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Order Structure on the Algebra of Permutations and of Planar Binary Trees

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Abstract. Let X_n be either the symmetric group on n letters, the set of planar binary n-trees or the set of vertices of the $(n-1)$-dimensional cube. In each case there exists a graded associative product on $\bigoplus_{n\geq 0} K[X_n]$. We prove that it can be described explicitly by using the weak Bruhat order on S_n, the left-to-right order on planar trees, the lexicographic order in the cube case.

Keywords: planar binary tree, order structure, weak Bruhat order, algebra of permutations, dendriform algebra

Introduction

Let S_n be the symmetric group acting on n letters. In [6] Malvenuto and Reutenauer showed that the shuffle product induces a graded associative product, denoted $*$, on the graded space $K[S_\infty] := \bigoplus_{n\geq 0} K[S_n]$ (here K is a field). By using the weak Bruhat order on S_n we give a closed formula for the product of basis elements as follows. Let $\sigma \in S_p$ and $\tau \in S_q$ be two permutations. We define two operations called respectively 'over' and 'under':

$$\sigma/\tau = \sigma \times \tau \in S_{p+q} \quad \text{and} \quad \sigma\backslash\tau = \xi_{p,q} \cdot \sigma \times \tau \in S_{p+q},$$

where $\xi_{p,q}$ is the permutation whose image is $(q+1\ \ q+2 \ldots q+p\ \ 1\ \ 2 \ldots q)$.

It turns out that $\sigma/\tau \leq \sigma\backslash\tau$ for the weak Bruhat order of S_{p+q}. We prove that the product $*$ on $K[S_\infty]$ is given on the generators by the sum of all permutations in between σ/τ and $\sigma\backslash\tau$:

$$\sigma * \tau = \sum_{\sigma/\tau \leq \omega \leq \sigma\backslash\tau} \omega. \tag{1}$$

Let Y_n be the set of planar binary trees with n interior vertices (so the number of elements in Y_n is the Catalan number $\frac{(2n)!}{n!(n+1)!}$). In [4] it is shown that there is a graded associative product on the graded space $K[Y_\infty] := \bigoplus_{n\geq 0} K[Y_n]$ induced by the "dendriform algebra" structure of $K[Y_\infty]$. We give a closed formula for the product of basis elements as follows. There is a partial order on Y_n induced by  $<$ . Let $u \in Y_p$ and $v \in Y_q$ be two planar binary trees. We define two operations called respectively 'over' and 'under' as follows. The element $u/v \in Y_{p+q}$ (resp. $u\backslash v \in Y_{p+q}$) is obtained by identifying the root of u with the left most leaf of v (resp. the right most leaf of u with the root of v). It turns out that $u/v \leq u\backslash v$ for the ordering of Y_{p+q}. We prove that the product $*$ on $K[Y_\infty]$ is given on the generators by

$$u * v = \sum_{u/v \leq t \leq u\backslash v} t. \tag{2}$$

Let us mention that these operations 'over' and 'under' on planar binary trees appear in the theory of renormalisation, cf. [2].

Observe that, since Y_n does not bear a group structure (unlike S_n), the product $*$ is defined in [5] by a recursive formula. So, a priori, the explicit computation of a product $u * v$ needs the computation of many terms. The above formula greatly simplifies this computation.

Let $Q_n = \{\pm 1\}^{n-1}$. There is a graded associative product on the graded vector space $K[Q_\infty] := \bigoplus_{n \geq 0} K[Q_n]$, where $K[Q_n]$ is identified with the Solomon descent algebra (cf. [5, 6]). It is in fact a Hopf algebra called the algebra of noncommutative symmetric functions, which is dual to the algebra of quasi-symmetric functions, cf. [3]. We give a closed formula for the product of basis elements as follows. There is a partial order on Q_n induced by $-1 < +1$. Let $\epsilon \in Q_p$ and $\delta \in Q_q$. We define two operations called respectively 'over' and 'under' as follows: $\epsilon/\delta := (\epsilon, -1, \delta) \in Q_{p+q}$ and $\epsilon\backslash\delta := (\epsilon, +1, \delta) \in Q_{p+q}$. It is immediate that $\epsilon/\delta \leq \epsilon\backslash\delta$ for the ordering of Q_{p+q}. We prove that the product $*$ on $K[Q_\infty]$ is given on the generators by

$$\epsilon * \delta = \sum_{\epsilon/\delta \leq \alpha \leq \epsilon\backslash\delta} \alpha = \epsilon/\delta + \epsilon\backslash\delta. \tag{3}$$

In [5] we constructed explicit maps

$$S_n \xrightarrow{\psi_n} Y_n \xrightarrow{\phi_n} Q_n$$

(see also [8], p. 24) and we observed that they are in fact restrictions of cellular maps from the cube to the Stasheff polytope and from the Stasheff polytope to the permutohedron respectively. Moreover, we showed that, after dualization and linear extension, the maps

$$K[Q_\infty] \xrightarrow{\phi^*} K[Y_\infty] \xrightarrow{\psi^*} K[S_\infty]$$

are injective homomorphisms of graded associative algebras. We take advantage of this result to deduce formulas (2) and (3) from formula (1).

The content of this paper is as follows. In the first part (Sections 1, 2 and 3) we deal with the partial orders on S_n, Y_n and Q_n respectively, and we show that the maps ψ_n and ϕ_n are compatible with the orders. In the second part (Sections 4, 5 and 6) we prove formulas (1), (2) and (3). In the case of the symmetric group and in the case of planar binary trees the algebras $K[S_\infty]$ and $K[Y_\infty]$ have a more refined structure: they are dendriform algebras (cf. [4]). We show that in both cases the products $\prec$ and $\succ$ can also be formulated in terms of the order structure.

In this paper we apply some results about Coxeter groups to the particular case of the symmetric groups. For the convenience of the reader we recall them in the Appendix.

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Convention. The vector space over $\mathbf{Q}$ generated by the set X is denoted $\mathbf{Q}[X]$. The linear dual $\mathbf{Q}[X]^*$ is identified with $\mathbf{Q}[X]$ under the identification of the basis with its dual. The image of a permutation $\sigma \in S_n$ is denoted by $(\sigma(1)\,\sigma(2)\ldots\sigma(n))$.

1. Weak Bruhat order on the symmetric group S_n

Let (W, S) be the Coxeter group $(S_n, \{s_1, \ldots, s_{n-1}\})$, where S_n is the symmetric group acting on $\{1, \ldots, n\}$, and s_i is the transposition of i and $i + 1$. We denote by $\cdot$ the group law of S_n and by 1_n the unit. In this section we compare the weak Bruhat order on S_n and the shuffles by applying the result of the Appendix. We also introduce in 1.9 the operations 'over' $/$ and 'under' $\backslash$ from $S_p \times S_q$ to S_{p+q} that are to be used in the Appendix.

For any permutation $\omega \in S_n$, its length $l(\omega)$ is the smallest integer k such that ω can be written as a product of k generators: $\omega = s_{i_1} \cdot s_{i_2} \cdot \ldots \cdot s_{i_k}$. Observe that the length of a permutation counts the number of inversions of its image.

By definition, a permutation $\sigma \in S_n$ has a *descent* at i, $1 \leq i \leq n-1$, if $\sigma(i) > \sigma(i+1)$. The set of *descents* of a permutation σ is $Desc(\sigma) := \{s_i \mid \sigma \text{ has a descent at } i\}$. Hence, for any subset $J \subseteq \{s_1, \ldots, s_{n-1}\}$ the set

$$X_J^n := \{\sigma \in S_n \mid l(\sigma \cdot s_i) > l(\sigma), \text{ for all } s_i \in J\}$$

described in the Appendix is the set of all permutations $\sigma \in S_n$ such that $Desc(\sigma) \subseteq \{s_1, \ldots, s_{n-1}\} \setminus J$.

In order to simplify the notation, we denote the subset $\{s_1, \ldots, s_{p-1}, s_{p+1}, \ldots, s_{p+q-1}\}$ of $\{s_1, \ldots, s_{p+q-1}\}$ by $\{s_p\}^c$. The set $X_{\{s_p\}^c}^{p+q}$ is the set of all (p, q)-shuffles of S_{p+q}, that is

$$Sh(p, q) := \{\sigma \in S_{p+q} \mid \sigma(1) < \cdots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \cdots < \sigma(p+q)\}.$$

There exists a canonical inclusion $\iota : S_p \times S_q \hookrightarrow S_{p+q}$, which maps the generator s_i of S_p to s_i in S_{p+q}, and the generator s_j of S_q to s_{j+p} in S_{p+q}. In other words we let a permutation of S_p act on $\{1, \ldots, p\}$ and we let a permutation of S_q act on $\{p+1, \ldots, p+q\}$. In what follows we identify $S_p \times S_q$ with its image in S_{p+q}.

Observe that, for $J = \{s_p\}^c$, the standard parabolic subgroup $W_{\{s_p\}^c}$ defined in the Appendix is precisely $S_p \times S_q$ in S_{p+q}.

Proposition A.2 of the Appendix takes the following form for the Coxeter group S_n:

Lemma 1.1 *Let $p, q \geq 1$.*

(a) *For any $\sigma \in S_{p+q}$ there exist unique elements $\xi \in Sh(p, q)$ and $\omega \in S_p \times S_q$ such that $\sigma = \xi \cdot \omega$.*

(b) *For any* $\xi \in Sh(p, q)$ *and any* $\omega \in S_p \times S_q$ *the length of* $\xi \cdot \omega \in S_{p+q}$ *is the sum*: $l(\xi \cdot \omega) = l(\xi) + l(\omega)$.

(c) *There exists a longest element in* $Sh(p, q)$, *denoted* $\xi_{p,q}$, *and* $\xi_{p,q} = (q+1 \;\; q+2 \; \ldots \; q+p \;\; 1 \;\; 2 \ldots q)$.

Definition 1.2 For $n \geq 1$, the *weak ordering* (also called *weak Bruhat order*) on S_n is defined as follows:

$$\omega \leq \sigma \text{ in } S_n, \quad \text{if there exists } \tau \in S_n \quad \text{such that } \sigma = \tau \cdot \omega \quad \text{with } l(\sigma) = l(\tau) + l(\omega).$$

The set of permutations S_n, equipped with the weak ordering is a partially ordered set, with minimal element 1_n, and maximal element the cycle $\omega_n^0 := (n \;\; n-1 \ldots 2 \;\; 1)$ (cf. [1]). For instance, for $n = 2$ we get

$$1_2 \to s_1$$

and for $n = 3$ we get

$$\begin{tikzcd} & 1_3 \arrow[dl] \arrow[dr] & \\ s_2 \arrow[d] & & s_1 \arrow[d] \\ s_1 s_2 \arrow[dr] & & s_2 s_1 \arrow[dl] \\ & s_1 s_2 s_1 & \end{tikzcd}$$

since $s_1 s_2 s_1 = s_2 s_1 s_2$. Here $a \to b$ means $a < b$.

For $n \geq 1$ and $1 \leq i \leq j \leq n-1$, let $c_{i,j} \in S_n$ be the permutation:

$$c_{i,j} := s_i \cdot s_{i+1} \cdot \ldots \cdot s_j.$$

Given $\omega \in Sh(p, q)$, it is easy to check that, if $\omega \neq 1_{p+q}$, then there exist integers $l \geq 0$, $1 \leq i_1 < i_2 < \cdots < i_{l+1} \leq p+q-1$, and $i_k \leq p+k-1$, for $1 \leq k \leq l+1$, such that:

$$\omega = c_{i_{l+1}, p+l} \cdot c_{i_l, p+l-1} \cdot \ldots \cdot c_{i_1, p}.$$

Under this notation one has

$$\xi_{p,q} = c_{q, p+q-1} \cdot c_{q-1, p+q-2} \cdot \ldots \cdot c_{1,p}.$$

Corollary A.4 of the Appendix and Lemma 1.1 imply the following result:

Lemma 1.3 *Let* $p, q \geq 1$ *be two integers. The longest element of the set* $Sh(p, q)$ (*all* (p, q)*-shuffles*) *is* $\xi_{p,q}$. *Moreover*, *one has*

$$Sh(p, q) = \{\omega \in S_{p+q} \mid \omega \leq \xi_{p,q}\}.$$

Lemma 1.4 *If $\sigma, \sigma' \in S_p$ and $\tau, \tau' \in S_q$ are permutations verifying $\sigma \le \sigma'$ and $\tau \le \tau'$, then $\sigma \times \tau \le \sigma' \times \tau'$.*

Proof: The permutations $\sigma \times \tau$ and $\sigma' \times \tau'$ belong to the subgroup $S_p \times S_q$ of S_{p+q}.

Since $\sigma \le \sigma'$ and $\tau \le \tau'$, there exist $\delta \in S_p$ and $\epsilon \in S_q$ such that $\sigma' = \delta \cdot \sigma$ and $\tau' = \epsilon \cdot \tau$, with $l(\sigma') = l(\delta) + l(\sigma)$ and $l(\tau') = l(\epsilon) + l(\tau)$.

One has $\sigma' \times \tau' = \delta \cdot \sigma \times \epsilon \cdot \tau = (\delta \times \epsilon) \cdot (\sigma \times \tau)$, with $l(\sigma' \times \tau') = l(\sigma') + l(\tau') = l(\delta \times \epsilon) + l(\sigma \times \tau)$. □

Lemma 1.5 *Let p and q be two nonnegative integers, and let $\sigma \in S_p$ and $\tau \in S_q$ be two permutations. If ω_1 and ω_2 are two elements of $Sh(p, q)$ such that $\omega_1 < \omega_2$, then*

$$\omega_1 \cdot (\sigma \times \tau) < \omega_2 \cdot (\sigma \times \tau).$$

Proof: The result follows immediately from Lemma 1.1. □

Definition 1.6 The *grafting* of $\sigma \in S_p$ and $\tau \in S_q$ is the permutation $\sigma \vee \tau \in S_{p+q+1}$ given by:

$$(\sigma \vee \tau)(i) := \begin{cases} \sigma(i) & \text{if } 1 \le i \le p, \\ p+q+1 & \text{if } i = p+1, \\ \tau(i-p-1)+p & \text{if } p+2 \le i \le p+q+1. \end{cases}$$

It is easily seen that,

$$\sigma \vee \tau = (\sigma \times \tau \times 1_1) \cdot s_{q+p} \cdot s_{q+p-1} \cdot \ldots \cdot s_{p+1},$$

for $\sigma \in S_p$ and $\tau \in S_q$.

Lemma 1.7 *If $\sigma \le \sigma'$ in S_p and $\tau \le \tau'$ in S_q, then $\sigma \vee \tau \le \sigma' \vee \tau'$ in S_{p+q+1}.*

Proof: Suppose $\sigma' = \epsilon \cdot \sigma$ and $\tau' = \delta \cdot \tau$, for some $\epsilon \in S_p$ and $\delta \in S_q$ such that $l(\sigma') = l(\epsilon) + l(\sigma)$ and $l(\tau') = l(\delta) + l(\tau)$. Clearly, $\sigma' \vee \tau' = (\epsilon \times \delta \times 1_1) \cdot (\sigma \vee \tau)$.

The permutations $\sigma \times \tau \times 1_1$ and $\sigma' \times \tau' \times 1_1$ belong to the subgroup $S_{p+q} \times S_1$ of S_{p+q+1}.

It is immediate to check that $l((s_{p+1} \cdot \ldots \cdot s_{p+q}) \cdot s_i) > l(s_{p+1} \cdot \ldots \cdot s_{p+q})$, for any $1 \le i \le p+q-1$, that is $s_{p+1} \cdot \ldots \cdot s_{p+q} \in Sh(p+q, 1)$. Since $s_{p+1} \ldots s_{p+q} = c_{p+1,p+q}$, by Lemma 1.1 one has $l((s_{p+1} \cdot \ldots \cdot s_{p+q}) \cdot \omega) = q + l(\omega)$, for any $\omega \in S_{p+q} \times S_1$. So

$$\begin{aligned} l(\sigma' \vee \tau') &= l((\sigma' \vee \tau')^{-1}) = l(\sigma') + l(\tau') + q \\ &= l(\epsilon) + l(\delta) + l(\sigma) + l(\tau) + q \\ &= l(\epsilon \times \delta \times 1_1) + l(\sigma \vee \tau). \end{aligned}$$ □

Proposition 1.8 *Let $\sigma \in S_n$ be a permutation such that $\sigma(i) = n$, for some $1 \le i \le n$. There exist unique elements $\sigma^l \in S_{i-1}$, $\sigma^r \in S_{n-i}$ and $\gamma \in Sh(i-1, n-i)$ such that:*

$$\sigma = (\gamma \times 1_1) \cdot (\sigma^l \vee \sigma^r).$$

Proof: Since $\sigma(i) = n$, the element σ may be written as $\sigma = \sigma' \cdot s_{n-1} \cdot s_{n-2} \cdot \ldots \cdot s_i$, with $\sigma' \in S_{n-1} \times S_1$ and $l(\sigma) = l(\sigma') + n - i$.

Lemma 1.1 implies that there exist unique elements $\epsilon \in Sh(i-1, n-i+1)$ and $\delta \in S_{i-1} \times S_{n-i+1}$, such that $\sigma' = \epsilon \cdot \delta$. Since the permutation s_{n-1} does not appear in a reduced expression of σ', the following assertions hold:

– the element ϵ is of the form $\epsilon = \gamma \times 1_1$ for some $\gamma \in Sh(i-1, n-i)$.
– the element δ belongs to $S_{i-1} \times S_{n-i} \times S_1$. So, $\delta = \sigma^l \times \sigma^r \times 1_1$, for unique $\sigma^l \in S_{i-1}$ and $\sigma^r \in S_{n-i}$.

Finally, we get that $\sigma = (\gamma \times 1_1) \cdot (\sigma^l \vee \sigma^r)$. The uniqueness of γ, σ^l and σ^r follows easily. $\square$

Definition 1.9 For $p, q \geq 1$, the operations '*over*' / and '*under*' \ from $S_p \times S_q$ to S_{p+q}, are defined as follows:

$$\sigma/\tau := \sigma \times \tau, \quad \text{and} \quad \sigma\backslash\tau := \xi_{p,q} \cdot (\sigma \times \tau),$$

for $\sigma \in S_p$ and $\tau \in S_q$.

Since $\sigma \times \tau \in S_p \times S_q$, for any $\sigma \in S_p$ and $\tau \in S_q$, and $\xi_{p,q} \in Sh(p,q)$, the following relation holds:

$$\sigma/\tau \leq \sigma\backslash\tau.$$

Lemma 1.10 *The operations / and \ are associative.*

Proof: Let $\sigma \in S_p$, $\tau \in S_q$ and $\delta \in S_r$. It is clear that

$$(\sigma \times \tau) \times \delta = \sigma \times \tau \times \delta = \sigma \times (\tau \times \delta).$$

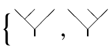
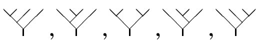
The formula above and the equality

$$\xi_{p+q,r} \cdot (\xi_{p,q} \times 1_r) = \xi_{p,q+r} \cdot (1_p \times \xi_{q,r})$$

imply that the operation \ is associative too. $\square$

2. Weak ordering on the set of planar binary trees

For $n \geq 1$, let Y_n denote the set of planar binary trees with n vertices:

$Y_0 = \{|\}$, $Y_1 = \{$  $\}$, $Y_2 = \{$  $\}$, $Y_3 = \{$  $\}$.

The *grafting* of a p-tree u and a q-tree v is the $(p+q+1)$-tree $u \vee v$ obtained by joining the roots of u and v to a new vertex and create a new root. For any tree t there exist unique trees t^l and t^r such that $t = t^l \vee t^r$. As a result we have $Y_n = \coprod_{p+q+1=n} Y_p \times Y_q$.

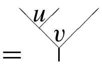
Definition 2.1 Let $\leq$ be the *weak ordering* on Y_n generated transitively by the following relations:

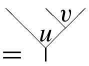
(a) if $u \leq u' \in Y_p$ and $v \leq v' \in Y_q$, then $u \vee v \leq u' \vee v'$ in Y_{p+q+1},
(b) if $u \in Y_p$, $v \in Y_q$ and $w \in Y_r$, then $(u \vee v) \vee w \leq u \vee (v \vee w)$.

The pair $(Y_n, \leq)$ is a poset.

Definition 2.2 The operations '*over*' / and '*under*' \ from $Y_p \times Y_q$ to Y_{p+q} are defined as follows:

– u/v is the tree obtained by identifying the root of u with the left most leaf of v,
– $u\backslash v$ is the tree obtained by identifying the right most leaf of u with the root of v,

$u/v =$ 

$u\backslash v =$ 

It is immediate to check that / and \ are associative.

Equivalently these operations can be defined recursively as follows:

– $t/| := t =: |\backslash t$ and $t\backslash| := t =: |/t$ for $t \in Y_n$,
– for $u = u^l \vee u^r$ and $v = v^l \vee v^r$ one has

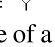
$$u/v := (u/v^l) \vee v^r, \quad \text{and} \quad u\backslash v := u^l \vee (u^r \backslash v).$$

Lemma 2.3 *For any trees $u \in Y_p$ and $v \in Y_q$ one has*

$$u/v \leq u\backslash v.$$

Proof: This is an immediate consequence of condition (b) of Definition 2.1. $\square$

The surjective map $\psi_n : S_n \to Y_n$ considered in [5] (see also [8] p. 24) is defined as follows:

– $\psi_0([\,]) = | \in Y_0$,
– $\psi_1(1_1) =$  $\in Y_1$,
– the image of a permutation $\sigma \in S_n$ is made of two sequences of integers: the sequence on the left of n and the sequence on the right of n in $(\sigma(1), \ldots, \sigma(n))$. These permutations are precisely the ones appearing in Proposition 1.8. Observe that one of them may be empty. By relabelling the integers in each sequence so that only consecutive integers (starting with 1) appear, one gets two permutations σ^l and σ^r. For instance (341625) gives the two sequences (341) and (25), which, after relabelling, give (231) and (12). Recursively $\psi_n(\sigma)$ is defined as $\psi_p(\sigma^l) \vee \psi_q(\sigma^r)$.

Notation For $n \geq 1$, let S_t be the subset of S_n such that a permutation $\sigma \in S_n$ belongs to S_t if and only if $\psi_n(\sigma) = t$, i.e.

$$S_t := \psi_n^{-1}(t) \subset S_n.$$

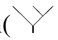
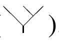
This subset admits the following description in terms of shuffles.

For $n = 1$, one has S $:= \{1_1\} = S_1$.
For $n \geq 2$, let $t = t^l \vee t^r$, with $t^l \in Y_q$ and $t^r \in Y_p$ and $q + p = n - 1$. We have

$$S_t = \{(\gamma \times 1_1) \cdot (\sigma \vee \tau) \mid \gamma \in Sh(p, q), \sigma \in S_{t^l} \text{ and } \tau \in S_{t^r}\},$$

for $1 \leq q \leq n - 2$. If $q = 0$, then $S_{|\vee t^r} = \{| \vee \sigma, \text{ for } \sigma \in S_{t^r}\}$. And, if $q = n - 1$, then $S_{t^l \vee |} = \{\sigma \vee |, \text{ for } \sigma \in S_{t^l}\}$.
For instance when $n = 2$, S $:= \{1_2\}$ and S $:= \{s_1\}$.

Definition 2.4 For $n \geq 0$, let Min and Max be the maps from Y_n into S_n defined as follows:

- For $n = 1$, Min() $:= 1_1 =:$ Max().
- For $n = 2$, Min() $:= 1_2 =:$ Max(), and Min() $:= s_1 =:$ Max().
- For $n \geq 3$, let $t = t^l \vee t^r$, with $t^l \in Y_q$ and $t^r \in Y_p$ and $p + q = n - 1$.

The permutations $\mathrm{Min}(t)$ and $\mathrm{Max}(t)$ are defined as follows:

– If $1 \leq q \leq n - 2$, then $\mathrm{Min}(t) := \mathrm{Min}(t^l) \vee \mathrm{Min}(t^r)$, and $\mathrm{Max}(t) := (\xi_{q,p} \times 1_1) \cdot (\mathrm{Max}(t^l) \vee \mathrm{Max}(t^r))$.
– If $q = 0$, then $\mathrm{Min}(t) := \xi_{n-1,1} \cdot (1_1 \times \mathrm{Min}(t^r))$ and $\mathrm{Max}(t) := \xi_{n-1,1} \cdot (1_1 \times \mathrm{Max}(t^r))$.
– If $q = n - 1$, then $\mathrm{Min}(t) := \mathrm{Min}(t^l) \times 1_1$ and $\mathrm{Max}(t) := \mathrm{Max}(t^l) \times 1_1$.

Clearly, $\mathrm{Min}(t)$ and $\mathrm{Max}(t)$ belong to S_t, for any tree $t \in Y_n$.

Theorem 2.5 *Let $n \geq 1$ and $t \in Y_n$. The following equality holds*:

$$S_t = \{\omega \in S_n \mid \mathrm{Min}(t) \leq \omega \leq \mathrm{Max}(t)\}.$$

Proof: Suppose $t = t^l \vee t^r$, with $t^l \in Y_q$, $t^r \in Y_q$ and $n = q + p + 1$.
Step 1. Let γ and γ' be elements of $Sh(p, q)$ such that $\gamma \leq \gamma'$. Suppose that $\sigma \leq \sigma'$ in S_p and $\tau \leq \tau'$ in S_q.
Lemma 1.7 implies that $\sigma \vee \tau \leq \sigma' \vee \tau'$. Now, $\gamma \times 1_1$ and $\gamma' \times 1_1$ belong to $Sh(p, q+1)$, and $\sigma \vee \tau$ and $\sigma' \vee \tau'$ are elements of $S_p \times S_{q+1}$; from Lemma 1.1 one gets,

$$(\gamma \times 1_1) \cdot (\sigma \vee \tau) \leq (\gamma' \times 1_1) \cdot (\sigma' \vee \tau').$$

For any $\gamma \in Sh(p, q)$, Lemma 1.3 states that $1_{p+q} \leq \gamma \leq \xi_{p,q}$. It follows that all $\omega \in S_t$ satisfies $\mathrm{Min}(t) \leq \omega \leq \mathrm{Max}(t)$.

Step 2. Conversely, let $\omega \in S_n$ be such that $\mathrm{Min}(t) \leq \omega \leq \mathrm{Max}(t)$.

Since $\mathrm{Min}(t) \leq \omega$, there exists $\omega_1 \in S_n$ such that $\omega = \omega_1 \cdot s_{p+q} \cdot \ldots \cdot s_{p+1}$, with $l(\omega) = l(\omega_1) + l(s_{p+q} \cdot \cdots \cdot s_{p+1}) = l(\omega_1) + q$.

By Lemma 1.1, there exist unique elements $\omega_2 \in Sh(p, q+1)$ and $\omega_3 \in S_p \times S_{q+1}$, such that $\omega_1 = \omega_2 \cdot \omega_3$, with $l(\omega_1) = l(\omega_2) + l(\omega_3)$.

Since $\omega \leq \mathrm{Max}(t)$, there exists $\delta \in S_n$ such that

$$(\xi_{p,q} \times 1_1) \cdot (\mathrm{Max}(t^l) \times \mathrm{Max}(t^r) \times 1_1) = \delta \cdot \omega_1,$$

with $l(\xi_{p,q}) + l(\mathrm{Max}(t^l)) + l(\mathrm{Max}(t^r)) = l(\delta) + l(\omega_1)$.

The permutation s_{p+q} does not appear in a reduced expression of ω_1. So, $\omega_2 \in Sh(p, q+1)$ and $\omega_2(n) = n$, which implies that $\omega_2 \leq \xi_{p,q} \times 1_1$.

On the other hand, the element $\omega_3 \in S_p \times S_{q+1}$ and s_{p+q} does not appear in a reduced decomposition of ω_3. So, $\omega_3 \in S_p \times S_q \times S_1$. Consequently ω_3 is of the form $\omega_3 = \sigma_4 \times \tau_4 \times 1_1$, for unique permutations $\sigma_4 \in S_p$ and $\tau_4 \in S_q$. Moreover, the inequalities

$$\begin{aligned}&(\mathrm{Min}(t^l) \times \mathrm{Min}(t^r) \times 1_1) \cdot s_{p+q} \cdot \ldots \cdot s_{p+1}\\ &\quad \leq \omega_2 \cdot (\sigma_4 \times \tau_4 \times 1_1) \cdot s_{p+q} \cdot \ldots \cdot s_{p+1}\\ &\quad \leq (\xi_{p,q} \times 1_1) \cdot (\mathrm{Max}(t^l) \times \mathrm{Max}(t^r) \times 1_1) \cdot s_{p+q} \cdot \ldots \cdot s_{p+1}\end{aligned}$$

imply

$$\begin{aligned}\mathrm{Min}(t^l) \times \mathrm{Min}(t^r) \times 1_1 &\leq \omega_2 \cdot (\sigma_4 \times \tau_4 \times 1_1)\\ &\leq (\xi_{p,q} \times 1_1) \cdot (\mathrm{Max}(t^l) \times \mathrm{Max}(t^r) \times 1_1).\end{aligned}$$

Since $1_n \leq \omega_2 \leq \xi_{p,q} \times 1_1$ in $Sh(p, q+1)$, by applying Lemma 1.5 we get

$$\mathrm{Min}(t^l) \times \mathrm{Min}(t^r) \leq \sigma_4 \times \tau_4 \leq \mathrm{Max}(t^l) \times \mathrm{Max}(t^r).$$

The elements σ_4 and τ_4 satisfy that $\mathrm{Min}(t^l) \leq \sigma_4 \leq \mathrm{Max}(t^l)$ and $\mathrm{Min}(t^r) \leq \tau_4 \leq \mathrm{Max}(t^r)$. A recursive argument states that $\sigma_4 \in S_{t^l}$ and $\tau_4 \in S_{t^r}$, and the proof is complete. □

Corollary 2.6 *The weak ordering of S_n induces a partial order $\leq_B$ on Y_n. This order is compatible with $\psi_n : S_n \to Y_n$:*

$$\sigma \leq \tau \Rightarrow \psi_n(\sigma) \leq_B \psi_n(\tau).$$

Proposition 2.7 *The order $\leq_B$ induced by the weak order on Y_n coincides with the order $\leq$ of Definition* 2.1.

Proof: We want to see that the order $\leq_B$ satisfies conditions (a) and (b) of Definition 2.1.

Given $t \in Y_n$ and $w \in Y_m$ recall that, for any $\sigma \in S_t$ and any $\tau \in S_w$, the permutation $\sigma \vee \tau$ belongs to $S_{t \vee w}$. Lemma 1.7 implies that $\leq_B$ verifies condition (a).

Let $t \in Y_n$, $u \in Y_r$ and $w \in Y_m$ be three trees. Suppose that $\sigma \in S_t$, $\delta \in S_u$ and $\tau \in S_w$. One has that $(\sigma \vee \delta) \vee \tau$ belongs to $S_{(t \vee u) \vee w}$, while $\sigma \vee (\delta \vee \tau)$ belongs to $S_{t \vee (u \vee w)}$. To prove condition (b), it suffices to check that $(\sigma \vee \delta) \vee \tau \leq \sigma \vee (\delta \vee \tau)$ in $S_{n+r+m+2}$.

Now, an easy calculation shows that:

$$(\sigma \vee \delta) \vee \tau = (\sigma \times \delta \times 1_1 \times \tau \times 1_1) \cdot s_{n+r+m+1} \cdot \ldots \cdot s_{n+r+2} \cdot s_{n+r} \cdot \ldots \cdot s_{n+1},$$

and

$$\sigma \vee (\delta \vee \tau) = (\sigma \times \delta \times \tau \times 1_2) \cdot s_{n+r+m} \cdot \ldots \cdot s_{n+r+1} \cdot s_{n+r+m+1} \cdot \ldots \cdot s_{n+1}.$$

We need to show that $(\sigma \times \delta \times 1_1 \times \tau \times 1_1) \cdot s_{n+r+m+1} \cdot \ldots \cdot s_{n+r+2}$ is smaller than $(\sigma \times \delta \times \tau \times 1_2) \cdot s_{n+r+m} \cdot \ldots \cdot s_{n+r+1} \cdot s_{n+r+m+1} \cdot \ldots \cdot s_{n+r+1}$. We use the relation

$$\begin{aligned} & s_{n+r+m} \cdot \ldots \cdot s_{n+r+1} \cdot s_{n+r+m+1} \cdot \ldots \cdot s_{n+r+1} \\ & \quad = s_{n+r+m+1} \cdot \ldots \cdot s_{n+r+1} \cdot s_{n+r+m+1} \cdot \ldots \cdot s_{n+r+2}. \end{aligned}$$

We have to prove that

$$(\sigma \times \delta \times 1_1 \times \tau \times 1_1) \leq (\sigma \times \delta \times \tau \times 1_2) \cdot s_{n+r+m+1} \cdot \ldots \cdot s_{n+r+1};$$

which is a consequence of the formula:

$$(1_1 \times \tau \times 1_1) \leq (\tau \times 1_2) \cdot s_{m+1} \cdot \ldots \cdot s_1, \quad \text{for any } \tau \in S_m,\ m \geq 1. \tag{2.6.1}$$

To prove (2.6.1) it suffices to check that

$$l(s_{m+1} \cdot \ldots \cdot s_1 \cdot (1_1 \times \tau \times 1_1)) = m + 1 + l(1_1 \times \tau \times 1_1).$$

This is clearly the case since $s_{m+1} \cdot \ldots \cdot s_1$ is in $Sh(1, m+1)$ and $1_1 \times \tau \times 1_1$ belongs to $S_1 \times S_{m+1}$. To end the proof, it suffices to observe that

$$(\tau \times 1_2) \cdot s_{m+1} \cdot \ldots \cdot s_1 = s_{m+1} \cdot \ldots \cdot s_1 \cdot (1_1 \times \tau \times 1_1), \quad \text{for any } \tau \in S_m,\ m \geq 0.$$

□

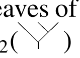
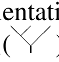
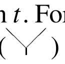
Corollary 2.8 *The map $\psi_n : S_n \to Y_n$ is a morphism of posets.*

Theorem 2.9 *Let $\sigma \in S_p$ and $\tau \in S_q$ be two permutations. The following equalities hold*:

$$\psi_{p+q}(\sigma/\tau) = \psi_p(\sigma)/\psi_q(\tau) \quad \text{and} \quad \psi_{p+q}(\sigma\backslash\tau) = \psi_p(\sigma)\backslash\psi_q(\tau).$$

Proof: In $\sigma/\tau = \sigma \times \tau$, under the map $S_p \times S_q \to S_{p+q}$, the symbols permuted by σ are strictly smaller and all to the left of the symbols permuted by τ. Hence under the definition of ψ_n as given after Lemma 2.3, one has $\psi_{p+q}(\sigma \times \tau) = \psi_p(\sigma)/\psi_q(\tau)$. The proof of the other case is symmetric. □

3. Weak ordering on the set of vertices of the hypercube

For $n \geq 2$, let $Q_n := \{+1, -1\}^{n-1}$ be the set of vertices of the hypercube. There is a surjective map $\phi_n : Y_n \to Q_n$, which is defined as follows. First we label the interior leaves from left to right by $1, 2, \ldots, n-1$. Second, we put $\phi_n(t) = (\epsilon_1, \ldots, \epsilon_{n-1})$, where ϵ_i is -1 when the stem of the ith leaf of t is right oriented (more precisely SW-NE), and $+1$ when it is left oriented (more precisely SE-NW). We take into account only the interior leaves of t, since the orientation of the two extreme ones does not depend on t. For instance $\phi_2($$) = (+1)$ and $\phi_2($$) = (-1)$. By convention $Q_1 = \{(-1)_1\}$ and $\phi_1($$) = (-1)_1$.

We consider Q_2 as the partially ordered set $Q_2 := \{-1 < +1\}$.

Definition 3.1 The set Q_n of vertices of the hypercube is a partially ordered set for the order:

$$\epsilon \leq \eta \quad \text{if and only if} \quad \epsilon_i \leq \eta_i, \quad \text{for all } 1 \leq i \leq n-1.$$

We denote by $(-1)_n$ the minimal element of Q_n, and by $(+1)_n$ its maximal element.

Definition 3.2 Given an element $\epsilon = (\epsilon_1, \ldots, \epsilon_{p-1}) \in Q_p$ and an element $\eta = (\eta_1, \ldots, \eta_{q-1}) \in Q_q$ the *grafting* of ϵ and η, denoted $\epsilon \vee \eta$, is the element of Q_{p+q+1} given by:

$$\epsilon \vee \eta := (\epsilon_1, \ldots, \epsilon_{p-1}, -1, +1, \eta_1, \ldots, \eta_{q-1}).$$

The operations over / and under \ from $Q_p \times Q_q$ to Q_{p+q} are defined by

$$\epsilon/\eta := (\epsilon_1, \ldots, \epsilon_{p-1}, -1, \eta_1, \ldots, \eta_{q-1}),$$
$$\epsilon\backslash\eta := (\epsilon_1, \ldots, \epsilon_{p-1}, +1, \eta_1, \ldots, \eta_{q-1}).$$

Remark 3.3 It is easily seen that the maps ϕ_n preserve the operations grafting $\vee$, over /, and under \.

Lemma 3.4 *Let t be an element of Y_n such that its ith leaf points to the right, for some $1 \leq i \leq n-1$. If w is another tree in Y_n such that $w \leq t$, then the ith leaf of w is right oriented too.*

Proof: The result is obvious for $n \leq 2$.

Since the order $\leq$ on Y_n is transitively generated by the relations given in Definition 2.1, it suffices to show that the assertion is true for the situations described in (a) and (b) of this Definition.

For (a): If $w = w^l \vee w^r$ and $t = t^l \vee t^r$, with $w^l \leq t^l$ and $w^r \leq t^r$, then the results is an immediate consequence of the inductive hypothesis.

For (b): Suppose $w = (u \vee v) \vee s$ and $t = u \vee (v \vee s)$, for some $u \in Y_p$, $v \in Y_q$ and $s \in Y_r$. If $q \geq 1$, then the kth leaf of w is oriented in the same direction that the kth leaf of t, for all $1 \leq k \leq n-1$.

If $q = 0$, then the kth leaf of w is oriented in the same direction that the kth leaf of t, for all $k \neq p+2$. And the $(p+2)$th leaf of w is right oriented, while $(p+2)$th leaf of t is left oriented.

□

Proposition 3.5 *For all $n \geq 1$ and all $\epsilon \in Q_n$ there exist two trees in Y_n, denoted* $\min(\epsilon)$ *and* $\max(\epsilon)$ *respectively, such that the inverse image of ϵ by $\phi_n : Y_n \to Q_n$ satisfies*:

$$\phi^{-1}(\epsilon) = \{t \in Y_n \mid \min(\epsilon) \leq t \leq \max(\epsilon)\}.$$

Proof: *Step 1.* The inverse image $\phi_n^{-1}((-1)_n)$ of the minimal element of Q_n is the minimal tree a_n of Y_n which has all its leaves pointing to the right. Similarly, the inverse image $\phi_n^{-1}((+1)_n)$ of the maximal element of Q_n is the maximal tree z_n of Y_n which has all its leaves pointing to the left. So, the theorem is obviously true for $\epsilon \in \{(-1)_n, (+1)_n\}$ if we define:

$$\min((-1)_n) := a_n =: \max((-1)_n), \text{ and } \min((+1)_n) := z_n =: \max((+1)_n).$$

If $\epsilon \notin \{(-1)_n; (+1)_n\}$, we define max and min recursively, as follows:

(a) If $\epsilon_1 = -1$ there exist $k \geq 1$ and $\epsilon' \in Q_{n-k}$ such that $\epsilon = (-1)_k/\epsilon'$. Define $\min(\epsilon) := a_k / \min(\epsilon')$.
If $\epsilon_1 = +1$, there exist $k \geq 2$ and $\epsilon' \in Q_{n-k}$ such that $\epsilon = (+1)_k/\epsilon'$. Define $\min(\epsilon) := z_k / \min(\epsilon')$.
(b) If $\epsilon_{n-1} = -1$, there exist $k \geq 2$ and $\epsilon' \in Q_{n-k}$ such that $\epsilon = \epsilon' \backslash (-1)_k$. Define $\max(\epsilon) := \max(\epsilon') \backslash a_k$.
If $\epsilon_{n-1} = +1$, there exist $k \geq 1$ and $\epsilon' \in Q_{n-k}$ such that $\epsilon = \epsilon' \backslash (+1)_k$. Define $\max(\epsilon) := \max(\epsilon') \backslash z_k$.

Step 2. It is easy to prove, by induction on n, that if $t \in \phi_n^{-1}(\epsilon)$, then $\min(\epsilon) \leq t \leq \max(\epsilon)$.

Conversely, let t be a tree such that $\min(\epsilon) \leq t \leq \max(\epsilon)$. Since $\min(\epsilon) \leq t$, Lemma 3.4 implies that the ith leaf of t is left oriented, for all i such that $\epsilon_i = +1$. Similarly, $t \leq \max(\epsilon)$ and Lemma 3.4 imply that the ith leaf of t is right oriented, for all i such that $\epsilon_i = -1$. So, t belongs to $\phi_n^{-1}(\epsilon)$. □

Corollary 3.6 *For $n \geq 2$, the order of Y_n induces a partial order $\leq_B$ on Q_n. This order is compatible with $\phi_n : Y_n \to Q_n$.*

Proposition 3.7 *The order $\leq_B$ of Q_n coincides with the order $\leq$ of Definition* 3.1.

Proof: If w and t are two trees in Y_n such that $w \leq t$, then Lemma 3.4 implies that $\phi_n(w) \leq \phi_n(t)$. It proves that if $\epsilon \leq_B \eta$ in Q_n, then $\epsilon \leq \eta$.
To prove that $\epsilon \leq \eta$ in Q_n implies that $\epsilon \leq_B \eta$, it suffices to show that

$$(\epsilon_1, \ldots, \epsilon_{p-1}, -1, \epsilon_{p+1}, \ldots, \epsilon_{n-1}) \leq_B (\epsilon_1, \ldots, \epsilon_{p-1}, +1, \epsilon_{p+1}, \ldots, \epsilon_{n-1}),$$

for all $1 \leq p \leq n-1$ and all elements $\epsilon_i \in \{-1, +1\}$, $1 \leq i \leq n-1$, $i \neq p$. Consider the element $\kappa := (\epsilon_1, \ldots, \epsilon_{p-1})$ in Q_p, and the element $\rho := (\epsilon_{p+1}, \ldots, \epsilon_{n-1}) \in Q_{n-p}$. Let $t \in Y_p$ be a tree in $\phi_p^{-1}(\kappa)$ and $w \in Y_{n-p}$ be a tree in $\phi_{n-p}^{-1}(\rho)$. It is easy to check that

$$\phi_n(t/w) = (\epsilon_1, \ldots, \epsilon_{p-1}, -1, \epsilon_{p+1}, \ldots, \epsilon_{n-1}) \quad \text{and}$$
$$\phi_n(t\backslash w) = (\epsilon_1, \ldots, \epsilon_{p-1}, +1, \epsilon_{p+1}, \ldots, \epsilon_{n-1}).$$

Since Lemma 2.3 states that $t/w \leq t\backslash w$ in Y_n, one gets the result. □

Corollary 3.8 *The map $\phi_n : Y_n \to Q_n$ is a morphism of posets.*
(See also [8], p. 24.)

4. The graded algebra of permutations $\mathbf{Q}[S_\infty]$

Consider the graded vector space $\mathbf{Q}[S_\infty] := \bigoplus_{n\geq 0}\mathbf{Q}[S_n]$, equipped with the shuffle product $*$ defined by:

$$\sigma * \tau := \sum_{x\in Sh(p,q)} x\cdot(\sigma \times \tau), \quad \text{for } \sigma \in S_p \quad \text{and} \quad \tau \in S_q.$$

In [6], C. Malvenuto and C. Reutenauer prove that $(\mathbf{Q}[S_\infty], *)$ is an associative algebra over $\mathbf{Q}$. We denote by $\overline{\mathbf{Q}[S_\infty]}$ the augmentation ideal.

Theorem 4.1 *Let $\sigma \in S_p$ and $\tau \in S_q$ be two permutations. The product $\sigma * \tau$ is the sum of all permutations $\omega \in S_{p+q}$ verifying*
$\sigma \times \tau \leq \omega \leq \xi_{p,q}\cdot(\sigma \times \tau)$, in other words:

$$\sigma * \tau = \sum_{\sigma/\tau\leq\omega\leq\sigma\backslash\tau} \omega.$$

Proof: Lemma 1.5 implies that

$$\sigma \times \tau \leq \delta\cdot(\sigma\times\tau) \leq \xi_{p,q}\cdot(\sigma\times\tau), \quad \text{for any } \delta \in Sh(p,q).$$

Suppose that $\omega \in S_{p+q}$ satisfies $\sigma \times \tau \leq \omega \leq \xi_{p,q}\cdot(\sigma\times\tau)$. Let $\omega_1 \in S_{p+q}$ be such that $\omega = \omega_1\cdot(\sigma\times\tau)$. It is obvious that $1_{p+q} \leq \omega_1$.

Since $\omega \leq \xi_{p,q}\cdot(\sigma\times\tau)$, the definition of the weak ordering implies that there exists $\epsilon \in S_{p+q}$ such that $\xi_{p,q}\cdot(\sigma\times\tau) = \epsilon\cdot\omega_1\cdot(\sigma\times\tau)$, with $l(\xi_{p,q}) = l(\epsilon) + l(\omega_1)$. It implies, by Lemma 1.3, that $\omega_1 \in Sh(p,q)$. This completes the proof of the Theorem. □

Definition 4.2 For $p, q \geq 0$, the subsets $Sh^1(p,q)$ and $Sh^2(p,q)$ of $Sh(p,q)$ are defined by:

$$Sh^1(p,q) := \{\omega \in Sh(p,q) \mid \omega(p+q) = p+q\}, \quad \text{and}$$
$$Sh^2(p,q) := \{\omega \in Sh(p,q) \mid \omega(p) = p+q\}.$$

Remark 4.3 The set $Sh(p, q)$ is the disjoint union of $Sh^1(p, q)$ and $Sh^2(p, q)$. Moreover, one has that

$$\begin{aligned} Sh^1(p,q) &= \{\omega \times 1_1 \mid \omega \in Sh(p, q-1)\} = Sh(p, q-1) \times 1_1; \quad \text{and} \\ Sh^2(p,q) &= \{(\omega \times 1_1)\cdot(1_{p-1} \vee 1_q) \mid \omega \in Sh(p-1, q)\} \\ &= (Sh(p-1, q) \times 1_1)\cdot(1_{p-1} \vee 1_q). \end{aligned}$$

Definition 4.4 The products $\prec$ and $\succ$ in $\overline{\mathbf{Q}[S_\infty]}$ are defined as follows:

$$\begin{aligned} \sigma \prec \tau &:= \sum_{\omega \in Sh^2(p,q)} \omega \cdot (\sigma \times \tau), \quad \text{and} \\ \sigma \succ \tau &:= \sum_{\omega \in Sh^1(p,q)} \omega \cdot (\sigma \times \tau), \end{aligned}$$

for $\sigma \in S_p$ and $\tau \in S_q$.

From Remark 4.3 one gets that the associative product $*$ of $\mathbf{Q}[S_\infty]$ satisfies

$$\sigma * \tau = \sigma \prec \tau + \sigma \succ \tau, \quad \text{for } \sigma, \tau \in \overline{\mathbf{Q}[S_\infty]}.$$

Proposition 4.5 *The operations $\prec$ and $\succ$ satisfy the relations*

(i) $(a \prec b) \prec c = a \prec (b \prec c) + a \prec (b \succ c)$,
(ii) $a \succ (b \prec c) = (a \succ b) \prec c$,
(iii) $a \succ (b \succ c) = (a \prec b) \succ c + (a \succ b) \succ c$,

for any $a, b, c \in \overline{\mathbf{Q}[S_\infty]}$. Hence $\overline{\mathbf{Q}[S_\infty]}$ is a dendriform algebra (as defined in [4] *).*

Proof: This is a consequence of the associativity property of the shuffle together with an inspection about the first element of the image of the permutations. □

The products $\prec$ and $\succ$ may also be described in terms of the order $\leq$ as follows.

Proposition 4.6 *For any $\sigma \in S_p$ and any $\tau \in S_q$, one has:*

$$\sigma \prec \tau = \sum_{(1_{p-1} \vee 1_q)\cdot(\sigma \times \tau) \leq \omega \leq \sigma \backslash \tau} \omega,$$

and

$$\sigma \succ \tau = \sum_{\sigma / \tau \leq \omega \leq (\xi_{p,q-1} \times 1_1)\cdot(\sigma \times \tau)} \omega\ .$$

Proof: Lemma 1.3 and Remark 4.3 imply that

$$Sh^1(p,q) = \{\omega \in S_{p+q} \mid \omega \leq \xi_{p,q-1} \times 1_1\}, \quad \text{and}$$
$$Sh^2(p,q) = \{\omega \in S_{p+q} \mid 1_{p-1} \vee 1_q \leq \omega \leq (\xi_{p-1,q} \times 1_1) \cdot (1_{p-1} \vee 1_q)\}.$$

The result follows immediately from Lemma 1.5. □

5. The graded algebra of planar binary trees $\mathbf{Q}[Y_\infty]$

The graded vector space $\mathbf{Q}[Y_\infty] := \bigoplus_{n\geq 0}\mathbf{Q}[Y_n]$ is a graded associative algebra for the product $*$ defined recursively as follows:

– $t * | = | * t := t$, for all $t \in Y_n,\ n \geq 1$,
– if $t = t^l \vee t^r$ and $w = w^l \vee w^r$, then

$$t * w := (t * w^l) \vee w^r + t^l \vee (t^r * w).$$

Moreover, the map $\psi^* : \mathbf{Q}[Y_\infty] \to \mathbf{Q}[S_\infty]$, defined by

$$\psi_n^*(t) := \sum_{\psi_n(\sigma)=t} \sigma,$$

is an algebra homomorphism (cf. [5]).

Theorem 5.1 *If t and w are two planar binary trees, then the product $t * w$ satisfies*

$$t * w = \sum_{t/w \leq u \leq t\backslash w} u.$$

Proof: Since the ordering $\leq$ on Y_n is induced by the weak ordering of S_n, the result is a straightforward consequence of Proposition 2.8 and Theorem 4.1. □

As in the case of the algebra $\mathbf{Q}[S_\infty]$, we may describe on $\overline{\mathbf{Q}[Y_\infty]} := \bigoplus_{n\geq 1}\mathbf{Q}[Y_n]$ two products $\prec$ and $\succ$, such that

$$t * w = t \prec w + t \succ w, \quad \text{for any } t, w \in \overline{\mathbf{Q}[Y_\infty]}.$$

Definition 5.2 Let $t \in Y_p$ and $w \in Y_q$. The elements $t \prec w$ and $t \succ w$ in $\overline{\mathbf{Q}[Y_\infty]}$ are given by:

$$t \prec w := t^l \vee (t^r * w), \quad \text{for } t = t^l \vee t^r,$$
$$t \succ w := (t * w^l) \vee w^r, \quad \text{for } w = w^l \vee w^r.$$

The space $\overline{\mathbf{Q}[Y_\infty]}$, equipped with the products $\prec$ and $\succ$ is a dendriform algebra (cf. [4, 5]). We prove now that $\overline{\psi^*} : \overline{\mathbf{Q}[Y_\infty]} \to \overline{\mathbf{Q}[S_\infty]}$ preserves $\prec$ and $\succ$.

Proposition 5.3 *The K-linear map $\overline{\psi^*} : \overline{\mathbf{Q}[Y_\infty]} \to \overline{\mathbf{Q}[S_\infty]}$ is a dendriform algebra homomorphism.*

Proof: We prove that $\psi^*(t \succ w) = \psi^*(t) \succ \psi^*(w)$, for any trees t and w. The proof that ψ^* preserves the product $\prec$ is analogous.

Recall that the associativity of the shuffle product is equivalent to the following equality

$$Sh(p, q + r) \cdot (1_p \times Sh(q, r)) = Sh(p + q, r) \cdot (Sh(p, q) \times 1_r).$$

Let $t \in Y_p$, and $w = w^l \vee w^r \in Y_{q+r+1}$ with $w^r \in Y_q$ and $w^l \in Y_r$. Recall from [4] that the right product is given by $t \succ w = (t * w^l) \vee w^r$. So,

$$\psi^*(t \succ w) = \psi^*((t * w^l) \vee w^r) = \sum_{\gamma \in Sh(p+q,r)} (\gamma \times 1_1) \cdot (\psi^*(t * w^l) \vee \psi^*(w^r)).$$

Since

$$\psi^*(t * w^l) = \psi^*(t) * \psi^*(w^l) = \sum_{\delta \in Sh(p,q)} \delta \cdot (\psi^*(t) \times \psi^*(w^l)),$$

one has by the preceding formula

$$\begin{aligned}\psi^*(t \succ w) &= \sum_{\gamma \in Sh(p+q,r)} \sum_{\delta \in Sh(p,q)} (\gamma \times 1_1) \cdot (\delta \times 1_{r+1}) \cdot ((\psi^*(t) \times \psi^*(w^l)) \vee \psi^*(w^r)) \\ &= \sum_{\omega \in Sh(p,q+r)} \sum_{\epsilon \in Sh(q,r)} (\omega \times 1_1) \cdot (1_p \times \epsilon \times 1_1) \\ &\quad \cdot ((\psi^*(t) \times \psi^*(w^l)) \vee \psi^*(w^r)).\end{aligned}$$

Since

$$\begin{aligned}(\psi^*(t) \times \psi^*(w^l)) \vee \psi^*(w^r) &= (\psi^*(t) \times \psi^*(w^l) \times \psi^*(w^r) \times 1_1) \cdot s_{p+q+r} \cdots s_{p+q} \\ &= \psi^*(t) \times (\psi^*(w^l) \vee \psi^*(w^r)),\end{aligned}$$

we get

$$\begin{aligned}\psi^*(t \succ w) &= \sum_{\omega \in Sh(p,q+r)} (\omega \times 1_1) \cdot \left(\psi^*(t) \times \sum_{\epsilon \in Sh(q,r)} (\epsilon \times 1_1) \cdot (\psi^*(w^l) \vee \psi^*(w^r))\right) \\ &= \sum_{\omega \in Sh(p,q+r)} (\omega \times 1_1) \cdot (\psi^*(t) \times \psi^*(w)) = \psi^*(t) \succ \psi^*(w).\end{aligned}$$

□

6. The graded algebra of the cube vertices $\mathbf{Q}[Q_\infty]$

Under taking the dual basis, the linear dual of the map $\phi_n \circ \psi_n$ gives a map $(\phi_n \circ \psi_n)^* : \mathbf{Q}[Q_n] \to \mathbf{Q}[S_n]$. Its image is the so-called *Solomon descent algebra*. The direct sum $\mathbf{Q}[Q_\infty] := \bigoplus_{n\geq 0} \mathbf{Q}[Q_n]$ is a graded subalgebra of $\mathbf{Q}[Y_\infty]$ and so of $\mathbf{Q}[S_\infty]$. Since, by Section 3, ψ_n is compatible with the orders and with the 'over' and 'under' operations, the same kind of arguments as in Section 5 implies the following result:

Theorem 6.1 *For any $\epsilon \in Q_p$ and any $\delta \in Q_q$, the product $*$ satisfies*:

$$\epsilon * \delta = \sum_{\epsilon/\delta \leq \alpha \leq \epsilon\backslash\delta} \alpha \ = \epsilon/\delta + \epsilon\backslash\delta.$$

Recall from Section 3 that

$$\epsilon/\delta := (\epsilon_1, \ldots, \epsilon_{p-1}, -1, \delta_1, \ldots, \delta_{q-1})$$
$$\epsilon\backslash\delta := (\epsilon_1, \ldots, \epsilon_{p-1}, +1, \delta_1, \ldots, \delta_{q-1}).$$

Since there is obviously no element between ϵ/δ and $\epsilon\backslash\delta$ the formula for the product on the generators takes the form $\epsilon * \delta = \epsilon/\delta + \epsilon\backslash\delta$, that is

$$\begin{aligned}(\epsilon_1, \ldots, \epsilon_{p-1}) * (\delta_1, \ldots, \delta_{q-1}) = {} & (\epsilon_1, \ldots, \epsilon_{p-1}, +1, \delta_1, \ldots, \delta_{q-1}) \\ & + (\epsilon_1, \ldots, \epsilon_{p-1}, -1, \delta_1, \ldots, \delta_{q-1}).\end{aligned}$$

Hence we recover exactly formula 4.6 of [5, p. 307].

Appendix. The weak Bruhat order on a Coxeter group

Let (W, S) be a finite Coxeter system (cf. [1]). So W is a finite group generated by the set S, with relations of the form

$$(s \cdot s')^{m(s,s')} = 1, \quad \text{for } s, s' \in S,$$

for certain positive integers $m(s, s')$, with $m(s, s) = 1$ for all $s \in S$.

For any element $w \in W$ the length $l(w)$ is the number of factors in a minimal expression of w in terms of elements in S. There exists a unique element of maximal length in W, denoted w^0.

Given a subset $J \subseteq S$, the *standard parabolic subgroup* W_J is the subgroup of W generated by J. Clearly, the pair (W_J, J) is a finite Coxeter system too.

Definition A.1 Let (W, S) be a finite Coxeter system and let J be a subset of S. The set X_J of elements of W that have no descent at J is defined as

$$X_J := \{w \in W \mid l(w \cdot s) > l(w), \quad \text{for all } s \in J\}.$$

A proof of the following classical result can be found for instance in [7], p. 258.

Proposition A.2 ([1] *Ch. IV, p. 37, Example* 3) *Let* (W, S) *be a finite Coxeter system, and let* J *be a subset of* S. *Every element of* W *can be written uniquely as* $w = x \cdot y$, *where* $x \in X_J$ *and* $y \in W_J$. *If* $x \in X_J$ *and* $y \in W_J$, *then* $l(x \cdot y) = l(x) + l(y)$.

Definition A.3 Let (W, S) be a finite Coxeter system, the *weak Bruhat order* on W is defined by:

$$x \le x' \text{ if } x = y \cdot x', \text{ with } l(x) = l(y) + l(x').$$

The group W equipped with the weak ordering is a finite poset with minimal element 1_W, and maximal element w^0.

Given a subset $J \subseteq S$, Proposition A2 implies that there exist unique elements $x_J^0 \in X_J$ and $w_J^0 \in W_J$ such that $w^0 = x_J^0 \cdot w_J^0$. It is easy to check that w_J^0 is the maximal element of (W_J, J), and that x_J^0 is the longest element of X_J.

Corollary A.4 *Let* (W, S) *be a finite Coxeter system and let* $J \subseteq S$, *then* X_J *is the subset of* W *characterized as follows*:

$$X_J = \{ w \in W \mid w \le x_J^0 \}.$$

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