# ORDERED DESARGUESIAN AFFINE HJELMSLEV PLANES

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# AFFINE HJELMSLEV PLANES

By

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A Thesis

Submitted to the School of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Master of Science

McMaster University

December 1975

MASTER OF SCIENCE (1975)
(Mathematics)

McMASTER UNIVERSITY
Hamilton, Ontario

TITLE: Ordered Desarguesian Affine Hjelmslev Planes

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SUPERVISOR: Dr. N. D. Lane

NUMBER OF PAGES: v, 48

ABSTRACT

The first chapter provides basic prerequisites. In the second chapter we demonstrate that an ordered Desarguesian A. H. plane is coordinatized by an ordered A. H. ring. We then show that given an ordered A. H. ring, one can construct an ordered Desarguesian A. H. plane. In the remaining chapters we give an example of such a structure and examine its relationship to the associated ordinary affine plane, the radical in the A. H. ring and the Archimedean axiom.

# ACKNOWLEDGEMENTS

I wish to extend my gratitude to my advisor Dr. N. D. Lane for his insight and encouragement during the preparation of this thesis.

I also wish to acknowledge the support of the National Research Council of Canada.

Finally, I wish to thank my family, particularly Mrs. Josephine Thomas who typed the manuscript.

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# INTRODUCTION

An affine Hjelmslev plane (or A. H. plane) may be described as a geometry where more than one line may pass through two distinct points. This is usually defined by a neighbouring relation and eight axioms. A Desarguesian A. H. plane may be coordinatized by an A. H. ring.

Hjelmslev himself studied ordered A. H planes using reflections and motions. In this thesis we will give a more rigourous account for Desarguesian A. H. planes using modern methods. In his paper [11] "On Ordered Geometries", P. Scherk discussed the equivalence of an ordering of a Desarguesian affine plane to an ordering of its division ring. We will define an ordered Desarguesian A. H. plane and follow Scherk's methods.

We begin by showing that the coordinate ring of such a plane is ordered. Then we construct an ordered A. H. plane from a given ordered A. H. ring.

The next section provides an example of such an ordered A. H. ring, and hence justifies our definition. We then briefly examine the ordinary affine plane associated with an ordered A. H. plane.

Next we examine the radical of an ordered A. H. ring and its relationship to the associated ordered A. H. plane. Finally, we prove a suggestion of Hjelmslev's that any Archimedean ordered

A. H. ring is a division ring, and thus any Archimedean ordered A. H. plane is an ordinary affine plane.

# CHAPTER 1

## Elementary Properties of Affine Hjelmslev Planes

1.1. $\mathcal{H} = [\mathbb{P}, \mathbb{L}, I, \|]$ is called an incidence structure with parallelism if:

1. $\mathbb{P}$ and $\mathbb{L}$ are sets.
2. $I \subseteq \mathbb{P} \times \mathbb{L}$.
3. $\| \subseteq L \times L$ is an equivalence relation (parallelism).

The elements of $\mathbb{P}$ are called points and are denoted by P, Q, R... The elements of $\mathbb{L}$ are called lines and are denoted by l, m, n... $(P, l) \in I$ is written $P\ I\ l$ and is read, "P is incident with l"; similarly, $(l, m) \in \|$ is written $l \| m$ and is read, "l is parallel to m". $P, Q\ I\ l$ will mean $P\ I\ l$ and $Q\ I\ l$, and $P\ I\ l, m$ will mean $P\ I\ l$ and $P\ I\ m$. We write $l \wedge m = \{P \in \mathbb{P} \mid P\ I\ l, m\}$ and $l \vee m = \{P \in \mathbb{P} \mid P\ I\ l \text{ or } P\ I\ m\}$.

1.2. Two points, P and Q, are neighbours (written $P \sim_{\mathbb{P}} Q$, or just $P \sim Q$) if there exist $l, m \in \mathbb{L}$, $l \neq m$, such that $P, Q\ I\ l, m$. Two lines, l and m, are neighbours (written $l \sim_{\mathbb{L}} m$, or just $l \sim m$) if for any $P\ I\ l$, there exists a $Q\ I\ m$ such that $P \sim Q$ and for any $Q\ I\ m$, there exists a $P\ I\ l$ such that $Q \sim P$. The non-neighbouring relationship will be denoted by $\not\sim$.

1.3. An incidence structure with parallelism is called an affine Hjelmslev plane (or an A. H. plane) if it satisfies the following axioms.

A1. For any $P, Q \in \mathbb{P}$, there exists $l \in \mathbb{L}$ such that $P, Q \, I \, l$. If $P \not\sim Q$, we write $l = PQ$.

A2. There exist $P_1, P_2, P_3 \in \mathbb{P}$ such that $P_i P_j \not\sim P_i P_k$ where $(i, j, k)$ is any permutation of $(1, 2, 3)$. $\{P_1, P_2, P_3\}$ is called a triangle.

A3. $\sim$ is transitive on $\mathbb{P}$.

A4. If $P \, I \, l, m$, then $l \not\sim m$ iff $|l \wedge m| = 1$.

A5. If $l \not\sim m$; $P, R \, I \, l$; $Q, R \, I \, m$ and $P \sim Q$, then $R \sim P, Q$.

A6. If $l \sim m$ and $n \not\sim l$ with $P \, I \, l, n$ and $Q \, I \, m, n$, then $P \sim Q$.

A7. If $l \parallel m$; $P \, I \, l, n$ and $l \not\sim n$, then $m \not\sim n$ and there exists a point $Q$ such that $Q \, I \, m, n$.

A8. For every $P \in \mathbb{P}$ and every $l \in \mathbb{L}$, there exists a unique line $L(P, l)$ such that $P \, I \, L(P, l)$ and $L(P, l) \parallel l$.

From A3 and the definition of the neighbour relation on L, it is obvious that $\sim$ is transitive on L also.

1.4. The set $\Pi_l = \{m \in \mathbb{L} \mid m \parallel l\}$ is a pencil of L. Two pencils, $\Pi_1$ and $\Pi_2$, are neighbours (written $\Pi_1 \sim \Pi_2$) if there exists $l_1 \in \Pi_1$ and $l_2 \in \Pi_2$, such that $l_1 \sim l_2$. In view of A8 it is clear that if $l \parallel m$ then either $l \wedge m = \emptyset$ or $l = m$. (However, the possibility that two non-parallel lines are disjoint is not excluded.)

1.5. Lemma. If $P \mathrel{I} l$, then there exists $Q \mathrel{I} l$ such that $P \not\sim Q$.

Proof. [7], Lemma 1. 1. 9.

1.6. Lemma. If $g \wedge h = \emptyset$ or $g \sim h$, then there exists $j$ such that $j \parallel h$; $j \sim g$ and $j \wedge g \neq \emptyset$.

Proof. [9], Lemma 2. 1.

1.7. Lemma. Let $\{P_1, P_2, P_3\}$ be a triangle, then for any $l \in \mathbb{L}$ there exists $P \in \{P_1, P_2, P_3\}$ such that $P \not\sim X$ for each $X \mathrel{I} l$.

Proof. [9], Lemma 2. 4.

1.8. Lemma. If $g \wedge h \neq \emptyset$, then there exists a point $S$ such that $S \not\sim X$ for all $X \mathrel{I} g \vee h$.

Proof. [9], Lemma 2. 9.

1.9. Lemma. Let $g_1 \parallel g_2$, then the following are equivalent:

1. $g_1 \sim g_2$.
2. There exist $P_i \mathrel{I} g_i$, $i = 1, 2$, such that $P_1 \sim P_2$.

Proof. [7], Lemma 1. 1. 10.

1.10. Lemma. If $P_i = g_i \wedge j$, $i = 1, 2$, and $g_1 \| g_2$, then the following are equivalent:

1. $g_1 \sim g_2$.
2. $P_1 \sim P_2$.

Proof. [7], Lemma 1. 1. 11.

1.11. A mapping $\sigma : \mathbb{P} \rightarrow \mathbb{P}$ is called a dilatation of $\mathcal{H}$ if $P, Q \ I \ l$ implies $\sigma P \ I \ L(Q, l)$, for each $l \in \mathbb{L}$. Let D be the set of all dilatations of $\mathcal{H}$ and let $D_p$ be the set of all dilatations of $\mathcal{H}$ with a fixed point P. For $P \in \mathbb{P}$, $O_p$ will denote the dilatation which maps each point of $\mathbb{P}$ into P, and $\mathbb{1}$ will denote the identity map.

1.12. A line l is called a trace of the dilatation $\sigma$ if there exists $P \ I \ l$ such that $\sigma P \ I \ l$. If l is a trace of $\sigma$, then $\sigma Q \ I \ l$ for all $Q \ I \ l$.

1.13. A dilatation $\hat{\tau}$ is called a quasitranslation if $\hat{\tau}$ has no fixed point, or $\hat{\tau} = 1$.

A quasitranslation $\tau$ is called a translation if for any trace, g, of $\tau$, and any line h, $h \| g$, then h is also a trace of $\tau$.

A pencil $\pi$ is called a direction of the translation $\tau$ if $\pi$ is a pencil of traces of $\tau$.

Let T be the set of all translations.

Let $D_\tau = \{\pi \mid \pi$ is a direction of $\tau\}$. Let $T_\pi = \{\tau \in T \mid \pi \in D_\tau\}$. The set $ND = \{\sigma \in D \mid \sigma P \sim P$ for each $P\}$ is called the set of neighbouring dilatations. $N = ND \cap T$ is called the set of neighbouring translations. It can be shown that if $\tau \in T \setminus N$, then $P \not\sim \tau P$ for any $P \in \mathbb{P}$, cf. [10], Theorem 3.6.

1.14. Lemma. Let $\tau \in T$, then the following are equivalent:

1. $\tau \notin N$.
2. If g and h are traces of $\tau$, then $h \parallel g$.
3. $|D_\tau| = 1$.

Proof. [9], Lemma 3. 11.

1.15. An A. H. plane $\mathcal{H}$ is called a translation plane (or a T.plane) if it satisfies the following axiom.

A9. T is a transitive group.

If T satisfies A9, then $\tau_{PQ}$ will denote the unique translation mapping P into Q. From [10], Theorem 3.7, T is abelian.

1.16. Let $\mathcal{H}$ be a T-plane. A trace preserving endomorphism is a map, $a: T \rightarrow T$, $(\tau \longrightarrow \tau^a)$ such that:

1. a is a group endomorphism of T.
2. $D_\tau \subseteq D_{\tau^a}$.

H will denote the set of all these endomorphisms. Then H is a ring with identity with respect to the operations:

$$\tau^{ab} = (\tau^b)^a, \quad \text{and} \quad \tau^{a+b} = \tau^a \tau^b.$$

The identity element, 1, and the zero, o of H are defined by $\tau^1 = \tau$ and $\tau^o = 1 \in T$, respectively, for all $\tau \in T$.

H* will denote the multiplicative monoid of H, and $\eta$ will denote the set of non-units of H.

1.17. Theorem. Let $\mathcal{H}$ be a translation plane, and let P be any point.

1. For each $a \in H$, there exists a unique dilatation $\sigma = \sigma(a) \in D_P$ such that $(\tau_{PS})^a = \tau_{P\,\sigma S}$ for each $\tau_{PS} \in T$.
2. The mapping $h_P$: $H^* \longrightarrow D_P$ defined by $h_P(a) = \sigma(a)$ is a monoid isomorphism.

In fact $a \in \eta$ iff $\sigma(a) \in M_P$, where $M_P = \{\sigma \in D_P \mid \sigma Q \sim P$, for all $Q\}$.

Proof. [9], Theorem 4.4.

1.18. Corollaries. [9], 4. 4. 1. - 4. 4. 6.

1. Each $a \in H$ is uniquely determined by its action on one $\tau \in T \setminus N$.
2. If $\tau \in T \setminus N$ and $a \in H$, then $\tau^a = 1$ implies $a = o$, and $\tau^a = \tau^b$ implies $a = b$.
3. $N^a \subseteq N$ for each $a \in H$.
4. Let $a \in H$, then the following are equivalent:
   a) There exists $\tau \in T \setminus N$ such that $\tau^a \in N$.
   b) $T^a \subseteq N$.
   c) $a \in \eta$.
5. H is a local ring with unique maximal ideal $\eta$.

1.19. $\mathcal{H}$ is called Desarguesian if $\mathcal{H}$ is a translation plane and satisfies the following axiom.

A10. If $\tau_1 \in T \setminus N$ and $\tau_2 \in T$ and $D_{\tau_1} \subseteq D_{\tau_2}$, then there exists $a \in H$ such that $\tau_1^{\,a} = \tau_2$.

We note that if O, A, B I 1 and $O \not\sim A$, then $B = \tau_{OA}^{\,b}(O)$ for some $b \in H$.

1.20. Axiom A10 is connected with the following one, which applies to any point P.

A10.P. For each collinear triple P, Q, R, with $P \not\sim Q$, there exists a dilatation $\sigma \in D_P$ such that $\sigma Q = R$.

1.21. Theorem. Let $\mathcal{H}$ be a translation plane, then the following are equivalent.

1. A10.
2. A10.P for every point P.
3. There exists P such that A10.P. holds.

Proof. [9], Theorem 5.3.

Lorimer and Lane have also shown, [9], Theorem 5.11, that A10.P is equivalent to a Desarguesian configurational condition for P.

1.22. Coordinates can be introduced in a Desarguesian affine Hjelmslev plane, in the following way.

Theorem. Let $\pi_1 \nparallel \pi_2$. Let $\tau_i \in T_{\pi_i} \setminus N$; $i = 1, 2$. Then for each $\tau \in T$ there exists a unique a, b $\in$ H such that $\tau = \tau_1^a \tau_2^b$.

Proof. [9], 6.1.

1.23. Theorem. Let $\mathcal{H}$ be a Desarguesian A. H. plane with the ring H, then H is a local ring with the properties:

1. $\eta$ is the set of two-sided zero divisors of H.
2. If a, b $\in$ H, then a $\in$ bH or b $\in$ aH.

Proof. [9], Theorem 6.9.

1.24. Such a local ring is called an affine Hjelmslev ring (or an A. H. ring). One can always construct an analytic model of a Desarguesian A. H. plane $\mathcal{H}$(H) over such a ring, H, ( [8], Section 3). The A. H. plane so constructed will have as local ring of trace preserving endomorphisms an A. H. ring H' isomorphic to H; [8], 3.10.

1.25 Lemma. If a $\in$ H, a $\neq$ o and O, A, B $\in$ $\mathbb{P}$ are not collinear, then, given A, B I l, $\tau_{OB}^a(O)$ I L( $\tau_{OA}^a(O)$, l); see Figure 1.

Proof. $\tau_{AB} = \tau_{OB}\tau_{AO} = \tau_{OB}\tau_{OA}^{-1}$.

Thus $\tau_{AB}^a = \tau_{OB}^a \tau_{OA}^{-a}$. Let $\tau_{OA}^a(O) = C$, $\tau_{OB}^a(O) = D$.

Then, $\tau_{AB}^a(C) = \tau_{OB}^a(\tau_{OA}^{-a}(C)) = \tau_{OB}^a(O) = D$.

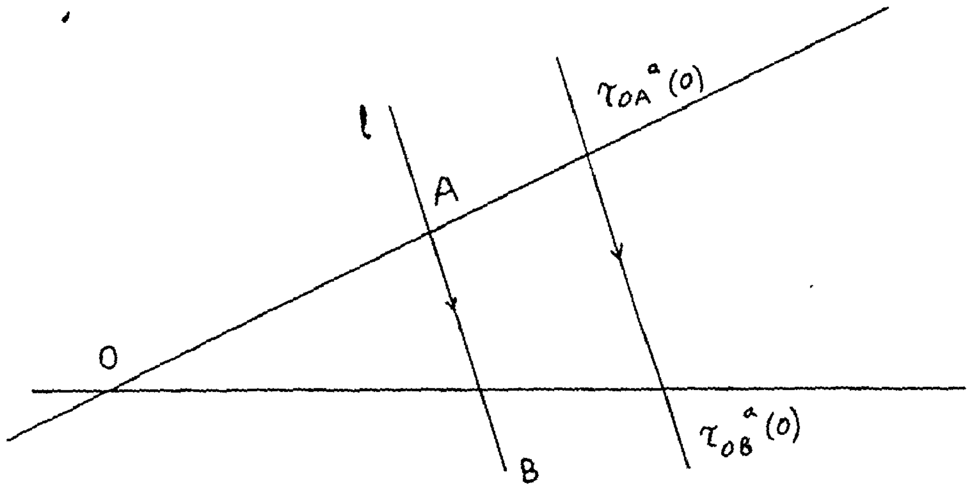



Figure 1.

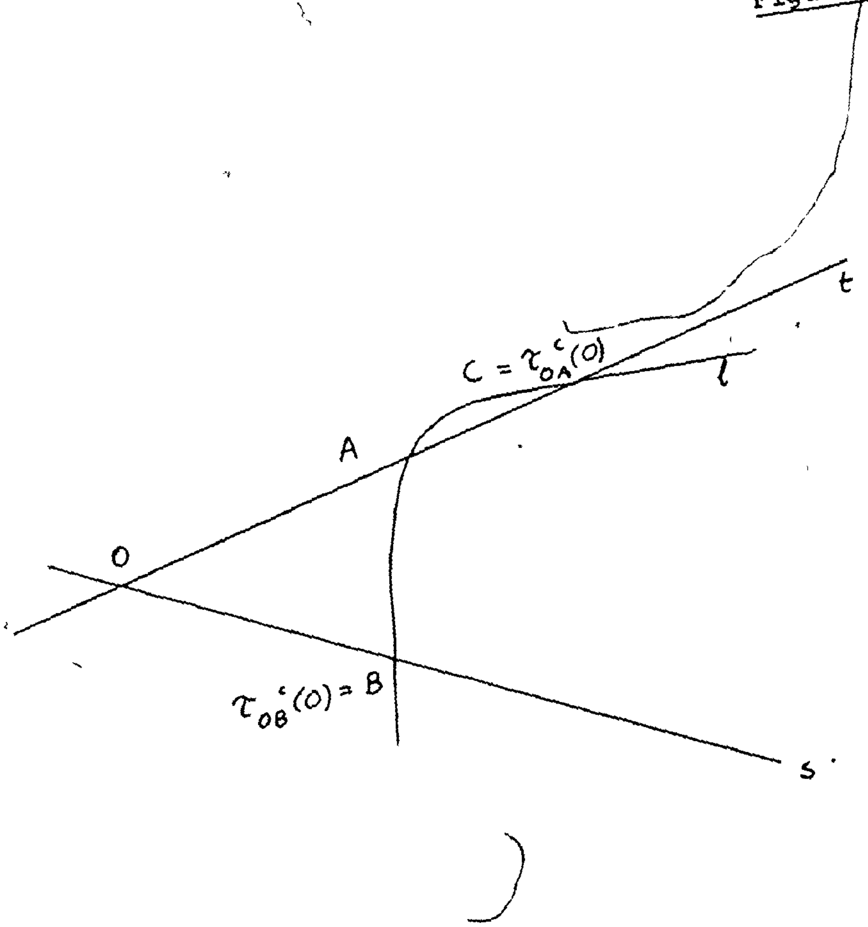



Figure 2.

Since l is a $\tau_{AB}$ trace, it is a $\tau_{AB}{}^{a}$ trace. Let m = L(C, l).

Then m is a $\tau_{AB}{}^{a}$ trace. Thus $\tau_{AB}{}^{a}(C)$ I m, but $\tau_{AB}{}^{a}(C) = D = \tau_{OB}{}^{a}(O)$.

<u>1.26. Note.</u> Since we have put no restrictions on O, A and B, it is possible that $L(\tau_{OA}{}^{a}(O), l) = l$ even if $a \neq 1$; cf. for example Figure 2. In a Desarguesian A. H. plane, suppose O, B I s and A $\not\sim$ X for any X I s. Then AB = l and l $\not\sim$ s. Let OA = t and suppose l $\sim$ t. Thus, by A4, $|l \wedge t| \neq 1$. Take another point, C, in $l \wedge t$; C $\neq$ A. Then, since O $\not\sim$ A, $\tau_{OA} \notin N$, and so $C = \tau_{OA}{}^{c}(O)$ for some $c \in H$, where $c \neq 1$.

By <u>1.25,</u> $\tau_{OB}{}^{c}(O)$ I L(C, l). However, C I l, thus $\tau_{OB}{}^{c}(O)$ I l and, since l $\not\sim$ s, $\tau_{OB}{}^{c}(O) = l \wedge s = B$. However, in our dicussion we shall always consider the situation where l $\not\sim$ s, t.

## CHAPTER 2

### The Construction of an Ordered A. H. Ring from a Geometrically Ordered Desarguesian A. H. plane

2.1. We shall assume that $\mathcal{H}$ is a given Desarguesian affine Hjelmslev plane. Thus, in addition to A1 to A8, $\mathcal{H}$ is a translation plane, and satisfies A10.

Following P. Scherk, [11], we shall introduce seven axioms of order which will enable us to say that a point on a line lies "between" two other points on the same line.

We shall show that this geometric ordering of $\mathcal{H}$ induces an algebraic ordering of the A. H. ring H of the trace preserving endomorphisms of the group T of translations of $\mathcal{H}$.

2.2. An A. H. ring H will be called ordered if there exists a subset, $H^+$, of H, such that:

1. Every $a \in H$ satisfies exactly one, of $a \in H^+$, $-a \in H^+$, $a = o$.
2. If $a, b \in H^+$, then $a + b \in H^+$.
3. If $a, b \in H^+$, and $b \notin \eta$, then $ab \in H^+$.

(Note: Chapter 4 contains an example of an A. H. ring in which condition 3 holds but $a, b \in H^+$ and $a \notin \eta$ does not imply $ab \in H^+$.)

2.3. An ordering of the affine Hjelmslev plane, $\mathcal{H}$, is a ternary relation on $\mathbb{P}$, which satisfies the following axioms.

O1. (A, B, C) implies that A, B, C are mutually distinct and collinear.

O2. (A, B, C) implies (C, B, A).

O3. If A, B and C are mutually disinct and collinear, then exactly one, of (A, B, C), (B, C, A) and (C, A, B) holds.

O4. If A, B, C, D are collinear, then (A, B, C) and (B, C, D) imply (A, B, D) and (A, C, D).

O5. If A, B, C, D, are collinear, then (A, B, C) and (A, C, D) imply (A, B, D) and (B, C, D).

O6. Two, of (B, A, C), (C, A, D) and (D, A, B) exclude the third, if A, B, C, D are collinear.

O7. Parallel projections preserve order.

2.4. In the context of affine Hjelmslev planes we shall assume that a parallel projection, from a line l to a line m, in the direction $\pi$ is a bijective mapping from the distinct points of l onto the distinct points of m; i.e. $X = l \wedge L(X, \pi)$ is taken into $X' = m \wedge L(X, \pi)$, for any X I l. Thus, by A7, $L(X, \pi) \not\sim l, m$.

2.5. These axioms are not independent, since O5 can be deduced from O1, O2, O3, O4 and O6. However, they are in a convenient form for application.

Theorem. O1, O2, O3, O4 and O6 imply O5.

Proof. Assume A, B, C and D are collinear, and (A, B, C) and (A, C, D). If (B, C, D) is false, then, by O3, either (C, D, B) or (D, B, C).

Case 1: Suppose (C, B, D). (using O2).

Then, by O6, (A, B, C) and (C, B, D) exclude (A, B, D). Hence, by O3, either (B, D, A) or (D, A, B).

However, by O4, (D, A, B) and (A, B, C) imply (D, A, C), which contradicts (A, C, D), by O3. On the other hand, (A, D, B) and (D, B, C) imply (A, D, C), which contradicts (A, C, D).

Thus (C, B, D) is false.

Case 2: Suppose (C, D, B).

Then (A, C, D) and (C, D, B) imply (A, C, B), by O4, which contradicts (A, B, C), by O3. Thus (C, D, B) is false.

Hence (B, C, D).

Finally, (A,B, C) and (B, C, D) imply (A, B, D), by O4.

2.6. B and C are said to lie on the same side of A if exactly one, of (A, B, C), (A, C, D) and B = C $\neq$ A, holds. This will be denoted $B,C|A$.

2.7. Theorem. If l is any line through A, then the property of lying on the same side of A, on l, is an equivalence relation on $(\mathbb{P} \setminus \{A\}) \wedge l$.

Proof. 1. If B $\neq$ A, then $B,B|A$.

2. If $B,C|A$, then $C,B|A$, by definition.

3. <u>Claim</u>: If $B,C|A$ and $C,D|A$, then $B,D|A$.

If $B = C$ or $C = D$, then $B,D|A$. If $B = D$, then $B,D|A$.

Hence we may assume that B,C and D are mutually distinct.

Since they are all different from A, there remain four possible cases.

<u>Case 1:</u> (A, D, C) and (B, C, A) imply (A, D, B), by O5.

<u>Case 2:</u> (A, B, C) and (A, C, D) imply (A, B, D).

<u>Case 3:</u> (A, D, C) and (A, B, C). Then, by O3, one, of $B = D \neq A$, (A, B, D), (B, D, A) and (B, A, D) holds. But (B, A, D) and (A, D, C) imply (B, A, C), by O4, which contradicts (A, B, C), by O3. Hence $B, D|A$.

<u>Case 4:</u> (A, C, D) and (B, C, A). Then, by O3, one, of $B = D \neq A$, (A, B, D), (B, D, A) and (D, A, B) holds. But (A, C, D) and (D, A, B) imply (C, A, B), by O5, which contradicts (A, C, B). Hence $B,D|A$.

Thus in every case $B,D|A$. The claim is proved.

<u>2.8. Lemma.</u> The property of lying on the same side of A is preserved under a parallel projection, as described in <u>2.4.</u>

<u>Proof.</u> Let X' be the image of X under a parallel projection from a line l to a line m. Let A, B, C I l and $B,C|A$. Thus one, of $B = C \neq A$, (B, C, A) and (C, B, A) holds. Hence, one, of $B' = C' \neq A'$, (B', C', A') and (C', A', B') holds, by O7 and A7. Thus $B', C'|A'$.

2.9. Lemma. If A, B, C, D are collinear, then (B, A, C) and (B, A, D) imply C,D|A.

Proof. By 06, (B, A, C) and (B, A, D) exclude (C, A, D). Hence $C = D \neq A$ or (C, D, A) or (D, C, A). Thus C,D|A.

2.10. Theorem. Translations preserve order, see Figure 3.

Proof. Let A, B, C I l and (A, B, C) and $\tau \in T$.
Then $\tau B$, $\tau C$ I $L(\tau A, l) = m$.

Case 1: l is not a $\tau$ trace.

a) $l \not\sim m$.

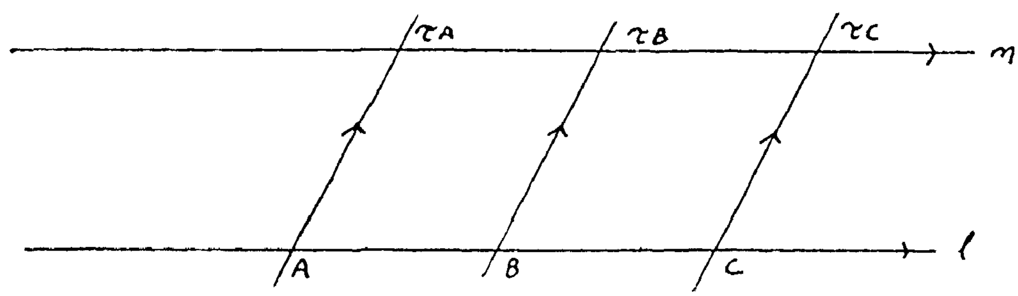
Then $A \not\sim X$ for X I m, by 1.9, so $L(A, \tau A) \not\sim m$, and $\tau A \not\sim Y$ for any Y I l, so $L(A, \tau A) \not\sim l$. Thus for any Y I l, $L(Y, L(A, \tau A)) \not\sim m, l$, by A7. Since $\tau$ is a translation, $\tau Y$ I $L(Y, L(A, \tau A)) \wedge m$. So $\tau$ can be considered to be generated by a parallel projection having a pencil of lines parallel to $L(A, \tau A)$. Thus 07 applies and so (A, B, C) implies $(\tau A, \tau B, \tau C)$.

b) $l \sim m$.

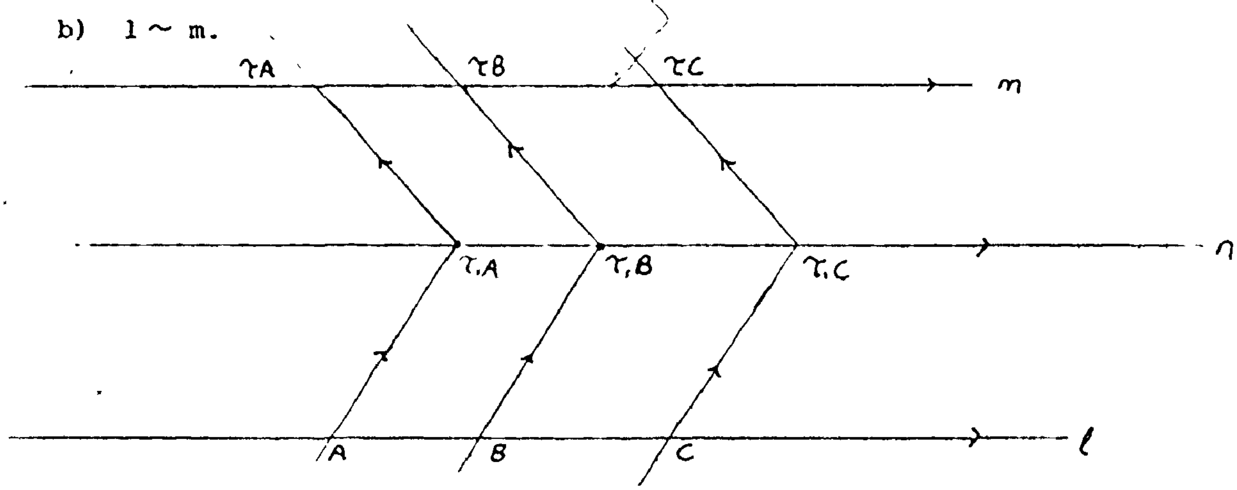
Then there exists a point $Z \not I$ l with $Z \not\sim Y$ for any Y I l, by 1.7. Take $n = L(Z, l)$. Then $n \not\sim l$ and so $n \not\sim m$.
Take some X I n such that neither t = AX, nor s = $\tau$AX, are $\tau$ traces. Then $\tau_1 = \tau_{AX}$ and $\tau_2 = \tau_{X \tau A}$ are as in Case 1 a) and $\tau = \tau_2 \tau_1$. Thus (A, B, C) implies $(\tau_1 A, \tau_1 B, \tau_1 C)$, which

Case 1: l is not a $\tau$ trace.

a) $l \not\sim m$.




b) $l \sim m$.




Case 2: l is a $\tau$ trace.

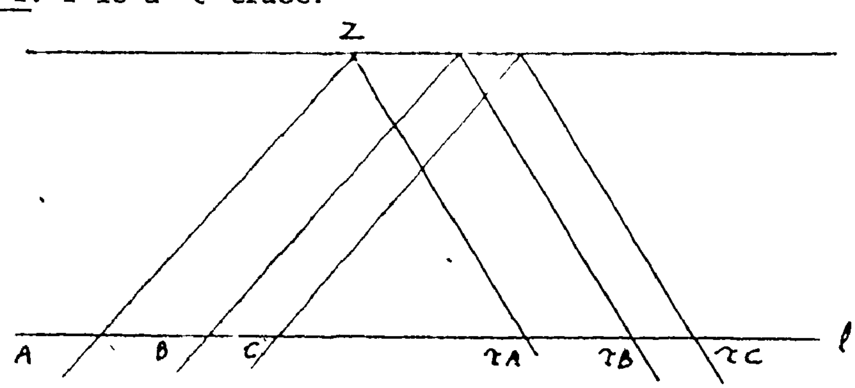



Figure 3.

implies $(\tau_2\tau_1 A,\ \tau_2\tau_1 B,\ \tau_2\tau_1 C) = (\tau A,\ \tau B,\ \tau C)$.

Case 2: $l$ is a $\tau$ trace.

Choose $Z$ a point which is not a neighbour of any point on $l$, then $Z \not I l$. Take $\tau_1 = \tau_{AZ}$ and define $\tau_2$ by $\tau_2(\tau_1 A) = \tau A$. Clearly $AZ$ and $Z\tau A$ are not $\tau$ traces. Then by Case 1, $(A, B, C)$ implies $(\tau_1 A,\ \tau_1 B,\ \tau_1 C)$ which implies $(\tau_2\tau_1 A,\ \tau_2\tau_1 B,\ \tau_2\tau_1 C) = (\tau A,\ \tau B,\ \tau C)$.

2.11. We define $a \in H^+(O, A)$ if $\tau_{OA}{}^a(O)$, $A|O$, where $O \not\sim A$. We say that $a$ is positive with respect to $O$ and $A$.

2.12. Theorem. For any choice of $O$ and $A$, $O \not\sim A$, $1 \in H^+(O, A)$.

Proof. $A$, $\tau_{OA}{}^1(O) = A$, lie on the same side of $O$.

2.13. Lemma. $H^+(O, A)$ does not depend on the choice of $A$; $A \not\sim O$, see Figure 4.

Proof. Let $B$ be any point such that $B \not\sim A$ and $B \not\sim O$.

Case 1: $B \not I OA = l$.

Let $BO = m$. Since $l \wedge m \neq \emptyset$, there exists a point $S$, such that $S \not\sim Z$ for any $Z\ I\ l \vee m$, by 1.8. Let $OS = n$. Then $n \not\sim l, m$. Since $O \not\sim A$, any $X\ I\ l$ can be expressed in the form $\tau_{OA}{}^x(O)$, for a unique $x \in H$. Then, by 1.25. $\tau_{OS}{}^x(O)\ I\ L(X, AS)$. Also $\tau_{OS}{}^x(O)\ I\ n$.

If $AS \sim n$, then, since $l \not\sim n$, A6 implies $O \sim A$; a contradiction.

Case 1: B $\not I$ OA = 1.

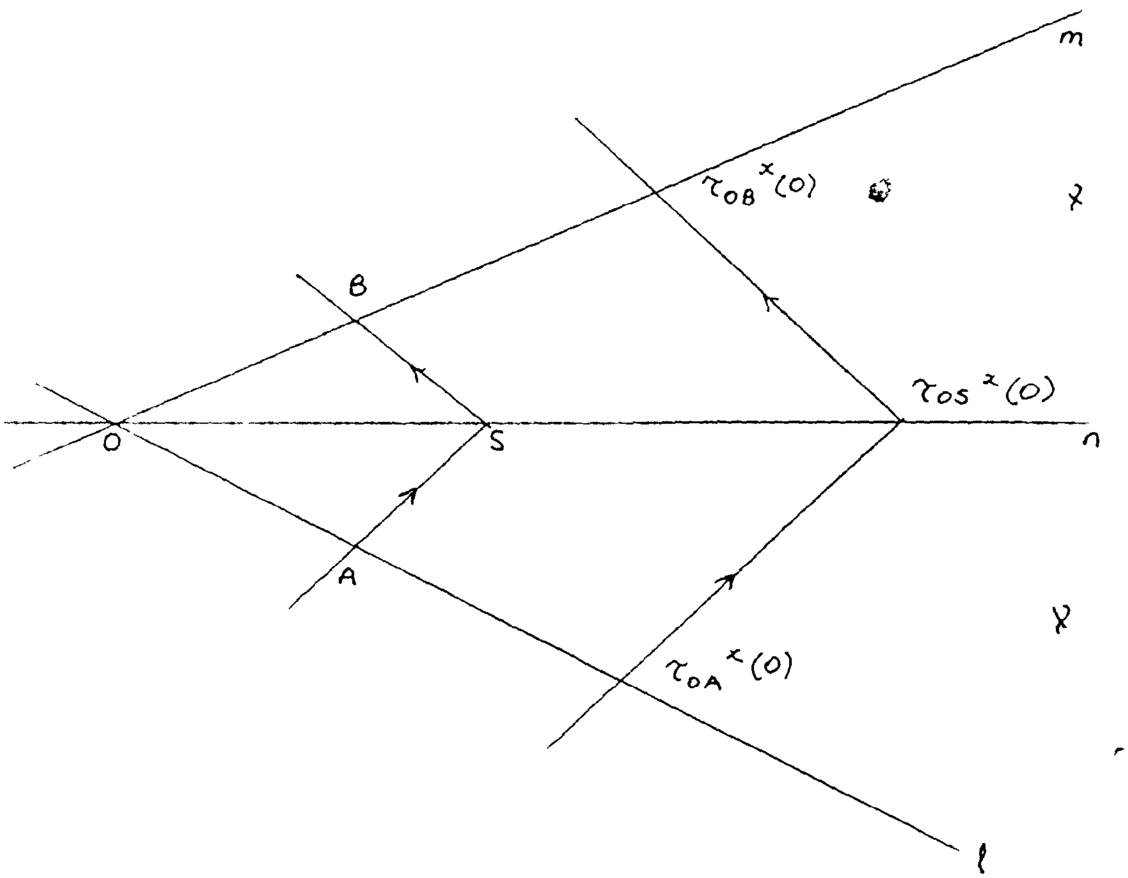



Figure 4.

Thus AS $\not\sim$ n and also AS $\not\sim$ 1. Then the map which takes $X = \tau_{OA}{}^{x}(O)$ I 1 into $\tau_{OS}{}^{x}(O)$ I n is a parallel projection, as in 2.4, and so A, $\tau_{OA}{}^{x}(O) \mid O$ implies S, $\tau_{OS}{}^{x}(O) \mid O$.

Similarly, since SB $\not\sim$ m and SB $\not\sim$ n, the map which takes $Y = \tau_{OS}{}^{y}(O)$ I n into $\tau_{OB}{}^{y}(O)$ I m is a parallel projection. Thus S, $\tau_{OS}{}^{y}(O) \mid O$ implies B, $\tau_{OB}{}^{y}(O) \mid O$.

Thus $a \in H^{+}(O, A)$ implies $a \in H^{+}(O, B)$ and symmetrically $a \in H^{+}(O, B)$ implies $a \in H^{+}(O, A)$.

Case 2: B I OA = 1.

Choose C, C $\not\sim$ X for any X I 1. Then A, $\tau_{OS}{}^{a}(O) \mid O$ iff C, $\tau_{OC}{}^{a}(O) \mid O$ iff B, $\tau_{OB}{}^{a}(O) \mid O$. Thus $a \in H^{+}(O, A)$ iff $a \in H^{+}(O, B)$.

2.14. Notation. We may now write $H^{+}(O, A)$ as $H^{+}(O)$.

2.15. Lemma. $H^{+}(O)$ does not depend on the choice of O, see Figure 5.

Proof. Let O' $\neq$ O; O, O' I 1.

Choose a point A which is not a neighbour of any point on 1.

Put $\tau_{OA}(O') = A'$. Then A' I L(A, 1) = m and $\tau_{OA} = \tau_{O'A'}$, thus OA $\parallel$ O'A'. Since A is not a neighbour of any point on 1, 1 $\not\sim$ m and also OA $\not\sim$ 1. Thus, by A7, O'A' $\not\sim$ 1.

Hence O7 applies to the projection $\pi$, parallel to 1, taking any

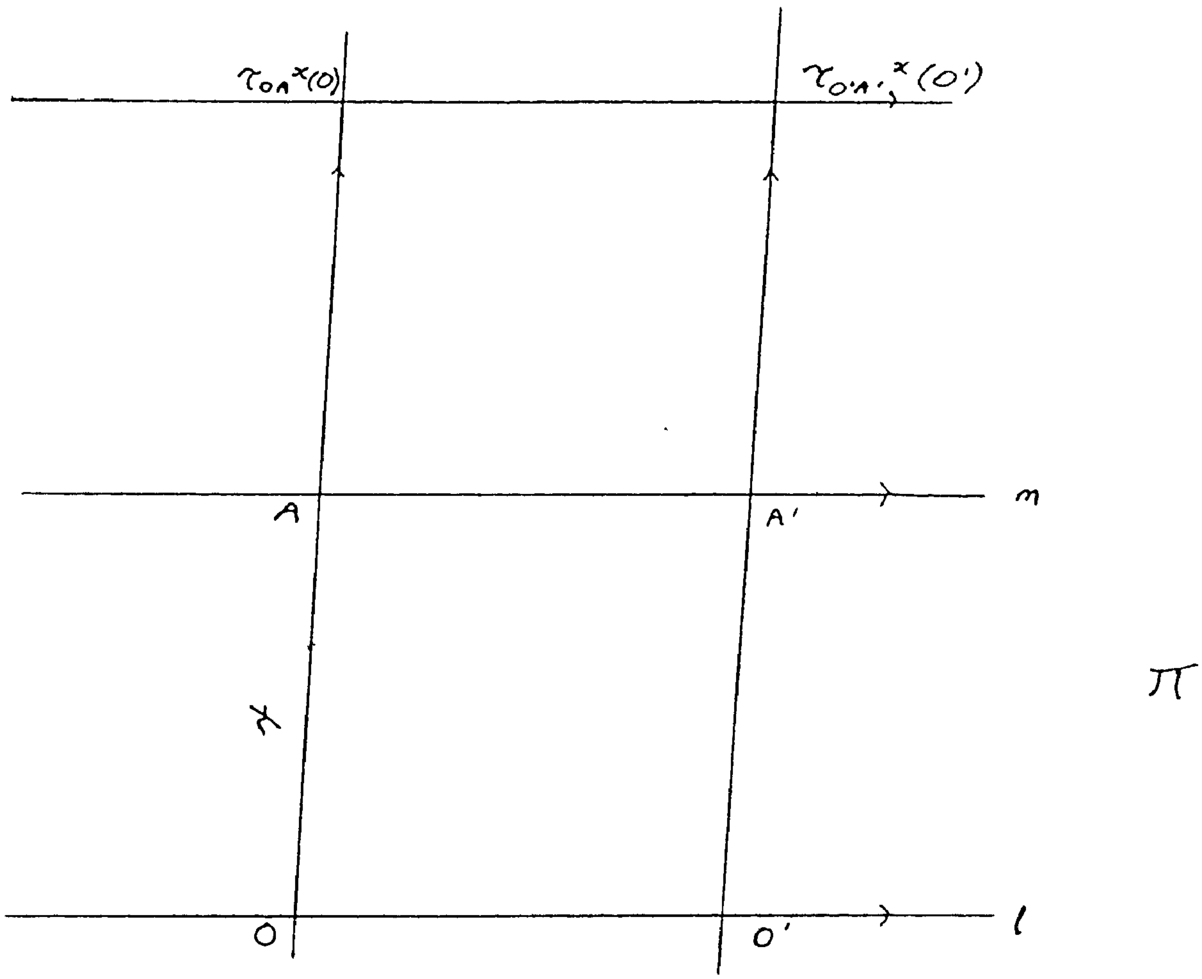



Figure 5.

$X = \tau_{OA}{}^{x}(O)$ I $OA$, for a unique $x \in H$, into $\tau_{O'A'}{}^{x}(O')$. Thus $A, \tau_{OA}{}^{a}(O) \mid O$ implies $A', \tau_{O'A'}{}^{a}(O') \mid O'$ i.e., $a \in H^{+}(O)$ implies $a \in H^{+}(O')$. Symmetrically $a \in H^{+}(O')$ implies $a \in H^{+}(O)$.

2.16. Notation. We may now write $H^{+}(O)$ as $H^{+}$.

2.17. Theorem. If $a, b \in H^{+}$ and $b \notin \eta$, then $ab \in H^{+}$.

Proof. Choose $O, A \in \mathbb{P}$ such that $O \not\sim A$.
Then $b \in H^{+}$ implies $A, \tau_{OA}{}^{b}(O) \mid O$. Put $B = \tau_{OA}{}^{b}(O)$.
Then $B \neq O$ and since $b \notin \eta$, $B \not\sim O$. (If $B \sim O$ then $\tau_{OB} \in N$, but $\tau_{OB} = \tau_{OA}{}^{b}$ so, by 1.18, $b \in \eta$, since $\tau_{OA} \notin N$; a contradiction.)
Then $a \in H^{+}$ implies $B, \tau_{OB}{}^{a}(O) \mid O$. But $\tau_{OB}{}^{a}(O) = (\tau_{OA}{}^{b})^{a}(O) = \tau_{OA}{}^{ab}(O)$. $A,B \mid O$ and $B, \tau_{OB}{}^{a}(O) \mid O$ imply $A, \tau_{OB}{}^{a}(O) \mid O$, i.e, $A, \tau_{OA}{}^{ab}(O) \mid O$, by 2.7. Thus $ab \in H^{+}$.

2.18. We may assume that every line of $\mathcal{H}$ is incident with at least three points, or O1 to O7 are satisfied trivially, and H has only two elements with no ordering. In fact, for every proper affine Hjelmslev plane, H has at least four elements and each line is incident with at least four points; cf. [3], Chapter 2.

2.19. Lemma. The characteristic of H is not equal to two.

Proof. Assume (O, A, B).

If we apply $\tau_{OA}$ to O, A and B, then 2.10 implies

$(\tau_{OA}(O), \tau_{OA}(A), \tau_{OA}(B))$, which equals $(A, \tau_{OA}{}^2(O), \tau_{OA}\tau_{OB}(O))$.

Put $\tau_{OA}\tau_{OB}(O) = C = \tau_{OB}\tau_{OA}(O)$. Now if we apply $\tau_{OB}$ to (O, A, B), then 2.10 implies $(\tau_{OB}(O), \tau_{OB}(A), \tau_{OB}(B))$, which equals $(B, C, \tau_{OB}{}^2(O))$.

If the characteristic of H is two, then $\tau_{OA}{}^2(O) = \tau_{OA}{}^0(O) = O$ and $\tau_{OB}{}^2(O) = O$. So (A, O, C) and (B, C, O) hold and these imply (A, O, B), by O4; a contradiction.

2.20. If C, B I l, we define O I l to be the midpoint of C and B on l if $\tau_{CO} = \tau_{OB}$.

2.21. Lemma. If Q, C, O, B, P I l and (P, B, C) and O is the midpoint on l of both Q and P, and C and B, then (P, B, Q).

Proof. Take T I l so that $O \not\sim T$.

Then $\tau_{OB} = \tau_{CO} = \tau_{OT}{}^a$ for a unique $a \in H$, and $\tau_{OP} = \tau_{QO} = \tau_{OT}{}^b$ for a unique $b \in H$, i.e., $B = \tau_{OT}{}^a(O)$, $C = \tau_{OT}{}^{-a}(O)$, $P = \tau_{OT}{}^b(O)$, $Q = \tau_{OT}{}^{-b}(O)$. Now choose $T' \not\sim X$ for any X I l, by 1.7.

Then T' $\not{I}$ l (see Figure 6). Construct a parallel projection from l to OT' = n, in the direction of TT' = m. Then $n \not\sim m$, or by A6, $O \sim T$.

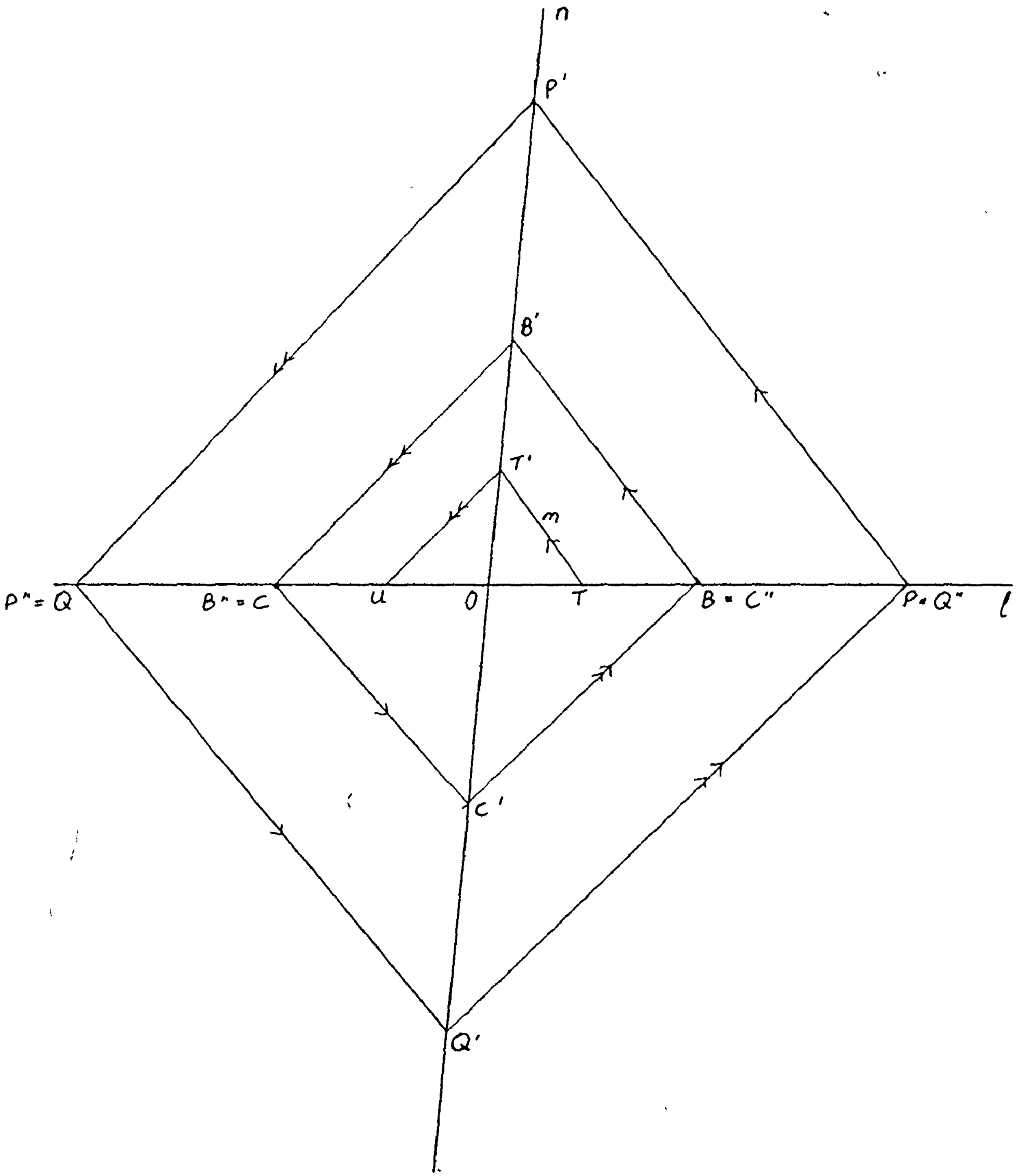



Figure 6.

Thus L(B, m) $\not\sim$ n and meets n at a unique point, say B', by A7. Similarly, C', P' and Q' are obtained uniquely. By 1.25, $\tau_{OT'}^{a}(O)$ I L( $\tau_{OT}^{a}(O)$, m). Thus, since B', C', P' and Q' are unique, B' = $\tau_{OT'}^{a}(O)$, C' = $\tau_{OT'}^{-a}(O)$, P' = $\tau_{OT'}^{b}(O)$ and Q' = $\tau_{OT'}^{-b}(O)$.

Now construct a second parallel projection from n to l, taking T' into U = $\tau_{OT}^{-1}(O)$. T'U $\not\sim$ l, by our choice of T', so A7 uniquely determines B", C", P", Q" and, as before, B" = $\tau_{OU}^{a}(O)$ etc. These parallel projections are as described in 2.4, since n $\not\sim$ m and n $\not\sim$ l and also T'U $\not\sim$ l and T'U $\not\sim$ m (if T'U $\sim$ m then U $\sim$ O, but $\tau_{OT} \notin$ N and $\tau_{OT} = \tau_{UO}$, thus U $\not\sim$ O). Thus O7 applies to both parallel projections. So (P, B, C) implies (P', B', C') which implies (P", B", C"). However, P" = $\tau_{OU}^{b}(O) = (\tau_{OT}^{-1})^{b}(O)$ = Q and B" = $\tau_{OU}^{a}(O)$ = C, C" = $\tau_{OU}^{-a}(O)$ = B. Thus we have (Q, C, B) which with (P, B, C) implies (Q, B, P).

<u>2.22. Theorem.</u> If a, b $\in H^{+}$, a $\neq$ b, then a + b $\in H^{+}$.

<u>Proof.</u> Choose O, A; O $\not\sim$ A.

Then a, b $\in H^{+}$ imply A, $\tau_{OA}^{a}(O) \mid$ O and A, $\tau_{OA}^{b}(O) \mid$ O.

Put $\tau_{OA}^{a}(O)$ = B, $\tau_{OA}^{b}(O)$ = C. Thus A, B | O and A, C | O, so B, C | O.

Also $a \neq b$ implies that $B \neq C$. Hence (O, B, C) or (O, C, B).

<u>Case 1:</u> (O, B, C).

O, A, B I l. Let E be the midpoint of B, C on l. Set $Q = \tau_{OE}(E) = \tau_{OE}^{2}(O)$. Thus E is also the midpoint of Q and O on l. Then $\tau_{OA}^{a+b} = \tau_{OA}^{a}\tau_{OA}^{b} = \tau_{OB}\tau_{OC} = \tau_{EB}\tau_{OE}\tau_{EC}\tau_{OE} = \tau_{OE}^{2} = \tau_{OQ}$. Hence $Q = \tau_{OA}^{a+b}(O)$. (O, B, C) implies (O, B, Q), by <u>2.21.</u> Thus B, Q | O and, since A, B | O, A, Q | O, i.e., A, $\tau_{OA}^{a+b}(O)$ O, i.e., $a + b \in H^{+}$.

<u>Case 2:</u> (O, C, B).

The proof follows symmetrically.

<u>2.23.</u> Let $H^{-}$ be the complement of $H^{+} \cup \{o\}$ in H. Thus $H = H^{+} \cup \{o\} \cup H^{-}$. The elements of $H^{-}$ are called <u>negative</u> elements of H.

<u>2.24. Theorem.</u> $a \in H^{+}$ implies $-a \in H^{-}$.

<u>Proof.</u> $a \in H^{+}$ implies $a \neq o$, hence $-a \neq o$. Since characteristic of H is not two, $a \neq -a$. If $-a \in H^{+}$, then $a+(-a) \in H^{+}$, by <u>2.22</u>; a contradiction.

<u>2.25. Lemma.</u> $a \in H^{-}$ iff ( $\tau_{OA}^{a}(O)$, O, A), where $O \neq A$.

Proof. If $a \in H$, then $a \in H^+ \cup H^-$ iff $a \neq o$ iff $\tau_{OA}{}^a \neq 1$ iff $\tau_{OA}{}^a(O) \neq O$ iff exactly one, of $\tau_{OA}{}^a(O) = A$, $(\tau_{OA}{}^a(O), A, O)$, $(A, \tau_{OA}{}^a(O), O)$, $(A, O, \tau_{OA}{}^a(O))$ holds. $a \in H^+$ is equivalent to the assertion that exactly one of the first three relations holds.

Thus, the last relation is equivalent to the $a \in H^-$.

2.26. Theorem. If $a, b \in H^-$, $a \neq b$, then $a + b \in H^-$.

Proof. Choose $O \neq A$.

Then $a, b \in H^-$ imply $(\tau_{OA}{}^a(O), O, A)$ and $(\tau_{OA}{}^b(O), O, A)$, which exclude $(\tau_{OA}{}^a(O), O, \tau_{OA}{}^b(O))$, by O6. Hence $\tau_{OA}{}^a(O), \tau_{OA}{}^b(O) \mid O$. As in 2.22, let $\tau_{OA}{}^a(O) = B$, $\tau_{OA}{}^b(O) = C$ so $(B, C, O)$ or $(O, B, C)$.

Case 1: $(O, B, C)$.

Letting $Q = \tau_{OE}(E)$, where $E$ is the midpoint on $l$ of $B$ and $C$, we have, as in 2.22, $(O, B, Q)$. Since $(B, O, A)$ we have $(Q, O, A) = (\tau_{OA}{}^{a+b}(O), O, A)$, i.e., $a + b \in H^-$.

Case 2: $(O, C, B)$.

The proof follows symmetrically.

2.27. Theorem. $a \in H^-$ implies $-a \in H^+$.

Proof. $a \in H^-$ implies $a \neq o$, hence $-a \neq o$. Since characteristic of $H$ is not two, $a \neq -a$. If $-a \in H^-$, then $-a + a \in H^-$, by 2.26; a contradiction.

<u>2.28. Theorem.</u> For all $a \in H$, exactly one, of $a \in H^+$, $a = o$, $-a \in H^+$ holds.

<u>2.29. Theorem.</u> $-1 \in H^-$.

<u>Proof.</u> By <u>2.12</u>, $1 \in H^+$. Hence, by <u>2,24</u>, $-1 \in H^-$.

<u>2.30. Theorem.</u> If $a \in H^+$, then $2a \in H^+$.

<u>Proof.</u> Choose $O \not\sim A$.

<u>Claim.</u> $\tau_{OA}{}^{a}(O)$, $\tau_{OA}{}^{2a}(O) \mid O$.

Since $O \not\sim A$, $\tau_{OA}{}^{a}(O) \neq O$, by <u>1.18</u>, and since characteristic of H is not equal to two $\tau_{OA}{}^{2a}(O) \neq O$.

If $\tau_{OA}{}^{2a}(O) = \tau_{OA}{}^{a}(O) \neq O$, then $\tau_{OA}{}^{a}(O)$, $\tau_{OA}{}^{2a}(O) \mid O$.

If $O$, $\tau_{OA}{}^{a}(O)$ and $\tau_{OA}{}^{2a}(O)$ are mutually distinct (and collinear), then, by O3, exactly one, of $(\tau_{OA}{}^{a}(O), \tau_{OA}{}^{2a}(O), O)$, $(\tau_{OA}{}^{2a}(O), \tau_{OA}{}^{a}(O), O)$ and $(\tau_{OA}{}^{2a}(O), O, \tau_{OA}{}^{a}(O))$ holds.

If $(\tau_{OA}{}^{2a}(O), O, \tau_{OA}{}^{a}(O))$, then, applying $\tau_{OA}{}^{-a}$, we get $(\tau_{OA}{}^{a}(O), \tau_{OA}{}^{-a}(O), O)$. But, since $-a \in H^-$, $(\tau_{OA}{}^{-a}(O), O, A)$.

Thus, by O4, $(\tau_{OA}{}^{a}(O), O, A)$; a contradiction to $a \in H^+$.

Thus $\tau_{OA}{}^{a}(O)$, $\tau_{OA}{}^{2a}(O) \mid O$ in all cases.

Since $A$, $\tau_{OA}{}^{a}(O) \mid O$, we have $A$, $\tau_{OA}{}^{2a}(O) \mid O$, i.e., $2a \in H^+$.

2.31. Thus, by 2.28; 2.22 and 2.30; 2.17, H is an ordered A. H. ring.

# CHAPTER 3

## The Construction of a Geometrically Ordered Desarguesian A. H. Plane from an Ordered A. H. Ring

3.1. Now, given an algebraically ordered A. H. ring H, we wish to construct a geometrically ordered Desarguesian affine Hjelmslev plane. Since Lorimer and Lane ([8], 3) have constructed a Desarguesian affine Hjelmslev plane $\mathcal{H}$ from an A. H. ring, it remains to show that the given ordering of H induces a geometric ordering of $\mathcal{H}(H)$. Since the A. H. ring of $\mathcal{H}$ is isomorphic to H, we may identify the two A. H. rings and assume that the A. H. ring of $\mathcal{H}$ is the given A. H. ring H. Using [8], 3.11, we can verify that H and H' are order isomorphic; cf. Appendix 1.

3.2. If A, B, C are mutually distinct and collinear with a line l, then there exist points O and E on l such that $O \not\sim E$. We define $(A, B, C)_{O,E}$ if $a < b < c$ or $c < b < a$, where $A = \tau_{OE}^{\ a}(O)$, $B = \tau_{OE}^{\ b}(O)$, $C = \tau_{OE}^{\ c}(O)$. B is said to lie between A and C.

3.3. Lemma. Order on a line l is independent of the choice of O, E on l, where $O \not\sim E$.

Proof. Let $O \not\sim E$ and $(A, B, C)_{O,E}$.

Let O', E' be any two points such that $O' \not\sim E'$.

Then $A = \tau_{OE}^{a}(O) = \tau_{O'E'}^{a'}(O')$, $B = \tau_{OE}^{b}(O) = \tau_{O'E'}^{b'}(O')$,

$C = \tau_{OE}^{c}(O) = \tau_{O'E'}^{c'}(O')$. Since $O \not\sim E$, there exists

$x \in H$, $x \neq o$ such that $\tau_{O'E'} = \tau_{OE}^{x}$ and $y \in H$ such that

$\tau_{OO'} = \tau_{OE}^{y}$. Then $x \notin \eta$ since $\tau_{O'E'} \notin N$.

$A = \tau_{OE}^{a}(O) = \tau_{O'E'}^{a'}(O') = \tau_{O'E'}^{a'}\tau_{OO'}(O) = (\tau_{OE}^{x})^{a'}(\tau_{OE})^{y}(O)$

$= \tau_{OE}^{a'x+y}(O)$. Thus $a = a'x+y$, since $\tau_{OE} \notin N$.

Assume $a' < b' < c'$, i.e., $b'-a'$, $c'-b' \in H^{+}$.

If $x \in H^{+}$, then $(b'-a')x$, $(c'-b')x \in H^{+}$ since $x \notin \eta$.

Hence $(b'x+y)-(a'x+y)$, $(c'x+y)-(b'x+y) \in H^{+}$, ie., $b-a$, $c-b \in H^{+}$,

i.e., $a < b < c$.

If $x \in H^{-}$, then $-x \in H^{+}$ and so $c < b < a$.

Similarly, one can deal with the case when $a' < b' < c'$.

Thus $(A, B, C)_{O,E}$ implies $(A, B, C)_{O',E'}$ for any $O' \not\sim E'$.

<u>3.4.</u> We are now in a position to prove O1 to O7, as in Chapter 2. We may now write $(A, B, C)_{O,E}$ as $(A, B, C)$.

<u>3.5.</u> <u>O1.</u> (A, B, C) implies that A, B, C are mutually distinct and collinear, by definition.

<u>3.6.</u> <u>O2.</u> Clearly (A, B, C) implies (C, B, A).

<u>3.7.</u> <u>O3.</u> A, B, C, mutually distinct and collinear implies exactly one, of (A, B, C), (B, C, A) and (C, A, B) holds.

<u>Proof.</u> Assume A, B, C I 1. If $O \not\sim E$ and O, E I 1 and $A = \tau_{OE}^{a}(O)$, $B = \tau_{OE}^{b}(O)$, $C = \tau_{OE}^{c}(O)$, then exactly one of:

1. $a < b < c$,
2. $b < c < a$,
3. $c < a < b$,
4. $c < b < a$,
5. $b < a < c$,
6. $a < c < b$, holds.

(A, B, C) is equivalent to 1 and 4.

(C, A, B) is equivalent to 3 and 5.

(B, C, A) is equivalent to 2 and 6.

Thus exactly one, of (A, B, C), (B, C, A) and (C, A, B) holds.

<u>3.8.</u> <u>O4.</u> (A, B, C) and (B, C, D) imply (A, B, D) and (A, C, D), if A, B, C, D are collinear.

<u>Proof.</u> If $O \not\sim E$ and $A = \tau_{OE}^{a}(O)$, $B = \tau_{OE}^{b}(O)$, $C = \tau_{OE}^{c}(O)$, $D = \tau_{OE}^{d}(O)$, then (A, B, C) is equivalent to $a < b < c$ or $c < b < a$. (B, C, D) is equivalent to $b < c < d$ or $d < c < b$. Hence (A, B, C) and (B, C, D) are equivalent to $a < b < c < d$ or $d < c < b < a$ which implies $a < b < d$ or $d < b < a$, i.e., (A, B, D). Similarly we obtain (A, C, D).

3.9. 05. (A, B, C) and (A, C, D) imply (B, C, D) and (A, B, D), if A, B, C, D are collinear.

Proof. The proof is similar to that of 04.

3.10. 06. Two, of (B, A, C), (C, A, D) and (D, A, B), exclude the third, if A, B, C and D are collinear.

Proof. Without loss of generality assume (B, A, C) and (C, A, D). This is equivalent to $b < a < c$ and $d < a < c$ or $c < a < b$ and $c < a < d$. In particular either $b < a$ and $d < a$ or $a < d$ and $a < b$. (D, A, B) is equivalent to $d < a < b$ or $b < a < d$, which is clearly excluded.

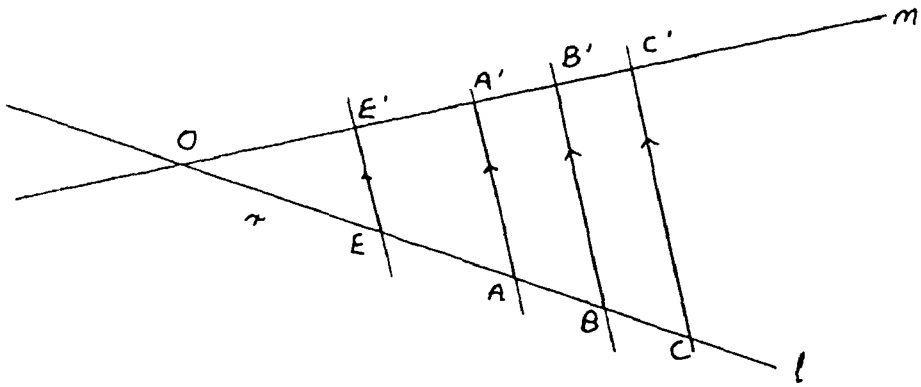
3.11. 07. If $\pi$ is a parallel projection, then it preserves order, see Figure 7.

Proof. Assume A, B, C I l and (A, B, C) and $\pi$ is a parallel projection which takes any X I l into X' I m such that $X = l \wedge L(X, \pi)$, $X' = m \wedge L(X, \pi)$.

Case 1: $l \quad m \neq \emptyset$.

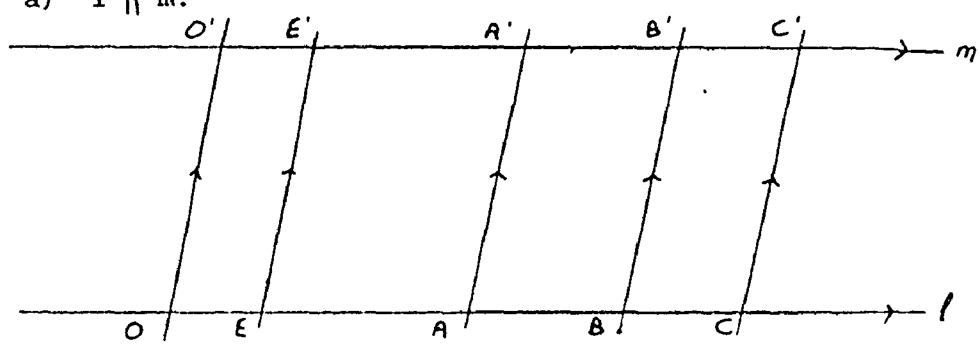
Say O I l, m. Now take E I l such that $O \not\sim E$.

Then $A = \tau_{OE}^{a}(O)$, $B = \tau_{OE}^{b}(O)$, $C = \tau_{OE}^{c}(O)$ for some a, b, c $\in$ H and $a < b < c$ or $c < b < a$. By 1.25, $\tau_{OE'}^{a}(O)$ I $L(\tau_{OE}^{a}(O), L(E, \pi))$. Thus $A' = \tau_{OE'}^{a}(O)$ and, since $O \not\sim E$, $L(O, \pi) \not\sim L(E, \pi)$, by 1.10, Case 1: $l \wedge m \neq \emptyset$.




Case 2: $l \wedge m = \emptyset$.

a) $l \parallel m$.




b) $l \nparallel m$.

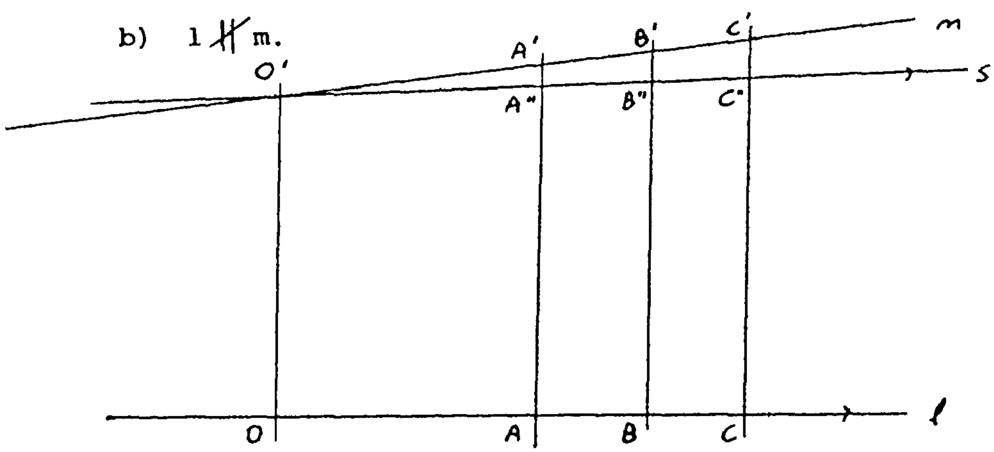



Figure 7.

so $O \not\sim E'$. Since $a < b < c$ or $c < b < a$ 3.2 implies that $(A', B', C')$.

Case 2: $l \wedge m = \emptyset$.

a) $l \parallel m$.

Take $E \; I \; l$, $O \not\sim E$. Then $\tau_{OE}(O') \; I \; L(E, \pi) \wedge m$, thus $E' = \tau_{OE}(O')$ and $O' \not\sim E'$. Since $A = \tau_{OE}{}^{a}(O)$ for some $a \in H$ and $\tau_{OE}{}^{a}(O') \; I \; L(A, \pi) \wedge m$, $A' = \tau_{OE}{}^{a}(O')$, $B' = \tau_{OE}{}^{b}(O')$, $C' = \tau_{OE}{}^{c}(O')$. Thus, as before, $(A, B, C)$ implies $(A', B', C')$.

b) $l \nparallel m$.

Then, by 1.6, there exists $s$, $s \parallel l$, $s \wedge m \neq \emptyset$ and $s \sim m$. Say $O' \; I \; s, m$. $L(X, \pi) \not\sim m$ for any $X \; I \; l$, thus $L(X, \pi) \not\sim s$. $L(O', \pi)$ meets $l$ at a unique point $O$, by the definition of a parallel projection, and $L(O', \pi) \not\sim l$. However, $O' \; I \; s$ and $L(O', \pi) \not\sim s$, so by A7, for any $X \; I \; l$ $L(X, \pi)$ meets $s$ at a unique point $X''$.

Thus, $(A, B, C)$ implies $(A'', B'', C'')$, by Case 2(a), which implies $(A', B', C')$, by Case 1.

3.12. By 3.5 to 3.11, $\mathcal{H}$ is a geometrically ordered Desarguesian affine Hjelmslev plane.

## CHAPTER 4

## An Example of an Ordered A. H. Ring

4.1. A projective Hjelmslev ring or P. H. ring is an A. H. ring which also satisfies:

3. If $a, b \in H$, then $a \in Hb$ or $b \in Ha$.

The following example of an A. H. ring which is not a P. H. ring is originally due to R. Baer, [2], and is examined in [9], 5.

4.2. Let F be a field, F' a proper subfield of F, and $\phi$ an isomorphism from F into F'. Take $H = F \times F$ and define addition and multiplication as follows:

$$(a, b) + (c, d) = (a + c, b + d),$$

$$(a, b) \cdot (c, d) = (ac, \phi(a)d + bc).$$

Then H is easily seen to be an A. H. ring, but not a P. H. ring, where $\eta = \{(o, a) \mid a \in F\}$.

4.3. We can take $F = Q(x)$, a simple transcendental extension of the rational numbers Q, $F' = Q(x^2)$ and $\phi$ the map from F into F' which takes x into $x^2$.

Q(x) can be regarded as the field of real-valued rational functions with rational coefficients, and can be made into an ordered field by defining:

$$Q(x)^{+} = \left\{ \frac{f(x)}{g(x)} \in Q(x) \middle| \ \exists \ \ x_{\bullet} \in Q \quad \frac{f(z)}{g(z)} > 0, \ \forall z < x \right\}.$$

Then clearly,

1. For any $a \in Q(x)$, either $a \in Q(x)^{+}$ or, $-a \in Q(x)^{+}$ or, $a = 0$.
2. $a \in Q(x)^{+}$, $b \in Q(x)^{+}$ implies $a + b \in Q(x)^{+}$.
3. $a \in Q(x)^{+}$, $b \in Q(x)^{+}$ implies $a \cdot b \in Q(x)^{+}$.

Now consider $H = Q(x) \times Q(x)$ and define a lexicographic order on H, i.e., $H^{+} = \{ \alpha = (a, b) \in H \mid$ either $a > 0$ or $a = 0$ and $b > 0 \}$.

Then clearly,

1. For any $\alpha \in H$, either $\alpha \in H^{+}$ or, $-\alpha \in H^{+}$ or, $\alpha = o$.
2. $\alpha \in H^{+}$, $\beta \in H^{+}$ implies $\alpha + \beta \in H^{+}$.

Also, 3. $\alpha, \beta \in H^{+}$, $\beta \notin \eta$ implies $\alpha\beta \in H^{+}$.

<u>Proof of 3.</u>

<u>Case 1:</u> , .

Let $\alpha = (a, b)$, $\beta = (c, d)$. Then $a > 0$ and $c > 0$.

Thus, $\alpha\beta = (ac, \phi(a)d + bc) \in H^{+}$, since $ac > 0$.

<u>Case 2:</u> , .

Let $\alpha = (0, b)$, $\beta = (c, d)$. Then $b, c > 0$.

Thus, $\alpha\beta = (0 \cdot c, \ \phi(0)d + bc)$. But, $\phi(0) = 0$, thus $\alpha\beta = (0, bc) \in H^{+}$, since $bc > 0$.

Thus H is an ordered A. H. ring.

<u>4.4.</u> It is interesting to note that $\alpha, \beta \in H^{+}$, $\alpha \notin \eta$, $\beta \in \eta$ does not imply that $\alpha\beta \in H^{+}$.

Assume $\alpha = (a, b)$, $\beta = (0, d)$, $a, d > 0$. Thus $\alpha, \beta \in H^{+}$.

Then, $\alpha\beta = (a \cdot 0, \ \phi(a)d + b \cdot 0) = (0, \ \phi(a)d)$. However, $\phi(a)$ is not necessarily greater than zero in $Q(x^2)$. For example, take $a = f(x)$, where $f(x) = -x$. Then $\phi(a) = f(x^2) = -x^2$ and clearly $\phi(a) < 0$. Thus $\phi(a)d < 0$, and so $\alpha\beta \in H^-$.

## CHAPTER 5

## The Ordinary Affine Plane Associated with an Ordered Desarguesian A. H. Plane

5.1. With every affine Hjelmslev plane, $\mathcal{H}$, we may associate a structure $\bar{\mathcal{H}} = [\bar{\mathbb{P}}, \bar{\mathbb{L}}, \bar{I}, \bar{\|}]$ by the quotient maps $\chi_{\mathbb{P}}$ and $\chi_{\mathbb{L}}$ of $\sim_{\mathbb{P}}$ and $\sim_{\mathbb{L}}$. We define $\bar{P}\ \bar{I}\ \bar{l}$ iff there exists $S \in \mathbb{P}$ such that $P \sim S$ and $S\ I\ l$. If $\bar{\|}$ is the parallelism relation for ordinary affine planes, then $\bar{\mathcal{H}}$ is an ordinary affine plane; cf. [8], 1.2. In fact, A. H. planes may be defined in terms of their associated ordinary affine plane; cf. [9], 2.2.

5.2. Once a Desarguesian A. H. plane has been coordinatized, we see that if $P = (a, b)$, $Q = (c, d)$, then $P \sim Q$ iff $a-c \in \eta$ and $b-d \in \eta$ ; cf. [8], 6.2,3.

5.3. If $\mathcal{H}$ is a Desarguesian A. H. plane, then it is clear that $\bar{\mathcal{H}}$ is the ordinary Desarguesian affine plane coordinatized by the division ring $H/\eta$ . If H is an ordered A. H. ring, then $H/\eta$ is an ordered division ring and thus $\bar{\mathcal{H}}$ is an ordered affine plane; cf. [11].

## CHAPTER 6

## The Radical of an Ordered A. H. Ring

6.1. Theorem. * Let H be an ordered A. H ring and $\eta$ be the radical (maximal ideal) of H.

1. If $r \in \eta$, then $-1 < r < 1$.
2. If $c \in \eta$, then $-c < b < c$ implies $b \in \eta$.

Proof. 1. Suppose $r \in H^+$ and $r > 1$. Then $r - 1 \notin \eta$; cf. [6],pg.75. Since $r \in \eta$, there exists r' such that $r'r = o$. We may assume, without loss of generality, that $r' \in H^+$. Thus $r'(r - 1) \in H^+$. However, $r'(r - 1) = -r'$; a contradiction. Thus $r < 1$. If $r < -1$, then $-r > 1$; again a contradiction.

2. Consider, for instance, $o < b < c$ and suppose $b \notin \eta$. Then $c - b \in H^+$ and $c - b \notin \eta$. Since $c \in \eta$, there exists c' such that $c'c = o$. We may assume, without loss of generality, that $c' \in H^+$. Then $c'b \in H^+$. But $-c'b = c'(c - b) \in H^+$; a contradiction. Thus $b \in \eta$. If $-c < b < o$, then $o < -b < c$; again a contradiction.

---

* Dr. J. W. Lorimer brought this result to my attention.

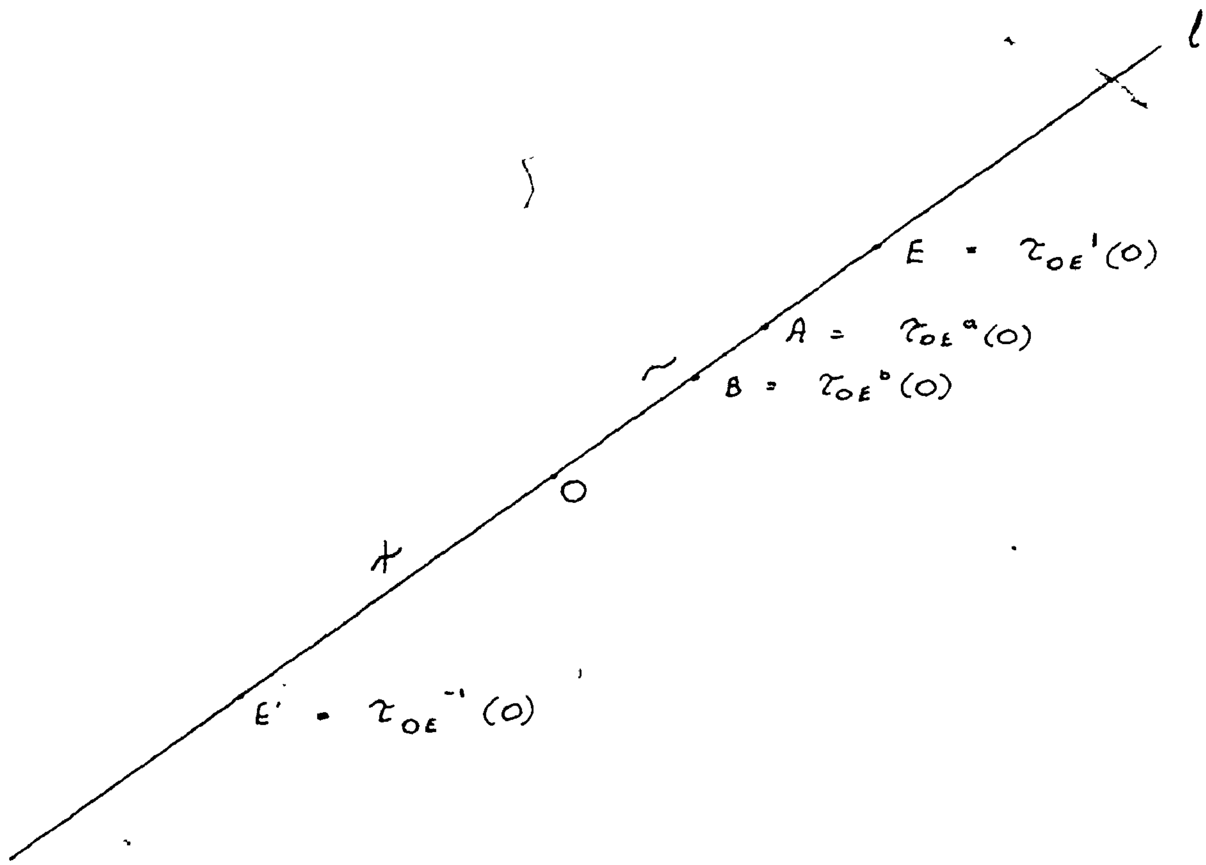



Figure 8.

6.2. Geometrically, this means that if O is any point on a line l, in an ordered Desarguesian A. H. plane, then all the neighbouring points of O lie between E and E', where E is any non-neighbour of O and E' = $\tau_{OE}^{-1}(O)$. If A is any neighbour of O, then all points between A and O are also neighbours of O. See Figure 8.

## CHAPTER 7

## The Archimedean Axiom and Desarguesian A. H. Planes

7.1. An ordered A. H. ring H is called Archimedean if for any a, b $\in H^+$ there exists $n \in \mathbb{Z}$ such that $na > b$. This is clearly equivalent to the Archimedean ordering of the plane $\mathcal{H}(H)$, i.e., if (O, A, B) then there exists $n \in \mathbb{Z}$ such that (O, B, $\tau_{OA}{}^n(O)$). Since if O, A, B I l, then O $\not\sim$ T for some T I l where A = $\tau_{OT}{}^a(O)$ and B = $\tau_{OT}{}^b(O)$ and A, T $\not|$ O and B, T $\not|$ O; cf. [1], pg.78.

7.2. Hjelmslev and Klingenberg suggested, [4], pg.17 and [5], pg.406, that any Archimedean ordered A. H. ring is automatically a division ring. This is indeed true. Thus any Archimedean ordered Desarguesian A. H. plane is an ordinary Desarguesian affine plane.

7.3. Theorem. Let H be an Archimedean ordered A. H. ring. Then H is a division ring, i.e., $\eta = \{o\}$.

Proof. Let H be Archimedean and assume H is not a division ring. Then, for any a, b $\in H^+$ there exists $n \in \mathbb{Z}$ such that $na > b$, i.e., $na - b \in H^+$. Take a, b $\in H^+$, $a \in \eta$, $b \notin \eta$. Thus there exists $a' \in H$ such that $a'a = o$. We may assume that $a' \in H^+$.

Hence $a'b \in H^+$. Since $na - b \notin \eta$, $-a'b = a'(na - b) \in H^+$, a contradiction.

Note. By means of ternary rings, Dr. J. W. Lorimer has shown that any non-Desarguesian proper A. H. plane cannot be Archimedean ordered.

## APPENDIX 1

The A. H. ring, H', of trace preserving endomorphisms of an A. H. plane $\mathcal{H}$(H) is order isomorphic to H. This is clear when we consider the following theorem, originally due to Klingenberg.

Theorem. Let H' be the A. H. ring of trace preserving endomorphisms of $\mathcal{H}$(H) and let $\eta$ be its unique maximal ideal. Then; cf. [8], 3.10.

1. $c' \in H'$ iff there exists $c \in H$ such that

   $$\tau_{O\,(a,b)}^{\;c'} = \tau_{O\,(ca,cb)} \text{ for all } (a,b) \quad H \times H.$$

   Thus $\tau_{O\,(1,0)}^{\;c'} = \tau_{O\,(c,0)}$.

2. The mapping $H \to H'$ $(c \to c')$ is a ring isomorphism.

3. $c' \in \eta'$ iff $c \in \eta$.

If $c \in H^+$ then consider the points $O = (0,0)$, $E = (1,0)$, $C = (c,0)$. $c \in H^+$ implies that $c = 1$, or $0 < 1 < c$, or $0 < c < 1$, i.e. $C = E$, or $(O, E, C)$, or $(O, C, E)$. So $E, C \mid O$, but $C = (c,0)$ and $(c,0) = \tau_{O\,(c,0)}(O) = \tau_{O\,(1,0)}^{\;c'}(O) = \tau_{OE}^{\;c'}(O)$. Thus $c' \in H'^+$.

If $c' \in H'^+$ then $E, C \mid O$ where $C = \tau_{OE}^{\;c'}(O) = \tau_{O\,(c,0)}(O)$. Thus $0 < 1 < c$, or $0 < c < 1$, or $c = 1$, i.e. $c \in H^+$.

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CANADIAN THESES ON MICROFICHE | THÈSES CANADIENNES SUR MICROFICHE

NAME OF AUTHOR/*NOM DE L'AUTEUR* Vincent Thomson

TITLE OF THESIS/*TITRE DE LA THÈSE* An Energy Dispersive Method for Resonance Neutron Capture Gamma Ray Spectroscopy

UNIVERSITY/*UNIVERSITÉ* McMaster

DEGREE FOR WHICH THESIS WAS PRESENTED/ *GRADE POUR LEQUEL CETTE THÈSE FUT PRÉSENTÉE* Ph.D.

YEAR THIS DEGREE CONFERRED/*ANNÉE D'OBTENTION DE CE DEGRÉ* 1976

NAME OF SUPERVISOR/*NOM DU DIRECTEUR DE THÈSE* Dr. T. J. Kennett



DATED/*DATÉ* January 21, 1976 SIGNED/*SIGNÉ* V. Thomson

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AN ENERGY DISPERSIVE METHOD
FOR RESONANCE NEUTRON CAPTURE
GAMMA RAY SPECTROSCOPY

# AN ENERGY DISPERSIVE METHOD FOR RESONANCE NEUTRON CAPTURE GAMMA RAY SPECTROSCOPY

by

Vincent J. Thomson

A Thesis

Submitted to the Faculty of Graduate Studies

in Partial Fulfilment of the Requirements

for the Degree

Doctor of Philosophy

McMaster University

October, 1975

DOCTOR OF PHILOSOPHY (1975) (Physics)

McMASTER UNIVERSITY Hamilton, Ontario

TITLE: An Energy Dispersive Method for Resonance Neutron Capture Gamma Ray Spectroscopy

AUTHOR: Vincent J. Thomson, B.Sc. (University of Windsor)

SUPERVISOR: Professor T.J. Kennett

NUMBER OF PAGES: x, 93

ABSTRACT:

A method for the determination of partial radiation cross sections based on the energy dispersion of the capture process was undertaken. A description of the experimental reactor facility and details of the method has been presented. Measurements of gamma rays following neutron capture in isotopes of silicon, chromium and nickel revealed resonances which were analyzed for resonance parameters. The characteristics of the resonance decay properties for the different isotopes were discussed.