

Chapter 4

Ordered Fuzzy Numbers: Definitions and Operations

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Abstract We outline basic notions and assumptions related to the Ordered Fuzzy Number (OFN) model. Definitions of mathematical operations, several interpretations of their results, as well as additional OFN parameters are presented. Some of them, such as inclination or order, are specific to OFNs, whereas others are equivalent to those present in the well-known convex fuzzy number model. An important aspect of this part is also a discussion of algebraic properties of the OFN model.

4.1 Introduction

In previous works, we noted that original fuzzy arithmetic operations could have some limitations. Those limitations were recognized and addressed in different ways [17, 25]. Many researchers agree that calculations involving fuzzy numbers should accumulate uncertainty, by compliance with the meaning of a fuzzy number as a distribution of possibilities [26, 27]. Indeed, this assumption occurs in almost all interpretations of fuzziness [4, 5]. Although it seems to be natural for many applications, we would like to point out that in some scenarios it would be truly useful to derive crisper information from fuzzier inputs, that is, to reverse the uncertainty accumulation process. We believe that by allowing such reversing one would obtain a kind of general mathematical model of fuzzy numbers, which – depending on practical needs – can possess more or less constrained properties.

Following the above way of thinking, we have been seeking a framework that would include standard fuzzy numbers as special cases. One of the possibilities for building such a broader space of fuzzy numbers is to rely on the Ordered Fuzzy

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Number (OFN) model [13, 15], that is, proceed with decomposition of membership functions of fuzzy numbers onto ordered components. By treating those components as an ordered pair, it becomes possible to model the fuzzy number opposites better, canceling each other out instead of growing their fuzziness. In this chapter, we look at such capabilities of OFNs from a mathematical perspective and discuss basic properties and examples of OFN arithmetic operations in order to illustrate computational straightforwardness and representative richness of the considered approach. We also pay attention to the role of those objects in the space of OFNs that do not correspond to fuzzy numbers understood as standard fuzzy sets.

The chapter is organized as follows. In Sect. 4.2, we recall preliminary concepts of the OFN model. In Sect. 4.3, we discuss how to redefine fundamental notions of fuzzy sets and numbers for OFNs. In Sect. 4.4, we focus on OFNs that are not expressible by means of standard fuzzy numbers and, in particular, we show how to transform them into a convex fuzzy set format. In Sect. 4.5, we outline basic arithmetic operations and discuss the corresponding algebraic properties of the OFN model. In Sect. 4.6, we discuss the notion of the OFN's direction from the perspective of practical applications. Section 4.7 concludes this part.

4.2 The Ordered Fuzzy Number Model

As already recalled in Chap. 3, the considered idea of an alternative way of looking at fuzzy arithmetics is based on the notion:

Definition 4.1 An **Ordered Fuzzy Number (OFN)** A is an ordered pair

$$A = (f_A, g_A) \tag{4.1}$$

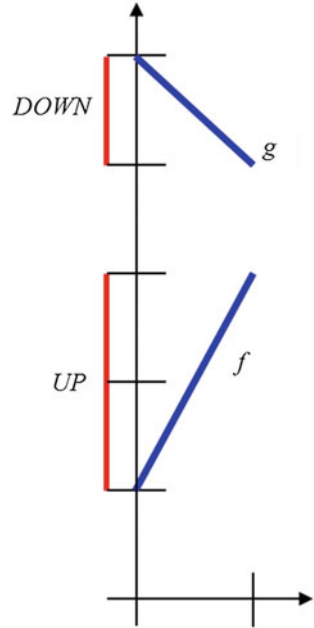
of continuous functions $f_A, g_A : [0, 1] \rightarrow \mathbb{R}$, called the **up part**, and the **down part** of A , respectively.

It follows from the continuity of f_A and g_A that their images are bounded intervals. Let us denote them as $UP_A = f_A([0, 1])$ and $DOWN_A = g_A([0, 1])$ (see Fig. 4.1).

Considering functions f_A and g_A as an ordered pair is the crucial difference in comparing to standard fuzzy numbers that can be represented by means of so-called *L-R notation* [4]. By *L* and *R* one means left (increasing) and right (decreasing) components of a membership function of a given fuzzy number. Such components can be inverted to form functions assigning real values to the elements of the unit interval $[0, 1]$ [16]. Standard arithmetic operations on such represented fuzzy numbers are then defined over the pairs of increasing components and the pairs of decreasing components. This means that in the classical framework it is impossible to add, for example, an increasing component of a fuzzy number A to a decreasing component of another fuzzy number B . In the case of OFNs, such operations are allowed.

As discussed in Chap. 3, shapes related to pairs (f_A, g_A) and (g_A, f_A) are the same. However, they differ by something that can be interpreted as *direction*. This

Fig. 4.1 An example of OFN



new kind of information can be additionally marked graphically with arrows. It can be seen in Fig. 4.2, which illustrates the following operation.

Definition 4.2 Reversal of direction of $A = (f_A, g_A)$ consists in replacing its up part (f_A) and down part (g_A) with each other. The resulting OFN $A|^- = (f_{A|^-}, g_{A|^-})$ is defined as follows, for each $\alpha \in [0, 1]$,

$$f_{A|^-}(\alpha) = g_A(\alpha) \quad g_{A|^-}(\alpha) = f_A(\alpha) \tag{4.2}$$

$A|^-$ is called an *OFN of reversed direction* or a *reversed OFN*.

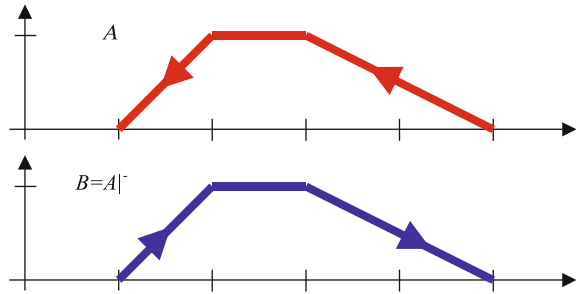
To distinguish between different types of OFN directions, let us also introduce the following characteristics. Herein, parameters $s_A = f_A(0)$, $e_A = g_A(0)$ (s / e stands for start/end) and $1_A^- = f_A(1)$, $1_A^+ = g_A(1)$ ($- / +$ stands for *reaching/leaving* a precise component of a fuzzy number) are useful.

Definition 4.3 For a given $A = (f_A, g_A)$, we say that:

- Direction is *strictly neutral*, if $f_A = g_A$; that is, each element belongs equally to the up and down parts of A .
- Direction is *strictly positive* for $f_A \neq g_A$, if $1_A^- < 1_A^+$, else if $1_A^- = 1_A^+$ and $s_A < e_A$.
- Direction is *strictly negative* for $f_A \neq g_A$, if $1_A^- > 1_A^+$, else if $1_A^- = 1_A^+$ and $s_A > e_A$.

The above rules define some specific cases of *strict* direction. Certainly, one can notice that the reversal operation changes directions; that is, if A is strictly positive

Fig. 4.2 Reversal operation



(negative), then $A|^-$ is strictly negative (positive). However, one can also imagine pairs (f_A, g_A) that do not follow any such neutral/positive/negative characteristics. For example, one can consider a situation where $f_A \neq g_A$ and at the same time there is $s_A = 1_A^- = 1_A^+ = e_A$. Although such pairs do not possess a strict direction, they can play an important role in fuzzy arithmetic operations.

Another notion that was introduced within the OFN model is *inclination*. Its role is to make a general comparison between the up and down parts of a given $A = (f_A, g_A)$. Below let us refer to the so-called mean inclination:

Definition 4.4 The **mean inclination** of an OFN $A = (f_A, g_A)$ is defined as the function $i_{m_A} : [0, 1] \rightarrow \mathbb{R}$:

$$i_{m_A} = (f_A + g_A)/2 \tag{4.3}$$

Let us note that the mean inclination of OFNs with symmetrical shapes is a constant function. It does not depend on direction, in particular, there is $i_{m_A} = i_{m_{A|^-}}$. Inclination can be used, for example, in a defuzzification process. For more detailed investigation related to defuzzification methods in the OFN model, refer to [2, 10] and some further chapters in this book. Below we recall just one example.

Definition 4.5 Let OFN $A = (f_A, g_A)$ be given. The result of the **center of mean inclination** defuzzification is the real number x_A calculated as follows.

$$x_A = (x_{min} + x_{max})/2 \tag{4.4}$$

where

$$x_{min} = \min\{i_{m_A}(\alpha) : \alpha \in [0, 1]\} \quad x_{max} = \max\{i_{m_A}(\alpha) : \alpha \in [0, 1]\} \tag{4.5}$$

4.3 Basic Notions for OFNs

There is a huge variety of pairs (f_A, g_A) , wherein only a part of them is going to correspond to standard fuzzy numbers, whereas the others may require deeper interpretation. From the point of view of arithmetic operations, those other numbers –

called *improper* OFNs (see Sect. 4.4) – can be treated as abstract objects aimed at transforming standard inputs into standard outputs. Thus, it is important to understand the general characteristics of both proper and improper OFNs.

4.3.1 Standard Representation of OFNs

One of the most basic special cases refers to OFNs $A = (f_A, g_A)$ with monotonic functions. If f_A and g_A are both monotonic, then intervals UP_A and $DOWN_A$ retain the following dependencies.

$$UP_A = [\min\{s_A, 1_A^-\}, \max\{s_A, 1_A^-\}] \quad DOWN_A = [\min\{1_A^+, e_A\}, \max\{1_A^+, e_A\}] \quad (4.6)$$

Furthermore, for monotonic f_A and g_A , it is possible to determine their inverse functions from \mathbb{R} to $[0, 1]$. Inverse functions are defined in a nontrivial way within the corresponding intervals UP_A and $DOWN_A$. To obtain a kind of continuous shape, we connect them with a plot of a constant function equal to 1 over interval $CONST_A = [\min\{1_A^-, 1_A^+\}, \max\{1_A^-, 1_A^+\}]$ (Fig. 4.3). Thus we have three functions that can be used to represent monotonic pairs (f_A, g_A) in a form more comparable to standard convex fuzzy numbers recalled in Chap. 1.

This form (or a view) is called a *standard representation*. The three considered functions η_A^{UP} , η_A^{CONST} , η_A^{DOWN} are defined as follows, for $x \in \mathbb{R}$:

$$\begin{aligned} \eta_A^{UP}(x) &= \begin{cases} f_A^{-1}(x) & \text{for } x \in UP_A \\ 0 & \text{otherwise} \end{cases} \\ \eta_A^{CONST}(x) &= \begin{cases} 1 & \text{for } x \in CONST_A \\ 0 & \text{otherwise} \end{cases} \\ \eta_A^{DOWN}(x) &= \begin{cases} g_A^{-1}(x) & \text{for } x \in DOWN_A \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (4.7)$$

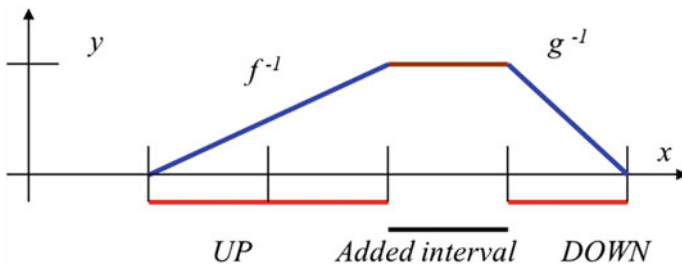


Fig. 4.3 OFN presented in the standard form corresponding to convex fuzzy numbers

The above functions can be called the *up part*, the *constant part*, and the *down part*. A fuzzy number represented in such a way can be interpreted analogously to the standard model of convex fuzzy numbers, as outlined in Chap. 1. Properties of the UP_A and $DOWN_A$ intervals (equalities 4.6) can be rewritten as

$$\eta_A^{UP}(s_A) = 0 \quad \eta_A^{UP}(1_A^-) = 1 \quad \eta_A^{DOWN}(1_A^+) = 1 \quad \eta_A^{DOWN}(e_A) = 0 \quad (4.8)$$

Similar transformation might also be possible for nonmonotonic up/down parts, although in such a case we would need to proceed with inversion of curves rather than functions. This would surely lead us towards the already-mentioned improper OFN objects. We also need to remember that even pairs of monotonic functions can correspond to improper OFNs. For example, if f_A and g_A are both increasing or both decreasing, then the above-considered three-component representation of A is still straightforward although it does not correspond to a standard fuzzy number.

4.3.2 OFN Support

Let us extend the concept of support – one of the most fundamental fuzzy set notions – onto the realm of OFNs. The extension is presented for both a general case and for OFNs $A = (f_A, g_A)$ with monotonic functions f_A and g_A . This way, just as before, the monotonicity of components is used as an illustrative special case, for which it is easier to interpret basic concepts inherited from fuzzy set theory.

A support is an important parameter in analyzing and modeling convex fuzzy numbers. In many practical situations, calculations involving fuzzy numbers are actually focused on their ranges of maximum membership and nonzero membership, which correspond to their supports. It is therefore also useful to introduce this notion for the OFN model. The following definition extends the classical case:

Definition 4.6 For OFN $A = (f_A, g_A)$, the **support** $supp_A$ is an interval calculated as follows.

$$supp_A = \{f_A(\alpha) : \alpha \in (0, 1]\} \cup CONST_A \cup \{g_A(\alpha) : \alpha \in (0, 1]\} \quad (4.9)$$

Let us note that $supp_A$ is almost equal to the set-theoretic sum of intervals UP_A , $CONST_A$, and $DOWN_A$. The only difference is that – depending on the shapes of functions f_A and g_A – we sometimes need to subtract elements s_A and/or e_A .

Going further, if we consider OFNs with monotonic up and down parts, the support can be equivalently introduced by a simpler formula. Namely, we can use the bounds of particular intervals UP_A , $CONST_A$, $DOWN_A$ of an OFN $A = (f_A, g_A)$ to determine its support $supp_A$:

$$supp_A = \begin{cases} (x_1, x_2) & \text{for } x_1 \notin CONST_A \text{ and } x_2 \notin CONST_A \\ [x_1, x_2] & \text{for } x_1 \in CONST_A \text{ and } x_2 \notin CONST_A \\ (x_1, x_2] & \text{for } x_1 \notin CONST_A \text{ and } x_2 \in CONST_A \\ [x_1, x_2] & \text{for } x_1 \in CONST_A \text{ and } x_2 \in CONST_A \end{cases} \quad (4.10)$$

where

$$x_1 = \min\{s_A, e_A, 1_A^-, 1_A^+\} \quad x_2 = \max\{s_A, e_A, 1_A^-, 1_A^+\} \quad (4.11)$$

Certainly this is just one of the possibilities to generalize the standard notion of a support onto the case of OFNs. Depending on interpretation, one might also think about defining $supp_A$ as an ordered interval [8] or as an interval (ordered or nonordered) taking into account only the quantities of s_A and e_A . Nevertheless, any reformulation of support should be equivalent to standard support whenever an OFN can be interpreted as a standard convex fuzzy number.

4.3.3 OFN Membership Function

Let us attempt to redefine the notion of a fuzzy membership function for OFNs in general. The following way of doing it is, as in the case of support, one of many possibilities. In principle, we propose that a fuzzy membership of a number $x \in \mathbb{R}$ in an OFN $A = (f_A, g_A)$ should be the highest context $\alpha \in [0, 1]$, in which x occurs, that is, such that $f_A(\alpha) = x$ or $g_A(\alpha) = x$.

Definition 4.7 A **membership function** of OFN $A = (f_A, g_A)$, denoted by $\mu_A : \mathbb{R} \rightarrow [0, 1]$, is defined for $x \in \mathbb{R}$ as

$$\mu_A(x) = \begin{cases} 1 & \text{for } x \in CONST_A \\ 0 & \text{for } x \notin supp_A \\ \max\{\alpha \in [0, 1] : f_A(\alpha) = x \vee g_A(\alpha) = x\} & \text{otherwise} \end{cases} \quad (4.12)$$

If the up and down parts correspond to monotonic functions, then we can redefine the membership function of an OFN as below:

$$\mu_A(x) = \begin{cases} 1 & \text{for } x \in [\min\{1_A^-, 1_A^+\}, \max\{1_A^-, 1_A^+\}] \\ 0 & \text{for } x \notin supp_A \\ \max\{f_A^{-1}(x), g_A^{-1}(x)\} & \text{otherwise} \end{cases} \quad (4.13)$$

The above form remains consistent with Definition 4.7, however, it starts resembling standard membership functions. If f_A and g_A have disjoint images, then $\max\{f_A^{-1}(x), g_A^{-1}(x)\}$ can be replaced by $f_A^{-1}(x)$ and $g_A^{-1}(x)$ within intervals $UP_A \setminus [1_A^-, 1_A^+]$ and $DOWN_A \setminus [1_A^-, 1_A^+]$, respectively. Moreover, if the up part is increasing, the down part is decreasing, and $f_A \leq g_A$, then a formula in Definition 4.7 becomes the exact representation of a standard fuzzy number's shape.

The formula in Definition 4.7 possesses very interesting characteristics also for improper OFNs $A = (f_A, g_A)$, regardless of whether their corresponding functions f_A and g_A are monotonic. Namely, for an arbitrary OFN, its membership function can be decomposed onto three fragments: strictly increasing, constant, and strictly decreasing. Some of those fragments may not correspond to continuous functions but the resulting membership function remains piecewise continuous as expected for standard fuzzy numbers. The mechanism introduced in Definition 4.7 is sometimes referred to as the so-called *MAX-choice* principle (see Sect. 4.4).

Thanks to the above observation, OFN membership functions can be employed in practice similarly to those of convex fuzzy numbers. As elaborated in Chap. 2, fuzzy memberships play the role of important numeric counterparts of linguistic rules in control systems. On the other hand, in further chapters we show that sometimes it is worth mixing classical approaches to constructing fuzzy controllers with those based on fuzzy arithmetics. From this perspective, generalizations of fuzzy membership functions for the OFN model can be especially helpful.

4.3.4 Real Numbers as OFN Singletons

In the case of convex fuzzy numbers, a real number $x \in \mathbb{R}$ is represented by the characteristic function χ_x , which equals 1 for x and 0 otherwise. In the OFN model, representing real numbers is easy and intuitive as well.

Definition 4.8 A real number $x \in \mathbb{R}$ is represented in terms of OFNs by a pair $\mathbf{x} = (f_x, g_x)$, where f_x and g_x are defined as the function constantly equal to x :

$$f_x = g_x = x \quad (4.14)$$

Thus, an OFN representing real number x forms a unit at the level x . After transforming it to a standard view it is a vertical segment. It also coincides with the meaning of singleton in the case of standard fuzzy numbers. Therefore, the name *singleton* is used for real numbers represented in the OFN model as well.

The support of a singleton according to Definition 4.6 is the interval $[x, x]$, that is, a single point x . Such a singleton always has *strict* neutral direction (see Definition 4.3). This is because the whole OFN is covered by both its parts, up and down.

4.4 Improper OFNs

The OFN model suggests looking at imprecision from a new perspective. The key aspect is related to the notion of direction. It can be interpreted in a practical way. Some examples of interpretations of that new aspect can be found in Sect. 4.6. OFNs

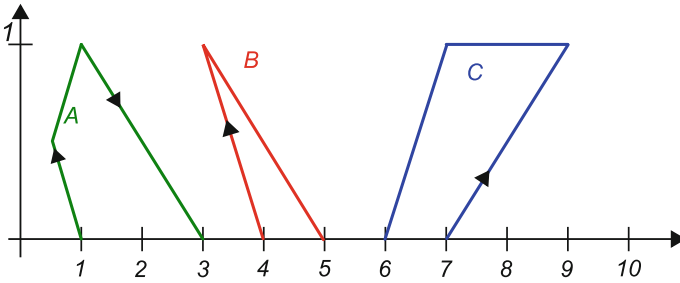


Fig. 4.4 Examples of improper OFNs

are considered there as results of observations in time. Indeed, the time can be a natural (although not the only) interpretation of direction.

The consequence of the new approach to modeling imprecise numbers is a kind of inconsistency with standard fuzzy numbers. Namely, there are OFNs, which do not have a membership function in the sense of a convex fuzzy set. We have already mentioned such improper OFNs several times. A subspace of improper OFNs can be characterized in many ways. Let us formalize them as

Definition 4.9 We say that $A = (f_A, g_A)$ is an **improper OFN**, if:

1. f_A or g_A is not monotonic.¹
2. UP_A , $DOWN_A$, and/or $CONST_A$ overlap.²

In general, when one of the parts of an OFN is neither monotonic nor constant, then such an OFN is improper. However, as already mentioned, there exist also improper OFNs with monotonic parts. For some examples, let us refer to Fig. 4.4. Let us note that, as earlier, we use arrows to express graphically OFNs' direction.

Improper OFNs cannot be represented as convex fuzzy sets. Still, it does not mean that they are of no use. When we interpret the direction as time of a measurement, then even improper OFNs turn out to represent important information about the observed processes (see Sect. 4.6.3). Moreover, it may turn out that during a chain of arithmetic calculations some intermediate results are improper, even though both inputs and outputs are interpretable as standard fuzzy numbers. In such cases, it would be quite unreasonable to abandon the whole computational process because, after all, the most important aspect is the interpretation of the final results.

If for a given scenario interpretation of an improper OFN is important, we can – depending on practical needs – utilize one of the available defuzzification mechanisms (e.g., the one outlined at the end of Sect. 4.2) or proceed with fuzzy membership derivation described in Definition 4.7. As mentioned before, we can refer to that

¹For simplicity, in this chapter we do not distinguish between the cases of increasing/decreasing and nondecreasing/nonincreasing functions. In this definition, we refer to that second case.

²By overlapping of intervals, for example, UP_A and $DOWN_A$, we mean that there are elements $x, y, z \in \mathbb{R}$, $x < y < z$, such that $x, z \in UP_A$ and $y \in DOWN_A$, or $x, z \in DOWN_A$ and $y \in UP_A$.

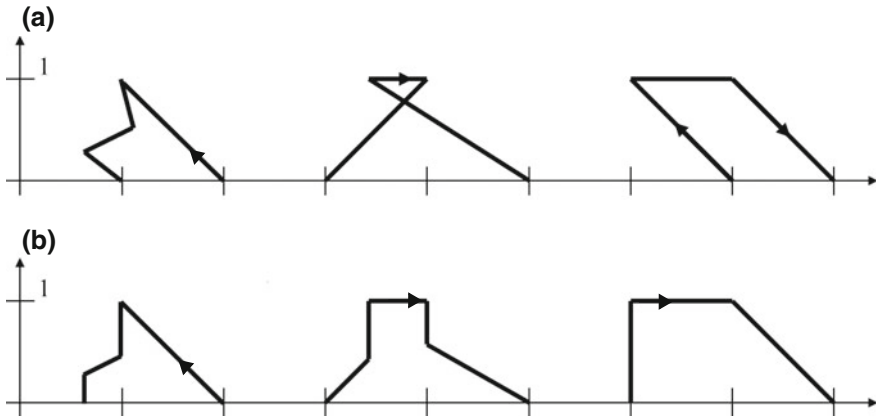


Fig. 4.5 (a) Improper OFNs. (b) The same OFNs interpreted using the MAX-choice principle

derivation as the MAX-choice principle. This is because, for each given argument, it labels it with the maximal element out of all relevant values.

Figure 4.5 presents an effect of applying the MAX-choice principle for three examples of improper OFNs. Such a solution enables us to build a convex fuzzy set interpretation of an arbitrary OFN. The obtained sets are normal. This fulfills demands of standard scenarios of applying fuzzy numbers. Moreover, proper outputs of the considered procedure can still be interpreted as OFNs, if necessary. However, let us emphasize that the MAX-choice should be used only if transformation of an improper OFN to the proper one is truly needed. In particular, if there are any further calculations planned over a given improper OFN, then its transformation would mean losing information that might be potentially important later.

4.5 Basic Operations on OFNs

This section presents examples of calculations on OFNs. Their main idea is to operate separately on the up and down parts. Such an approach allows us to conduct computations on OFNs directly on the universe of real numbers. It follows from the fact that we are now working with functions from $[0, 1]$ to \mathbb{R} .

4.5.1 Addition and Subtraction

Definition 4.10 Let OFNs $A = (f_A, g_A)$, $B = (f_B, g_B)$, and $C = (f_C, g_C)$ be given. We can say that:

- C is the sum of A and B , denoted $C = A + B$, if for every $\alpha \in [0, 1]$ there is:

$$f_A(\alpha) + f_B(\alpha) = f_C(\alpha) \quad g_A(\alpha) + g_B(\alpha) = g_C(\alpha) \quad (4.15)$$

- C is the result of subtracting B from A , denoted $C = A - B$, if there is:

$$f_A(\alpha) - f_B(\alpha) = f_C(\alpha) \quad g_A(\alpha) - g_B(\alpha) = g_C(\alpha) \quad (4.16)$$

Adding numbers A and B of the same direction (see Definition 4.3), which are described by linear up/down functions, allows us to obtain the same results as in the case of operations on standard fuzzy numbers [4]. Figure 4.6 illustrates adding two OFNs and transforming the obtained result to the standard form.

More examples are presented in Figs. 4.7 and 4.8. Figure 4.8 shows a result that is an improper OFN. Let us also note that, as we show later, the results of adding OFNs with opposite directions do not need to be improper.

Figures 4.9 and 4.10 illustrate some examples of subtraction. In Fig. 4.10, again, we get an improper OFN as a result.

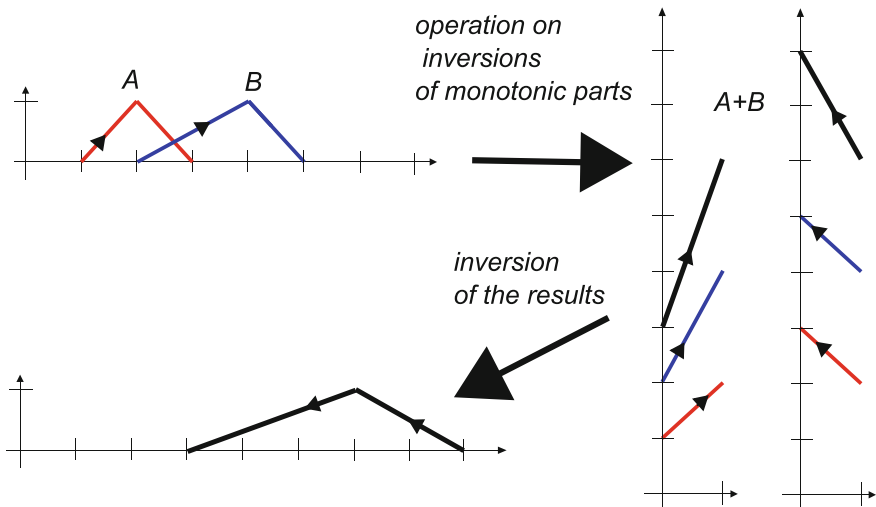


Fig. 4.6 A general mechanism of adding two OFNs

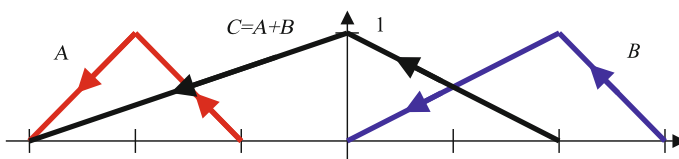


Fig. 4.7 Adding two OFNs with negative direction

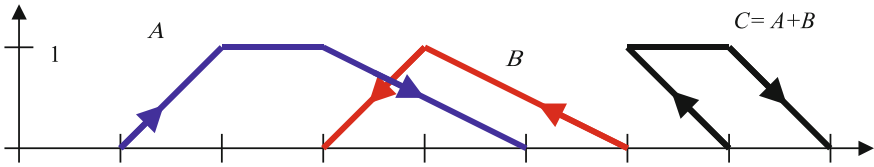


Fig. 4.8 Improper OFN as a result of adding OFNs

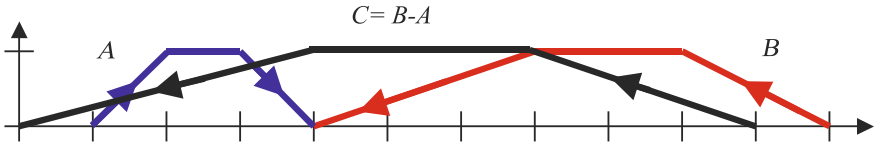


Fig. 4.9 A proper result of subtraction of OFNs

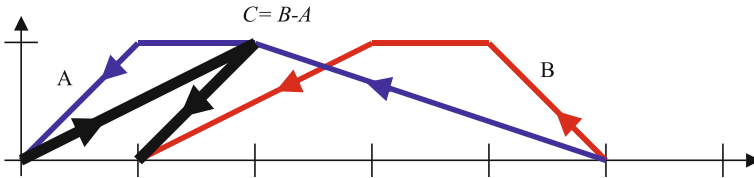


Fig. 4.10 An improper result of subtraction of OFNs

As discussed before, subtracting A should be the same as adding a number $-A$ that is opposite to A . The ability of expressing $-A$ for each A is, in our opinion, one of the biggest advantages of the OFN model, from the perspectives of both finer mathematical properties and managing imprecision during arithmetic calculations in practice. Let us recall that such a number can be introduced as $-A = (f_{-A}, g_{-A})$, defined as follows.

$$f_{-A} = -f_A \quad g_{-A} = -g_A \tag{4.17}$$

In particular, if $B = A$, then the number $-A$ is added to A . Then, for every $\alpha \in [0, 1]$, we obtain $f_C(\alpha) = f_A(\alpha) - f_A(\alpha) = 0$ and $g_C(\alpha) = g_A(\alpha) - g_A(\alpha) = 0$. Therefore, the result of operation of the form $A - A$ is the singleton representing 0, that is, the pair $\mathbf{0} = (f_{\mathbf{0}}, g_{\mathbf{0}})$ defined using formula (4.14) for $x = 0$ (see Fig. 4.11).

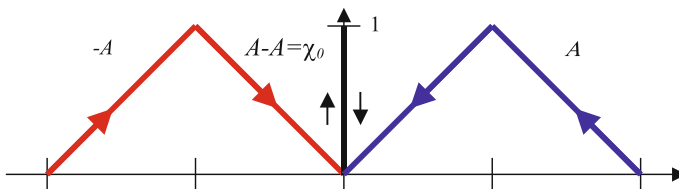


Fig. 4.11 Singleton zero as a result of an OFN operation

In general, when considering different examples of operations on OFNs, we may obtain results that have no clear interpretation in a standard framework. However, whenever required, the previously discussed MAX-choice principle can be utilized to derive such an interpretation. Moreover, improper OFNs – such as those visible in Figs. 4.8 and 4.10 – still contain significant information and they can serve as inputs to further operations that might ultimately result in proper OFNs.

Finally, we are now able to work with direction, an additional component of specification that illustrates a position of the OFN’s up part in relation to its down part. It is totally up to us whether that new component is used only as a purely mathematical property or it corresponds to a nontrivial real-world context.

4.5.2 Multiplication and Division

Definition 4.11 Let three OFNs $A = (f_A, g_A)$, $B = (f_B, g_B)$, and $C = (f_C, g_C)$ be given. We can say that:

- C is the result of multiplication of A and B , denoted $C = A \cdot B$, if for every $\alpha \in [0, 1]$ there is:

$$f_A(\alpha) \cdot f_B(\alpha) = f_C(\alpha) \quad g_A(\alpha) \cdot g_B(\alpha) = g_C(\alpha) \quad (4.18)$$

- C is the result of A divided by B , denoted $C = A/B$, if there is:

$$f_A(\alpha)/f_B(\alpha) = f_C(\alpha) \quad g_A(\alpha)/g_B(\alpha) = g_C(\alpha) \quad (4.19)$$

Multiplication of two OFNs is shown in Fig. 4.12. Certainly, division A/B can be formulated only under the constraint that B does not contain 0; that is, for every $\alpha \in [0, 1]$ we have $f_B(\alpha) \neq 0$ and $g_B(\alpha) \neq 0$.

As in the case of addition and subtraction operations, division should be expressible as multiplication by an inverse number. For a given $A = (f_A, g_A)$, its inverse

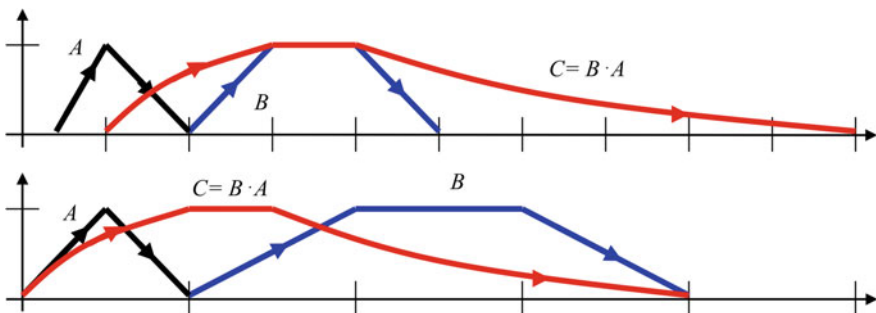


Fig. 4.12 Examples of multiplication of OFNs

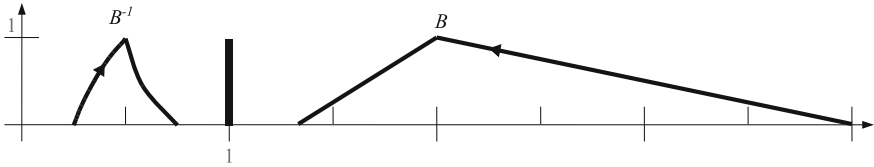


Fig. 4.13 Inversion of fuzzy number B

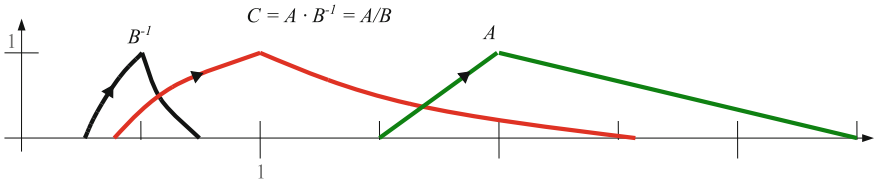


Fig. 4.14 Multiplication of A by B^{-1}

$A^{-1} = (f_{A^{-1}}, g_{A^{-1}})$ is defined as follows.

$$f_{A^{-1}} = 1/f_A \quad g_{A^{-1}} = 1/g_A \tag{4.20}$$

The division procedure is shown in Figs. 4.13 and 4.14. First, we determine an inverse number for B . As we can see, inverse numbers have opposite directions. Next, the inversion of B is multiplied by A and we obtain a result of dividing A by B .

Although a real (precise) zero represented by the pair $\mathbf{0} = (f_0, g_0)$ is a neutral element for addition, the neutral element for multiplication is the pair $\mathbf{1} = (f_1, g_1)$. An important property of the model presented in this chapter is the fact that multiplying any OFN by its inverse number allows us to obtain exactly the neutral element for multiplication as a result.

This makes it possible to analyze OFNs from a more formal mathematical perspective, analogously to some previous works on fuzzy number extensions [6]. Let us notice that the space of OFNs is isomorphic to a linear space of real two-dimensional vector-valued functions defined on the closed interval $[0, 1]$, with a norm specified as

$$\|A\| = \max\left(\sup_{\alpha \in [0,1]} |f_A(\alpha)|, \sup_{\alpha \in [0,1]} |g_A(\alpha)|\right) \tag{4.21}$$

It is topologically a Banach space. The neutral element of addition is $\mathbf{0}$. We also have a Banach algebra with the unity $\mathbf{1}$ [19]. Hence, although in this chapter we focus on more basic operations, further studies on advanced mathematical characteristics of OFNs are surely possible.

4.5.3 General Model of Operations

The most natural and intuitive way to introduce a general pattern of calculations on OFNs is to define them pairwise on their up and down parts. Let us formulate such a pattern below. It represents all previous operations in a short form.

Definition 4.12 Let OFNs $A = (f_A, g_A)$, $B = (f_B, g_B)$, and $C = (f_C, g_C)$ be given. The sum $C = A + B$, subtraction $C = A - B$, product $C = A \cdot B$, and division $C = A/B$ are defined by the following formula holding for every $\alpha \in [0, 1]$.

$$f_C(\alpha) = f_A(\alpha) \star f_B(\alpha) \quad g_C(\alpha) = g_A(\alpha) \star g_B(\alpha) \quad (4.22)$$

where \star replaces operations $+$, $-$, \cdot , and $/$. Moreover, A/B is determined only if $B = (f_B, g_B)$ does not contain zero values.

In fact, \star can represent any transformation of two OFNs under specific constraints. Such transformations have already been discussed and analyzed, including various examples in [9, 14]. Let us consider two further examples:

Definition 4.13 Let $A = (f_A, g_A)$, $B = (f_B, g_B)$, and $C = (f_C, g_C)$ be three OFNs. We can say that:

- C is the result of exponentiation of A raised to the power of B , denoted $C = A^B$, if for every $\alpha \in [0, 1]$ there is:

$$f_C(\alpha) = f_A(\alpha)^{f_B(\alpha)} \quad g_C(\alpha) = g_A(\alpha)^{g_B(\alpha)} \quad (4.23)$$

- C is the result of the logarithm of A with respect to base B , denoted $C = \log_B(A)$, if for every $\alpha \in [0, 1]$ there is:

$$f_C(\alpha) = \log_{f_B(\alpha)}(f_A(\alpha)) \quad g_C(\alpha) = \log_{g_B(\alpha)}(g_A(\alpha)) \quad (4.24)$$

Of course, as in the case of adequate operations for real numbers, the same restrictions should be applied with OFNs. During exponentiation, when the exponent is not an integer, the main limitation is exclusion as a base of those OFNs that contain negative values. In the case of logarithms, OFNs can contain only nonnegative values and, in addition, the base of the logarithm cannot include 1.

In summary, the OFN model grants flexibility of a wide range of calculations on imprecise data in a similar way as in the case of real numbers representing crisp data. It retains fuzzy quantitative characteristics, but without the necessity to grow imprecision. While using the OFN model, one should certainly remember that its foundations are slightly different from the case of Zadeh's fuzzy sets. In particular, improper OFNs can appear. However, despite their unusual shapes, improper OFNs can still contain important information needed for calculations. In particular, in examples related to data processing in Sect. 4.6.3, such objects are an important part of the analysis of information available in practice.

The above flexibility can be important for applications, where users expect a support for multiple types of operations. For example, in relational database systems, SQL statements need to include a number of arithmetic expressions. In the literature, one can find interesting examples of fuzzy-like histograms summarizing data contents that are employed to optimize database performance [20]. There are also database implementations aimed at acceleration of arithmetic calculations by means of interval ranges of values occurring for particular columns in particular data clusters [24]. Such solutions could be reconsidered by extending the currently utilized interval and trapezoidal summary structures with a concept of OFN-related direction. Namely, direction could be used to express a trend of values observed on data rows consecutively loaded into a database.

4.5.4 Solving Equations

This subsection shows how easy and flexible the calculations with OFNs can be. For standard fuzzy numbers, solving simple equations is often quite inaccurate. Surely, such solutions are expected if one follows the previously mentioned assumptions about accumulation of uncertainty during fuzzy arithmetic operations. However, one might also consider fuzzy equations for other purposes, for example, in order to set up some indirect embedded constraints for fuzzy variables. In such a case, it should also be possible to reverse a degree of uncertainty, that is, to obtain an equation's result that would be *less fuzzy* than the equation's coefficients.

Examples in this subsection refer to solving an equation $X = A + B$, where A and B are known fuzzy numbers. Attention should be paid to the following two possibilities: B having a greater support than A (Fig. 4.15), as well as A having a greater support than B (Fig. 4.16).

In the framework of standard fuzzy numbers, for the first of the above possibilities, there is a solution although it cannot be obtained by a simple arithmetic operation. However, as for the second possibility above, the solution does not exist because there is no such standard fuzzy number that could be added to A to obtain an outcome with a *narrower* support.

With use of the OFN model, both options are resolved in the same manner by simple calculation of $X = B - A$, which is presented by Figs. 4.17 and 4.18. In particular, in Fig. 4.18, we can observe that X has the opposite direction to A .

Thanks to the freedom of algebraic operations on OFNs, we can also relatively easily deal with equations involving fuzzy polynomials defined by analogy to polynomials over real numbers. By a fuzzy polynomial P we mean a function that transforms each given OFN $X = (f_X, g_X)$ into OFN specified as follows.

$$P(X) = A_n X^n + \dots + A_1 X + A_0 \quad (4.25)$$

where X is treated as a fuzzy *variable* and A_0, A_1, \dots, A_n are called *coefficients*.

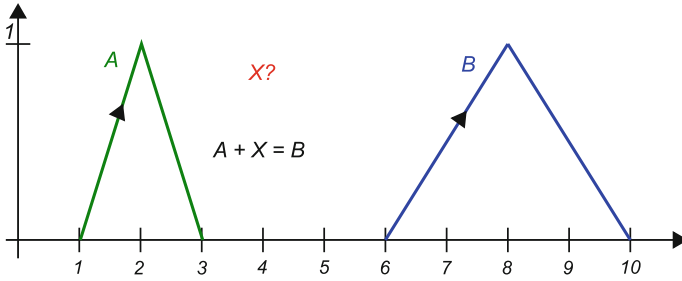


Fig. 4.15 Equation $A + X = B$ with the right-hand side B being *wider* than its left-hand side component A

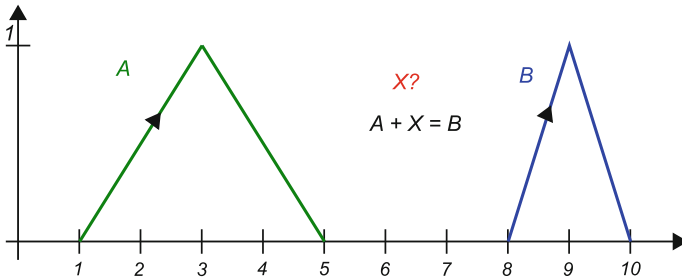


Fig. 4.16 Equation $A + X = B$ with the right-hand side B being *narrower* than its left-hand side component A

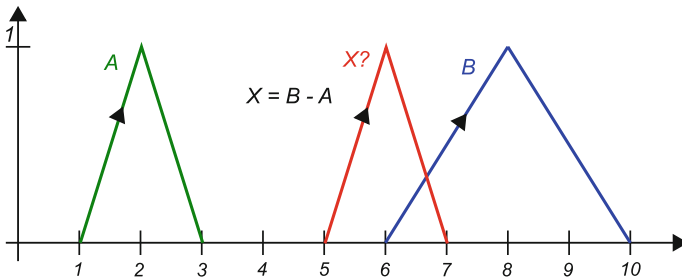


Fig. 4.17 Solution of the equation illustrated by Fig. 4.15

Finally, it is worth emphasizing that this subsection is just a brief introduction to the problem of solving equations within the OFN model. For more detailed investigation we refer to Chap. 9, where OFN-based complex equations are considered for some applications in economy. More advanced examples related to utilization of OFNs in differential equations can be found in [11]. On the other hand, it is important to compare further the expressive power of the OFN model with other approaches with regard to fuzzy equation-solving methods [1].

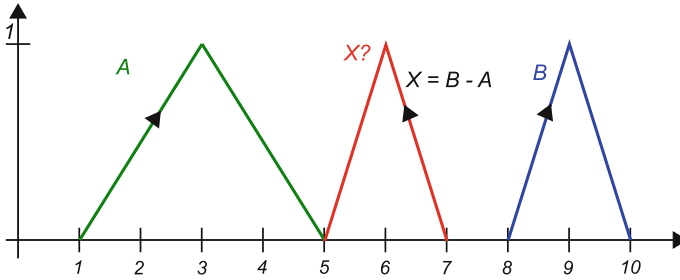


Fig. 4.18 Solution of the equation illustrated by Fig. 4.16

4.6 Interpretations of OFNs

The OFN model enables us to establish a quite efficient computational framework. It provides a new look at imprecision, however, it also has other consequences that need to be considered in applications. First of all, there is some inconsistency between OFNs and standard fuzzy numbers. This is understandable because OFNs are a larger class of objects. This aspect was commented on, for example, in [15, 21].

A potential for practical usage of OFNs also corresponds to their direction, an additional kind of information that is not represented by standard fuzzy numbers. This aspect turned out to be useful in many real-world scenarios, such as representing trends in control processes, expressing diversity of opinions in social networks, modeling dynamics of financial data, and simulating brain functions [3, 7, 18, 23].

Interpretation of objects represented by the OFN model was discussed and analyzed, for example, in [12, 22]. Here we present some revised aspects of that analysis. A common use of fuzzy sets is to represent the imprecise data, wherein fuzzy numbers are dedicated to imprecise quantitative data. The OFN model is primarily created for representing and processing fuzzy quantities as well. Let us remember it when drawing further intuitions related to applications.

4.6.1 Direction as a Trend

Interpretation of OFNs comprises adapting a general idea of standard fuzzy numbers, with an addition of direction. By using OFNs, we can describe trends of imprecise quantitative values observed in real-world processes. The up and down parts of OFNs can be related, for example, to the experts’ opinions about dynamic changes of the analyzed values. In the following subsections, we refer to a couple of possible cases of direction interpretations.

When using OFNs, we have two options. We can utilize their direction just for arithmetic purposes, or we can assign them with more complex information. In Fig. 4.19, we can see OFN $A = (f_A, g_A)$, which represents a linguistic variable *slow*

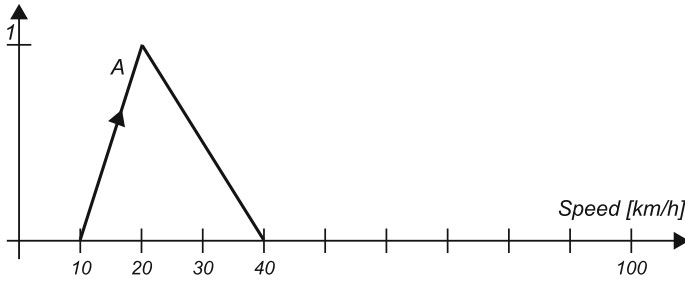


Fig. 4.19 Fuzzy number *about 20* with a growing trend

corresponding to the speed of a vehicle. Formally, *slow* is a fuzzy set rather than a fuzzy number. On the other hand, the domain of possible velocities is an interval of real numbers, thus in practice *slow* will usually be represented by a triangular (or trapezoidal) convex fuzzy number (or fuzzy interval). Now, let us use the linguistic term *about 20km/h* instead of *slow*. By modeling it with the use of OFN $A = (f_A, g_A)$, we can interpret A 's direction to say that it is *about 20 in the speed-up process*. Thus, A 's direction extends application options, without diminishing the importance of standard fuzzy number interpretation. After all, information of a form *about 20km/h in the speed-up process* is an extension of *about 20km/h*.

4.6.2 Validity of Operations

The analysis of a fuzzy arithmetic model should also include verification as to whether the results of calculations are consistent with real-world expectations.

According to the trend-based interpretation, let OFNs A and B visible in Fig. 4.20 represent opinions prepared by an expert about two units of a financial company: A 's income is at a level of 3 million, with an upward trend, as well as B 's income is at a level of 6 million, with a downward trend.

By using OFNs, the expert can actually describe not only a value and a trend but also an escalation of that trend. We have two OFNs with different *spreads* between their up and down components; indeed, object A is *wider* than B . By making the up part of B range from 7 to 6 million, the expert considers a potential of changes within 1 million. On the other hand, the up part of A ranges from 1 to 3, thus A could be recognized as a process that is more dynamic than in the case of B .

Yet another aspect is the considered OFNs' direction, which informs that B is a decreasing process and A is an increasing one.

In reality, we would expect here the total income at a level of 9 million. If we use the OFN model and add numbers A and B together, then we get the anticipated results. However, as we can see in Fig. 4.21, we also have additional information that seems to be consistent with our expectations as well. Namely, the obtained result

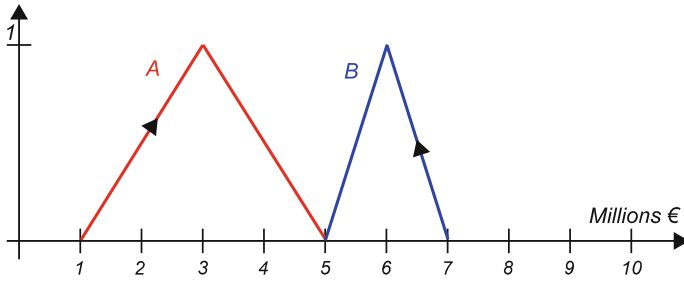


Fig. 4.20 OFNs that describe an income for two units of a financial company

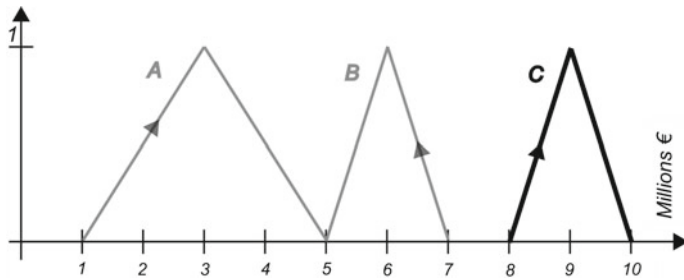


Fig. 4.21 A result of adding both incomes

shows that the trend is growing. Indeed, an increasing process related to *A* is more dynamic than a decreasing process related to *B*. However, because of *B*, the overall increasing trend is less dynamic than for *A*.

The above example shows how interpretation of the OFNs' direction can correspond to intuitions behind real-world observations. Such correspondence is important not only from a viewpoint of mathematical properties but also becomes useful at an operational level. In general, we believe that it can open new opportunities in front of applications of fuzzy numbers. Part III of this book presents more ideas about utilization of OFNs and their direction components.

4.6.3 The Meaning of Improper OFNs

Figure 4.22 illustrates a situation that is somewhat alternative to the example of income analysis considered in Sect. 4.6.2. As before, *A* and *B* are incomes of units of a financial company. However, now their sum is an improper OFN.

In this particular example, objects *A* and *B* are not symmetrical. They model a change in the income process. *A* represents an increase, which is slowing down. *B* represents a decreasing income, which is going to drop down even more. Thus we can expect the future dynamics of a sum of both incomes to be directed towards a

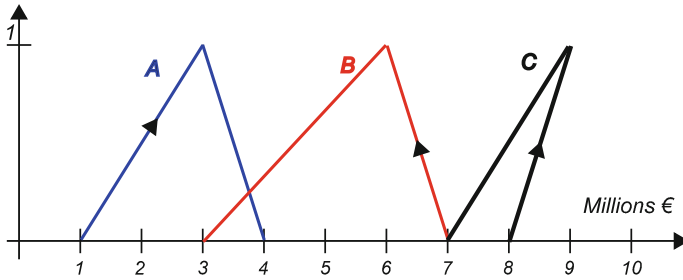


Fig. 4.22 Interpretation of improper OFN

decreasing trend. Even for A – which is still growing – the future expectations (down part) present less potential than the past (up part).

In summary, C represents a collapse of the upward trend of income for the whole company. This way, C contains information about a reversing trend. It shows that the idea of using OFNs to model changes of values is valid and intuitive.

Analogous analysis could also be conducted for other examples of potential usage of OFNs. As mentioned in Sect. 4.5.3, one of them could be related to operating with data summaries within a massive data-processing framework [20, 24]. In such a case, an improper OFN could mean that at the beginning of a data load process the values of a given column tend to increase to a certain most representative level, but later they begin to decrease again. In such a scenario, the trend-based interpretation of OFNs is related to a natural flow of data being loaded to a database system rather than the time of actual observation or measurement.

4.7 Summary and Further Intuitions

The main technical differences between the OFN model and standard fuzzy numbers refer to inverting and ordering local components of a fuzzy membership function. Such ordering has deep consequences for forming mathematical properties and implementing the model. It provides both (1) computational characteristics allowing us to define opposite and inverse OFNs (with respect to addition and multiplication, resp.), and (2) additional information, called direction, that is not present in previous approaches and that can be useful in practice.

Usually, one is focused on a result of actions rather than their inputs. Thus, while dealing with functions, one tends to concentrate on their output values. In the case of fuzzy membership functions, outputs take a form of elements of the interval $[0, 1]$, interpreted as truth values or degrees of compatibility. As for fuzzy numbers, the primary goal should be to model real numbers. In the case of OFNs $A = (f_A, g_A)$, as well as an ordering representing their up parts and down parts, we indeed focus on the target real numbers induced by functions $f_A, g_A : [0, 1] \rightarrow \mathbb{R}$.

The emphasis on degrees of truth is proper while working with qualitative approaches corresponding to fuzzy logic. However, for quantitative approaches, such as those referring to fuzzy numbers, processing with the elements of \mathbb{R} seems to be a good idea. Hence, OFNs should be considered primarily as an alternative to methods based on standard fuzzy arithmetics, rather than fuzzy logic.

Surely, the outcomes of calculations obtained within the OFN model may in some situations be harder to interpret than in the case of far more popular convex fuzzy numbers. However, we hope that this chapter provided the readers with appropriate tools to let such an interpretation be sufficiently straightforward.

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