ORDER NORMS ON BOUNDED PARTIALLY ORDERED SETS

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Abstract. In this paper, we extend the domains of affirmation and negation operators, and more important, of triangular (semi)norms and (semi)conorms from the unit interval to bounded partially ordered sets. The fundamental properties of the original operators are proven to be conserved under this extension. This clearly shows that they are essentially based upon order-theoretic notions. Consequently, a rather general order-theoretic invariance study of these operators is undertaken. Also, in a brief algebraic excursion, the notion of weak invertibility of these operators is introduced, and the relation with the order-theoretic concept of residuals is studied. The importance of these results for fuzzy set theory and possibility theory is briefly discussed.

1. Introduction

In this paper, we introduce and study *affirmation* and *negation operators*, *triangular (semi)norms* and *triangular (semi)conorms* on *bounded partially ordered sets*. These operators are generalizations of the well-known classes of operators defined on the real unit interval [10,14,19,20,21,23] that play an important role in fuzzy set theory [24]. To be more precise, affirmation and negation operators on the real unit interval are used for the pointwise definition of respectively hedge and complement operators for fuzzy sets; and triangular norms and conorms play an analogous role in the pointwise definition of union and intersection operators for fuzzy sets and generalized disjunction and conjunction operators in fuzzy logic [8,15]. The generalization of these classes of operators to bounded posets is very important for the definition of useful hedge, complement, union and intersection operators for the L-fuzzy sets, introduced by Goguen [11]. These general fuzzy-set-theoretical operators need not necessarily be truth-functional (for more detail, we refer to the doctoral dissertation of one of us [7]). Furthermore, our generalized triangular seminorms and semiconorms can be used to introduce general classes of fuzzy integrals, which in turn play a central role in a measure- and integral-theoretic approach to possibility theory. For a more detailed account of this approach, we refer to [6,7]. Finally, we feel that these generalized operators are important and interesting enough in their own right to be studied in a general order-theoretic context.

Triangular norms and conorms were introduced in 1963 by Schweizer and Sklar [19,20], within the framework of probabilistic metric spaces. More specifically, they are based on a notion used by Menger [16] in order to extend the *triangle inequality* in the definition of metric spaces towards probabilistic metric spaces. In fuzzy set theory, they were introduced for the first time by Alsina, Trillas and Valverde [1], and Prade [17], who used them for the definition of new classes of fuzzy union and intersection operators. They are isotonic, associative and commutative binary operators on the real unit interval, that furthermore satisfy a few special boundary conditions. Taking into account their probabilistic origins, it is hardly surprising that they have up to now only been defined on $[0,1]$. Upon inspection, we find however that none of their defining properties are typical only of the real unit interval. These properties can therefore easily be generalized to define certain classes of operators on *bounded partially ordered sets*.

Triangular seminorms and semiconorms on the real unit interval were first introduced in an paper by Suárez García and Gil Álvarez [21] concerning fuzzy integrals. With triangular norms respectively conorms they share the isotonicity and boundary conditions, but they need not be commutative nor

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associative. The generalization of these operators to bounded partially ordered sets can therefore follow the same general lines as the extension of the domain of triangular norms and conorms.

In sections 2 and 3, it is shown that not only the above-mentioned generalization is possible, but that the generalized operators have the same fundamental properties as their counterparts on the real unit interval. The proofs of these properties are not always trivial extensions of the proofs of their counterparts, because the potential incomparability of elements of *partially* ordered sets tends to complicate matters. The results of section 3 force us to the conclusion that triangular (semi)norms and (semi)conorms are essentially order-theoretic notions. This conclusion leads to the study of the order-theoretic invariance of order norms in section 4. In section 5, we study some aspects of the behaviour of order norms as algebraic operators. It is among other things shown that—disregarding trivial exceptions—an order norm can not be used to form a group, because it is not invertible. The notion of weak invertibility is introduced, and the connection with the well-known concept of residuals [2] is studied. The results in this paragraph are very important for the development of a general theory of seminormed and semiconormed fuzzy integrals, as is shown in [7] and will be published in detail elsewhere.

Unless explicitly stated otherwise, we shall in the sequel denote by $(L, \leq)$ a *bounded partially ordered set*. The *smallest element* of $(L, \leq)$ will be denoted by ℓ, the *greatest element* by u. We also assume that $\ell \neq u$. Using the familiar interval notation, we shall also at times write $[\ell, u]$ instead of L, $]\ell, u[$ instead of $L \setminus \{\ell, u\}$, ...

2. Affirmation and Negation Operators

Definition 2.1 (Affirmation and Negation Operators).

(1) *An affirmation operator a on $(L, \leq)$ is an order-preserving permutation of $(L, \leq)$, i.e., a is a permutation of L, and a and a^{-1} are isotonic.*
(2) *A negation operator n on $(L, \leq)$ is an order-reversing permutation of $(L, \leq)$, i.e., n is a permutation of L, and n and n^{-1} are antitonic.*

An affirmation operator on $(L, \leq)$ is an order-automorphism of $(L, \leq)$, whereas a negation operator on $(L, \leq)$ is a dual order-automorphism of $(L, \leq)$. This means that on an arbitrary bounded poset $(L, \leq)$, there always exists at least one affirmation operator, namely the identity transformation. On the other hand, the existence of a negation operator on an arbitrary poset $(L, \leq)$ is not always guaranteed. A necessary and sufficient condition for the existence of a negation operator on $(L, \leq)$ is that there exist an order-isomorphism between the structure $(L, \leq)$ and its dual structure $(L, \geq)$. In other words, $(L, \leq)$ must be self-dual. It goes without saying that not every bounded poset satisfies this requirement. In the sequel, whenever we use the phrase 'Let n be an arbitrary negation operator on $(L, \leq)$', it is implicitly assumed that such a negation operator can indeed be defined on $(L, \leq)$.

We want to stress that an isotonic permutation is not necessarily order-preserving, and that an antitonic permutation is need not be order-reversing. Let us also mention the following immediate properties of affirmation and negation operators, the proofs of which are left implicit.

Proposition 2.1 (Boundary Values).

(1) *Let a be an arbitrary affirmation operator on $(L, \leq)$. Then $a(\ell) = \ell$ and $a(u) = u$.*
(2) *Let n be an arbitrary negation operator on $(L, \leq)$. Then $n(\ell) = u$ and $n(u) = \ell$.*

Proposition 2.2.

(1) *When a is an affirmation operator on $(L, \leq)$, a^{-1} is an affirmation operator on $(L, \leq)$ as well.*
(2) *When n is a negation operator on $(L, \leq)$, n^{-1} is a negation operator on $(L, \leq)$ as well.*

Proposition 2.3 (Duality Principle).

(1) *An affirmation operator on $(L, \leq)$ is an affirmation operator on the dual structure $(L, \geq)$.*
(2) *A negation operator on $(L, \leq)$ is a negation operator on the dual structure $(L, \geq)$.*

Proposition 2.4.

(1) *An affirmation operator a on $(L, \leq)$ is strictly isotonic, i.e., $(\forall(\lambda, \mu) \in L^2)(\lambda < \mu \Rightarrow a(\lambda) < a(\mu))$.*
(2) *A negation operator n on $(L, \leq)$ is strictly antitonic, i.e., $(\forall(\lambda, \mu) \in L^2)(\lambda < \mu \Rightarrow n(\lambda) > n(\mu))$.*

We want to stress that our definition of a negation operator on $(L, \leq)$ is more restricted than the immediate generalization of the existing definitions of negation operators on $[0,1]$ would allow for. In our opinion, Weber gives the most general definition of a negation operator on $[0,1]$: for him, a negation operator is an antitonic transformation of $[0,1]$ that exchanges the boundaries 0 and 1. Less general definitions are for instance given by Kerre [13,14], for whom negation operators are continuous and strictly antitonic transformations of $[0,1]$ that exchange the boundaries 0 and 1; and by Dubois and Prade [10], who only consider strictly antitonic and involutive transformations of $[0,1]$ that exchange 0 and 1. We want to remark that this last definition is highly redundant. Indeed, an antitonic and involutive transformation of $[0,1]$ can easily be proven to be a strictly antitonic permutation of $[0,1]$, that necessarily exchanges 0 and 1. In our more general framework, we indeed have the following proposition.

Proposition 2.5. *An involutive and antitonic transformation of $(L, \leq)$ is a negation operator on $(L, \leq)$.*

Proposition 2.6.

(1) *On a finite chain there always exists a unique affirmation operator, that is the identity transformation.*
(2) *On a finite chain there always exists a unique negation operator. This negation operator is necessarily involutive.*

We conclude this section with a few examples.

Example 2.1. Consider the set $X = \{a, b\}$ and its power set $\mathcal{P}(X) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. $(\mathcal{P}(X), \subseteq)$ is a complete (and therefore bounded) Boolean lattice. The classical set-theoretical complement operator on X is one of the two negation operators that can be defined on $(\mathcal{P}(X), \subseteq)$. The other negation operator is the permutation of $\mathcal{P}(X)$ that exchanges $\emptyset$ and X, and leaves $\{a\}$ and $\{b\}$ unchanged.

Example 2.2 (Two-Valued Logic). Consider the set $\mathcal{T} = \{\mathit{true}, \mathit{false}\}$ of the truth-values of classical logic, together with the total order relation $\leq$, defined by $\mathit{false} < \mathit{true}$. Then $(\mathcal{T}, \leq)$ is a finite chain. The only affirmation operator on $(\mathcal{T}, \leq)$ is the identity transformation of $\mathcal{T}$, and the only negation operator is the (involutive) negation (of truth-values) in classical two-valued logic.

Example 2.3 (Three-Valued Logic). Consider the chain of three elements, that is unique up to an order-isomorphism. The identity transformation is the only affirmation operator on this chain. We can also define a unique negation operator on this chain, that is involutive, exchanges its boundaries and maps the middle element onto itself. This operator is the negation operator in a number of important three-valued logics. In particular, we mention the three-valued systems of Kleene and Lukasiewicz [18].

Example 2.4. Consider the complete (bounded) subchain $([0,1], \leq)$ of the chain of the real numbers. The permutation c of $[0,1]$, defined as

$$c\colon [0,1] \to [0,1]\colon x \mapsto 1 - x,$$

is antitonic and involutive, and therefore is a negation operator on $([0,1], \leq)$. Let furthermore f be an antitonic permutation of $([0,1], \leq)$. Then

$$c_f \stackrel{\text{def}}{=} f^{-1} \circ c \circ f$$

is an involutive negation operator on $([0,1], \leq)$ as well. Trillas [22] has shown that all involutive and continuous negation operators on $([0,1], \leq)$ can be thus characterized.

3. Order Norms

3.1 Basic Definitions.

Definition 3.1 (Triangular Seminorms and Semiconorms). *A triangular seminorm (shortly t-seminorm) P on $(L, \leq)$ is a $L^2 - L$ mapping satisfying:*

(1) *boundary conditions: $(\forall \lambda \in L)(P(u, \lambda) = P(\lambda, u) = \lambda)$;*
(2) *isotonicity: $(\forall (\lambda_1, \lambda_2, \mu_1, \mu_2) \in L^4)(\lambda_1 \leq \lambda_2$ and $\mu_1 \leq \mu_2 \Rightarrow P(\lambda_1, \mu_1) \leq P(\lambda_2, \mu_2))$.*

A triangular semiconorm (shortly t-semiconorm) Q on $(L, \leq)$ is a $L^2 - L$ mapping satisfying:

1. *boundary conditions:* $(\forall \lambda \in L)(Q(\ell, \lambda) = Q(\lambda, \ell) = \lambda)$;
2. *isotonicity:* $(\forall(\lambda_1, \lambda_2, \mu_1, \mu_2) \in L^4)(\lambda_1 \leq \lambda_2$ *and* $\mu_1 \leq \mu_2 \Rightarrow Q(\lambda_1, \mu_1) \leq Q(\lambda_2, \mu_2))$.

The set of the triangular seminorms on $(L, \leq)$ will be denoted by $\mathcal{N}_s(L, \leq)$. The set of the triangular semiconorms on $(L, \leq)$ will be denoted by $\mathcal{C}_s(L, \leq)$.

Definition 3.2 (Triangular Norms and Conorms).

1. *A triangular norm (shortly t-norm) T on $(L, \leq)$ is a triangular seminorm on $(L, \leq)$ that is commutative and associative. The set of the triangular norms on $(L, \leq)$ will be denoted by $\mathcal{N}(L, \leq)$.*
2. *A triangular conorm (shortly t-conorm) S on $(L, \leq)$ is a triangular semiconorm on $(L, \leq)$ that is commutative and associative. The set of the triangular conorms on $(L, \leq)$ will be denoted by $\mathcal{C}(L, \leq)$.*

Definition 3.3 (Order Norms). *Triangular seminorms, semiconorms, norms and conorms on $(L, \leq)$ are collectively called order norms on $(L, \leq)$. The set of the order norms on $(L, \leq)$ will be denoted by $\mathcal{O}_s(L, \leq)$. The set of the triangular norms and conorms on $(L, \leq)$ will be denoted by $\mathcal{O}(L, \leq)$.*

Of course, we have the following inclusions and equalities:

$$\begin{aligned}
&\mathcal{N}(L, \leq) \subseteq \mathcal{N}_s(L, \leq) \subset L^{L^2} \\
&\mathcal{C}(L, \leq) \subseteq \mathcal{C}_s(L, \leq) \subset L^{L^2} \\
&\mathcal{O}_s(L, \leq) = \mathcal{N}_s(L, \leq) \cup \mathcal{C}_s(L, \leq) \\
&\mathcal{O}(L, \leq) = \mathcal{N}(L, \leq) \cup \mathcal{C}(L, \leq),
\end{aligned}$$

where L^{L^2} is the set of the $L^2 - L$ mappings.

In the sequel, we shall denote by P, Q, T and S respectively an arbitrary t-seminorm, t-semiconorm, t-norm and t-conorm on $(L, \leq)$, unless explicitly stated to the contrary.

3.2 Important Properties. In what follows, we give a few important properties of our generalized triangular (semi)norms and (semi)conorms. Most of these properties are generalizations of results that have been proven for the homologous operators on $([0, 1], \leq)$ (see [8,13,14,23]). We shall therefore omit the proofs of these generalizations. We shall explicitly state the proof of such a generalization only when it is rendered more complicated by the potential incomparability of the elements of $(L, \leq)$.

The definitions explicitly fix the values of order norms on two of the four 'boundaries of L^2'. It is easily proven that they also leave no choice for the values order norms take on the two opposite boundaries.

Proposition 3.1 (Additional Boundary Properties).

1. $(\forall \lambda \in L)(P(\ell, \lambda) = P(\lambda, \ell) = \ell)$.
2. $(\forall \lambda \in L)(Q(u, \lambda) = Q(\lambda, u) = u)$.
3. $(\forall \lambda \in L)(T(\ell, \lambda) = T(\lambda, \ell) = \ell)$.
4. $(\forall \lambda \in L)(S(u, \lambda) = S(\lambda, u) = u)$.

Proposition 3.1 allows us to answer an important question regarding the possible coincidence of t-(semi)norms and t-(semi)conorms. Indeed, let P and Q be respectively an arbitrary t-seminorm and an arbitrary t-semiconorm on $(L, \leq)$, and suppose that $P = Q$. From the defining boundary conditions and the additional boundary properties we deduce, for arbitrary λ in L, that $\lambda = Q(\ell, \lambda) = P(\ell, \lambda) = \ell$, which implies that $L = \{\ell\}$—and therefore also $u = \ell$. We have however excluded this 'pathological' case from the beginning. A triangular (semi)norm can therefore *never* coincide with a triangular (semi)conorm, or equivalently,

$$\mathcal{N}_s(L, \leq) \cap \mathcal{C}_s(L, \leq) = \emptyset \qquad \text{and} \qquad \mathcal{N}(L, \leq) \cap \mathcal{C}(L, \leq) = \emptyset.$$

Triangular (semi)norms and (semi)conorms are, in other words, distinct notions. They can however be considered as dual notions, in the sense of the following proposition. This duality also leads to another connection between t-(semi)norms and t-(semi)conorms that will be studied in the next section.

Proposition 3.2 (Duality Principle).

(1) *A t-seminorm on $(L, \leq)$ is a t-semiconorm on the dual structure $(L, \geq)$.*
(2) *A t-semiconorm on $(L, \leq)$ is a t-seminorm on the dual structure $(L, \geq)$.*
(3) *A t-norm on $(L, \leq)$ is a t-conorm on the dual structure $(L, \geq)$.*
(4) *A t-conorm on $(L, \leq)$ is a t-norm on the dual structure $(L, \geq)$.*

Proof. It is obvious that the isotonicity, associativity and commutativity conditions are dually invariant. In the dual structure $(L, \geq)$, ℓ is the greatest element and u is the smallest element. This implies that the boundary conditions for t-(semi)norms are the duals of the boundary conditions for t-(semi)conorms and vice versa. $\square$

The following definition will prove very useful in the sequel.

Definition 3.4. *Let us define the following binary operators on $(L, \leq)$:*

(1) $Z\colon L^2 \to L\colon (\lambda, \mu) \mapsto Z(\lambda, \mu)$, *where* $Z(\lambda, \mu) = \begin{cases} \lambda & ; \quad \mu = u \\ \mu & ; \quad \lambda = u \\ \ell & ; \quad \textit{elsewhere}; \end{cases}$

(2) $Z^*\colon L^2 \to L\colon (\lambda, \mu) \mapsto Z^*(\lambda, \mu)$, *where* $Z^*(\lambda, \mu) = \begin{cases} \lambda & ; \quad \mu = \ell \\ \mu & ; \quad \lambda = \ell \\ u & ; \quad \textit{elsewhere}. \end{cases}$

Furthermore, if $(L, \leq)$ is a bounded lattice, the following definitions make sense:

(3) $\frown\colon L^2 \to L\colon (\lambda, \mu) \mapsto \lambda \frown \mu$, *where* $\lambda \frown \mu = \inf(\lambda, \mu)$;
(4) $\smile\colon L^2 \to L\colon (\lambda, \mu) \mapsto \lambda \smile \mu$, *where* $\lambda \smile \mu = \sup(\lambda, \mu)$.

The operators Z and Z^* are generalizations of well-known binary operators defined on the real unit interval (see, for instance, [9,13,14]). $\frown$ and $\smile$ are the classical operators 'meet' and 'join', that can be defined on arbitrary—not necessarily bounded—lattices. For the most important properties of the latter operators, we refer to [2]. The operators $\frown$, $\smile$, Z and Z^* play an important role in order norm theory. In the following propositions, we prove among other things that they belong to the set $\mathcal{O}_s(L, \leq)$ of the order norms on $(L, \leq)$. At the same time, they constitute a natural boundary for this set.

Proposition 3.3 (Special Order Norms).

(1) *Z is a triangular (semi)norm on $(L, \leq)$.*
(2) *Z^* is a triangular (semi)conorm on $(L, \leq)$.*

Moreover, if $(L, \leq)$ is a bounded lattice, the following holds.

(3) *$\frown$ is a triangular (semi)norm on $(L, \leq)$.*
(4) *$\smile$ is a triangular (semi)conorm on $(L, \leq)$.*

Proof. The proof of (3) and (4) is immediate, taking into account the properties of meet and join in lattices. We shall now prove (1). The proof of (2) is analogous. By definition, Z satisfies the boundary conditions of triangular norms. It is also obvious that Z is isotonic and commutative. We proceed to show that Z is associative. Let (λ, μ, ν) be an arbitrary element of L^3. Then, by definition,

$$Z(Z(\lambda, \mu), \nu) = \begin{cases} Z(\lambda, \mu) & ; \quad \nu = u \\ \nu & ; \quad Z(\lambda, \mu) = u \\ \ell & ; \quad \text{elsewhere} \end{cases} = \begin{cases} \lambda & ; \quad \mu = \nu = u \\ \mu & ; \quad \lambda = \nu = u \\ \nu & ; \quad \lambda = \mu = u \\ \ell & ; \quad \text{elsewhere} \end{cases}$$

and an analogous line of reasoning yields the same result for $Z(\lambda, Z(\mu, \nu))$. $\square$

This also means that the existence of triangular (semi)norms and (semi)conorms on an arbitrary bounded poset $(L, \leq)$ is always guaranteed, or in other words,

$$\emptyset \subset \mathcal{N}(L, \leq) \subseteq \mathcal{N}_s(L, \leq) \qquad \text{and} \qquad \emptyset \subset \mathcal{C}(L, \leq) \subseteq \mathcal{C}_s(L, \leq).$$

Proposition 3.4.

(1) $(\forall(\lambda,\mu)\in L^2)(Z(\lambda,\mu)\leq P(\lambda,\mu))$.
(2) $(\forall(\lambda,\mu)\in L^2)(Q(\lambda,\mu)\leq Z^*(\lambda,\mu))$.

Furthermore, if $(L,\leq)$ is a bounded lattice, the following holds.

(3) $(\forall(\lambda,\mu)\in L^2)(P(\lambda,\mu)\leq \lambda\frown\mu)$.
(4) $(\forall(\lambda,\mu)\in L^2)(\lambda\smile\mu\leq Q(\lambda,\mu))$.

Of course, analogous results are valid for arbitrary triangular norms and conorms.

In the literature, the following alternative definitions for the triangular norm Z and conorm Z^* on the real unit interval $[0,1]$ often appear (see, for instance, [23]):

$$Z\colon[0,1]^2\to[0,1]\colon\quad(\lambda,\mu)\mapsto\begin{cases}\lambda\frown\mu & ;\quad \lambda\smile\mu=1\\ 0 & ;\quad \lambda\smile\mu<1\end{cases}$$
$$Z^*\colon[0,1]^2\to[0,1]\colon\quad(\lambda,\mu)\mapsto\begin{cases}\lambda\smile\mu & ;\quad \lambda\frown\mu=0\\ 1 & ;\quad \lambda\frown\mu>0.\end{cases}$$

We could try and use these alternative definitions to introduce the operators Z and Z^* on arbitrary bounded lattices—not posets. They would of course coincide with ours in the case $(L,\leq)=([0,1],\leq)$, or more generally, whenever $(L,\leq)$ is a bounded chain. However, this would no longer necessarily be the case for arbitrary bounded lattices. Nevertheless, one readily verifies that the extended alternative definitions also lead to triangular norms and conorms, that therefore satisfy the ‘boundary properties’ of proposition 3.4. An example will clarify this.

Example 3.1. Consider the universe $X=\{a,b,c\}$ and the complete Boolean lattice $(\mathcal{P}(X),\subseteq)$ with smallest element $\emptyset$ and greatest element X, where of course, $\mathcal{P}(X)$ is the power class of X. Consider the elements $\{a,b\}$ and $\{b,c\}$ of this lattice. We have that $\{a,b\}\cup\{b,c\}=X$ and $\{a,b\}\cap\{b,c\}=\{b\}$. The extension of the alternative definition of Z to $(\mathcal{P}(X),\subseteq)$ would yield $Z(\{a,b\},\{b,c\})=\{a,b\}\cap\{b,c\}=\{b\}\supset\emptyset$, since $\{a,b\}\cup\{b,c\}=X$, whereas our definition yields $Z(\{a,b\},\{b,c\})=\emptyset$ because $\{a,b\}\subset X$ and $\{b,c\}\subset X$.

Order norms may be considered as natural extensions of the lattice-theoretic operators $\frown$ and $\smile$. The following propositions tell us however that a number of properties are reserved for meet and join, and cannot be shared with other order norms. These propositions are generalizations of results that have been proven for the homologous operators on $([0,1],\leq)$ (see, for instance, [8,13,14,23]). As stated before, we shall only explicitly state the proofs of these propositions when they are rendered more complicated by the potential incomparability of the elements of $(L,\leq)$.

Proposition 3.5 (Distributivity Implies Absorption).

(1) $(\forall(\lambda,\mu,\nu)\in L^3)(P(\lambda,Q(\mu,\nu))=Q(P(\lambda,\mu),P(\lambda,\nu)))\Rightarrow(\forall(\lambda,\mu)\in L^2)(Q(P(\lambda,\mu),\lambda)=\lambda)$
and
$(\forall(\lambda,\mu,\nu)\in L^3)(P(Q(\nu,\mu),\lambda)=Q(P(\nu,\lambda),P(\mu,\lambda)))\Rightarrow(\forall(\lambda,\mu)\in L^2)(Q(\lambda,P(\mu,\lambda))=\lambda)$.
(2) $(\forall(\lambda,\mu,\nu)\in L^3)(Q(\lambda,P(\mu,\nu))=P(Q(\lambda,\mu),Q(\lambda,\nu)))\Rightarrow(\forall(\lambda,\mu)\in L^2)(P(Q(\lambda,\mu),\lambda)=\lambda)$
and
$(\forall(\lambda,\mu,\nu)\in L^3)(Q(P(\nu,\mu),\lambda)=P(Q(\nu,\lambda),Q(\mu,\lambda)))\Rightarrow(\forall(\lambda,\mu)\in L^2)(P(\lambda,Q(\mu,\lambda))=\lambda)$.
(3) $(\forall(\lambda,\mu,\nu)\in L^3)(T(\lambda,S(\mu,\nu))=S(T(\lambda,\mu),T(\lambda,\nu)))\Rightarrow(\forall(\lambda,\mu)\in L^2)(S(T(\lambda,\mu),\lambda)=\lambda)$.
(4) $(\forall(\lambda,\mu,\nu)\in L^3)(S(\lambda,T(\mu,\nu))=T(S(\lambda,\mu),S(\lambda,\nu)))\Rightarrow(\forall(\lambda,\mu)\in L^2)(T(S(\lambda,\mu),\lambda)=\lambda)$.

Proposition 3.6 (Absorption Implies Idempotence).

(1) $(\forall(\lambda,\mu)\in L^2)(Q(P(\lambda,\mu),\lambda)=\lambda)\Rightarrow(\forall\lambda\in L)(Q(\lambda,\lambda)=\lambda)$
and
$(\forall(\lambda,\mu)\in L^2)(Q(\lambda,P(\mu,\lambda))=\lambda)\Rightarrow(\forall\lambda\in L)(Q(\lambda,\lambda)=\lambda)$.
(2) $(\forall(\lambda,\mu)\in L^2)(P(Q(\lambda,\mu),\lambda)=\lambda)\Rightarrow(\forall\lambda\in L)(P(\lambda,\lambda)=\lambda)$
and
$(\forall(\lambda,\mu)\in L^2)(P(\lambda,Q(\mu,\lambda))=\lambda)\Rightarrow(\forall\lambda\in L)(P(\lambda,\lambda)=\lambda)$.
(3) $(\forall(\lambda,\mu)\in L^2)(S(T(\lambda,\mu),\lambda)=\lambda)\Rightarrow(\forall\lambda\in L)(S(\lambda,\lambda)=\lambda)$.
(4) $(\forall(\lambda,\mu)\in L^2)(T(S(\lambda,\mu),\lambda)=\lambda)\Rightarrow(\forall\lambda\in L)(T(\lambda,\lambda)=\lambda)$.

We want to stress that in proving propositions 3.5 and 3.6, use is made only of the *defining boundary conditions* and *additional boundary properties* for order norms. This means that these propositions are more generally valid for any couple of binary operators on L, one of which satisfies the defining boundary conditions and additional boundary properties for t-seminorms and the other of which satisfies the defining boundary conditions and additional boundary properties for t-semiconorms.

Proposition 3.7 (Idempotence). *Let $(L, \leq)$ be a bounded lattice.*

(1) *P is idempotent if and only if $P = \frown$.*
(2) *Q is idempotent if and only if $Q = \smile$.*
(3) *T is idempotent if and only if $T = \frown$.*
(4) *S is idempotent if and only if $S = \smile$.*

Proof. It suffices to prove (1) and (2). Let us for example show that (1) holds. The proof of (2) is analogous. From the properties of meets in lattices, we may conclude that $\frown$ is an idempotent t-seminorm. Conversely, let the t-seminorm P on $(L, \leq)$ be idempotent, i.e., $(\forall \lambda \in L)(P(\lambda, \lambda) = \lambda)$. Consider an arbitrary couple (λ, μ) in L^2. There are three possibilities.

a. $\lambda \geq \mu$. It then follows from the isotonicity of P that $P(\lambda, \mu) \geq P(\mu, \mu) = \mu = \lambda \frown \mu$. Hence, $P(\lambda, \mu) = \lambda \frown \mu$, taking into account proposition 3.4.
b. $\lambda \leq \mu$. It then follows from the isotonicity of P that $P(\lambda, \mu) \geq P(\lambda, \lambda) = \lambda = \lambda \frown \mu$. Hence, $P(\lambda, \mu) = \lambda \frown \mu$, taking into account proposition 3.4.
c. $\lambda \parallel \mu$, i.e. λ and μ are incomparable. We already know from proposition 3.4 that $P(\lambda, \mu) \leq \lambda \frown \mu$. Furthermore, $\lambda \frown \mu$ is comparable to μ, and therefore a. and b. imply that $P(\lambda \frown \mu, \mu) = (\lambda \frown \mu) \frown \mu = \lambda \frown \mu$. Should $P(\lambda, \mu) < \lambda \frown \mu$, it would immediately follow that $P(\lambda, \mu) < P(\lambda \frown \mu, \mu)$. But on the other hand, since $\lambda \frown \mu \leq \lambda$ and P is isotonic, we also have that $P(\lambda, \mu) \geq P(\lambda \frown \mu, \mu)$. From this contradiction we conclude that $P(\lambda, \mu) = \lambda \frown \mu$. □

Propositions 3.5–3.7 imply among other things that t-(semi)norms can only be distributive w.r.t. a t-(semi)conorm if the latter is the join operator, and dually, that t-(semi)conorms can only be distributive w.r.t. a t-(semi)norm if the latter is the meet operator. This conclusion can in fact be considered as a first step towards the material in section 5.

3.3 Potential Properties of Order Norms.

Let us now study a few interesting potential properties of order norms, some of which (definitions 3.5 and 3.6) are generalizations of properties of triangular norms and conorms on the real unit interval (see, for instance, [13]). On the other hand, definitions 3.7 and 3.8 introduce new properties that are not without interest, as we shall point out in section 6. In proposition 3.8, we study in more detail how these potential properties are related.

Definition 3.5.

(1) *A triangular seminorm P on $(L, \leq)$ is called Archimedean iff $(\forall \lambda \in]\ell, u[)(P(\lambda, \lambda) < \lambda)$.*
(2) *A triangular semiconorm Q on $(L, \leq)$ is called Archimedean iff $(\forall \lambda \in]\ell, u[)(Q(\lambda, \lambda) > \lambda)$.*

Of course, these definitions in particular also apply to triangular norms respectively conorms.

Definition 3.6.

(1) *A triangular seminorm P on $(L, \leq)$ is called positive (or is said to have no zero divisors) iff*

$$(\forall (\lambda, \mu) \in L^2)(P(\lambda, \mu) = \ell \Rightarrow (\lambda = \ell \text{ or } \mu = \ell)).$$

(2) *A triangular semiconorm Q on $(L, \leq)$ is called positive (or is said to have no zero divisors) iff*

$$(\forall (\lambda, \mu) \in L^2)(Q(\lambda, \mu) = u \Rightarrow (\lambda = u \text{ or } \mu = u)).$$

Of course, these definitions in particular also apply to triangular norms respectively conorms.

Definition 3.7.

(1) *A triangular seminorm P on $(L, \leq)$ is called strongly resolving on the left iff*

$$(\forall(\lambda_1, \lambda_2, \mu) \in L^3)(P(\lambda_1, \mu) \leq P(\lambda_2, \mu) \Rightarrow (\lambda_1 \leq \lambda_2 \textit{ or } \mu = \ell)),$$

strongly resolving on the right iff

$$(\forall(\lambda_1, \lambda_2, \mu) \in L^3)(P(\mu, \lambda_1) \leq P(\mu, \lambda_2) \Rightarrow (\lambda_1 \leq \lambda_2 \textit{ or } \mu = \ell))$$

and strongly resolving iff it is strongly resolving on the left and on the right.

(2) *A triangular semiconorm Q on $(L, \leq)$ is called strongly resolving on the left iff*

$$(\forall(\lambda_1, \lambda_2, \mu) \in L^3)(Q(\lambda_1, \mu) \leq Q(\lambda_2, \mu) \Rightarrow (\lambda_1 \leq \lambda_2 \textit{ or } \mu = u)),$$

strongly resolving on the right iff

$$(\forall(\lambda_1, \lambda_2, \mu) \in L^3)(Q(\mu, \lambda_1) \leq Q(\mu, \lambda_2) \Rightarrow (\lambda_1 \leq \lambda_2 \textit{ or } \mu = u))$$

and strongly resolving iff it is strongly resolving on the left and on the right.

Of course, these definitions in particular also apply to triangular norms and conorms respectively. Furthermore, if a triangular (co)norm is strongly resolving on the left or on the right, it is strongly resolving.

Definition 3.8.

(1) *A triangular seminorm P on $(L, \leq)$ is called resolving on the left iff*

$$(\forall(\lambda_1, \lambda_2, \mu) \in L^3)(P(\lambda_1, \mu) = P(\lambda_2, \mu) \Rightarrow (\lambda_1 = \lambda_2 \textit{ or } \mu = \ell)),$$

resolving on the right iff

$$(\forall(\lambda_1, \lambda_2, \mu) \in L^3)(P(\mu, \lambda_1) = P(\mu, \lambda_2) \Rightarrow (\lambda_1 = \lambda_2 \textit{ or } \mu = \ell)),$$

and resolving iff it is resolving on the left and on the right.

(2) *A triangular semiconorm Q on $(L, \leq)$ is called resolving on the left iff*

$$(\forall(\lambda_1, \lambda_2, \mu) \in L^3)(Q(\lambda_1, \mu) = Q(\lambda_2, \mu) \Rightarrow (\lambda_1 = \lambda_2 \textit{ or } \mu = u)),$$

resolving on the right iff

$$(\forall(\lambda_1, \lambda_2, \mu) \in L^3)(Q(\mu, \lambda_1) = Q(\mu, \lambda_2) \Rightarrow (\lambda_1 = \lambda_2 \textit{ or } \mu = u)),$$

and resolving iff it is resolving on the left and on the right.

Of course, these definitions in particular also apply to triangular norms respectively conorms. Furthermore, if a triangular (co)norm is resolving on the left or on the right, it is resolving.

Proposition 3.8.

(1) *If a triangular (semi)(co)norm on $(L, \leq)$ is strongly resolving on the left, then it is resolving on the left.*
(2) *If a triangular (semi)(co)norm on $(L, \leq)$ is strongly resolving on the right, then it is resolving on the right.*
(3) *If a triangular (semi)(co)norm on $(L, \leq)$ is strongly resolving, then it is resolving.*
(4) *If a triangular (semi)(co)norm on $(L, \leq)$ is resolving on the left or on the right, then it is positive and Archimedean.*

Proof. We shall prove (1) and (4) for triangular seminorms and semiconorms respectively. The proofs of (1) and (4) for other order norms, and the proofs of (2) and (3) are similar. First, assume that the triangular seminorm P on $(L, \leq)$ is strongly resolving on the left. Let λ_1, λ_2 and μ be arbitrary elements of L, and assume that $P(\lambda_1, \mu) = P(\lambda_2, \mu)$. This implies that $P(\lambda_1, \mu) \leq P(\lambda_2, \mu)$ and $P(\lambda_2, \mu) \leq P(\lambda_1, \mu)$. From the assumption, it follows that

$$\begin{cases} \lambda_1 \leq \lambda_2 \text{ or } \mu = \ell \\ \lambda_2 \leq \lambda_1 \text{ or } \mu = \ell, \end{cases}$$

which implies that $\lambda_1 = \lambda_2$ or $\mu = \ell$. This means that P is resolving on the left if it is strongly resolving on the left.

Next, assume that the triangular semiconorm Q is resolving on the left, i.e.,

$$(\forall(\lambda_1, \lambda_2, \mu) \in L^3)(Q(\lambda_1, \mu) = Q(\lambda_2, \mu) \Rightarrow (\lambda_1 = \lambda_2 \text{ or } \mu = u)).$$

Putting $\lambda_1 = \lambda$ and $\lambda_2 = u$ in this expression yields, taking into account the additional boundary properties for t-semiconorms,

$$(\forall(\lambda, \mu) \in L^2)(Q(\lambda, \mu) = u \Rightarrow (\lambda = u \text{ or } \mu = u)),$$

which means that Q is positive. Finally, in order to prove that Q is Archimedean, let us assume *ex absurdo* that it is not. This implies that there exists at least one μ in $]\ell, u[$ satisfying $Q(\mu, \mu) = \mu$. Taking into account the defining boundary conditions of triangular semiconorms, this leads to $Q(\mu, \mu) = \mu = Q(\ell, \mu)$, and since Q is assumed to be resolving on the left, this implies that $\mu = \ell$ or $\mu = u$, a contradiction. If Q is resolving on the right, a similar proof can be given. □

Using this proposition and the commutativity of triangular norms and conorms, we immediately conclude the following: if a triangular (co)norm is strongly resolving, then it is resolving, and if it is resolving, then it is positive and Archimedean.

3.4 Examples. We conclude this section with two examples that shed some light on the relation between our order norms and the conjunction and disjunction operators in important logics.

Example 3.2 (Two-Valued Logic). Consider the finite (Boolean) chain $(\mathcal{T}, \leq)$, already mentioned in example 2.2. This structure is very special, because

- as stated before, it has only one negation operator, that is the negation $\neg$ of (the truth-values of) classical logic;
- it has only one triangular (semi)norm, that is the conjunction $\wedge$ of (the truth-values of) classical logic and coincides with the join (or Boolean multiplication) of $(\mathcal{T}, \leq)$;
- it has only one triangular (semi)conorm, that is the disjunction $\vee$ of (the truth-values of) classical logic and coincides with the meet (or Boolean addition) of $(\mathcal{T}, \leq)$.

Example 3.3 (Three-Valued Logic). Consider the chain K_3 with three elements 0, $\frac{1}{2}$, 1, such that $0 < \frac{1}{2} < 1$. In example 2.3 we have already mentioned that this chain has a unique negation operator. Let us also verify which order norms can be defined on this chain. Since the defining boundary conditions and additional boundary properties determine the values an order norm takes on the 'boundaries of L^2', there is only freedom in the choice of the image of the couple $(\frac{1}{2}, \frac{1}{2})$. Taking into account the isotonicity and the boundary conditions, only the values 0 and $\frac{1}{2}$ are possible for t-(semi)norms, whereas for t-(semi)conorms only the values $\frac{1}{2}$ and 1 need be considered. Furthermore taking into account propositions 3.3 and 3.4, we see that there are only two t-(semi)norms

- $\frown$, corresponding with the choice $\frac{1}{2}$ for the image of $(\frac{1}{2}, \frac{1}{2})$
- Z, corresponding with the choice 0 for the image of $(\frac{1}{2}, \frac{1}{2})$;

and that there are only two t-(semi)conorms

- $\smile$, corresponding with the choice $\frac{1}{2}$ for the image of $(\frac{1}{2}, \frac{1}{2})$
- Z^*, corresponding with the choice 1 for the image of $(\frac{1}{2}, \frac{1}{2})$.

Among these order norms especially $\frown$ and $\smile$ are used as a conjunction respectively disjunction operator in three-valued logics. We specifically have in mind the ternary systems of Kleene, Lukasiewicz, Gödel (or Heyting), ... [18].

4. Order-Theoretic Invariance

In this section, a denotes an arbitrary affirmation operator on $(L, \leq)$, and n denotes an arbitrary negation operator on $(L, \leq)$. Furthermore, if λ is an arbitrary element of L, we shall denote by $\underline{\lambda}$ the constant $L^2 - \{\lambda\}$ mapping.

We start the discussion by defining an important class of $L^{L^2} - L^{L^2}$ mappings.

Definition 4.1. *Let f be permutation of L. Then the mapping $\mathcal{T}_f \colon L^{L^2} \to L^{L^2}$ is defined as follows. For an arbitary $L^2 - L$ mapping g:*

$$\mathcal{T}_f(g) \stackrel{\text{def}}{=} f^{-1} \circ g \circ (f \circ \mathrm{pr}_1, f \circ \mathrm{pr}_2) = f^{-1} \circ g \circ (f \times f),$$

where, of course, pr_1 and pr_2 are first and second coordinate projection operators, and $\times$ is the direct product operator of mappings.

Proposition 4.1. *Let f_1 and f_2 be arbitrary permutations of L. Then $\mathcal{T}_{f_1 \circ f_2} = \mathcal{T}_{f_2} \circ \mathcal{T}_{f_1}$. Furthermore, $\mathcal{T}_{\mathbf{1}_L} = \mathbf{1}_{L^{L^2}}$, where in general $\mathbf{1}_A$ stands for the identity transformation of the set A. Finally, for an arbitrary permutation f of L, $\mathcal{T}_f$ is a permutation of L^{L^2}, and $\mathcal{T}_{f^{-1}} = (\mathcal{T}_f)^{-1}$.*

Proof. Let g be an arbitrary element of L^{L^2} and let λ and μ be arbitrary elements of L. Then

$$\begin{aligned}
\mathcal{T}_{f_1 \circ f_2}(g)(\lambda, \mu) &= (f_1 \circ f_2)^{-1}(g((f_1 \circ f_2)(\lambda), (f_1 \circ f_2)(\mu))) \\
&= f_2^{-1}(f_1^{-1}(g(f_1(f_2(\lambda)), f_1(f_2(\mu))))) \\
&= f_2^{-1}(\mathcal{T}_{f_1}(g)(f_2(\lambda), f_2(\mu))) \\
&= \mathcal{T}_{f_2}(\mathcal{T}_{f_1}(g))(\lambda, \mu),
\end{aligned}$$

which implies that $\mathcal{T}_{f_1 \circ f_2} = \mathcal{T}_{f_2} \circ \mathcal{T}_{f_1}$. Furthermore, $\mathcal{T}_{\mathbf{1}_L}(g) = \mathbf{1}_L^{-1} \circ g \circ (\mathbf{1}_L \circ \mathrm{pr}_1, \mathbf{1}_L \circ \mathrm{pr}_2) = g$, whence indeed $\mathcal{T}_{\mathbf{1}_L} = \mathbf{1}_{L^{L^2}}$. Finally, if we put $f_1 \stackrel{\text{def}}{=} f$ and $f_2 \stackrel{\text{def}}{=} f^{-1}$, we find on the one hand that $\mathcal{T}_{f \circ f^{-1}} = \mathcal{T}_{\mathbf{1}_L} = \mathbf{1}_{L^{L^2}}$ and on the other hand, using the relation just proven, that $\mathcal{T}_{f \circ f^{-1}} = \mathcal{T}_{f^{-1}} \circ \mathcal{T}_f$ and $\mathcal{T}_{f^{-1} \circ f} = \mathcal{T}_f \circ \mathcal{T}_{f^{-1}}$. Therefore,

$$\mathcal{T}_{f^{-1}} \circ \mathcal{T}_f = \mathcal{T}_f \circ \mathcal{T}_{f^{-1}} = \mathbf{1}_{L^{L^2}}.$$

From this, we may conclude that $\mathcal{T}_f$ is a permutation of L^{L^2}, and that $(\mathcal{T}_f)^{-1} = \mathcal{T}_{f^{-1}}$. □

This proposition implies that the algebraic structure $(\{\, \mathcal{T}_f \mid f \text{ is a permutation of } L \,\}, \circ\,)$ is a group. Furthermore, let f be an arbitrary permutation of L and let be g be an arbitrary $L^2 - L$ mapping, i.e., a binary operator on L. Then of course, $\mathcal{T}_f(g)$ is a binary operator on L as well. Almost by definition, f is an isomorphism between the algebraic structures (L, g) and $(L, \mathcal{T}_f(g))$. Also, definition 4.1 and proposition 4.1 tell us that between the operators f, g and $\mathcal{T}_f(g)$ generalized de Morgan laws are valid. Whenever n is a negation operator, we shall say that $\mathcal{T}_n(g)$ is the *dual operator* of g w.r.t. n. The reason for this terminology is made clear by theorem 4.1.

Definition 4.2. *Let us define the following relation on L^{L^2}. For arbitrary $L^2 - L$ mappings g_1 and g_2:*

$$g_1 \preceq g_2 \Leftrightarrow (\forall (\lambda, \mu) \in L^2)(g_1(\lambda, \mu) \leq g_2(\lambda, \mu)).$$

Of course, '$\preceq$' is a partial order relation on L^{L^2}. More specifically, $(L^{L^2}, \preceq)$ is a bounded poset with greatest element $\underline{u}$ and smallest element $\underline{\ell}$. Moreover, if $(L, \leq)$ is a bounded lattice, $(L^{L^2}, \preceq)$ is a bounded lattice as well, and if $(L, \leq)$ is a complete lattice, $(L^{L^2}, \preceq)$ too will be a complete lattice. Also, in general, $(\mathcal{O}_s(L, \leq), \preceq)$ and $(\mathcal{O}(L, \leq), \preceq)$ are bounded partially ordered sets with smallest element Z and greatest element Z^*. If $(L, \leq)$ is a bounded lattice, $(\mathcal{N}_s(L, \leq), \preceq)$ and $(\mathcal{N}(L, \leq), \preceq)$ are bounded partially ordered sets with smallest element Z and greatest element $\frown$, and $(\mathcal{C}_s(L, \leq), \preceq)$ and $(\mathcal{C}(L, \leq), \preceq)$ are bounded partially ordered sets with smallest element $\smile$ and greatest element Z^*.

Theorem 4.1. *Let f be an arbitrary permutation of L. Then:*

(1) *$\mathcal{T}_f$ is an order-automorphism of (affirmation operator on) $(L^{L^2}, \preceq)$ if and only if f is an order-automorphism of (affirmation operator on) $(L, \leq)$.*
(2) *$\mathcal{T}_f$ is a dual order-automorphism of (negation operator on) $(L^{L^2}, \preceq)$ if and only if f is a dual order-automorphism of (negation operator on) $(L, \leq)$.*

Proof. We shall give the proof of (2). The proof of (1) is analogous. First, assume that f is a dual order-automorphism of $(L, \leq)$. Consider two arbitrary elements g_1 and g_2 of L^{L^2}. Then, by definition,

$$g_1 \preceq g_2 \Leftrightarrow (\forall(\lambda, \mu) \in L^2)(g_1(\lambda, \mu) \leq g_2(\lambda, \mu))$$

and since f is a permutation of L

$$\Leftrightarrow (\forall(\lambda, \mu) \in L^2)(g_1(f(\lambda), f(\mu)) \leq g_2(f(\lambda), f(\mu)))$$

and since f and f^{-1} are order-reversing (see proposition 2.2)

$$\Leftrightarrow (\forall(\lambda, \mu) \in L^2)(\mathcal{T}_f(g_1)(\lambda, \mu) \geq \mathcal{T}_f(g_2)(\lambda, \mu))$$
$$\Leftrightarrow \mathcal{T}_f(g_2) \preceq \mathcal{T}_f(g_1).$$

This means that the permutation $\mathcal{T}_f$ is order-reversing, and hence is a dual order-automorphism of $(L^{L^2}, \preceq)$.

Conversely, assume that $\mathcal{T}_f$ is a dual order-automorphism of $(L^{L^2}, \preceq)$. Consider two arbitrary elements λ and μ of L. Then,

$$\lambda \leq \mu \Leftrightarrow \underline{\lambda} \preceq \underline{\mu} \Leftrightarrow \mathcal{T}_f(\underline{\mu}) \preceq \mathcal{T}_f(\underline{\lambda}) \Leftrightarrow f^{-1}(\mu) \leq f^{-1}(\lambda).$$

This means that f^{-1} and therefore also f (see proposition 2.2) is order-reversing. □

Proposition 4.2. *Let g be an arbitrary $L^2 - L$ mapping. Then:*

(1) $g \in \mathcal{N}_s(L, \leq) \Leftrightarrow \mathcal{T}_a(g) \in \mathcal{N}_s(L, \leq) \Leftrightarrow \mathcal{T}_n(g) \in \mathcal{C}_s(L, \leq)$;
(2) $g \in \mathcal{C}_s(L, \leq) \Leftrightarrow \mathcal{T}_a(g) \in \mathcal{C}_s(L, \leq) \Leftrightarrow \mathcal{T}_n(g) \in \mathcal{N}_s(L, \leq)$;
(3) $g \in \mathcal{N}(L, \leq) \Leftrightarrow \mathcal{T}_a(g) \in \mathcal{N}(L, \leq) \Leftrightarrow \mathcal{T}_n(g) \in \mathcal{C}(L, \leq)$;
(4) $g \in \mathcal{C}(L, \leq) \Leftrightarrow \mathcal{T}_a(g) \in \mathcal{C}(L, \leq) \Leftrightarrow \mathcal{T}_n(g) \in \mathcal{N}(L, \leq)$.

Proof. As an example, we shall give the proof of (3). The proof of (4) is completely analogous. The proofs of (1) and (2) are in principle contained in the proofs of (3) and (4). Let T be an arbitrary t-norm on $(L, \leq)$ and let S be an arbitrary t-conorm on $(L, \leq)$. We first prove that $\mathcal{T}_n(S)$ is a t-norm on $(L, \leq)$.
a. *Boundary conditions.* For arbitrary λ in L we have that

$$\mathcal{T}_n(S)(u, \lambda) = n^{-1}(S(n(u), n(\lambda))) = n^{-1}(S(\ell, n(\lambda))) = n^{-1}(n(\lambda)) = \lambda.$$

b. *Isotonicity.* Consider an arbitrary $(\lambda_1, \lambda_2, \mu_1, \mu_2)$ in L^4 and assume that $\lambda_1 \leq \lambda_2$ and $\mu_1 \leq \mu_2$. It follows from the antitonicity of n that

$$n(\lambda_1) \geq n(\lambda_2) \text{ and } n(\mu_1) \geq n(\mu_2),$$

and, since S is isotonic

$$S(n(\lambda_1), n(\mu_1)) \geq S(n(\lambda_2), n(\mu_2)).$$

Furthermore, since n^{-1} is antitonic

$$\mathcal{T}_n(S)(\lambda_1, \mu_1) \leq \mathcal{T}_n(S)(\lambda_2, \mu_2).$$

c. *Associativity.* For arbitrary (λ, μ, ν) in L^3 we have that

$$\begin{aligned}\mathcal{T}_n(S)(\lambda, \mathcal{T}_n(S)(\mu, \nu)) &= n^{-1}(S(n(\lambda), n(\mathcal{T}_n(S)(\mu, \nu))))\\ &= n^{-1}(S(n(\lambda), S(n(\mu), n(\nu))))\\ &= n^{-1}(S(S(n(\lambda), n(\mu)), n(\nu)))\\ &= n^{-1}(S(n(\mathcal{T}_n(S)(\lambda, \mu)), n(\nu)))\\ &= \mathcal{T}_n(S)(\mathcal{T}_n(S)(\lambda, \mu), \nu).\end{aligned}$$

d. *Commutativity.* For arbitrary (λ, μ) in L^2, we obtain

$$\mathcal{T}_n(S)(\lambda, \mu) = n^{-1}(S(n(\lambda), n(\mu))) = n^{-1}(S(n(\mu), n(\lambda))) = \mathcal{T}_n(S)(\mu, \lambda).$$

This proves that $\mathcal{T}_n(S)$ is a t-norm on $(L, \leq)$. In an analogous way, we can prove that $\mathcal{T}_a(S)$ and $\mathcal{T}_n(T)$ are t-conorms on $(L, \leq)$, and that $\mathcal{T}_a(T)$ is a t-norm on $(L, \leq)$.

Now, assume that $g \in \mathcal{N}(L, \leq)$. Using the results just proven, we may conclude that $\mathcal{T}_n(g) \in \mathcal{C}(L, \leq)$ and that $\mathcal{T}_a(g) \in \mathcal{N}(L, \leq)$. Conversely, assume that $\mathcal{T}_n(g) \in \mathcal{C}(L, \leq)$. Since n^{-1} is also a negation operator on $(L, \leq)$, we find, using proposition 4.1, that

$$\mathcal{N}(L, \leq) \ni \mathcal{T}_{n^{-1}}(\mathcal{T}_n(g)) = (\mathcal{T}_{n^{-1}} \circ \mathcal{T}_n)(g) = \mathcal{T}_{n \circ n^{-1}}(g) = \mathbf{1}_{L^{L^2}}(g) = g.$$

In an analogous way, we can prove that if $\mathcal{T}_a(g) \in \mathcal{N}(L, \leq)$, then $g = \mathcal{T}_{a^{-1}}(\mathcal{T}_a(g)) \in \mathcal{N}(L, \leq)$. □

This also means that:

- the restriction $\mathcal{T}_a|\mathcal{O}_s(L, \leq)$ is an order-automorphism of $(\mathcal{O}_s(L, \leq), \preceq)$;
- the restriction $\mathcal{T}_a|\mathcal{N}_s(L, \leq)$ is an order-automorphism of $(\mathcal{N}_s(L, \leq), \preceq)$;
- the restriction $\mathcal{T}_a|\mathcal{C}_s(L, \leq)$ is an order-automorphism of $(\mathcal{C}_s(L, \leq), \preceq)$;
- the restriction $\mathcal{T}_n|\mathcal{O}_s(L, \leq)$ is a dual order-automorphism of $(\mathcal{O}_s(L, \leq), \preceq)$;
- the restriction $\mathcal{T}_n|\mathcal{N}_s(L, \leq)$ is a dual order-isomorphism between $(\mathcal{N}_s(L, \leq), \preceq)$ and $(\mathcal{C}_s(L, \leq), \preceq)$;

and analogous observations can be made for triangular norms and conorms instead of triangular seminorms and semiconorms. The fact that $\mathcal{T}_n$ transforms a triangular (semi)norm into an triangular (semi)conorm and vice versa, is of course closely related with the duality principle of proposition 3.2, since the negation operator n transforms the bounded poset $(L, \leq)$ into its dual structure $(L, \geq)$. As another immediate consequence, we find, taking into account proposition 2.1 and theorem 4.1, that

- $\mathcal{T}_a(Z) = Z$ and $\mathcal{T}_a(Z^*) = Z^*$;
- $\mathcal{T}_n(Z) = Z^*$ and $\mathcal{T}_n(Z^*) = Z$;

and when $(L, \leq)$ is a bounded lattice,

- $\mathcal{T}_a(\frown) = \frown$ and $\mathcal{T}_a(\smile) = \smile$;
- $\mathcal{T}_n(\frown) = \smile$ and $\mathcal{T}_n(\smile) = \frown$.

These invariance and dual invariance properties are once more immediate generalizations of de Morgan's laws. It is also immediately verified that $\mathcal{T}_a$ and $\mathcal{T}_n$ preserve the distributivity, absorption and idempotence properties, that play a role in propositions 3.5–3.7. It is also easily proven that $\mathcal{T}_a$ preserves the potential properties of order norms, discussed in section 3.3. Furthermore, $\mathcal{T}_n$ transforms these properties for t-(semi)norms into their counterparts for t-(semi)conorms and vice versa. As an example, we explicitly state that a triangular seminorm P on $(L, \leq)$ is strongly resolving on the left if and only if its dual t-semiconorm $\mathcal{T}_n(P)$ on $(L, \leq)$ w.r.t. n is strongly resolving on the left.

5. An Algebraic Excursion

In this section, we make the following assumptions. By $(C, \leq)$ we mean a *complete lattice*, i.e., the infimum and supremum of arbitrary subsets of L exist. The smallest element $\sup \emptyset$ of L is denoted by ℓ, the greatest element $\inf \emptyset$ by u. It is also assumed that $\ell \neq u$. As before, $(L, \leq)$ denotes a bounded partially ordered set. Since in what follows no confusion can arise between $(C, \leq)$ and $(L, \leq)$, we shall also denote the smallest and greatest element of $(L, \leq)$ by ℓ respectively u. Finally, we shall denote by P

an arbitrary triangular seminorm on $(L, \leq)$ or $(C, \leq)$, depending on the context. Similarly, Q stands for an arbitrary triangular semiconorm, T for an arbitrary triangular norm and S for an arbitrary triangular conorm on $(L, \leq)$ or $(C, \leq)$, again depending on the context.

Propositions 3.1 and 3.2 lead us to a few algebraic considerations concerning partially ordered groupoids and monoids, shortly *po*-groupoids and *po*-monoids (see, for instance, [2]). We can indeed consider a triangular seminorm P on $(L, \leq)$ as a binary multiplication, and investigate the algebraic structure $(L, \leq, P)$. The isotonicity and defining boundary conditions for t-seminorms imply that this structure is a *po*-groupoid with *identity* u. The additional boundary properties, together with the fact that ℓ is the smallest element of L, imply that this structure has a (unique) *zero* ℓ. Since all elements of L are 'smaller than or equal to' the identity u of the multiplication P, the structure $(L, \leq, P)$ is integral. Taking into account $(\forall(\lambda, \mu) \in L^2)(P(\lambda, \mu) \leq \lambda)$, all elements of L are ideal.

Similary, the isonicity, commutativity, assocativity and the defining boundary conditions for a triangular norm T on $(L, \leq)$ imply that the structure $(L, \leq, T)$ is a commutative *po*-monoid (with identity u). The additional boundary properties, together with the fact that ℓ is the smallest element of L, imply that this structure has a (unique) *zero* ℓ. For analogous reasons as before, the structure $(L, \leq, T)$ is integral and all elements of L are ideal.

For triangular (semi)conorms however, matters seem at first sight to be a bit more complicated. Indeed, the absorbing element u of a t-semiconorm Q on $(L, \leq)$ cannot be a zero for the structure $(L, \leq, Q)$, because it is not the smallest element of $(L, \leq)$. It is, however, the smallest element of $(L, \geq)$. Using proposition 3.2 and the conclusions derived above, we immediately find that $(L, \geq, Q)$ is a *po*-groupoid with identity ℓ and zero u, and that for an arbitrary triangular conorm S on $(L, \leq)$ the structure $(L, \geq, S)$ is a commutative *po*-monoid with (identity ℓ and) zero u. We may indeed conclude the following.

Proposition 5.1.

(1) $(L, \leq, P)$ *is an integral po-groupoid with zero* ℓ *and identity* u*, all elements of which are ideal.*
(2) $(L, \leq, T)$ *is an integral commutative po-monoid with zero* ℓ *and identity* u*, all elements of which are ideal.*
(3) $(L, \geq, Q)$ *is an integral po-groupoid with zero* u *and identity* ℓ*, all elements of which are ideal.*
(4) $(L, \geq, S)$ *is an integral commutative po-monoid with zero* u *and identity* ℓ*, all elements of which are ideal.*

In what follows, we shall say that an arbitrary $C^2 - C$ operator g is completely distributive w.r.t. sup, iff for arbitrary λ in C and for an arbitrary family $(\mu_j \mid j \in J)$ of elements of C:

$$\begin{cases} g(\lambda, \sup_{j\in J} \mu_j) = \sup_{j\in J} g(\lambda, \mu_j) \\ g(\sup_{j\in J} \mu_j, \lambda) = \sup_{j\in J} g(\mu_j, \lambda). \end{cases}$$

If g is completely distributive w.r.t. sup, then g is isotonic and $(\forall\lambda \in C)(g(\lambda, \ell) = g(\ell, \lambda) = \ell)$. This means that triangular seminorms are among the operators that *can* be completely distributive w.r.t. sup. For the complete distributivity of g w.r.t. inf an analogous definition and analogous (dual) conclusions can be formulated. This leads to the following definition.

Definition 5.1 (Complete Lattices with Order Norm).

(1) *The combination* $(C, \leq, P)$ *of a complete lattice* $(C, \leq)$ *provided with a t-seminorm* P *on* $(C, \leq)$ *that is completely distributive w.r.t.* sup *will be called a complete lattice with t-seminorm.*
(2) *The combination* $(C, \leq, T)$ *of a complete lattice* $(C, \leq)$ *provided with a t-norm* T *on* $(C, \leq)$ *that is completely distributive w.r.t.* sup *will be called a complete lattice with t-norm.*
(3) *The combination* $(C, \geq, Q)$ *of a complete lattice* $(C, \geq)$ *provided with a t-semiconorm* Q *on* $(C, \leq)$ *that is completely distributive w.r.t.* inf *will be called a complete lattice with t-semiconorm.*
(4) *The combination* $(C, \geq, S)$ *of a complete lattice* $(C, \geq)$ *provided with a t-conorm* S *on* $(C, \leq)$ *that is completely distributive w.r.t.* inf *will be called a complete lattice with t-conorm.*

In proposition 5.1 we have already shed some light on the algebraic significance of order norms in general. Upon inspection, we also find a connection between the structures introduced in the previous definition and notions in the theory of complete lattice groupoids (*cl*-groupoids) and complete lattice monoids (*cl*-monoids) (see, for instance, [2]).

Proposition 5.2.

(1) *A complete lattice with t-seminorm $(C, \leq, P)$ is an integral cl-groupoid with zero ℓ and identity u, all elements of which are ideal.*
(2) *A complete lattice with t-norm $(C, \leq, T)$ is a integral commutative cl-monoid with zero ℓ and identity u, all elements of which are ideal.*
(3) *A complete lattice with t-semiconorm $(C, \geq, Q)$ is an integral cl-groupoid with zero u and identity ℓ, all elements of which are ideal.*
(4) *A complete lattice with t-conorm $(C, \geq, S)$ is an integral commutative cl-monoid with zero u and identity ℓ, all elements of which are ideal.*

Proof. We first give a proof of (1). Since P is isotonic, we can consider P as a multiplicative operator on $(C, \leq)$. The defining boundary conditions imply that u is an identity for P, and the additional boundary properties together with the fact that ℓ is the smallest element of C imply that ℓ is a zero for P. From the complete distributivity of P w.r.t. sup it follows that $(C, \leq, P)$ is a *cl*-groupoid with zero ℓ and identity u. Since all elements of C are 'smaller than or equal to' the identity u of the multiplication P, the structure $(C, \leq, P)$ is integral. Taking into account $(\forall(\lambda, \mu) \in C^2)(P(\lambda, \mu) \leq \lambda)$ all elements of C are ideal.

The proof of (2) is now immediate, since a triangular norm is in particular a triangular seminorm, and since the multiplicative operator T is commutative and associative.

Although the proof of (3) follows from the proof of (4) by applying the duality principle, we shall give a more explicit proof, that also explains the notation $(C, \geq, Q)$. The t-semiconorm Q on $(C, \leq)$ is isotonic and may therefore be considered as as a multiplicative operator not only on $(C, \leq)$, but also on $(C, \geq)$. In both cases ℓ is the identity. However, it is only when Q is considered as a multiplicative operator on $(C, \geq)$ that on the one hand u is a zero for Q, and on the other hand all elements of C are integral and ideal.

The proof of (4) follows from that of (2), using the duality principle. □

We also have the following immediate order-theoretic invariance and dual invariance properties.

Proposition 5.1. *Let g be an arbitrary $C^2 - C$ mapping. Let a be an arbitrary affirmation operator on $(L, \leq)$ and let n be an arbitrary negation operator on $(L, \leq)$.*

(1) *$(C, \leq, g)$ is a complete lattice with t-(semi)norm if and only if $(C, \leq, \mathcal{T}_a(g))$ is a complete lattice with t-(semi)norm.*
(2) *$(C, \leq, g)$ is a complete lattice with t-(semi)norm if and only if $(C, \geq, \mathcal{T}_n(g))$ is a complete lattice with t-(semi)conorm.*

5.1 Invertibility.

Order norms can indeed in general be considered as multiplicative operators. This raises the following question: is it possible to define *inverses* for these operators, and for instance in the case of triangular norms and conorms to arrive at commutative multiplicative group structures? The following definition is a first step towards an answer to this question.

Definition 5.2 (Inverses). *Let λ and μ be elements of L. An element α of L is called a left-inverse for P of λ w.r.t. μ iff $P(\alpha, \lambda) = \mu$. An element β of L is called a right-inverse for P of λ w.r.t. μ iff $P(\lambda, \beta) = \mu$. An element γ of L is called an inverse for P of λ w.r.t. μ iff $P(\gamma, \lambda) = P(\lambda, \gamma) = \mu$. The set of the left-inverses for P of λ w.r.t. μ is denoted by $\mathcal{L}_P(\lambda, \mu)$, the set of the right-inverses by $\mathcal{R}_P(\lambda, \mu)$ and the set of the inverses by $\mathcal{I}_P(\lambda, \mu)$. For t-semiconorms we have analogous definitions.*

Since t-norms respectively t-conorms are in particular t-seminorms respectively t-semiconorms, definition 5.2 also applies to the former operators. Furthermore, with λ and μ arbitrary elements of L:

(1) $\mathcal{L}_T(\lambda, \mu) = \mathcal{R}_T(\lambda, \mu) = \mathcal{I}_T(\lambda, \mu)$;
(2) $\mathcal{L}_S(\lambda, \mu) = \mathcal{R}_S(\lambda, \mu) = \mathcal{I}_S(\lambda, \mu)$.

Let us furthermore remark that if an order norm g on $(L, \leq)$ is resolving on the left (respectively right), then if an element λ of L has a left-inverse (respectively right-inverse) α w.r.t. an element μ of L, this left-inverse (respectively right-inverse) is *unique*, i.e., $\mathcal{L}_g(\lambda, \mu) = \{\alpha\}$ (respectively $\mathcal{R}_g(\lambda, \mu) = \{\alpha\}$).

The following proposition is of crucial importance for this discussion.

Proposition 5.2. *Let λ and μ be elements of L. Then:*

1. $\neg(\lambda \geq \mu) \Rightarrow (\mathcal{L}_P(\lambda, \mu) = \mathcal{R}_P(\lambda, \mu) = \mathcal{I}_P(\lambda, \mu) = \emptyset)$;
2. $\neg(\lambda \leq \mu) \Rightarrow (\mathcal{L}_Q(\lambda, \mu) = \mathcal{R}_Q(\lambda, \mu) = \mathcal{I}_Q(\lambda, \mu) = \emptyset)$;
3. $\neg(\lambda \geq \mu) \Rightarrow (\mathcal{I}_T(\lambda, \mu) = \emptyset)$;
4. $\neg(\lambda \leq \mu) \Rightarrow (\mathcal{I}_S(\lambda, \mu) = \emptyset)$.

Proof. Of course, it suffices to prove (1) and (2). We shall give the proof of (1). The proof of (2) is analogous. Let λ and μ be arbitrary elements of L such that $\neg(\lambda \geq \mu)$. Let α be an arbitrary element of L. It follows from the isotonicity and boundary conditions of P that $P(\lambda, \alpha) \leq \lambda$ and $P(\alpha, \lambda) \leq \lambda$. $\neg(\lambda \geq \mu)$ means $\lambda < \mu$ or $\lambda \parallel \mu$. In both cases it follows that $P(\lambda, \alpha) \neq \mu$ and $P(\alpha, \lambda) \neq \mu$, whence $\alpha \notin \mathcal{R}_P(\lambda, \mu)$ and $\alpha \notin \mathcal{L}_P(\lambda, \mu)$ and therefore also $\alpha \notin \mathcal{I}_P(\lambda, \mu)$. $\square$

This proposition precludes the various forms of invertibility for order norms. Indeed, let in general M be a multiplicative operator on a set V with zero o and unit e. Then M is called *left-invertible* iff

$$(\forall v \in V \setminus \{o\})(\exists v' \in V)(M(v', v) = e),$$

right-invertible iff

$$(\forall v \in V \setminus \{o\})(\exists v'' \in V)(M(v, v'') = e),$$

and *invertible* iff

$$(\forall v \in V \setminus \{o\})(\exists v''' \in V)(M(v, v''') = M(v''', v) = e).$$

As the following corollary of proposition 5.2 shows, order norms on $(L, \leq)$ cannot satisfy these forms of invertibility, unless $(L, \leq)$ is a Boolean chain (of length two).

Corollary 5.1. *Let λ be an element of L. Then:*

1. $(\lambda < u) \Rightarrow (\mathcal{L}_P(\lambda, u) = \mathcal{R}_P(\lambda, u) = \mathcal{I}_P(\lambda, u) = \emptyset)$;
2. $(\lambda > \ell) \Rightarrow (\mathcal{L}_Q(\lambda, \ell) = \mathcal{R}_Q(\lambda, \ell) = \mathcal{I}_Q(\lambda, \ell) = \emptyset)$;
3. $(\lambda < u) \Rightarrow (\mathcal{I}_T(\lambda, u) = \emptyset)$;
4. $(\lambda > \ell) \Rightarrow (\mathcal{I}_S(\lambda, \ell) = \emptyset)$.

5.2 Weak Invertibility.

From the previous discussion we may conclude the following: whenever $(L, \leq)$ is not order-isomorphic to the Boolean chain $(\mathcal{T}, \leq)$ of the truth-values of classical logic, the order norms defined on $(L, \leq)$ cannot be invertible. However, proposition 5.2 leads us to the definition of another, and in a certain sense weaker, form of invertibility, that is very useful in possibility theory, as is shown in [7].

Definition 5.3 (Weak Invertibility). *A triangular seminorm P on $(L, \leq)$ is called*

1. *weakly left-invertible iff* $(\forall(\lambda, \mu) \in L^2)(\lambda \geq \mu \Rightarrow \mathcal{L}_P(\lambda, \mu) \neq \emptyset)$;
2. *weakly right-invertible iff* $(\forall(\lambda, \mu) \in L^2)(\lambda \geq \mu \Rightarrow \mathcal{R}_P(\lambda, \mu) \neq \emptyset)$;
3. *weakly invertible iff* $(\forall(\lambda, \mu) \in L^2)(\lambda \geq \mu \Rightarrow \mathcal{I}_P(\lambda, \mu) \neq \emptyset)$.

A triangular semiconorm Q on $(L, \leq)$ is called

1. *weakly left-invertible iff* $(\forall(\lambda, \mu) \in L^2)(\lambda \leq \mu \Rightarrow \mathcal{L}_Q(\lambda, \mu) \neq \emptyset)$;
2. *weakly right-invertible* $(\forall(\lambda, \mu) \in L^2)(\lambda \leq \mu \Rightarrow \mathcal{R}_Q(\lambda, \mu) \neq \emptyset)$;
3. *weakly invertible iff* $(\forall(\lambda, \mu) \in L^2)(\lambda \leq \mu \Rightarrow \mathcal{I}_Q(\lambda, \mu) \neq \emptyset)$.

Since t-norms respectively t-conorms are in particular also t-seminorms respectively t-semiconorms, the above definitions also apply to the former operators. Their commutativity furthermore implies that weak left- and right-invertibility coincide with weak invertibility. This leads to:

1. a triangular norm T on $(L, \leq)$ is weakly invertible if and only if $(\forall(\lambda, \mu) \in L^2)(\lambda \geq \mu \Rightarrow \mathcal{I}_T(\lambda, \mu) \neq \emptyset)$;
2. a triangular conorm S on $(L, \leq)$ is weakly invertible if and only if $(\forall(\lambda, \mu) \in L^2)(\lambda \leq \mu \Rightarrow \mathcal{I}_S(\lambda, \mu) \neq \emptyset)$.

We shall now show that in the case of *complete lattices with order norm*, a *criterion* for weak invertibility can be derived. It appears that the introduction of the notions *left-residual*, *right-residual* and *residual*, well-known in lattice theory [2], is an important step in establishing this criterion. Residuals are a generalization of *relative pseudo-complements*, a central notion in the theory of Brouwerian lattices (see, for instance, [2]): residuals reduce to relative pseudo-complements when the multiplicative operator is the meet of the lattice under consideration. Brouwerian lattices are closely related with *Heyting algebras*, that provide a logical framework for the constructivist foundation of mathematics, advocated by L. Brouwer in the beginning of this century (see, for instance, [2,12]). In a Heyting algebra the relative pseudo-complement is used as an implication operator. As is shown in [7], the residuals introduced below also have a causal connotation, since they emerge in a very natural way in the development of a theory of conditional possibility.

Let us now apply the definitions of residuals found in the literature [2] to the structures we are concerned with in this section.

Definition 5.4 (Residuals). *Since $(C,\leq)$ is a complete lattice, the following definitions make sense. Let λ and μ be arbitrary elements of C. We introduce the following notions.*

(1) *The left-residual $\lambda \lhd_P \mu$ for P of λ by μ: $\lambda \lhd_P \mu \stackrel{\text{def}}{=} \sup\{\,\nu \mid \nu \in C \text{ and } P(\nu,\mu) \leq \lambda\,\}$; and the right-residual $\mu \rhd_P \lambda$ for P of λ by μ: $\mu \rhd_P \lambda \stackrel{\text{def}}{=} \sup\{\,\nu \mid \nu \in C \text{ and } P(\mu,\nu) \leq \lambda\,\}$.*
(2) *The left-residual $\lambda \lhd_Q \mu$ for Q of λ by μ: $\lambda \lhd_Q \mu \stackrel{\text{def}}{=} \inf\{\,\nu \mid \nu \in C \text{ and } Q(\nu,\mu) \geq \lambda\,\}$; and the right-residual $\mu \rhd_Q \lambda$ for Q of λ by μ: $\mu \rhd_Q \lambda \stackrel{\text{def}}{=} \inf\{\,\nu \mid \nu \in C \text{ and } Q(\mu,\nu) \geq \lambda\,\}$.*
(3) *The residual $\lambda \vartriangle_T \mu$ for T of λ by μ: $\lambda \vartriangle_T \mu \stackrel{\text{def}}{=} \lambda \lhd_T \mu = \mu \rhd_T \lambda$.*
(4) *The residual $\lambda \vartriangle_S \mu$ for S of λ by μ: $\lambda \vartriangle_S \mu \stackrel{\text{def}}{=} \lambda \lhd_S \mu = \mu \rhd_S \lambda$.*

We may consider $\lhd_P$, $\rhd_P$, $\lhd_Q$, $\rhd_Q$, $\vartriangle_T$ and $\vartriangle_S$ as *binary operators* on C, i.e. $C^2 - C$ mappings that map an arbitrary element (λ,μ) of C^2 to the corresponding residual. As an example, let us give the explicit definition of the right-residual operator $\rhd_P$.

$$\rhd_P\colon C^2 \to C\colon (\lambda,\mu) \mapsto \lambda \rhd_P \mu,$$

where of course

$$\lambda \rhd_P \mu = \sup\{\,\nu \mid \nu \in C \text{ and } P(\lambda,\nu) \leq \mu\,\}.$$

The definitions of the other residual operators are similar. Using these definitions, we deduce the following invariance and dual invariance laws.

Proposition 5.3. *Let f be an affirmation or a negation operator on $(C,\leq)$.*

(1) $\mathcal{T}_f(\lhd_P) = \lhd_{\mathcal{T}_f(P)}$ *and* $\mathcal{T}_f(\rhd_P) = \rhd_{\mathcal{T}_f(P)}$.
(2) $\mathcal{T}_f(\lhd_Q) = \lhd_{\mathcal{T}_f(Q)}$ *and* $\mathcal{T}_f(\rhd_Q) = \rhd_{\mathcal{T}_f(Q)}$.
(3) $\mathcal{T}_f(\vartriangle_T) = \vartriangle_{\mathcal{T}_f(T)}$.
(4) $\mathcal{T}_f(\vartriangle_S) = \vartriangle_{\mathcal{T}_f(S)}$.

Proof. It suffices to prove (1) and (2). We shall give the proof of (1) when $f = n$ is a negation operator. The rest of the proof is completely analogous. Let λ and μ be arbitrary elements of C. Then, taking into account proposition 4.2,

$$\begin{aligned}
\lambda \lhd_{\mathcal{T}_n(P)} \mu &= \inf\{\,\nu \mid \nu \in C \text{ and } \mathcal{T}_n(P)(\nu,\mu) \geq \lambda\,\}\\
&= \inf\{\,\nu \mid \nu \in C \text{ and } n^{-1}(P(n(\nu),(\mu))) \geq \lambda\,\}\\
&= \inf\{\,\nu \mid \nu \in C \text{ and } P(n(\nu),n(\mu)) \leq n(\lambda)\,\}
\end{aligned}$$

and, since n is a permutation of C,

$$\begin{aligned}
&= \inf\{\,n^{-1}(\nu) \mid \nu \in C \text{ and } P(\nu,n(\mu)) \leq n(\lambda)\,\}\\
&= \inf n^{-1}(\{\,\nu \mid \nu \in C \text{ and } P(\nu,n(\mu)) \leq n(\lambda)\,\})\\
&= n^{-1}(\sup\{\,\nu \mid \nu \in C \text{ and } P(\nu,n(\mu)) \leq n(\lambda)\,\})\\
&= n^{-1}(n(\lambda) \lhd_P n(\mu))\\
&= \mathcal{T}_n(\lhd_P)(\lambda,\mu). \quad \square
\end{aligned}$$

The following theorem is the main result of this subsection and allows us to deduce a criterion for weak invertibility in the context of complete lattices with order norms.

Theorem 5.1 (Cancellation Law). *Let $(C, \leq, P)$, $(C, \geq, Q)$, $(C, \leq, T)$, $(C, \geq, S)$ be complete lattices with t-seminorm, t-semiconorm, t-norm and t-conorm respectively. Let λ and μ be arbitrary elements of C. Then:*

1. $(\exists \nu \in C)(P(\nu, \lambda) = \mu) \Leftrightarrow P(\mu \lhd_P \lambda, \lambda) = \mu$ *and* $(\exists \nu \in C)(P(\lambda, \nu) = \mu) \Leftrightarrow P(\lambda, \lambda \rhd_P \mu) = \mu$*;*
2. $(\exists \nu \in C)(Q(\nu, \lambda) = \mu) \Leftrightarrow Q(\mu \lhd_Q \lambda, \lambda) = \mu$ *and* $(\exists \nu \in C)(Q(\lambda, \nu) = \mu) \Leftrightarrow Q(\lambda, \lambda \rhd_Q \mu) = \mu$*;*
3. $(\exists \nu \in C)(T(\nu, \lambda) = \mu) \Leftrightarrow T(\mu \vartriangle_T \lambda, \lambda) = \mu$*;*
4. $(\exists \nu \in C)(S(\nu, \lambda) = \mu) \Leftrightarrow S(\mu \vartriangle_S \lambda, \lambda) = \mu$*.*

Proof. It suffices to prove (1) and (2). We shall give the proof of (1). The proof of (2) is similar. Let λ and μ be arbitrary elements of C. We shall only prove the first equivalence. The proof of the second equivalence is analogous.

First, assume that $(\exists \nu \in C)(P(\nu, \lambda) = \mu)$. This implies that $\mathcal{L}_P(\lambda, \mu) \neq \emptyset$. Let us use the following notation:

$$\mathcal{L}'_P(\lambda, \mu) \stackrel{\text{def}}{=} \{\, \nu \mid \nu \in C \text{ and } P(\nu, \lambda) \leq \mu \,\}.$$

Then of course $\mu \lhd_P \lambda = \sup \mathcal{L}'_P(\lambda, \mu)$. Furthermore taking into account the complete distributivity of P w.r.t. sup and the fact that $\mathcal{L}_P(\lambda, \mu) \neq \emptyset$ and $\mathcal{L}'_P(\lambda, \mu) \neq \emptyset$, we may conclude that $\sup \mathcal{L}_P(\lambda, \mu) \in \mathcal{L}_P(\lambda, \mu)$ and $\sup \mathcal{L}'_P(\lambda, \mu) \in \mathcal{L}'_P(\lambda, \mu)$. Taking into account $\mathcal{L}_P(\lambda, \mu) \subseteq \mathcal{L}'_P(\lambda, \mu)$ and therefore also $\sup \mathcal{L}_P(\lambda, \mu) \leq \sup \mathcal{L}'_P(\lambda, \mu)$, the isotonicity of P implies that $P(\sup \mathcal{L}_P(\lambda, \mu), \lambda) \leq P(\sup \mathcal{L}'_P(\lambda, \mu), \lambda)$. Therefore, we have on the one hand, since $\sup \mathcal{L}_P(\lambda, \mu) \in \mathcal{L}_P(\lambda, \mu)$ that $\mu \leq P(\mu \lhd_P \lambda, \lambda)$. On the other hand, since $\mu \lhd_P \lambda \in \mathcal{L}'_P(\lambda, \mu)$, we may write $\mu \geq P(\mu \lhd_P \lambda, \lambda)$, whence $\mu = P(\mu \lhd_P \lambda, \lambda)$.

Conversely, from $P(\mu \lhd_P \lambda, \lambda) = \mu$ we immediately deduce that $\mu \lhd_P \lambda \in \mathcal{L}_P(\lambda, \mu)$, and therefore $\mathcal{L}_P(\lambda, \mu) \neq \emptyset$. $\square$

Corollary 5.6. *Let λ and μ be elements of C.*

1. *If the equation $P(\nu, \lambda) = \mu$ in ν admits a solution, then $\mu \lhd_P \lambda$ is the greatest solution (w.r.t. the order relation $\leq$), and if the equation $P(\lambda, \nu) = \mu$ in ν admits a solution, then $\lambda \rhd_P \mu$ is the greatest solution (w.r.t. the order relation $\leq$).*
2. *If the equation $Q(\nu, \lambda) = \mu$ in ν admits a solution, then $\mu \lhd_Q \lambda$ is the smallest solution (w.r.t. the order relation $\leq$), and if the equation $Q(\lambda, \nu) = \mu$ in ν admits a solution, then $\lambda \rhd_Q \mu$ is the smallest solution (w.r.t. the order relation $\leq$).*
3. *If the equation $T(\nu, \lambda) = \mu$ in ν admits a solution, then $\mu \vartriangle_T \lambda$ is the greatest solution (w.r.t. the order relation $\leq$).*
4. *If the equation $S(\nu, \lambda) = \mu$ in ν admits a solution, then $\mu \vartriangle_S \lambda$ is the smallest solution (w.r.t. the order relation $\leq$).*

Also, we remind the reader of the fact that if, for instance, P is resolving on the left and the equation $P(\nu, \lambda) = \mu$ in ν admits a solution, then $\mu \lhd_P \lambda$ is the *only* solution there is, i.e., $\mathcal{L}_P(\lambda, \mu) = \{\mu \lhd_P \lambda\}$. Similar remarks apply for right-inverses and inverses, and of course, for order norms in general.

From theorem 5.1 and definition 5.3, we deduce the following result.

Proposition 5.4 (Criterion for Weak Invertibility).

1. *P is weakly left-invertible if and only if $(\forall (\lambda, \mu) \in C^2)(\lambda \geq \mu \Rightarrow P(\mu \lhd_P \lambda, \lambda) = \mu)$ and P is weakly right-invertible if and only if $(\forall (\lambda, \mu) \in C^2)(\lambda \geq \mu \Rightarrow P(\lambda, \lambda \rhd_P \mu) = \mu)$.*
2. *Q is weakly left-invertible if and only if $(\forall (\lambda, \mu) \in C^2)(\lambda \leq \mu \Rightarrow Q(\mu \lhd_Q \lambda, \lambda) = \mu)$ and Q is weakly right-invertible if and only if $(\forall (\lambda, \mu) \in C^2)(\lambda \leq \mu \Rightarrow Q(\lambda, \lambda \rhd_Q \mu) = \mu)$.*
3. *T is weakly invertible if and only if $(\forall (\lambda, \mu) \in C^2)(\lambda \geq \mu \Rightarrow T(\mu \vartriangle_T \lambda, \lambda) = \mu)$.*
4. *S is weakly invertible if and only if $(\forall (\lambda, \mu) \in C^2)(\lambda \leq \mu \Rightarrow S(\mu \vartriangle_S \lambda, \lambda) = \mu)$.*

5.3 Examples.

We conlude this section with a few examples in order to make the discussion above more concrete and to show that the structures studied in this section are not as esoteric as they might seem at first sight.

Example 5.1. Let $(B, \leq)$ be an arbitrary complete Boolean lattice. It is well-known that (see, for example, [2]) $\frown$ is completely distributive w.r.t. sup and that $\smile$ is completely distributive w.r.t. inf. If the triangular norm $\frown$ on $(B, \leq)$ is considered as a multiplicative operator on $(B, \leq)$, then $(B, \leq, \frown)$ is a complete lattice with t-norm. If $\smile$ is considered as a multiplicative operator on $(B, \geq)$, then $(B, \geq, \smile)$ is a complete lattice with t-conorm.

Example 5.2. Let $(K, \leq)$ be an arbitrary finite chain. An arbitrary t-(semi)norm on $(K, \leq)$ forms together with $(K, \leq)$ a complete chain with t-(semi)norm. An arbitrary t-(semi)conorm on $(K, \leq)$ forms together with $(K, \leq)$ a complete chain with t-(semi)conorm. The reason for this is of course that in finite chains arbitrary *isotonic* operators are completely distributive w.r.t. sup and inf.

Example 5.3 (Two-Valued Logic). Consider the finite complete Boolean chain $(\mathcal{T}, \leq)$, mentioned in examples 2.2 and 3.2. Taking into account the previous example, we have that

(1) $(\mathcal{T}, \leq, \wedge)$ is a complete chain with t-norm;
(2) $(\mathcal{T}, \geq, \vee)$ is a complete chain with t-conorm.

Let λ and μ be arbitrary elements of $\mathcal{T}$. Then

$$\begin{aligned}
\mu \vartriangle_\wedge \lambda &= \begin{cases} \mathit{false} & ; \quad \lambda = \mathit{true} \text{ and } \mu = \mathit{false} \\ \mathit{true} & ; \quad \text{elsewhere} \end{cases} \\
(\mu \vartriangle_\wedge \lambda) \wedge \lambda &= \mu \wedge \lambda \\
\mathcal{I}_\wedge(\lambda, \mu) &= \begin{cases} \emptyset & ; \quad \lambda = \mathit{false} \text{ and } \mu = \mathit{true} \\ \{\mathit{false}\} & ; \quad \lambda = \mathit{true} \text{ and } \mu = \mathit{false} \\ \{\mathit{true}\} & ; \quad \lambda = \mu = \mathit{true} \\ \mathcal{T} & ; \quad \lambda = \mu = \mathit{false}. \end{cases}
\end{aligned}$$

This means that $\wedge$ is not only weakly invertible, but also invertible. The inverse of *true* (w.r.t. the identity *true*) for the operator $\wedge$ is *true*. The zero *false* of course has no inverse w.r.t. *true* for $\wedge$. Dually, we have that

$$\begin{aligned}
\mu \vartriangle_\vee \lambda &= \begin{cases} \mathit{true} & ; \quad \lambda = \mathit{false} \text{ and } \mu = \mathit{true} \\ \mathit{false} & ; \quad \text{elsewhere} \end{cases} \\
(\mu \vartriangle_\vee \lambda) \vee \lambda &= \mu \vee \lambda \\
\mathcal{I}_\vee(\lambda, \mu) &= \begin{cases} \emptyset & ; \quad \lambda = \mathit{true} \text{ and } \mu = \mathit{false} \\ \{\mathit{true}\} & ; \quad \lambda = \mathit{false} \text{ and } \mu = \mathit{true} \\ \{\mathit{false}\} & ; \quad \lambda = \mu = \mathit{false} \\ \mathcal{T} & ; \quad \lambda = \mu = \mathit{true}. \end{cases}
\end{aligned}$$

This means that $\vee$ is not only weakly invertible, but also invertible. The inverse of *false* (w.r.t. the identity *false*) for the operator $\vee$ is *false*. The zero *true* of course has no inverse w.r.t. *false* for $\vee$. Furthermore, one readily verifies that

$$\mu \vartriangle_\wedge \lambda = \neg\lambda \vee \mu = \lambda \Rightarrow \mu,$$

where $\Rightarrow$ is the implication operator of (the truth-values of) classical logic, and also

$$\mu \vartriangle_\vee \lambda = \neg\lambda \wedge \mu = \neg(\neg\mu \vee \lambda) = \neg(\mu \Rightarrow \lambda).$$

Finally, proposition 5.3 here takes the following form—with $T = \wedge$, $n = \neg$ and $\mathcal{T}_n(T) = \mathcal{T}_\neg(\wedge) = \vee$: for arbitrary λ and μ in $\mathcal{T}$

$$\lambda \vartriangle_\vee \mu = \lambda \mathcal{T}_\neg(\vartriangle_\wedge)\mu$$

or equivalently

$$\neg(\lambda \Rightarrow \mu) = \neg(\neg\mu \Rightarrow \neg\lambda)$$

or equivalently

$$\lambda \Rightarrow \mu = \neg\mu \Rightarrow \neg\lambda.$$

This means that the invariance and dual invariance properties of proposition 5.3 can be interpreted as generalized contrapositivity laws.

Example 5.4. Let $(C, \leq)$ be a completely distributive lattice. Then $(C, \leq, \frown)$ is a complete lattice with t-norm and $(C, \geq, \smile)$ is a complete lattice with t-conorm. Also, taking into account the consistence property of meets and joins in lattices (see, for instance, [2]), $\frown$ and $\smile$ are weakly invertible. In particular, if the order relation '$\leq$' is total, we may write, for arbitrary λ and μ in C:

$$\mu \vartriangle_{\frown} \lambda = \begin{cases} u & ; \quad \lambda \leq \mu \\ \mu & ; \quad \lambda > \mu \end{cases}$$

$$(\mu \vartriangle_{\frown} \lambda) \frown \lambda = \begin{cases} \lambda & ; \quad \lambda < \mu \\ \mu & ; \quad \lambda \geq \mu \end{cases} = \lambda \frown \mu$$

$$\mathcal{I}_{\frown}(\lambda, \mu) = \begin{cases} \emptyset & ; \quad \lambda < \mu \\ [\lambda, u] & ; \quad \lambda = \mu \\ \{\mu\} & ; \quad \lambda > \mu \end{cases}$$

and dually

$$\mu \vartriangle_{\smile} \lambda = \begin{cases} \ell & ; \quad \lambda \geq \mu \\ \mu & ; \quad \lambda < \mu \end{cases}$$

$$(\mu \vartriangle_{\smile} \lambda) \smile \lambda = \begin{cases} \lambda & ; \quad \lambda > \mu \\ \mu & ; \quad \lambda \leq \mu \end{cases} = \lambda \smile \mu$$

$$\mathcal{I}_{\smile}(\lambda, \mu) = \begin{cases} \emptyset & ; \quad \lambda > \mu \\ [\ell, \lambda] & ; \quad \lambda = \mu \\ \{\mu\} & ; \quad \lambda < \mu. \end{cases}$$

Example 5.5. Consider the real unit interval $[0, 1]$, together with the natural ordering '$\leq$' of the reals. $([0, 1], \leq)$ is a complete chain. Cappelle et. al. [3] have shown that an isotonic operator on $([0, 1], \leq)$ is completely distributive w.r.t. sup respectively inf if and only if this operator has *lower-semicontinuous* respectively *upper-semicontinuous* partial mappings. $([0, 1], \leq)$ can therefore be enriched with an arbitrary t-(semi)norm on $([0, 1], \leq)$ with lower-semicontinuous partial mappings in order to form a complete chain with t-(semi)norm, and can be enriched with an arbitrary t-(semi)conorm on $([0, 1], \leq)$ with upper-semicontinuous partial mappings to form a complete chain with t-(semi)conorm. More specifically, the product operator $\times$ on $[0, 1]$ is a continuous triangular norm on $([0, 1], \leq)$, and therefore has lower-semicontinuous partial mappings. For arbitrary elements a and b of $[0, 1]$:

$$b \vartriangle_{\times} a = \begin{cases} 1 & ; \quad a = 0 \\ b/a & ; \quad a \neq 0 \text{ and } b \leq a \\ 0 & ; \quad a \neq 0 \text{ and } b > a \end{cases}$$

$$(b \vartriangle_{\times} a) \times a = \begin{cases} b & ; \quad b \leq a \\ 0 & ; \quad b > a \end{cases}$$

$$\mathcal{I}_{\times}(a, b) = \begin{cases} \{b/a\} & ; \quad b \leq a \\ \emptyset & ; \quad b > a, \end{cases}$$

which implies that $\times$ is weakly invertible. Moreover, $\times$ is strongly resolving and therefore also resolving, positive and Archimedean. Not surprisingly therefore, $\mathcal{I}_{\times}(a, b)$ is a singleton whenever $b \leq a$.

6. Conclusion

In this paper, we have studied some order-theoretic and algebraic aspects of affirmation operators, negation operators, triangular (semi)norms and (semi)conorms, defined on bounded partially ordered sets. Our results show that these operators and their fundamental properties can be defined and investigated in a framework that is essentially order-theoretic, and give us an indication of their order-theoretic and algebraic importance. To conclude this paper, we give a brief discussion of a few applications of the ideas developed above.

Consider the set $\mathcal{F}_{(L,\leq)}(X)$ of the mappings from a universe X into a bounded poset $(L,\leq)$, or, in the lingo of fuzzy set theory, the set of the $(L,\leq)$-fuzzy sets on X [11]. We can define a partial order relation $\sqsubseteq$ on $\mathcal{F}_{(L,\leq)}(X)$ in the following familiar way. For arbitrary $(L,\leq)$-fuzzy sets g_1 and g_2 on X:

$$g_1 \sqsubseteq g_2 \Leftrightarrow (\forall x \in X)(g_1(x) \leq g_2(x)).$$

Of course, $(\mathcal{F}_{(L,\leq)}(X), \sqsubseteq)$ is a bounded partially ordered set. Triangular norms and conorms on $(L,\leq)$ can be used to define pointwise (or truth-functional) associative, commutative and isotonic intersection and union operators for $(L,\leq)$-fuzzy sets, that satisfy certain boundary conditions. But, since $(\mathcal{F}_{(L,\leq)}(X), \sqsubseteq)$ is a bounded partially ordered set, we can also consider triangular norms and conorms on $(\mathcal{F}_{(L,\leq)}(X), \sqsubseteq)$ and use them to define intersection and union operators for $(L,\leq)$-fuzzy sets that are associative, commutative and isotonic, but *not necessarily truth-functional* (for more detail, we again refer to [7]). One of us has given a justification for the introduction of fuzzy-set-theoretical operators that are not truth-functional in [4,5].

Triangular seminorms and semiconorms on a bounded poset $(L,\leq)$ can also be used to defined general classes of seminormed and semiconormed $(L,\leq)$-fuzzy integrals that are extremely important in a measure- and integral-theoretic approach to possibility and necessity theory (analogous to the well-known measure and integral-theoretic treatment of probability), as we have shown in [6] and in a much more general way in [7]. This approach gives a consistent solution to many extant problems in possibility theory. We mention among other things the introduction of conditional possibility, its uniqueness, a consistent definition of possibilistic independence of events and of possibilistic variables, ... The algebraic structures studied in section 5 play a central role in our treatment of this matter. For example, the equations mentioned in corollary 5.6 are special cases of the defining integral equations for conditional possibility. Of course, the link we have shown to exist between the unicity of the solutions of these equations and the fact that the associated order norms are resolving (on the left or on the right), has in this context immediate and interesting consequences. By the way, the fact that the product operator $\times$ on $[0,1]$ is weakly invertible and resolving (see example 5.5) is intimately linked with the existence and the (almost sure) unicity of conditional *probability* distributions.

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