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ORDERED INCIDENCE GEOMETRY AND THE GEOMETRIC FOUNDATIONS OF CONVEXITY THEORY [1]

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An **Ordered Incidence Geometry**, that is a geometry with certain axioms of **incidence** and **order**, is proposed as a minimal setting for the fundamental convexity theorems, which usually appear in the context of a linear vector space, but require only incidence, order (and for separation, **completeness**), and none of the linear structure of a vector space.

INTRODUCTION

We study the following question: What are the relevant geometric assumptions for convexity, especially for separation theorems? For answers, i.e. convexity theorems under different axioms see the survey by Danzer, Grünbaum and Klee [7], Ky Fan's generalization of the Krein-Milman Theorem [9], the Helly-type theorems of Levi [15] and Grünbaum [10], and the separation theorem of Ellis [8]. Axiom systems for convexity geometries were given by Prenowitz and Jantosciak [20] and Bryant and Webster ([3],[4],[5]) using **joins** (intervals joining pairs of points) as primitives.

Our objective is to develop convexity geometry using **affine sets** as primitives, in analogy with the classical (Hilbert) approach to Euclidean geometry. The notions used in separation theorems: **convex sets**, **hyperplanes** and **sides** of a hyperplane, are described here in terms of: **affine sets**, their **incidence** properties and **order** relations. The geometry is called **Ordered Incidence Geometry**, since for two or three dimensions the axioms resemble Hilbert's incidence and order axioms [11], see also [25].

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An application to sub-$\mathcal{F}$ functions, [1], is given in §8.

The order of development allows using standard arguments (e.g. [14]) in proofs, which are mostly omitted. An infinite dimensional case is discussed in [23].

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1. AXIOMS

An **Ordered Incidence Geometry** (abbreviated **OIG**), G, is a triple

$$G = \{X,\ \mathcal{A},\ \dim\ \}$$

and an order relation (**betweeness**), endowed with ten axioms given below. Here

X is the **space** of elements (**points**),

$\mathcal{A}$ is a family of subsets of X, called the **affine sets** of G,

"dim" a function : $\mathcal{A} \to$ { integers } $\cup \{\infty\}$, the **dimension**.

An affine set $A \in \mathcal{A}$ is called a k-**affine** if $\dim A = k$. In particular, we use the terms: **point** for a 0-affine[2], **line** for a 1-affine, **plane** for a 2-affine. By convention $\dim \emptyset = -1$.

AXIOM 1. $\mathcal{A}$ contains X, $\emptyset$, and all singletons $\{x\}$, $x \in X$.
AXIOM 2 (Intersection Axiom). $\mathcal{A}$ is closed under arbitrary intersections.

DEFINITION 1. For $S \subset X$, the **affine hull** of S is

$$\mathbf{a}(S) = \cap\{A : A \in \mathcal{A}, S \subset A\}$$

which by Axiom 2 is an affine set. We use this to define a hyperplane (needed in §7):
DEFINITION 2. A set $H \in \mathcal{A}$ is a **hyperplane** if exists $x \in X \setminus H$ such that $\mathbf{a}(H \cup \{x\}) = X$.

The next three axioms express **monotonicity properties** of dimension.
AXIOM 3. If $A,\ B \in \mathcal{A},\ A \subset B$ then $\dim A \leq \dim B$.

[2] It should be clear from the context whether a "point" is an element of X or of $\mathcal{A}$.

AXIOM 4. If $x \in \mathbf{X}$, $A \in \mathcal{A}, x \notin A$ then $\dim \mathbf{a}(A \cup \{x\}) = \dim A + 1$.[3]
AXIOM 5. For A, B, $H \in \mathcal{A}$,

$$\text{if } \begin{cases} B,\ H \subset A,\ \dim H = \dim A - 1 \\ B \cap H \neq \emptyset, B \end{cases} \quad \text{then } \dim\, (B \cap H) = \dim B - 1$$

Points lying on the same line are called **collinear.** For the geometry to have more than one line we need:
AXIOM 6. There exist three non-collinear points.

The remaining 4 axioms define the order relation **betweeness.** For distinct collinear points a, b, c we denote by abc the fact that b is between a and c. The set of all points between a and b is called the **open segment** or **open interval** joining a, b and is denoted by (a, b).
AXIOM 7. abc is equivalent to cba.
AXIOM 8. If $a \neq c$ then there exist points b, d such that abc and acd.
AXIOM 9. If a, b, c are distinct and collinear then one and only one of them is between the other two.
AXIOM 10 (The Pasch Axiom[4]). If a, b, c are distinct, and if L is a line in $\mathbf{a}(\{a, b, c\})$ with a, b, c not in L and $L \cap (a, b) \neq \emptyset$, then either $L \cap (a, c) \neq \emptyset$ or $L \cap (b, c) \neq \emptyset$.[5]

2. IMMEDIATE CONSEQUENCES

The results in this section are immediate consequences of the axioms. Some could be taken as alternative axioms for the same geometry.

COROLLARY 1. If $A \in \mathcal{A}$ and $S \subset A$ then $\mathbf{a}(S) \subset A$. □
This reduces to Hilbert's Incidence Axiom I,6 [11] for $\dim A = 2$ and $\#S = 2$, where $\#$ denotes **number of elements.**

COROLLARY 2. If A is a k-affine, $A \neq \emptyset$, then there is a subset S of A with $\#(S) = k + 1$ and $\dim \mathbf{a}(S) > k - 1$ □
For $k = 1, 2$ and 3, Corollary 2 reduces to Hilbert's Incidence Axioms I,3

[3] Thus, for $\dim \mathbf{X} < \infty$, hyperplanes are $(\dim \mathbf{X} - 1)$-affines.
[4] [18], §2, Kernsatz IV.
[5] If a, b, c are collinear, Axiom 10 can be stated as follows: Let $h \in (a, b), h \neq c$. Then either $h \in (a, c)$ or $h \in (b, c)$. (The point h is the intersection of the lines $\mathbf{a}(\{a, b, c\})$ and L.)

and I,8 specifying the existence of (i) two distinct points on any line, (ii) three non-collinear points on any plane, and (iii) four non-coplanar points in the 3-dimensional space.

COROLLARY 3. IF $S \subset \mathbf{X}$, $x \in \mathbf{X}$ then $\mathbf{a}(S \cup \{x\}) = \mathbf{a}(\mathbf{a}(S) \cup \{x\})$. □

COROLLARY 4. If $S \subset \mathbf{X}$, $\#S = k+1$, then $\dim \mathbf{a}(S) \leq k$. □

COROLLARY 5. Let A, B be k-affines, $S \subset A \cap B$. Then either $\mathbf{a}(S) = A = B$ or $\dim \mathbf{a}(S) < k$. □

COROLLARY 6. $A, B \in \mathcal{A}$, $A \subset B \implies \dim A = \dim B$ iff $A = B$. □

A converse of Corollary 2 is:

COROLLARY 7. If $\emptyset \neq S \subset \mathbf{X}$, $\#S = k+1$, $\dim \mathbf{a}(S) > k-1$ then there is a unique k-affine containing S. □

For $k = 1$ and 2, Corollary 7 reduces to Hilbert's Incidence Axioms I,1-2 and I,4-5, respectively [11].

The following corollary states roughly that if $\mathbf{a}(S)$ is "overdetermined" by S, then certain points of S are "affine combinations" of others.

COROLLARY 8. If $S \subset \mathbf{X}$, $\#S = k+1$, $\dim \mathbf{a}(S) \leq k-1$ then there is an $x \in S$ such that $x \in \mathbf{a}(S \setminus \{x\})$. □

In plane geometry it is well known that the Pasch Axiom is equivalent to a Plane Separation Axiom, ([16], Chapter 12). This holds also here.

DEFINITION 3. Let A, H be affine sets, $H \subset A$, $\emptyset \neq H \neq A$. Then H **separates** A if for any two points $x, y \in A \setminus H$ such that

$$(x, y) \cap H \neq \emptyset \tag{1}$$

there is no point $z \in A \setminus H$ such that[6],

$$(x, z) \cap H = \emptyset \text{ and } (y, z) \cap H = \emptyset \tag{2}$$

THEOREM 1. Let $A, H \in \mathcal{A}$, $\dim A \geq 1$, $H \subset A$, $\dim H = \dim A - 1$. Then H separates A.

PROOF. The case $\dim A = 1$ follows from Axiom 10 (footnote 5). Let $\dim A \geq 2$, and suppose H does not separate A, i.e. there are distinct points $x, y, z \in A \setminus H$ satisfying (1) and (2). It follows from Axiom 10 that

[6] I.e. x and y cannot be, at the same time, on "opposite" sides of H and on the "same" side of H.

x, y, z are non-collinear. Let P be the plane through x, y, z. The intersection $P \cap H$ is

(i) H if dim $A = 2$ (i.e. $P = A$),

(ii) a line if dim $A > 2$ (by Axiom 5),

so that, in either case, $P \cap H$ is a line, say L. Since L intersects (x, y), it follows from Axiom 10 that L also intersects (x, z) or (y, z), violating (2). □
Conversely, Axiom 10 follows from Theorem 1, i.e. the two are equivalent.

3. MODELS

Concrete models of Ordered Incidence Geometries include: The **real Euclidean n-dimensional space** ($\mathbf{X} = R^n$, with "affine sets" and "dimension" given their standard vector space meanings), the **Poincare half-plane incidence plane** ($\mathbf{X}$ is the upper half-plane, and 1-affines are the restrictions to $\mathbf{X}$ of (i) vertical lines, and (ii) circles with centers on the x-axis) and the **Moulton incidence plane**, [16]. The last two models are special cases of **Beckenbach geometries**, defined below.

DEFINITION 4 ([1], see also ([21], §§84-85). Let (a, b), (c, d) be open intervals (not necessarily bounded) in R. A family $\mathcal{F}$ of continuous functions $F : (a, b) \to (c, d)$ is a **Beckenbach family (B-family** for short) if for any two points (x_1, y_1), (x_2, y_2) with $a < x_1 < x_2 < b$, $y_1, y_2 \in (c, d)$ there exists a unique $F \in \mathcal{F}$, denoted by F_{12}, such that

$$F_{12}(x_i) = y_i, \quad (i = 1, 2) \tag{3}$$

DEFINITION 5. Let (a, b), (c, d) be as above and let $\mathcal{F}$ be a B-family of functions : $(a, b) \to (c, d)$. The **Beckenbach geometry (B-geometry)**, $\mathbf{G}_{\mathcal{F}}$, determined by $\mathcal{F}$, is a two dimensional geometry with $\mathbf{X} = (a, b) \times (c, d)$, and the 1-affine through any pair of points (x_1, y_1), (x_2, y_2) in $\mathbf{X}$ is,

(i) the vertical line $x = x_1$ if $x_1 = x_2$,

(ii) the graph of F_{12} (defined by (3)) if $x_1 \neq x_2$.

For a B-geometry Axioms 1-9 are easily verified, and Axiom 10 follows from:
LEMMA 1 (Beckenbach [1]). Let $a < x_0 < b$ and let F_α, F_β be two distinct members of $\mathcal{F}$ such that $F_\alpha(x_0) = F_\beta(x_0)$. Then

$F_\alpha(x) > F_\beta(x)$ for all x in (a, b) on one side of x_0,

$F_\alpha(x) < F_\beta(x)$ for all x in (a, b) on the other side of x_0. □

The next four examples use B-families in the form $\mathcal{F} = \{F(x\,;\alpha,\beta)\}$, where α, β are real parameters.

EXAMPLE 1. $\mathcal{F}_1 = \{\sqrt{\beta - (x-\alpha)^2} : \beta > 0,\ \alpha \in R\}$ with $(a,b) = R$, $(c,d) = (0,\infty)$, and $\mathbf{G}_{\mathcal{F}_1}$ gives the Poincare half-plane incidence plane.

EXAMPLE 2. $\mathcal{F}_2 = \{\alpha\phi_1(x) + \beta\phi_2(x) + \phi_3(x)\}$ where:
(i) ϕ_i are given continuous functions, and
(ii) $\phi_2(x) > 0$ on (a,b).
A necessary and sufficient condition for $\mathcal{F}_2$ to be a B-family is that ϕ_1/ϕ_2 is strictly monotone. For example, with $(a,b) = R$, $\mathcal{F} = \{\alpha e^x + \beta e^{-x}\}$ is a B-family, while $\mathcal{F} = \{\alpha x^2 + \beta\}$ is not.

EXAMPLE 3. $\mathcal{F}_3(x;\alpha,\beta) = \phi(\alpha,x) - \beta$ with ϕ differentiable in α for all x. A necessary and sufficient condition for $\mathcal{F}_3$ to be a B-family is that $\partial\phi/\partial\alpha$ is a srictly monotone function of x. For $\phi(\alpha,x) = \alpha x$, $\mathbf{G}_{\mathcal{F}_3}$ is the Euclidean plane geometry.

EXAMPLE 4. $\mathcal{F}_4(x;\alpha,\beta) = a(\alpha)u(x) + b(\alpha)v(x) - \beta$, where a and u are strictly increasing, b and v strictly decreasing. For example, $\mathcal{F} = \{\cosh(\alpha + x) - \beta : \alpha,\beta \in R\}$.

4. TRIANGLES

Three non-collinear points $\{a,b,c\}$ constitute a **triangle** Δabc. The basic properties of triangles ([18], §2) are sampled in the following two pairs of lemmas, each pair consisting of a result and a (sort of) converse:

LEMMA 2. For any $u \in (a,c)$ and $v \in (u,b)$, there is a point $w \in (b,c)$ such that $v \in (a,w)$. □
LEMMA 3. For any $u \in (a,c)$ and $w \in (b,c)$ there is a point v in the intersection $(a,w) \cap (b,u)$. □

LEMMA 4. Let $u \in (a,c)$, $w \in (b,c)$. Then for any $v \in (u,w)$ there is a $z \in (a,b)$ such that $v \in (c,z)$. □
LEMMA 5. Let $z \in (a,b)$. Then for any $v \in (c,z)$ there exist two points $u \in (a,c)$, $w \in (b,c)$ such that $v \in (u,w)$. □

5. LINEAL HULLS

In the Euclidean geometry R^n a set A is **affine** if and only if

$$A = \{\sum \lambda_i x_i : \ x_i \in A, \ \sum \lambda_i = 1\} \tag{4}$$

i.e. A coincides with the set of **affine combinations** of its elements. The analogous representation in an OIG (where algebraic constructions such as (4) are not available) is given in Theorem 2. First we require:

DEFINITION 6. For a given subset S of $\mathbf{X}$, the **lineal hull** of S, is

$$\ell(S) = \cup\{\overline{xy} : \ x, y \in S\}$$

the union of lines through pairs of points in S. By convention, $\ell(\emptyset) = \emptyset$ and $\ell(\{x\}) = \{x\}, \ \forall x \in \mathbf{X}$. We also use the abbreviation $\ell^{(2)}(S) = \ell(\ell(S))$.

LEMMA 6. If A is affine, $x \notin A$, then $\mathbf{a}(A \cup \{x\}) = \ell^{(2)}(A \cup \{x\})$.
PROOF. Use Corollary 1 and Axiom 8. □

THEOREM 2. S is an affine set if and only if $S = \ell(S)$. □

6. CONVEX SETS

The basic properties of convex sets are developed in this section.

DEFINITION 7. A set $S \subset \mathbf{X}$ is:
(i) **star shaped** at x if for all $y \in S$, $(x, y) \subset S$,
(ii) **convex** if for any two points $x, y \in S$, $(x, y) \subset S$.
DEFINITION 8. For any set $S \subset \mathbf{X}$, the **convex hull** of S, conv (S), is the intersection of all convex sets containing S.
DEFINITION 9. For any set $S \subset \mathbf{X}$,
(i) the **core** of S is

$$\text{core } S = \{x \in S : \forall y \in \mathbf{X}, \ y \neq x, \ \exists z \in (x, y) \text{ such that } (x, z) \subset S\} \tag{5}$$

(ii) the **relative core** of S, relcore S, is defined by (5) with

"$\forall y \in \mathbf{a}(S)$" replacing "$\forall y \in \mathbf{X}$"

(iii) the set **linearly accessible** from S, is

$$\text{lina } S = \{y \in \mathbf{X} : \exists x \in S \text{ such that } (x,y) \subset S\}$$

(iv) the **closure** of S is cl $S = S \cup \text{lina } S$.

LEMMA 7. Let a set S with a nonempty relative core[7] be star shaped at p. Then $x \in \text{relcore } S \Longrightarrow (p,x) \subset \text{relcore } S$ i.e. relcore S is also star shaped at p. □

THEOREM 3. Let S be a convex set, $y \in \text{lina } S$, $x \in \text{relcore } S$, $y \neq x$. Then $(x,y) \subset \text{relcore } S$.
PROOF. Use Axioms 8, 10 and Lemma 7. □

THEOREM 4. If S is convex, then relcore S and cl S are convex. □

This section ends with a result of a topological nature, Theorem 5, that a (nonempty) finite-dimensional convex set has a nonempty relative core.
DEFINITION 10. For any $S \subset \mathbf{X}$, the **dimension** of S is defined as the dimension of its affine hull, dim S = dim $\mathbf{a}(S)$.
DEFINITION 11. An n-**simplex** is the convex hull of a set S with

$$\#S = n+1, \ \dim S = n$$

We prove first that a simplex has a nonempty relative core.
LEMMA 8. Let $\Delta_n = \text{conv}\,\{x_1, \ldots, x_{n+1}\}$ be an n-simplex. Then

$$\text{relcore } \Delta_n \neq \emptyset$$

PROOF. By induction on n. For $n = 1$ the result follows from Axiom 8. In the inductive step use Axioms 4, 5 and 10. □

LEMMA 9. If $\emptyset \neq C$ is a convex set, dim $C < \infty$, then dim C = dim $\Delta_{\max}$ where $\Delta_{\max}$ is a maximal dimensional simplex contained in C.
PROOF. The existence of a maximal $\Delta_{\max}$ follows since C is finite-dimensional. Then use Axiom 4 to prove $\mathbf{a}(C) = \mathbf{a}(\Delta_{\max})$. □

THEOREM 5. If C is a nonempty convex set, dim $C < \infty$, then

$$\text{relcore } C \neq \emptyset. \quad \Box$$

[7] In the finite-dimensional case, relcore $S \neq \emptyset$ for any convex set S, see Theorem 5.

7. SEPARATION

The main result here is Theorem 7 stating conditions under which two disjoint convex sets can be separated by a hyperplane.

From Theorem 1 it follows that any hyperplane separates the space in the sense of Definition 3. We elaborate on this statement in the following:
LEMMA 10. Given a hyperplane H in $\mathbf{X}$, there exist unique nonempty convex sets H^+, H^- such that

(a) H, H^+, H^- are disjoint, and

(b) $\mathbf{X} = H \cup H^+ \cup H^-$.

PROOF. Select any $x_0 \in \mathbf{X} \setminus H$ and define:

$$H^+ = \{y \notin H : (x_0, y) \cap H = \emptyset\}, \tag{6}$$

$$H^- = \{y \notin H : (x_0, y) \cap H \neq \emptyset\}. \tag{7}$$

Then (a) and (b) are obvious. Nonemptyness, convexity and uniqueness [8] of H^+ and H^- use standard arguments. □
One can similarly obtain:
LEMMA 11. Let H, H^+, H^- be as in Lemma 10. Then

(i) $H^+ = \text{core } H^+$, $H^- = \text{core } H^-$

(ii) $H = \text{lina } H^+ \cap \text{lina } H^-$

(iii) $H \cup H^+ = \text{cl } H^+$, $H \cup H^- = \text{cl } H^-$. □

DEFINITION 12. A convex set C is

(i) **open** if $C = \text{core } C$,

(ii) **relatively open** if $C = \text{relcore } C$,

(iii) **closed** if $C = \text{cl } C$.

The following definitions are suggested by Lemma 11.
DEFINITION 13. Let H, H^+, H^- be as in Lemma 10. Then

(i) H^+, H^- are the **open halfspaces** of (i.e. corresponding to) H.

(ii) $H \cup H^+$, $H \cup H^-$ are the **closed halfspaces** of H.

DEFINITION 14. Let $A, B \subset \mathbf{X}$, and let H be a hyperplane. Then

(i) H **separates** A and B if A and B are contained in opposite closed halfspaces of H.

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[8] Uniqueness means that the (unordered) pair $\{H^+, H^-\}$ is independent of the particular x_0 used in (6),(7).

(ii) Moreover, if $A \cup B$ is not a subset of H then H **separates** A and B **properly.**

The next lemma implies a converse of Theorem 1: The only affine sets with the separation property (Definition 3) are hyperplanes.
LEMMA 12. If A, B, C are affine sets, $A \subset B$ and $A \neq B$, and if B separates C, then A does not separate C.
PROOF. Suppose A separates C and let A^+, A^- be the "opposite sides" of A in C, i.e.

$$C = A \cup A^+ \cup A^-$$

where A, A^+, A^- are disjoint,
x, y in A^+ or in $A^- \implies (x,y) \cap A = \emptyset$,
$x \in A^+,\ y \in A^- \implies (x,y) \cap A \neq \emptyset$.
Given that B separates C, let

$$C = B \cup B^+ \cup B^-$$

be the analogous decomposition of C with respect to B.
Now let $x \in B^+$. Then $x \notin A$, and without loss of generality let $x \in A^+$. Any other point in B^+ must also be in A^+ for if $x \neq y \in B^+, y \in A^-$ then (x, y) intersects A but not B, a contradiction. Therefore

$$B^+ \subset A^+ \tag{8}$$

and similarly,

$$B^- \subset A^- \tag{9}$$

The inclusions (8) and (9) lead to contradiction. □

DEFINITION 15. A **convex pair** in $\mathbf{X}$ is an unordered pair $\{C, D\}$ of nonempty convex sets with

$$\mathbf{X} = C \cup D,\ C \cap D = \emptyset$$

A classical result of Mazur [17] and Kakutani [13], (see also [24], Theorem 2.3 and references therein), can be stated for OIG as follows:
THEOREM 6. If A, B are disjoint convex sets in $\mathbf{X}$, then there exists a convex pair $\{C, D\}$ with

$$A \subset C,\ B \subset D \tag{10}$$

PROOF. We first prove an auxilliary result:
(a) If S is a nonempty convex set, and $x_0 \notin S$, then the set

$$K(S, x_0) = \{x : x \in [x_0, y]\,,\ y \in S\}$$

is convex. Here we denote by $[a, b]$ the **closed segment** joining a, b:

$$[a, b] = (a, b) \cup \{a\} \cup \{b\}$$

Given (a), the proof of ([24], Theorem 2.3) can be followed, using Zorn's Lemma to obtain $\{C, D\}$ as a maximal element of the set (partially ordered by inclusion) of disjoint convex sets $\{C, D\}$ satisfying (10). □

Convex pairs are used to prove existence of hyperplanes separating disjoint convex sets. First we require:

DEFINITION 16. An affine set is **openly decomposable** if it is the union of two disjoint, relatively open, nonempty convex sets.

DEFINITION 17. A geometry $\mathbf{G} = \{\mathbf{X}, \mathcal{A}, \dim\}$ is a **Complete Ordered Incidence Geometry** (**COIG** for short) if it satisfies, in addition to Axioms 1-10 (of §1), the following:
AXIOM 11 (Completeness Axiom). No line in $\mathbf{G}$ is openly decomposable.

This usage of completeness is standard, see also [5]. An example of a non-complete OIG is the **rational Euclidean n-dimensional space** with $\mathbf{X} = Q^n$, the set of rational n-tuples, and $\mathcal{A}$ and "dim" given their standard vector space meanings.

Lemma 14 below requires the following property (seemingly stronger than completeness): No affine set (of any dimension) is openly decomposable. We show this to be equivalent to completeness.
LEMMA 13. Let $\mathbf{G} = \{\mathbf{X}, \mathcal{A}, \dim\}$ be a COIG. Then no affine set is openly decomposable.
PROOF. Let $A \in \mathcal{A}$ be openly decomposable, i.e.

$$A = C_1 \cup C_2 \tag{11}$$

where C_1, C_2 are disjoint, nonempty, relatively open, convex sets. From (11) follows $A = \mathbf{a}(C_1) \cup \mathbf{a}(C_2)$ and consequently $A = \mathbf{a}(C_1) = \mathbf{a}(C_2)$, showing that, restricted to A, the relative cores of C_1 and C_2 can be taken as cores, i.e.

$$C_i = \text{core } C_i, \quad i = 1, 2 \tag{12}$$

Choose any two points $x_i \in C_i$ and let L be the line $\overline{x_1 x_2}$. From (12) follows the existence of points $z_i \in C_i$, $(i = 1, 2)$, such that

$$(x_i, z_i) \subset L, \quad i = 1, 2$$

Extending the two (relatively) open segments (x_i, z_i) beyond x_i, $(i = 1, 2)$, we get the intervals (unbounded on one side):

$$I_i = (x_i, z_i) \cup \{y \in L : x_i \in (z_i, y)\}, \quad i = 1, 2$$

By Zorn's Lemma, the set of such intervals has a maximal element $\{I_1, I_2\}$, and consequently $L = I_1 \cup I_2$, violating Axiom 11. □

Given a convex pair $\{C, D\}$, the following lemma gives conditions for the existence of a hyperplane with opposite sides $\{\text{core } C, \text{core } D\}$.

LEMMA 14. If $\{C, D\}$ is a convex pair in $\mathbf{X}$, then the set H defined by

$$H = \text{cl } C \cap \text{cl } D$$

satisfies:

(a) $H \cap \text{core } C = \emptyset = H \cap \text{core } D$

If the geometry is complete then:

(b) $H \neq \emptyset$ and $X = H \cup \text{core } C \cup \text{core } D$

(c) If either core $C \neq \emptyset$ or dim $\mathbf{X} < \infty$, then H is a hyperplane. □

Combining the above results, we finally prove:

THEOREM 7. (The Separation Theorem). Let $\mathbf{G} = \{\mathbf{X}, \mathcal{A}, \dim\}$ be a COIG, and let A, B be disjoint convex sets in $\mathbf{X}$. Then a hyperplane H properly separating A and B exists if:

(a) core $A \neq \emptyset$, in which case $H \cap \text{core } A = \emptyset$,

or if

(b) dim $\mathbf{X} < \infty$.

PROOF. Let $\{C, D\}$ and H be given by Theorem 6 and Lemma 14. Then H separates A and B in the sense that

$$A \subset H \cup \text{core } C = H \cup H^+, \; B \subset H \cup \text{core } D = H \cup H^- \tag{13}$$

To prove proper separation (Definition 14(ii)) we show that

$$A \cup B \not\subset H \tag{14}$$

(a) If core $A \neq \emptyset$ then, by (10), core $A \subset$ core $C \neq \emptyset$ and (14) follows from Lemma 14(a).
(b) Let dim $\mathbf{X} < \infty$, core $A = \emptyset$ (otherwise it is case (a) again), and

$$A \cup B \subset H$$

We restrict the discussion to H which we denote by H_1. In H_1 there is a hyperplane H_2 (i.e. dim $H_2 =$ dim $H_1 - 1$) separating A and B in the sense of (13). Now there are two cases:

(i) H_2 separates A and B properly, (ii) $A \cup B \subset H_2$.

In case (ii) we repeat the process: Restrict to H_2, find a hyperplane H_3 (in H_2) separating A and B, etc. From dim $H_{i+1} =$ dim $H_i - 1$ it follows that after finitely many repetitions, an affine set H_i is reached in which one of the sets A, B has a nonempty core, i.e.

$$\dim H_i = \max \{\dim A, \dim B\}$$

and, by part (a), it is case (i), (although case (i) may occur sooner.) Suppose then that case (i) is reached after k successive restrictions, a situation described by

$$\mathbf{X} = H_0 \supset H_1 \supset H_2 \supset \cdots \supset H_{k+1}$$

where H_{i+1} separates A and B in H_i ($i = 0, \ldots, k$), $A \cup B \subset H_k$ and $A \cup B \not\subset H_{k+1}$. Reversing our steps we construct a sequence of affine sets $\{\bar{H}_i\}$

$$H_{k+1} = \bar{H}_{k+1} \subset \bar{H}_k \subset \cdots \subset \bar{H}_1 \subset \bar{H}_0 = \mathbf{X} \tag{15}$$

where $\bar{H}_1$ separates A and B properly in $\mathbf{X}$. A sequence (15) is defined recursively as follows:

For $i = k, \cdots, 1$
  choose any $x_i \in H_{i-1} \setminus \mathrm{a}(A \cup B)$
  define $\bar{H}_i = \mathrm{a}(\bar{H}_{i+1} \cup \{x_i\})$ □

REMARKS. (a) If core $A \neq \emptyset$, the assumption "$A \cap B = \emptyset$" in Theorem 7 can be replaced by "core $A \cap B = \emptyset$".
(b) To show that completeness is needed in Theorem 7, consider the rational line Q, in which the sets $\{x : x < \sqrt{2}\}$ and $\{x : x > \sqrt{2}\}$ cannot be separated

by a hyperplane (the hyperplanes of Q are its points).

8. APPLICATIONS TO FUNCTIONS ON THE REAL LINE: SUB-$\mathcal{F}$ FUNCTIONS AND FENCHEL DUALITY.

We use the terminology of §3. Let $\mathcal{F}$ be a given B-family on the interval (a, b) (we take $(c, d) = R$), and let $\mathbf{G}_{\mathcal{F}}$ be the associated B-geometry.

DEFINITION 18 (Beckenbach [1]). A function $f : (a, b) \to R$ is sub-$\mathcal{F}$ if for any two points $a < x_1 < x_2 < b$ and $F_{12} \in \mathcal{F}$ defined by $F_{12}(x_i) = f(x_i), \quad i = 1, 2,$

$$f(x) \leq F_{12}(x) \quad \text{for all } x_1 \leq x \leq x_2 \tag{16}$$

f is **super-$\mathcal{F}$** if the reverse inequality holds in (16). Sub-$\mathcal{F}$ functions are generalizations of convex functions. Indeed, for the family $\mathcal{F}$ of affine functions, **sub-$\mathcal{F}$** and **super-$\mathcal{F}$** become **convex** and **concave** (in the ordinary sense), respectively. Sub-$\mathcal{F}$ functions have been applied to 2nd order differential inequalities, e.g. [6], [12] and [19].
In this section we study the geometric properties of sub-$\mathcal{F}$ functions (in the geometry $\mathbf{G}_{\mathcal{F}}$) and establish a Fenchel duality theorem (Theorem 8).
As usual, denote the **epigraph** and **hypograph** of f by

$$\text{epi } f = \{\begin{pmatrix} x \\ \mu \end{pmatrix} : \mu \geq f(x)\} \tag{17}$$

$$\text{hypo } f = \{\begin{pmatrix} x \\ \mu \end{pmatrix} : \mu \leq f(x)\} \tag{18}$$

Here also, "convexity of a set" and "convexity of a function" are related:
LEMMA 15. A function $f : (a, b) \to R$ is:
(a) sub-$\mathcal{F}$ iff epi f is convex.
(b) super-$\mathcal{F}$ iff hypo f is convex. □

The following characterization of the core of epi f is useful.
LEMMA 16. Let $a < a_1 < b_1 < b$, let f be sub-$\mathcal{F}$ on (a_1, b_1), and

$$A = \left\{ \begin{pmatrix} x \\ \mu \end{pmatrix} : \begin{matrix} a_1 \leq x \leq b_1 \\ \mu \geq f(x) \end{matrix} \right\}$$

Then

$$\text{core } A = \left\{ \begin{pmatrix} x \\ \mu \end{pmatrix} : \begin{array}{c} a_1 < x < b_1 \\ \mu > f(x) \end{array} \right\} \quad \square$$

REMARK. The separation theorem (Theorem 7), applied to the sets

$$A = \text{core (epi } f) \quad \text{and } B = \left\{ \begin{pmatrix} x \\ f(x) \end{pmatrix} \right\}$$

for given $x \in (a,b)$, shows that a sub-$\mathcal{F}$ function is supported at each point in (a,b) by a function $F \in \mathcal{F}$, [19]. Indeed, the support property is necessary and sufficient for f to be sub-$\mathcal{F}$.

From here on we specialize to the B-families of §3, Example 3,

$$\mathcal{F} = \{F(x) = \phi(\alpha, x) - \beta : \alpha, \beta \in R\} \tag{19}$$

where ϕ is differentiable in α, x and

$$\frac{\partial \phi}{\partial \alpha} \quad \text{is an increasing function of } x. \tag{20}$$

As in [2] define the **dual family**[9]

$$\mathcal{F}^* = \{F^*(\alpha) = \phi(\alpha, x) - \beta : x, \beta \in R\} \tag{21}$$

which is a B-family if [10]

$$\frac{\partial \phi}{\partial x} \quad \text{is an increasing function of } \alpha. \tag{22}$$

The **effective domain** of a function f, [22], is denoted by dom f.

DEFINITION 19. Given $f : (a,b) \to R$,

(i) the (**convex**) **conjugate** of f, f^*, is

$$f^*(\alpha) = \sup_{x \,\in\, \text{dom } f} \{\phi(\alpha, x) - f(x)\} \tag{23}$$

(ii) the (**concave**) **conjugate** of f, f_*, is

$$f_*(\alpha) = \inf_{x \,\in\, \text{dom } f} \{\phi(\alpha, x) - f(x)\} \tag{24}$$

[9]Note that in $\mathcal{F}^*$ the argument is α (one of the parameters of $\mathcal{F}$) and the parameters are x, β. Thus any pair $\{x, \beta\}$ determines a unique $F^* = F^*(\,\cdot\,; x, \beta)$ in $\mathcal{F}^*$.

[10]The conditions (20) and (22) guarantee that both $\mathcal{F}$ and $\mathcal{F}^*$ are B-families.

In convex analysis, the conjugate f^* is convex, regardless of f. The analogous result here is:
LEMMA 17. For any function $f : (a, b) \to R$,
(a) f^* is sub-$\mathcal{F}^*$,
(b) f_* is super-$\mathcal{F}^*$. □

A duality theorem of Fenchel type (see also [22], Theorem 31.1) now follows. A (somewhat weaker) Fenchel duality theorem was proved in [2] for $\mathcal{F}$-convex functions : $R^n \to R$.
THEOREM 8. Let
f be a sub-$\mathcal{F}$ function: $(a, b) \to R$,
g be a super-$\mathcal{F}$ function: $(a, b) \to R$,
and consider the pair of problems[11]

$$\inf \{f(x) - g(x) : x \in \text{dom } f \cap \text{dom } g\} \tag{P}$$

$$\sup \{g_*(\alpha) - f^*(\alpha) : \alpha \in \text{dom } f^* \cap \text{dom } g_*\} \tag{D}$$

If[12]

$$\text{int dom } f \cap \text{int dom } g \neq \emptyset \tag{25}$$

then

$$\inf \text{(P)} = \max \text{(D)}$$

PROOF. The proof is similar to the proof of ([22], Thorem 31.1). From (23) and (24),

$$g(x) + g_*(\alpha) \leq \phi(\alpha, x) \leq f(x) + f^*(\alpha), \quad \forall x, \alpha$$

so that

$$f(x) - g(x) \geq g_*(\alpha) - f^*(\alpha), \quad \forall x, \alpha$$

proving

$$\inf \text{(P)} \geq \sup \text{(D)} \tag{26}$$

In particular, inf (P) $= -\infty \Rightarrow$ sup (D) $= -\infty$. Let inf (P) $> -\infty$, and denote

$$\begin{aligned} \gamma &:= \inf \text{(P)} \\ &= \sup \{\beta : f(x) \geq g(x) + \beta, \ \forall x\} \end{aligned} \tag{27}$$

[11] The difference $f - g$ was shown in ([2], Theorem 4) to be unimodal
[12] dom f and dom g are intervals in (a, b), and "int" denotes the interior of a real interval.

By (26) it suffices to show the existence of $\alpha \in \text{dom } f^* \cap \text{dom } g_*$ such that

$$g_*(\alpha) - f^*(\alpha) \geq \gamma \tag{28}$$

Define two sets

$$A = \text{epi } f, \quad B = \text{hypo } (g + \gamma) = \left\{ \begin{pmatrix} x \\ \mu \end{pmatrix} : \mu \leq g(x) + \gamma \right\}$$

Note that for families $\mathcal{F}$ of type (19),

$$f \text{ is } \left\{ \begin{array}{c} \text{sub} - \mathcal{F} \\ \text{super} - \mathcal{F} \end{array} \right\} \Longrightarrow (f + \gamma) \text{ is } \left\{ \begin{array}{c} \text{sub} - \mathcal{F} \\ \text{super} - \mathcal{F} \end{array} \right\}, \quad \forall \gamma \in R$$

Therefore, using Lemma 15, A and B are convex in $\mathbf{G}_{\mathcal{F}}$. Now $\{\text{core } A\} \cap B = \emptyset$ by Lemma 16, and by theorem 7 there is a hyperplane H separating core A and B, and therefore separating A and B. In the geometry $\mathbf{G}_{\mathcal{F}}$ hyperplanes are the lines of Definition 5. By (25), the separating line H cannot be vertical, and is therefore of the form

$$H = \left\{ \begin{pmatrix} x \\ \mu \end{pmatrix} : \mu = \phi(x, \alpha^*) - \beta^* \right\}$$

for some pair of parameters α^*, β^*. Since H separates A and B,

$$\begin{aligned} f(x) &\geq \phi(\alpha^*, x) - \beta^* \geq g(x) + \gamma, \quad \forall x \\ \therefore \ \beta^* &\geq \sup_x \{\phi(\alpha^*, x) - f(x)\} = f^*(\alpha^*) \\ \therefore \ \gamma + \beta^* &\leq \inf_x \{\phi(\alpha^*, x) - g(x)\} = g_*(\alpha^*) \end{aligned}$$

And finally, $\gamma \leq g_*(\alpha^*) - f^*(\alpha^*)$, proving (28). □

The following example illustrates Theorem 8. Here (P) is a convex program, and there are infinitely many possible Fenchel duals, corresponding to the various decompositions of the objective function $f - g$, and the choice of the underlying family $\mathcal{F}$. One such dual is (D) below.

EXAMPLE 5. Let the primal problem be

$$\inf_{x \geq 0} (e^x + e^{-x}) \tag{P}$$

with optimal solution $x^* = 0$. We choose

$$\begin{aligned} f(x) &= e^x + e^{-x}, \quad \text{dom } f = R, \\ g(x) &= -\delta(x \mid R^+) = \left\{ \begin{array}{ll} 0 & , x \geq 0 \\ -\infty & , \text{otherwise} \end{array} \right., \quad \text{dom } g = R^+ \end{aligned}$$

Consider the family $\mathcal{F} = \{F(x) = \cosh(\alpha + x) - \beta\}$, (§3, Example 4). Here $\mathcal{F} = \mathcal{F}^*$ and (20), (22) are satisfied. Since $f \in \mathcal{F}$, f is sub-$\mathcal{F}$. Also (since $\mathcal{F}$ consists of convex functions), the indicator function g is (strictly) super-$\mathcal{F}$. The conjugates can be computed to give:

$$\begin{aligned} f^*(\alpha) &= \begin{cases} -\sqrt{(e^{\alpha}-2)(e^{-\alpha}-2)} & \text{if } |\alpha| \le \log\, 2 \\ \infty & \text{otherwise} \end{cases} \\ g_*(\alpha) &= \cosh(\max\{0,\alpha\}) \end{aligned}$$

so that dom $f^* = [-\log\, 2,\ \log\, 2]$, dom $g_* = R$ and the dual program is

$$\sup_{|\alpha| \le \log\, 2} \{\cosh(\max\{0,\alpha\}) + \sqrt{(e^{\alpha}-2)(e^{-\alpha}-2)}\} \qquad \text{(D)}$$

It can be verified that the optimal solution of (D) is $\alpha^* = 0$, and inf (P) = sup (D) = 2.

9. THÉ THEOREMS OF RADON AND HELLY.

The (closely related) theorems of Radon, Helly and Caratheodory (see e.g. [7]) hold also for OIG.

LEMMA 18. If $S \subset \mathbf{X}$, $\#S = n+2$, dim $S = n$, then there is a subset $T \subset S$, $\#T = n$, such that $S = T \cup \{x\} \cup \{y\}$ and $\mathbf{a}(T) \cap [x,y] \neq \emptyset$. □

THEOREM 9 (Radon). Let $S \subset \mathbf{X}$, $\#S \geq n+2$, dim $S = n$. Then S can be partitioned into $S = S_1 \cup S_2$ where $S_1 \cap S_2 = \emptyset$ and conv $S_1 \cap$ conv $S_2 \neq \emptyset$.

PROOF. Enough to consider the case $\#S = n+2$. We prove by induction on n. For $n = 1$, the theorem follows from the order axioms. For $n = 2$ and the inductive step, use Lemma 18. □

The proof of [24] can now be used, verbatim:

THEOREM 10 (Helly). Let dim $\mathbf{X} = n$ and let S be any family of convex sets in $\mathbf{X}$, $\#S = k > n+1$. If every $n+1$ sets in S have a nonempty intersection, then S has a nonempty intersection. □

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