine spaces, and let* D *and* D' *be non-empty follower-saturated subsets of* E *and* E', *respectively. Assume that the direction-cone of* E *is the sum of its extreme rays, and that each such extreme ray is engaged in the set of all. Then every order-isomorphism* $\varphi\colon D \to D'$ *is the restriction of an affine mapping from* E *to* E'.

Suppose for a moment that E is finite-dimensional; by Proposition 2, the direction-cone of E is the sum of its extreme rays if (and only if) the order is closed. We can obtain the conclusion of Corollary A1 with this assumption weakened, at the cost of a minor restriction on D and D' as follows.

COROLLARY A2. – *Let E and E' be finite-dimensional directed affine spaces with genuine orders, and let D and D' be non-empty follower-saturated subsets of E and E' respectively such that each is either open or closed. Assume that each extreme ray of the direction-cone of the closure of the order of E is engaged in the set of all such extreme rays. Then every order-isomorphism $\varphi: D \to D'$ is the restriction of an affine mapping from E to E'.*

PROOF. – Consider the closures of the orders of E and E'; they also direct E and E', respectively, and by Proposition 2 the direction-cone of the closure of the order of E is the sum of its extreme rays. The assumption on D and D' ensures that each is follower-saturated with respect to the corresponding closure order. Since the closure of an order is defined in purely order-theoretic terms, a mapping $\varphi: D \to D'$ that is an order-isomorphism with respect to the original orders is also an order-isomorphism with respect to the closure orders. The conclusion then follows from Corollary A1 applied to the closure orders. □

The general case, in which not all rays of $\mathfrak{R}$ are assumed to be engaged in $\mathfrak{R}$, is much more complicated, and will be analysed in Sections 7 and 8 (Theorem B and its corollaries). One interesting consequence of that analysis deserves mention here.

COROLLARY B2. – *Let E and E' be directed affine spaces, and assume that the direction-cone of E is the sum of its extreme rays. Then there exist non-empty follower-saturated subsets D and D' of E and E', respectively, and an order-isomorphism from D to D' if and only if there exists an affine order-isomorphism from E to E'.*

REMARK 5. – Corollary B2 implies that the complete description of all order-isomorphisms $\varphi: D \to D'$ for given directed affine spaces E and E' reduces to a determination of whether there exists an affine order-isomorphism from E to E' and a complete description of all order-isomorphisms between two follower-saturated subsets of E.

We shall see in Section 8 (Proposition 4) that unless all extreme rays are engaged there are D and D' as above and order-isomorphisms $\varphi: D \to D'$ that are *not* the restrictions of affine mappings.

REMARK 6. – Assume that the direction-cone K of the directed affine space E is the sum of its extreme rays. To say that the extreme ray R is disengaged in the set $\mathfrak{R}$ of all extreme rays means, geometrically, that K is the sum, hence the convex hull, of R and a cone in a supplementary subspace of codimension 1. This may be judged to be a rather exceptional situation, so that Corollary A1 describes a kind of general case; in particular, Corollary A1 certainly applies whenever $\dim E > 2$ and K is rotund or smooth, and this includes the case of space-time of Special Relativity with its causal order. Among the exceptional situations is the one in which E is finite-dimensional and K is a closed cone with a simplicial cross-section: in this case, every ray of $\mathfrak{R}$ is disengaged in $\mathfrak{R}$, and the order of E is a lattice-order. In particular, if $\dim E = 2$ we must have this special situation, and Corollary A1 does not apply; if $\dim E = 3$, then Corollary A1 fails to apply *only* in this special case, when $\mathfrak{R}$ has exactly three members.

REMARK 7. – Let E be a directed affine space whose direction-cone is the sum of its extreme rays. If each extreme ray is engaged in the set of all, Corollary A1 implies that every order-automorphism of E is in fact an affine automorphism; this means that the affine structure of E is completely determined by its order structure. We shall see in Section 8 (Proposition 4) that this conclusion does not hold if there are disengaged extreme rays.

5. – A result about three half-lines.

The following result is valid in an arbitrary affine space.

PROPOSITION 3. – *Let three pairwise disjoint half-lines be given such that every point of each lies on a line that meets the other two. Then all three half-lines are parallel to one plane. Moreover, if two of them lie in one plane, the third also lies in that plane.*

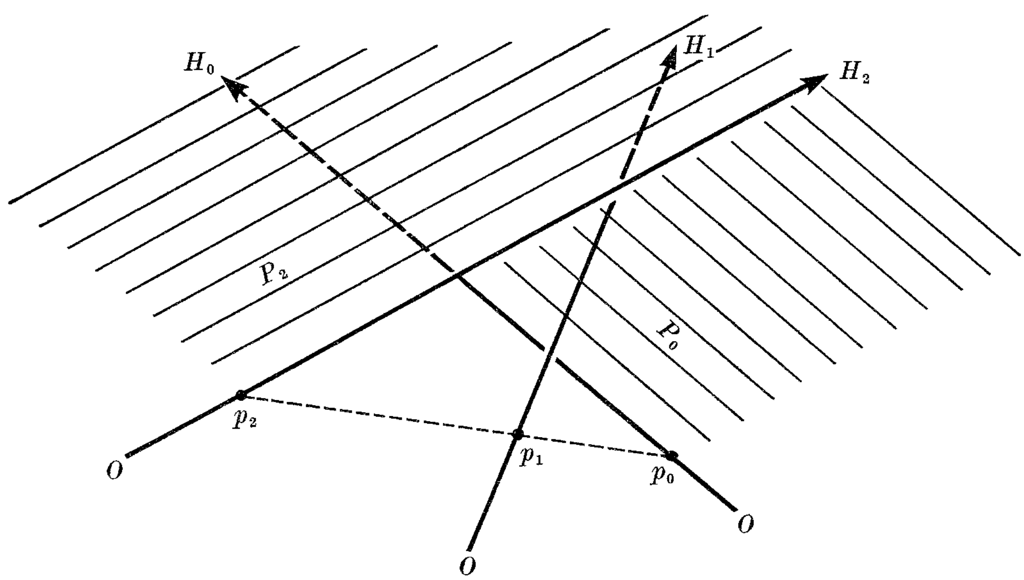

Figure 1

PROOF. – The assumption ensures that each of the half-lines lies in the affine span of the union of the other two. It follows that if two of the half-lines lie in one plane the third also lies in that plane; and that otherwise the dimension of the affine span of the union of any two, and hence of all three, half-lines is 3. We assume this latter alternative in the rest of the proof, and we let F be the three-dimensional affine span of the union of the half-lines.

By the assumption, we may and do choose a line that meets the three half-lines H_0, H_1, H_2 at points p_0, p_1, p_2, respectively (see Figure 1). We may and do select our numbering so that p_1 lies between p_0 and p_2. Let P_0 be the plane through H_0 parallel to H_2, and let P_2 be the plane through H_2 parallel to H_0. Then P_0 and P_2 are distinct parallel planes, and a line meeting both H_0 and H_2 cannot lie in P_0 or in P_2. It therefore follows from the assumption that H_1 cannot meet P_0 or P_2: for if H_1 met P_0 at x, say, then the line through x that meets H_0 and H_2 would lie in P_0.

Now H_1 contains the point p_1, which lies in F between the planes P_0 and P_2. Since H_1 does not meet P_0 or P_2, it must lie in the «strip» of F between P_0 and P_2. It is evident that this can happen only if H_1 is parallel to P_0 and to P_2; and H_0 and H_2 are parallel to these planes by construction. □

6. – Proof of Theorem A.

We assume in this section that E and E' are affine spaces with respective translation spaces V and V', and that $\mathfrak{R}$ and $\mathfrak{R}'$ are collections of rays in V and V', respectively. We assume that $\mathfrak{R}$ and $\mathfrak{R}'$ satisfy (R1), (R2), (R3), and that D and D' are non-empty subsets of E and E', respectively, that satisfy (D). (These are the assumptions of Theorem A except for the «engagedness» condition.)

Let $\varphi: D \to D'$ be a given injection with Property (H). We prove several lemmas.

LEMMA 1. – *Let $R, S \in \mathfrak{R}$, $R \neq S$. Then.*

$$\varphi(x+u+v) - \varphi(x+u) = \varphi(x+v) - \varphi(x) \qquad \textit{for all } x \in D,\ u \in R,\ v \in S.$$

PROOF. – The equality is trivially valid if $u = 0$ or $v = 0$; we therefore may and do assume that $u \neq 0$ and $v \neq 0$. By (R1) we have $S \neq -R$ and hence $v \notin R - R = \operatorname{Sp} R$. It follows that $x + jv + R$, $j = 0, 1, 2$, are three distinct parallel half-lines that meet the half-line $x + S$ only at the points $x + jv$, $j = 0, 1, 2$, respectively. Since φ has Property (H), $\varphi_>(x + jv + R)$, $j = 0, 1, 2$, and $\varphi_>(x + S)$ are half-lines and, since φ is injective, the three half-lines $\varphi_>(x + jv + R)$, $j = 0, 1, 2$, are pairwise disjoint and meet the half-line $\varphi_>(x + S)$ only at the points $\varphi(x + jv)$, $j = 0, 1, 2$, respectively (see Figure 2). This means that the ray

$$S' := \varphi_>(x + S) - \varphi(x) \in \mathfrak{R}'$$

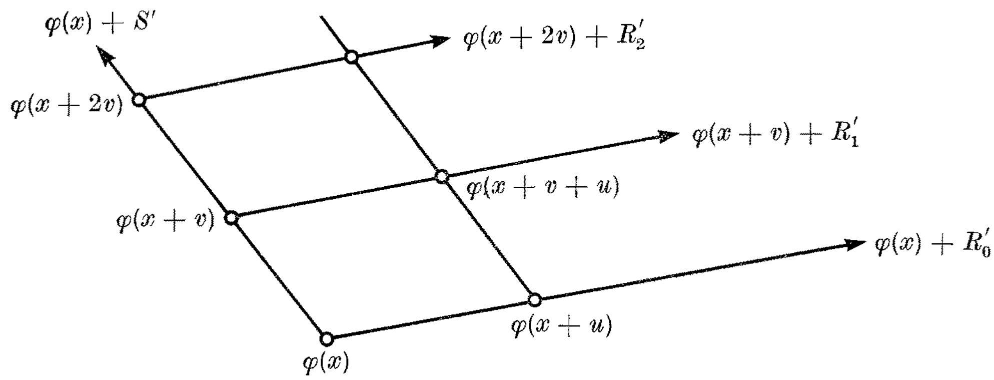

Figure 2

must be distinct from each of the rays

$$R_j' := \varphi_>(x + jv + R) - \varphi(x + jv) \in \mathfrak{R}' , \qquad j = 0, 1, 2.$$

We claim that the three half-lines $\varphi_>(x + jv + R)$, $j = 0, 1, 2$, satisfy the hypothesis of Proposition 3. Indeed, if z is a point on $\varphi_>(x + kv + R)$, where k is 0 or 1 or 2, then $z = \varphi(x + kv + w)$ for a suitable $w \in R$, and hence z lies on the half-line $\varphi_>(x + w + S)$, which meets all three half-lines under consideration at $\varphi(x + w + jv)$, respectively. Proposition 3 implies that these three half-lines must be parallel to one plane, which means that $\dim \mathrm{Sp}\,(R_0' \cup R_1' \cup R_2') \leqq 2$. Since R_0', R_1', $R_2' \in \mathfrak{R}'$, this is consistent with (R3) only if at least two of R_0', R_1', R_2' are equal, which means that at least two of the three half-lines are parallel, and hence lie in one plane. Using Proposition 3 again, we conclude that all three half-lines lie in one plane. This plane also includes the half-line $\varphi_>(x + S)$. It follows that $\dim \mathrm{Sp}\,(R_0' \cup R_1' \cup S') \leqq 2$. Since $S' \in \mathfrak{R}'$ and S' is distinct from both R_0' and R_1', this is consistent with (R3) only if $R_0' = R_1'$. It follows that the line through $\varphi(x + v)$ and $\varphi(x + v + u)$ is parallel to the line through $\varphi(x)$ and $\varphi(x + u)$.

If we interchange R and S, and u and v, in the preceding argument, we also conclude that the line through $\varphi(x + u)$ and $\varphi(x + u + v)$ is parallel to the line through $\varphi(x)$ and $\varphi(x + v)$. Therefore $\varphi(x)$, $\varphi(x + v)$, $\varphi(x + u + v)$, $\varphi(x + u)$ are consecutive vertices of a parallellogram. The conclusion of the lemma is an algebraic formulation of this assertion. □

LEMMA 2. – *Let $\mathcal{F}$ be a finite subset of $\mathfrak{R}$ and let $u_R \in R$ be given for each $R \in \mathcal{F}$. Then*

$$\varphi\Big(x + \sum_{R\in\mathcal{F}} u_R\Big) = \varphi(x) + \sum_{R\in\mathcal{F}} (\varphi(x + u_R) - \varphi(x)) \tag{6.1}$$

for all $x \in D$.

PROOF. – We proceed by induction. The assertion is trivial for $\mathcal{F} = \emptyset$. Suppose, then, that $\mathcal{F}$ is a non-empty finite subset of $\mathfrak{R}$, and that the assertion is valid when $\mathcal{F}$ is replaced by any proper subset $\mathcal{F}'$ of $\mathcal{F}$. Choose $S \in \mathcal{F}$ and set $\mathcal{F}' := \mathcal{F} \setminus \{S\}$. The induction hypothesis yields

$$\varphi\Big(x + \sum_{R\in\mathcal{F}} u_R\Big) = \varphi\Big(x + u_S + \sum_{R\in\mathcal{F}'} u_R\Big) = \varphi(x + u_S) + \sum_{R\in\mathcal{F}'} (\varphi(x + u_S + u_R) - \varphi(x + u_S)) .$$

By Lemma 1, we have $\varphi(x + u_S + u_R) - \varphi(x + u_S) = \varphi(x + u_R) - \varphi(x)$ for all $R \in \mathcal{F}'$, and this yields (6.1), the desired result. □

LEMMA 3. – *Let $x, y \in D$ be given. Suppose*

$$y - x = \sum_{S\in\mathfrak{R}} (p_S - q_S) , \tag{6.2}$$

where $(p_S|S \in \mathfrak{R})$ and $(q_S|S \in \mathfrak{R})$ are families with only finitely many non-zero terms and satisfying p_S, $q_S \in S$ for all $S \in \mathfrak{R}$. Then

$$\varphi(x + p_R + u) - \varphi(x + p_R) = \varphi(y + q_R + u) - \varphi(y + q_R) \tag{6.3}$$

for each $R \in \mathfrak{R}$ and each $u \in R$.

PROOF. – Let $R \in \mathfrak{R}$ be given. Using Lemma 2, we find that

$$\varphi\Big(x + u + \sum_{S\in\mathfrak{R}} p_S\Big) = \varphi(x + p_R + u) + \sum_{S\in\mathfrak{R}\setminus\{R\}} \big(\varphi(x + p_S) - \varphi(x)\big) \tag{6.4}$$

$$\varphi\Big(y + u + \sum_{R\in\mathfrak{R}} q_S\Big) = \varphi(y + q_R + u) + \sum_{S\in\mathfrak{R}\setminus\{R\}} \big(\varphi(y + q_S) - \varphi(y)\big) \tag{6.5}$$

hold for all $u \in R$. It follows from (6.2) that the left-hand sides of (6.4) and (6.5) are equal. Hence we have

$$\varphi(y + q_R + u) - \varphi(x + p_R + u) = \sum_{S\in\mathfrak{R}\setminus\{R\}} \Big(\big(\varphi(x + p_S) - \varphi(x)\big) - \big(\varphi(y + q_S) - \varphi(y)\big)\Big) .$$

Since the right-hand side does not depend on u, neither does the left-hand side; (6.3) is an immediate consequence. □

LEMMA 4. – *There is a mapping $\Phi\colon \mathfrak{R} \to \mathfrak{R}'$ such that*

$$\varphi_>(x + R) = \varphi(x) + \Phi(R) \qquad \textit{for every } x \in D,\ R \in \mathfrak{R}.$$

PROOF. – Let $x, y \in D$ and $R \in \mathfrak{R}$ be given. By (R2), $y - x$ has a representation of the form (6.2) with $(p_S|S \in \mathfrak{R})$, $(q_S|S \in \mathfrak{R})$ as described there. Property (H) implies that $\varphi_>(x+R)$, $\varphi_>(x+p_R+R)$, $\varphi_>(y+R)$, $\varphi_>(y+q_R+R)$ are half-lines with directions in $\mathfrak{R}'$; the first of these half-lines includes the second, and the third the fourth; and Lemma 3 asserts that the second and fourth have the same direction. It follows that all four have the same direction; in particular,

$$\varphi_>(x+R)-\varphi(x)=\varphi_>(y+R)-\varphi(y)\in\mathfrak{R}' .$$

Since $x, y\in D$ and $R\in\mathfrak{R}$ were arbitrary, this proves the existence of the required mapping $\Phi\colon\mathfrak{R}\to\mathfrak{R}'$. $\square$

LEMMA 5. – *If the ray $R\in\mathfrak{R}$ is engaged in $\mathfrak{R}$, then there is a mapping $\omega_R\colon R\to\Phi(R)$ such that*

$$\varphi(x+u)-\varphi(x)=\omega_R(u)\quad \textit{for all } x\in D,\ u\in R. \tag{6.6}$$

This mapping satisfies

$$\omega_R(tu)=t\omega_R(u)\quad \textit{for all } u\in R,\ t\in\mathbb{P}. \tag{6.7}$$

PROOF. – Since R is engaged in $\mathfrak{R}$, (R2) implies

$$V=\operatorname{Sp}\cup(\mathfrak{R}\diagdown\{R\})=\sum_{S\in\mathfrak{R}\diagdown\{R\}}(S-S) .$$

Hence, for each $x, y\in D$, $y-x$ has a representation of the form (6.2) with $p_R=q_R=0$. Lemmas 3 and 4 then give

$$\varphi(x+u)-\varphi(x)=\varphi(y+u)-\varphi(y)\in\Phi(R)\quad \text{for all } u\in R.$$

It follows that there is a mapping $\omega_R\colon R\to\Phi(R)$ satisfying (6.6).

Now let $u\in R\diagdown\{0\}$ be given. Since φ is injective, we have $\omega_R(u)\in\Phi(R)\diagdown\{0\}$, so that $\Phi(R)=\mathbb{P}\omega_R(u)$; and there is a unique function $\sigma\colon\mathbb{P}\to\mathbb{P}$ such that $\omega_R(tu)=\sigma(t)\,\omega_R(u)$ for all $t\in\mathbb{P}$; of course $\sigma(1)=1$. From (6.6), with some fixed $x\in D$,

$$\begin{aligned}\omega_R(su+tu)&=\big(\varphi(x+su+tu)-\varphi(x+tu)\big)+\big(\varphi(x+tu)-\varphi(x)\big)=\\&=\omega_R(su)+\omega_R(tu)\quad \text{for all } s, t\in\mathbb{P};\end{aligned}$$

hence $\sigma(s+t)=\sigma(s)+\sigma(t)$ for all $s, t\in\mathbb{P}$. In particular, σ is isotone. A standard argument from elementary analysis shows that $\sigma(t)=t\sigma(1)=t$ for all $t\in\mathbb{P}$, and (6.7) follows. $\square$

We denote by $\mathfrak{R}_e$ the set of all rays in $\mathfrak{R}$ that are engaged in $\mathfrak{R}$, and we set

$$K_e:=\sum_{R\in\mathfrak{R}_e}R,\qquad V_e:=\operatorname{Sp}\cup\mathfrak{R}_e=\operatorname{Sp}K_e=K_e-K_e .$$

LEMMA 6. – *There is a linear mapping* $\lambda_e\colon V_e \to V'$ *such that*

$$\varphi(x+v)-\varphi(x)=\lambda_e(v) \quad \textit{for all } x\in D,\ v\in K_e\,.$$

PROOF. – Let $u\in K_e$ be given. Then $u=\sum_{R\in\mathcal{F}} u_R$ for some finite set $\mathcal{F}\subset\mathfrak{R}_e$ and some family $(u_R|R\in\mathcal{F})$ with $u_R\in R$ for each $R\in\mathcal{F}$. By Lemmas 2 and 5 we then have

$$\varphi(x+tu)-\varphi(x)=\sum_{R\in\mathcal{F}}\omega_R(tu)=t\sum_{R\in\mathcal{F}}\omega_R(u)=t\big(\varphi(x+u)-\varphi(x)\big) \quad \text{for all } x\in D,\ t\in\mathbb{P}.$$

Since the middle links in this chain of equalities do not depend on x, neither do the ends. Since $u\in K_e$ was arbitrary, there exists a mapping $\varkappa\colon K_e\to V'$ satisfying

$$\varphi(x+u)-\varphi(x)=\varkappa(u) \quad \text{for all } x\in D,\ u\in K_e, \tag{6.8}$$

$$\varkappa(tu)=t\varkappa(u) \quad \text{for all } u\in K_e,\ t\in\mathbb{P}. \tag{6.9}$$

From (6.8) with x chosen in D,

$$\varkappa(u+v)=\big(\varphi(x+u+v)-\varphi(x+v)\big)+\big(\varphi(x+v)-\varphi(x)\big)=\varkappa(u)+\varkappa(v) \quad \text{for all } u,\ v\in K_e\,. \tag{6.10}$$

Since $V_e=K_e-K_e$, it follows easily from (6.9), (6.10) that $\varkappa$ has a unique linear extension $\lambda_e\colon V_e\to V'$. Since $\varkappa$ is the restriction of λ_e to K_e, the conclusion follows from (6.8). □

PROOF OF THEOREM A. – If all rays of $\mathfrak{R}$ are engaged in $\mathfrak{R}$, then $\mathfrak{R}_e=\mathfrak{R}$ and $K_e-K_e=V_e=\operatorname{Sp}\cup\mathfrak{R}=V$. If $x,\ y\in D$, then $y-x=u-v$ for suitable $u,\ v\in K_e$, and Lemma 6 implies

$$\varphi(y)-\varphi(x)=\big(\varphi(x+u)-\varphi(x)\big)-\big(\varphi(y+v)-\varphi(y)\big)=$$
$$=\lambda_e(u)-\lambda_e(v)=\lambda_e(u-v)=\lambda_e(y-x)\,.$$

Since $x,\ y\in D$ were arbitrary, φ is the restriction to D (with codomain adjusted to D') of an affine mapping from E to E' with gradient λ_e. □

7. – The general case.

In this section, we shall determine the form of an injection $\varphi\colon D\to D'$ satisfying (H) without assuming that all rays in $\mathfrak{R}$ are engaged in $\mathfrak{R}$. Our description will depend on some arbitrary choices; these could be avoided at the cost of introducing extraneous machinery.

Throughout this section, all the assumptions made in the first paragraph of Section 6 remain in force. We set

$$K := \sum_{R\in\mathfrak{R}} R\,; \tag{7.1}$$

this is consistent with the discussion in Section 4, and (R2) implies $V = K - K$. As in Section 6, $\mathfrak{R}_e$ is the set of all rays of $\mathfrak{R}$ that are engaged in $\mathfrak{R}$, and $K_e := \sum_{R\in\mathfrak{R}_e} R$, $V_e := \operatorname{Sp}\bigcup\mathfrak{R}_e = K_e - K_e$. We set $\mathfrak{R}_d := \mathfrak{R}\setminus\mathfrak{R}_e$, the set of all rays of $\mathfrak{R}$ that are disengaged in $\mathfrak{R}$, and choose $b_R \in R\setminus\{0\}$ for each $R \in \mathfrak{R}_d$.

Lemma 7. – *There are unique mappings $\pi_e\colon V \to V_e$ and $\beta_R\colon V \to \mathbb{R}$ for each $R \in \mathfrak{R}_d$, such that*

$$v = \pi_e(v) + \sum_{R\in\mathfrak{R}_d} \beta_R(v)\, b_R \qquad \textit{for every } v \in V$$

(*all but finitely many summands are* 0). *All these mappings are linear.*

Proof. – By the definition of « disengaged » and by the fact that $V = K - K$, we find that $(b_R | R \in \mathfrak{R}_d)$ is an independent family and $\operatorname{Sp}\{b_R | R \in \mathfrak{R}_d\}$ is a supplement of V_e in V. $\square$

To formulate our results succinctly, we introduce some terminology. $\mathbb{R}$ has a natural order, so that a set $J \subset \mathbb{R}$ is follower-saturated if and only if it satisfies $J + \mathbb{P} = J$. A *rescaling* is a real-valued strictly isotone function σ whose domain $\operatorname{Dom}\sigma$ and range $\operatorname{Rng}\sigma$ are follower-saturated: it is an order-isomorphism as a mapping from $\operatorname{Dom}\sigma$ to $\operatorname{Rng}\sigma$. A rescaling σ is *normalized* if $0 \in \operatorname{Dom}\sigma$ and $\sigma(0) = 0$ and $\sigma(1) = 1$.

We now choose $x_0 \in D$, and set $I_R := (\beta_R)_>(D - x_0)$ for each $R \in \mathfrak{R}_d$; we observe that I_R is a follower-saturated subset of $\mathbb{R}$, on account of (D), and that it contains 0.

Lemma 8. – *Let σ_R be a normalized rescaling with* $\operatorname{Dom}\sigma_R = I_R$ *for each* $R \in \mathfrak{R}_d$. *Then the formula*

$$\psi_x(v) := \pi_e(v) + \sum_{R\in\mathfrak{R}_d} \Big(\sigma_R\big(\beta_R(x + v - x_0)\big) - \sigma_R\big(\beta_R(x - x_0)\big)\Big)\, b_R$$

defines a bijection $\psi_x\colon K \to K$ for every $x \in D$.

Proof. – For each $R \in \mathfrak{R}_d$ and each $s \in I_R$, the mapping $t \mapsto \sigma_R(s + t) - \sigma_R(s)$ is a bijection from $\mathbb{P}$ to $\mathbb{P}$ that maps 0 to 0. The conclusion is an immediate consequence of Lemma 7, (7.1), and this fact.

Lemma 9. – *Assume that $\varphi\colon D \to D'$ is an injection that satisfies* (H). *For every $R \in \mathfrak{R}_d$ there is a unique $b'_R \in V'$ and a unique normalized rescaling σ_R with* $\operatorname{Dom}\sigma_R = I_R$

such that

$$\varphi(x + tb_R) - \varphi(x) = \big(\sigma_R(\beta_R(x - x_0) + t) - \sigma_R(\beta_R(x - x_0))\big)\, b'_R \tag{7.2}$$

for all $x \in D$, $t \in \mathbb{P}$.

PROOF. – If $b'_R \in V'$ and $\sigma_R\colon I_R \to \mathbb{R}$ are such that $\sigma_R(0) = 0$, $\sigma_R(1) = 1$, and (7.2) holds, we have in particular, using Lemma 7,

$$b'_R = \big(\sigma_R(1) - \sigma_R(0)\big)\, b'_R = \varphi(x_0 + b_R) - \varphi(x_0)\,;$$

this shows the uniqueness of b'_R. We therefore *set*

$$b'_R := \varphi(x_0 + b_R) - \varphi(x_0)\,.$$

By Lemma 4 and the fact that φ is injective, $b'_R \in \Phi(R)\diagdown\{0\}$, and hence

$$\mathbb{P}b'_R = \Phi(R) \in \mathfrak{R}'\,. \tag{7.3}$$

Let $x, y \in D$ satisfy $\beta_R(x - x_0) = \beta_R(y - x_0)$. Then $\beta_R(y - x) = 0$, and Lemma 7 implies that $y - x$ has a representation (6.2) with $p_R = q_R = 0$. Lemmas 3 and 4 and (7.3) show that

$$\varphi(x + tb_R) - \varphi(x) = \varphi(y + tb_R) - \varphi(y) \in \Phi(R) = \mathbb{P}b'_R \quad \text{for all } t \in \mathbb{P}.$$

Since $x, y \in D$ with $\beta_R(x - x_0) = \beta_R(y - x_0)$ were arbitrary, this shows that there is a unique function $\varrho\colon I_R \times \mathbb{P} \to \mathbb{P}$ such that

$$\varphi(x + tb_R) - \varphi(x) = \varrho\big(\beta_R(x - x_0), t\big)\, b'_R \quad \text{for all } x \in D,\ t \in \mathbb{P}. \tag{7.4}$$

If $s \in I_R$, choose $x \in D$ such that $\beta_R(x - x_0) = s$ (this is possible, since $I_R = (\beta_R)_>(D - x_0)\big)$. For every $r \in \mathbb{P}$ we have $\beta_R(x + rb_R - x_0) = s + r$, by Lemma 7. Therefore (7.4) implies

$$\begin{aligned}\varrho(s + r, t)\, b'_R &= \varphi(x + rb_R + tb_R) - \varphi(x + rb_R) = \\ &= \big(\varphi(x + (r + t)\, b_R) - \varphi(x)\big) - \big(\varphi(x + rb_R) - \varphi(x)\big) = \\ &= \big(\varrho(s, r + t) - \varrho(s, r)\big)\, b'_R \quad \text{for all } t \in \mathbb{P}.\end{aligned}$$

We conclude that

$$\varrho(s + r, t) = \varrho(s, r + t) - \varrho(s, r) \quad \text{for all } s \in I_R,\ r, t \in \mathbb{P}.$$

This is a functional equation for ϱ. An easy analysis shows that there is a unique function $\sigma_R\colon I_R \to \mathbb{R}$ such that $\sigma_R(0) = 0$ and

$$(7.5) \qquad \varrho(s, t) = \sigma_R(s + t) - \sigma_R(s) \quad \text{for all } s \in I_R,\ t \in \mathbb{P}.$$

Combination of (7.4) and (7.5) yields (7.2). Since φ is injective, so is σ_R; since ϱ is non-negative-valued, σ_R is isotone, hence strictly isotone. (7.2) and (7.3) and Lemma 4 yield

$$\mathbb{P}b'_R = \Phi(R) = \varphi_>(x + R) - \varphi(x) \subset \big(\operatorname{Rng}\sigma_R - \sigma_R(\beta_R(x - x_0))\big)\, b'_R \quad \text{for each } x \in D,$$

so that $\sigma_R(s) + \mathbb{P} \subset \operatorname{Rng}\sigma_R$ for each $s \in \operatorname{Dom}\sigma_R$, and hence $\operatorname{Rng}\sigma_R + \mathbb{P} \subset \operatorname{Rng}\sigma_R$. Thus σ_R is a rescaling; $\sigma_R(0) = 0$ by definition; and (7.2) yields, by Lemma 7,

$$b'_R = \varphi(x_0 + b_R) - \varphi(x_0) = \big(\sigma_R(1) - \sigma_R(0)\big)\, b'_R = \sigma_R(1)\, b'_R\,,$$

so that $\sigma_R(1) = 1$, and σ_R is a normalized rescaling. □

LEMMA 10. – *Assume that $\varphi\colon D \to D'$ is an injection that satisfies* (H). *For all $x \in D$,*

$$\varphi(x) = \varphi(x_0) + \lambda_{\mathrm{e}}\big(\pi_{\mathrm{e}}(x - x_0)\big) + \sum_{R \in \mathfrak{R}_{\mathrm{d}}} \sigma_R\big(\beta_R(x - x_0)\big)\, b'_R\,,$$

where λ_{e}, b'_R, σ_R are as in Lemmas 6 *and* 9.

PROOF. – Let $x \in D$ be given. Then $\pi_{\mathrm{e}}(x - x_0) = u - v$ for suitable $u, v \in K_{\mathrm{e}}$, and $\beta_R(x - x_0) = s_R - t_R$ for suitable $s_R, t_R \in \mathbb{P}$ for each $R \in \mathfrak{R}_{\mathrm{d}}$, with $s_R = t_R = 0$ if $\beta_R(x - x_0) = 0$ (hence for all but finitely many $R \in \mathfrak{R}_{\mathrm{d}}$). By Lemma 7, $\beta_R(u) = \beta_R(v) = 0$ for all $R \in \mathfrak{R}_{\mathrm{d}}$, and

$$(7.6) \qquad x_0 + u + \sum_{R \in \mathfrak{R}_{\mathrm{d}}} s_R b_R = x + v + \sum_{R \in \mathfrak{R}_{\mathrm{d}}} t_R b_R\,.$$

Lemma 9 implies, for each $R \in \mathfrak{R}_{\mathrm{d}}$,

$$\varphi(x_0 + u + s_R b_R) - \varphi(x_0 + u) = \big(\sigma_R(\beta_R(u) + s_R) - \sigma_R(\beta_R(u))\big)\, b'_R = \sigma_R(s_R)\, b'_R$$

$$\varphi(x + v + t_R b_R) - \varphi(x + v) = \big(\sigma_R(\beta_R(x + v - x_0) + t_R) - \sigma_R(\beta_R(x + v - x_0))\big)\, b'_R = \big(\sigma_R(s_R) - \sigma_R(s_R - t_R)\big)\, b'_R\,.$$

By Lemmas 2 and 6, therefore,

$$(7.7) \qquad \varphi\Big(x_0 + u + \sum_{R \in \mathfrak{R}_{\mathrm{d}}} s_R b_R\Big) = \varphi(x_0 + u) + \sum_{R \in \mathfrak{R}_{\mathrm{d}}} \big(\varphi(x_0 + u + s_R b_R) - \varphi(x_0 + u)\big) = \varphi(x_0) + \lambda_{\mathrm{e}}(u) + \sum_{R \in \mathfrak{R}_{\mathrm{d}}} \sigma_R(s_R)\, b'_R$$

$$\varphi\Big(x+v+\sum_{R\in\mathfrak{R}_{\mathrm d}} t_R b_R\Big) = \varphi(x+v) + \sum_{R\in\mathfrak{R}_{\mathrm d}} \big(\varphi(x+v+t_R b_R) - \varphi(x+v)\big) = \tag{7.8}$$
$$= \varphi(x) + \lambda_{\mathrm e}(v) + \sum_{R\in\mathfrak{R}_{\mathrm d}} \big(\sigma_R(s_R) - \sigma_R(s_R - t_R)\big)\, b'_R\,.$$

Combining (7.6), (7.7), (7.8) and noting the linearity of $\lambda_{\mathrm e}$ and the choice of u, v, s_R, t_R, we obtain

$$\varphi(x) = \varphi(x_0) + \lambda_{\mathrm e}(u) - \lambda_{\mathrm e}(v) + \sum_{R\in\mathfrak{R}_{\mathrm d}} \sigma_R(s_R - t_R)\, b'_R =$$
$$= \varphi(x_0) + \lambda_{\mathrm e}\big(\pi_{\mathrm e}(x - x_0)\big) + \sum_{R\in\mathfrak{R}_{\mathrm d}} \sigma_R\big(\beta_R(x - x_0)\big)\, b'_R\,. \qquad \square$$

LEMMA 11. – *Assume that $\varphi\colon D \to D'$ is an injection that satisfies* (H). *If $\lambda_{\mathrm e}$ and b'_R are as in Lemmas* 6 *and* 9, *then the linear mapping $\lambda\colon V \to V'$ defined by*

$$\lambda(v) := \lambda_{\mathrm e}\big(\pi_{\mathrm e}(v)\big) + \sum_{R\in\mathfrak{R}_{\mathrm d}} \beta_R(v)\, b'_R \qquad \textit{for all } v\in V$$

is injective and satisfies $\lambda(u) = \lambda_{\mathrm e}(u)$ for all $u\in V_{\mathrm e}$, $\lambda(b_R) = b'_R$ for all $R\in\mathfrak{R}_{\mathrm d}$, and $\lambda_{>>}(\mathfrak{R}) \subset \mathfrak{R}'$.

PROOF. – Let σ_R be the normalized rescaling of Lemma 9 for each $R\in\mathfrak{R}_{\mathrm d}$. In the language of Lemma 8, Lemma 10 implies that

$$\varphi(x_0 + v) - \varphi(x_0 + u) = \lambda\big(\psi_{x_0}(v) - \psi_{x_0}(u)\big) \qquad \text{for all } u,\ v\in K.$$

Since $\psi_{x_0}\colon K \to K$ is surjective and $K - K = V$ and φ is injective, if follows that λ is injective.

Let $R\in\mathfrak{R}$ be given. If $R\in\mathfrak{R}_{\mathrm e}$, then $\lambda_>(R) = (\lambda_{\mathrm e})_>(R) = \varphi_>(x_0 + R) - \varphi(x_0)\in\mathfrak{R}'$ by Lemma 6 and Property (H); if $R\in\mathfrak{R}_{\mathrm d}$, then $\lambda_>(R) = \lambda_>(\mathbb{P}b_R) = \mathbb{P}\lambda(b_R) = \mathbb{P}b'_R\in\mathfrak{R}'$ by (7.3). Thus $\lambda_>(R)\in\mathfrak{R}'$ in either case, i.e., $\lambda_{>>}(\mathfrak{R})\subset\mathfrak{R}'$. $\square$

THEOREM B. – *The mapping $\varphi\colon D \to D'$ is an injection satisfying* (H) *if and only if*

$$\varphi(x) = \varphi(x_0) + \lambda\Big(\pi_{\mathrm e}(x - x_0) + \sum_{R\in\mathfrak{R}_{\mathrm d}} \sigma_R\big(\beta_R(x - x_0)\big)\, b_R\Big) \qquad \textit{for all } x\in D, \tag{7.9}$$

where $\lambda\colon V \to V'$ is an injective linear mapping such that $\lambda_{>>}(\mathfrak{R})\subset\mathfrak{R}'$ and σ_R is a normalized rescaling with $\operatorname{Dom}\sigma_R = I_R$ *for each $R\in\mathfrak{R}_{\mathrm d}$.*

PROOF. – *The condition is necessary.* Assume that $\varphi\colon D \to D'$ is an injection satisfying (H), and let $\lambda_{\mathrm e}$, b'_R, σ_R, and λ be as in Lemmas 6, 9, 11. Then $\lambda\colon V \to V'$ is an injective linear mapping satisfying $\lambda_{>>}(\mathfrak{R})\subset\mathfrak{R}'$ (Lemma 11), each σ_R is a normalized rescaling with the desired domain (Lemma 9), and Lemmas 10 and 11 together imply (7.9).

The condition is sufficient. Assume that $\varphi: D \to D'$ satisfies (7.9) with λ and σ_R as stated. If $x, y \in D$ are such that $\varphi(x) = \varphi(y)$, (7.9) and the fact that λ is injective yield

$$\pi_e(y - x) + \sum_{R \in \mathfrak{R}_d} \Big(\sigma_R\big(\beta_R(y - x_0)\big) - \sigma_R\big(\beta_R(x - x_0)\big)\Big)\, b_R = 0 .$$

By Lemma 7 and the fact each σ_R is injective we find $\pi_e(y - x) = 0$ and $\beta_R(y - x) = \beta_R(y - x_0) - \beta_R(x - x_0) = 0$ for every $R \in \mathfrak{R}_d$. By Lemma 7 again, $y - x = 0$. Thus φ is injective.

Let $x \in D$ and $R \in \mathfrak{R}$ be given. If $R \in \mathfrak{R}_e$, (7.9) and Lemma 7 yield

$$\varphi_>(x + R) - \varphi(x) = \lambda_>\big((\pi_e)_>(R)\big) = \lambda_>(R) \in \mathfrak{R}' .$$

If $R \in \mathfrak{R}_d$, (7.9), Lemma 7, and the fact that σ_R is a rescaling yield

$$\begin{aligned} \varphi_>(x + R) - \varphi(x) &= \varphi_>(x + \mathbb{P} b_R) - \varphi(x) = \\ &= \Big((\sigma_R)_>\big(\beta_R(x - x_0) + \mathbb{P}\big) - \sigma_R\big(\beta_R(x - x_0)\big)\Big)\, \lambda(b_R) = \\ &= \mathbb{P}\lambda(b_R) = \lambda_>(\mathbb{P} b_R) = \lambda_>(R) \in \mathfrak{R}' . \end{aligned}$$

Thus $\varphi_>(x + R) - \varphi(x) \in \mathfrak{R}'$ in either case, and φ satisfies (H). $\square$

8. – Non-affine order-isomorphisms.

We now return to the situation in which E and E' are directed affine spaces, with respective direction-cones K and K' and $\mathfrak{R}$ and $\mathfrak{R}'$ are the sets of extreme rays of K and K', respectively; and it is assumed that K is the sum of its extreme rays, so that (7.1) holds. (If E is finite-dimensional, this assumption holds if and only if the order is closed, as shown in Proposition 2.) As we noted in Section 4, $\mathfrak{R}$ and $\mathfrak{R}'$ then satisfy (R1), (R2), (R3). The definitions at the beginning of Section 7 are applicable.

COROLLARY B1. – *Assume that D and D' are non-empty follower-saturated subsets of E and E', respectively, and that $\varphi: D \to D'$ is an order-isomorphism. Choose $x_0 \in D$. Then*

$$\varphi(x) = \varphi(x_0) + \lambda\Big(\pi_e(x - x_0) + \sum_{R \in \mathfrak{R}_d} \sigma_R\big(\beta_R(x - x_0)\big)\, b_R\Big) \quad \textit{for all } x \in D, \tag{8.1}$$

where $\lambda: V \to V'$ is the gradient of an affine order-isomorphism from E to E' and σ_R is a normalized rescaling with $\mathrm{Dom}\, \sigma_R = I_R := (\beta_R)_>(D - x_0)$ *for every $R \in \mathfrak{R}_d$.*

Proof. – Since φ is an order-isomorphism and D and D' are follower-saturated,

$$\varphi_>(x_0 + K) = \varphi(x_0) + K' . \tag{8.2}$$

Now D and D' satisfy (D), and φ is an injection satisfying (H) (cf. Section 4). Therefore Theorem B is applicable, and φ satisfies (7.9), which is (8.1), with an injective linear mapping $\lambda\colon V \to V'$ and with normalized rescalings σ_R with the desired domains.

We claim that $\lambda_>(K) = K'$; this will complete the proof: indeed, the orders are directing, so K' spans V', hence λ is surjective, hence invertible; since it satisfies $\lambda_>(K) = K'$, it is the gradient of an affine order-isomorphism from E to E'.

To prove our claim, we observe that, in the language of Lemma 8, (8.1) implies

$$\varphi(x_0 + v) - \varphi(x_0) = \lambda(\psi_{x_0}(v)) \qquad \text{for all } v \in K; \tag{8.3}$$

and combining (8.2) and (8.3) with the fact that $\psi_{x_0}\colon K \to K$ is surjective, we find

$$K' = \varphi_>(x_0 + K) - \varphi(x_0) = \lambda_>((\psi_{x_0})_>(K)) = \lambda_>(K) . \qquad \square$$

Corollary B2, the statement of which will be found in Section 4, is now an immediate consequence of Corollary B1.

We have a strong converse to Corollary B1; in formulating it, we have to respect the restriction imposed by Corollary B2.

Proposition 4. – *Let $\lambda\colon V \to V'$ be the gradient of an affine order-isomorphism from E to E'. For every non-empty follower-saturated subset D of E, every $x_0 \in D$, every $x_0' \in E'$, and every family of normalized rescalings σ_R with* $\operatorname{Dom}\sigma_R = I_R := (\beta_R)_>(D - x_0)$ *for each $R \in \mathfrak{R}_d$, the formula*

$$\varphi(x) := x_0' + \lambda\Big(\pi_e(x - x_0) + \sum_{R\in\mathfrak{R}_d} \sigma_R(\beta_R(x - x_0))\, b_R\Big) \qquad \textit{for all } x \in D \tag{8.4}$$

defines an order-isomorphism $\varphi\colon D \to D'$ for a suitable non-empty follower-saturated subset D' of E'.

Proof. – Let D, x_0, x_0', and the σ_R be given, and let D' be the range of the mapping defined by (8.4). Then $\varphi\colon D \to D'$, as defined by (8.4), is surjective. We shall show that

$$\varphi_>(x + K) = \varphi(x) + K' \qquad \text{for all } x \in D; \tag{8.5}$$

this will prove that D' is follower-saturated, and that φ is an order-isomorphism provided it is injective; but the injectivity of φ follows exactly as in the proof of Theorem B (sufficiency). It thus remains to prove (8.5).

The assumption on λ means that λ is an invertible linear mapping that satisfies $\lambda_{>}(K) = K'$. Let $x \in D$ be fixed. In the language of Lemma 8, it follows from (8.4) that

$$\varphi(x+v) - \varphi(x) = \lambda(\psi_x(v)) \qquad \text{for all } v \in K.$$

Therefore the fact that $\psi_x\colon K \to K$ is surjective (Lemma 8) implies

$$\varphi_{>}(x+K) - \varphi(x) = \lambda_{>}((\psi_x)_{>}(K)) = \lambda_{>}(K) = K',$$

and, since $x \in D$ was arbitrary, (8.5) is proved. $\square$

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