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**Ordering properties of the largest order statistics from Kumaraswamy-G models under random shocks**

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## Abstract

In this paper we compare the maximums of two independent and heterogeneous samples each following Kumaraswamy-G distribution with the same and the different parent distribution functions using the concept of matrix majorization. The comparisons are particularly carried out with respect to usual stochastic ordering when each sampling unit experiences a random shock. The implications of the results are explained with an application.

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# 1 Introduction

Order statistics have a prominent role in reliability theory, life testing, actuarial science, auction theory, hydrology and many other related and unrelated areas. If $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the order statistics corresponding to the random variables $X_1, X_2, \ldots, X_n$, then the sample minimum and sample maximum correspond to the smallest and the largest order statistics $X_{1:n}$ and $X_{n:n}$ respectively. The results of stochastic comparisons of the order statistics (largely on the smallest and the largest order statistics) can be seen in Dykstra *et al.* (1997), Fang and Balakrishnan (2018), Fang and Zhang (2015), Zhao and Balakrishnan (2011), Torrado and Kochar (2015), Balakrishnan *et al.* (2014), Li and Li (2015), Kundu *et al.* (2016), Kundu and Chowdhury (2016, 2018), Chowdhury and Kundu (2017) and the references there in for a variety of parametric models. The assumption in the papers lies in the fact that each of the order statistics $X_{1:n}, X_{2:n}, \ldots X_{n:n}$ occurs with certainty and the comparison is carried out on the minimums or the maximums of the order statistics. Now, it may so happen that the order statistics experience random shocks which may or may not result in its occurence and it is of interest to compare two such systems stochastically with respect to vector or matrix majorization. The model could arise in the context of reliability and actuarial sciences as described next.

Let us assume a parallel system consists of $n$ independent components in working conditions. Each component of the system receives a shock which may cause the component to fail. Let the random variable (rv) $T_i$ denote lifetime of the $i$th component in the system which experiences a random shock at binging. Also suppose that $I_i$ denote independent Bernoulli rvs, independent of the $X_i$s, with $E(I_i) = p_i$, will be called shock parameter hereafter. Then, the random shock impacts the $i$th component ($I_i = 1$) with probability $p_i$ or doesn't impact the $i$th component ($I_i = 0$) with probability $1 - p_i$. Hence, the rv $X_i = I_i T_i$ corresponds to the lifetime of the $i$th component in a system under shock. Fang and Balakrishnan (2018) has compared two such systems with generalized Birnbaum-Saunders components. Similar comparisons are carried out in the context of insurance where largest or smallest claim amounts in a portfolio of risks are compared stochastically. One may refer to Barmalzan*et al.* (2017), and Balakrishnan *et al.* (2018 for more detail.

Kumaraswamy (1980) proposed a new two-parameter probability distribution, known as Kumaraswamy's distribution ($Kw$ distribution) on (0,1) with hydrological applications. The distribution does not seem to be popular in the statistical literature and has seen only limited use and development in the hydrological and related literature (see Sundar and Subbiah (1989), Fletcher and Ponnambalam (1996), Seifi *et al.* (1989) and

Ganji *et al.* (2006)). A recent paper by Jones (2008) has explored the background and genesis of the $Kw$ distribution and discussed it's similarities to the beta distribution along with a number of advantages in terms of tractability. A random variable $X$ is said to have $Kw$ distribution with parameters $(\alpha, \beta)$, written as $Kw(\alpha, \beta)$, if the cumulative distribution function (cdf) of $X$ is given by

$$K(x) = 1 - (1 - x^\alpha)^\beta, \ \ 0 < x < 1, \ \alpha > 0, \ \beta > 0,$$

where $\alpha$ and $\beta$ are the shape parameters. Generalizing this distribution, Cordeiro and de Castro (2011) have proposed a new family of generalized distributions, called Kumaraswamy generalized family of distributions (called $Kw$-G distribution). For a random variable $X$ with cdf $F(x)$, the distribution function $G(x)$ of the $Kw$-G random variable is defined by

$$G(x) = 1 - (1 - (F(x))^\alpha)^\beta, \ \ x \in \Re, \ \alpha > 0, \ \beta > 0. \tag{1.1}$$

The $Kw$-G distribution, written as $Kw$-G$(\alpha, \beta, F)$, is shown to be used for the censored data quite effectively and has the ability to fit skewed data better than any existing distributions. Each of the $Kw$-G distributions can be obtained from a specified parent cdf $F$, e.g. the $Kw$-Weibull ($Kw$-W), $Kw$-gamma ($Kw$-Ga) and $Kw$-Gumbel ($Kw$-Gu) distributions can be obtained by taking $F(x)$ as the cdf of the Weibull, gamma and Gumbel distributions, respectively. Recently, Kundu and Chowdhury (2018) have studied the stochastic properties of minimum order statistics for $Kw$-G$(\alpha, \beta, F)$ model. In this paper, we take the work a step forward and compare maximums of two independent heterogeneous samples from $Kw$-G random variables with both common ($F$) and different ($F_1$ and $F_2$) homogenous parent cdf when each of the units in the sample experiences a random shock.

The rest of the paper is organized as follows. In Section 2, we have given the required notations, definitions and some useful lemmas which are used throughout the paper. Results related to usual stochastic ordering between two independent heterogeneous samples with associated random shock are derived in Section 3. One application of the results is provided in Section 4. Finally, Section 5 concludes the paper.

Throughout the paper, the word increasing (resp. decreasing) and nondecreasing (resp. nonincreasing) are used interchangeably, and $\Re_+$ denotes the set of positive real numbers $\{x : 0 < x < \infty\}$. We also write $a \overset{sign}{=} b$ to mean that $a$ and $b$ have the same sign and $h^{-1}$ denotes inverse of the function $h$.

# 2 Notations, Definitions and Preliminaries

Let $X$ and $Y$ be two absolutely continuous random variables with survival functions $\overline{F}_X(\cdot)$ and $\overline{F}_Y(\cdot)$ respectively.

In order to compare different order statistics, stochastic orders are used for fair and reasonable comparison. Different kinds of stochastic orders are developed and studied in the literature. The following well known definitions may be obtained in Shaked and Shanthikumar (2007).

**Definition 2.1** *Let $X$ and $Y$ be two absolutely continuous rvs with respective supports $(l_X, u_X)$ and $(l_Y, u_Y)$, where $u_X$ and $u_Y$ may be positive infinity, and $l_X$ and $l_Y$ may be negative infinity. Then, $X$ is said to be smaller than $Y$ in usual stochastic (st) order, denoted as $X \leq_{st} Y$, if $\bar{F}_X(t) \leq \bar{F}_Y(t)$ for all $t \in (-\infty, \infty)$.*

It is well known that the results on different stochastic orders can be established on using majorization order(s). Let $I^n$ denote an $n$-dimensional Euclidean space where $I \subseteq \Re$. Further, let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in I^n$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in I^n$ be any two real vectors with $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ being the increasing arrangements of the components of the vector $\mathbf{x}$. The following definitions on vector majorization may be found in Marshall *et al.* (2008).

**Definition 2.2** *i) The vector $\mathbf{x}$ is said to majorize the vector $\mathbf{y}$ (written as $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if*

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)}, \ j = 1, \ 2, \ \ldots, n-1, \ \ and \ \ \sum_{i=1}^{n} x_{(i)} = \sum_{i=1}^{n} y_{(i)}.$$

*ii) The vector $\mathbf{x}$ is said to weakly supermajorize the vector $\mathbf{y}$ (written as $\mathbf{x} \overset{w}{\succeq} \mathbf{y}$) if*

$$\sum_{i=1}^{j} x_{(i)} \leq \sum_{i=1}^{j} y_{(i)} \quad for \ j = 1, 2, \ldots, n.$$

*iii) The vector $\mathbf{x}$ is said to weakly submajorize the vector $\mathbf{y}$ (written as $\mathbf{x} \succeq_w \mathbf{y}$) if*

$$\sum_{i=j}^{n} x_{(i)} \geq \sum_{i=j}^{n} y_{(i)} \quad for \ j = 1, 2, \ldots, n.$$

It is easy to show that $\mathbf{x} \overset{m}{\succeq} \mathbf{y} \Rightarrow \mathbf{x} \overset{w}{\succeq} \mathbf{y}$.

**Definition 2.3** *A function $\psi : I^n \to \Re$ is said to be Schur-convex (resp. Schur-concave) on $I^n$ if*

$$\mathbf{x} \overset{m}{\succeq} \mathbf{y} \text{ implies } \psi(\mathbf{x}) \geq (resp. \ \leq) \psi(\mathbf{y}) \text{ for all } \mathbf{x}, \ \mathbf{y} \in I^n.$$

The following definitions related to matrix majorization may be found in Marshall *et al.* [**?**].

**Definition 2.4**  *i) A square matrix $\Pi_n$, of order $n$, is said to be a permutation matrix if each row and column has a single entry as 1, and all other entries as zero.*

*ii) A square matrix $P = (p_{ij})$, of order $n$, is said to be doubly stochastic if $p_{ij} \geq 0$, for all $i, j = 1, ...n, \sum_{i=1}^{n} p_{ij} = 1, j = 1, ..., n$ and $\sum_{j=1}^{n} p_{ij} = 1, i = 1, ..., n$.*

*iii) A square matrix $T_n$, of order $n$, is said to be $T-$transform matrix if it has the form*

$$T_n = \lambda I_n + (1 - \lambda)\Pi_n; \ 0 \leq \lambda \leq 1,$$

*where $I_n$ is the identity matrix and $\Pi_n$ is the permutation matrix.*

**Definition 2.5** *Consider the $m \times n$ matrices $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ with rows $\mathbf{a_1}, ..., \mathbf{a_m}$ and $\mathbf{b_1}, ..., \mathbf{b_m}$, respectively.*

*i) A is said to be larger than B in chain majorization, denoted by $A >> B$, if there exists a finite set of $n \times n$ $T-$transform matrices $T_1, ..., T_k$ such that $B = AT_1T_2...T_k$.*

*ii) A is said to majorize B, denoted by $A > B$, if $A = BP$, where the $n \times n$ matrix $P$ is doubly stochastic. Since a product of $T-$transforms is doubly stochastic, it follows that $A >> B \Rightarrow A > B$.*

*iii) A is said to be larger than the matrix B in row majorization, denoted by $A >^{row} B$, if $\mathbf{a_i} \overset{m}{\succeq} \mathbf{b_i}$ for $i = 1, ..., m$. It is clear that $A > B \Rightarrow A >^{row} B$.*

*iv) A is said to be larger than the matrix B in row weakly majorization, denoted by $A >^w B$, if $\mathbf{a_i} \overset{w}{\succeq} \mathbf{b_i}$ for $i = 1, ..., m$. It is clear that $A >^{row} B \Rightarrow A >^w B$.*

Thus it can be written that

$$A >> B \Rightarrow A > B \Rightarrow A >^{row} B \Rightarrow A >^w B.$$

**Notation 2.1** *Let us introduce the following notations.*

*(i)* $\mathcal{D}_+ = \{(x_1, x_2, \ldots, x_n) : x_1 \geq x_2 \geq \ldots \geq x_n > 0\}.$

*(ii)* $\mathcal{E}_+ = \{(x_1, x_2, \ldots, x_n) : 0 < x_1 \leq x_2 \leq \ldots \leq x_n\}.$

*(iii)* $\mathcal{U}_n = \left\{ (\boldsymbol{x}, \boldsymbol{y}) = \begin{bmatrix} x_1 & x_2 & \cdot & \cdot & \cdot & x_n \\ y_1 & y_2 & \cdot & \cdot & \cdot & y_n \end{bmatrix} : (x_i - x_j)(y_i - y_j) \geq 0; i, j = 1, 2, \ldots n \right\}.$

*(iv)* $\mathcal{V}_n = \left\{ (\boldsymbol{x}, \boldsymbol{y}) = \begin{bmatrix} x_1 & x_2 & \cdot & \cdot & \cdot & x_n \\ y_1 & y_2 & \cdot & \cdot & \cdot & y_n \end{bmatrix} : (x_i - x_j)(y_i - y_j) \leq 0; i, j = 1, 2, \ldots n \right\}.$

Let us first introduce the following lemmas which will be used in the next section to prove the results.

**Lemma 2.1** (*Lemma 3.1 of Kundu* et al. *(2016)*) *Let* $\varphi : \mathcal{D}_+ \to \Re$ *be a function, continuously differentiable on the interior of* $\mathcal{D}_+$. *Then, for* $\mathbf{x}, \mathbf{y} \in \mathcal{D}_+$,

$$\mathbf{x} \overset{m}{\succeq} \mathbf{y} \text{ implies } \varphi(\mathbf{x}) \geq (\text{resp. } \leq) \varphi(\mathbf{y})$$

*if, and only if,*

$$\varphi_{(k)}(\mathbf{z}) \text{ is decreasing (resp. increasing) in } k = 1, 2, \ldots, n,$$

*where* $\varphi_{(k)}(\mathbf{z}) = \partial\varphi(\mathbf{z})/\partial z_k$ *denotes the partial derivative of* $\varphi$ *with respect to its kth argument.* $\square$

**Lemma 2.2** (*Lemma 3.3 of Kundu* et al. *(2016)*) *Let* $\varphi : \mathcal{E}_+ \to \Re$ *be a function, continuously differentiable on the interior of* $\mathcal{E}_+$. *Then, for* $\mathbf{x}, \mathbf{y} \in \mathcal{E}_+$,

$$\mathbf{x} \overset{m}{\succeq} \mathbf{y} \text{ implies } \varphi(\mathbf{x}) \geq (\text{resp. } \leq) \varphi(\mathbf{y})$$

*if, and only if,*

$$\varphi_{(k)}(\mathbf{z}) \text{ is increasing (resp. decreasing) in } k = 1, 2, \ldots, n,$$

*where* $\varphi_{(k)}(\mathbf{z}) = \partial\varphi(\mathbf{z})/\partial z_k$ *denotes the partial derivative of* $\varphi$ *with respect to its k-th argument.* $\square$

**Lemma 2.3** (*Theorem A.8 of Marshall* et al. *(2011) p.p. 87*) *Let* $S \subseteq \Re^n$. *Further, let* $\varphi : S \to \Re$ *be a function. Then for* $\mathbf{x}, \mathbf{y} \in S$,

$$\mathbf{x} \succeq_w \mathbf{y} \implies \varphi(\mathbf{x}) \geq (\text{resp. } \leq)\varphi(\mathbf{y})$$

*if, and if,* $\varphi$ *is both increasing (resp. decreasing) and Schur-convex (resp. Schur-concave) on* $S$. *Similarly,*

$$\mathbf{x} \overset{w}{\succeq} \mathbf{y} \implies \varphi(\mathbf{x}) \geq (\text{resp. } \leq)\varphi(\mathbf{y})$$

*if, and if,* $\varphi$ *is both decreasing (resp. increasing) and Schur-convex (resp. Schur-concave) on* $S$.

# 3 Main Results

For $i = 1, 2, \ldots, n$, let $T_i$ (resp. $W_i$) be $n$ independent nonnegative rvs following $Kw$-G distribution as given in (??). Under random shock, let us assume $X_i = T_i I_i$ and $Y_i = W_i I_i^*$, the cdf of $X_i$ and $Y_i$ are given by

$$F_{X_i}(x) = 1 - P(T_i I_i \geq t) = 1 - P(T_i I_i \geq t \mid I_i = 1)P(I_i = 1) = 1 - p_i \left(1 - (F(x))^{\alpha_i}\right)^{\beta_i}$$

and

$$F_{Y_i}(x) = 1 - P(W_i I_i^* \geq t) = 1 - P(W_i I_i^* \geq t \mid I_i^* = 1)P(I_i^* = 1) = 1 - p_i^* \left(1 - (F(x))^{\gamma_i}\right)^{\delta_i}$$

respectively, where $E(I_i) = p_i$ and $E(I_i^*) = p_i^*$.

If $F_{X_{n:n}}(\cdot)\left(G_{Y_{n:n}}(\cdot)\right)$ and $\overline{F}_{X_{1:n}}(\cdot)\left(\overline{G}_{Y_{1:n}}(\cdot)\right)$ be the cdf and the survival function of $X_{n:n}(Y_{n:n})$ and $X_{1:n}(Y_{1:n})$ respectively, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ and $\boldsymbol{\delta} = (\delta_1, \delta_2, \ldots, \delta_n)$, then from (??) it can be written that

$$F_{X_{n:n}}(x) = \prod_{i=1}^{n} F_{X_i}(x) = \prod_{i=1}^{n} \left[1 - p_i \left(1 - (F(x))^{\alpha_i}\right)^{\beta_i}\right], \tag{3.1}$$

and

$$G_{Y_{n:n}}(t) = \prod_{i=1}^{n} F_{Y_i}(x) = \prod_{i=1}^{n} \left[1 - p_i^* \left(1 - (F(x))^{\gamma_i}\right)^{\delta_i}\right].$$

The first two theorems show that usual stochastic ordering exists between the largest order statistics of two independent and heterogeneous samples with associated random shocks for fixed $\boldsymbol{\alpha}$. Theorem ?? (Theorem ??) guarantees that the largest order statistic of sample 1 is stochastically larger than that of sample 2 with common heterogeneous shape parameter vectors (common $\boldsymbol{\alpha}$), when the shock parameter vector (shape parameter vector $\boldsymbol{\beta}$) of sample 1 majorizes that of sample 2.

**Theorem 3.1** *For $i = 1, 2, \ldots, n$, let $T_i$ and $W_i$ be two sets of mutually independent random variables with $T_i \sim Kw-G(\alpha_i, \beta_i, F)$ and $W_i \sim Kw-G(\alpha_i, \beta_i, F)$. Further, suppose that $I_i$ $(I_i^*)$ be a set of independent Bernoulli rvs, independent of $T_i$'s ($W_i$'s) with $E(I_i) = p_i$ $(E(I_i^*) = p_i^*), i = 1, 2, ..., n$. If $h : [0, 1] \rightarrow \Re_+$ is a differentiable, decreasing, and strictly convex function, then $\boldsymbol{h(p)} \overset{w}{\succeq} \boldsymbol{h(p^*)}$ implies $X_{n:n} \geq_{st} Y_{n:n}$ if $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{V}_n$ and $(\boldsymbol{\beta}, \boldsymbol{h(p)}), (\boldsymbol{\beta}, \boldsymbol{h(p^*)}) \in \mathcal{U}_n$, where $\boldsymbol{h(p)} = (h(p_1), h(p_2), \ldots, h(p_n))$.*

**Proof:** In view of the expression (??)

$$F_{X_{n:n}}(x) = \prod_{i=1}^{n} \left[1 - h^{-1}(u_i)\left(1 - (F(x))^{\alpha_i}\right)^{\beta_i}\right] = \Psi(\boldsymbol{u}, \boldsymbol{\alpha}, \boldsymbol{\beta})(say),$$

where $h(p_i) = u_i$. Differentiating $\Psi(\boldsymbol{u})$ partially, with respect to $u_i$, we get

$$\frac{\partial \Psi}{\partial u_i} = \left[ -\frac{dh^{-1}(u_i)}{du_i} \left(1 - (F(x))^{\alpha_i}\right)^{\beta_i} \right] \prod_{k \neq i=1}^{n} \left[1 - h^{-1}(u_k)\left(1 - (F(x))^{\alpha_k}\right)^{\beta_k}\right] \geq 0,$$

since $h(u)$ is decreasing in $u$. So, $\Psi$ is increasing in each $u_i$. Again,

$$\frac{\partial \Psi}{\partial u_i} - \frac{\partial \Psi}{\partial u_j} \overset{sign}{=} \left[ \left\{ -\frac{dh^{-1}(u_j)}{du_j}\left(1 - (F(x))^{\alpha_j}\right)^{\beta_j} \right\} \left\{ 1 - h^{-1}(u_i)\left(1 - (F(x))^{\alpha_i}\right)^{\beta_i} \right\} \right]$$
$$- \left[ \left\{ -\frac{dh^{-1}(u_i)}{du_i}\left(1 - (F(x))^{\alpha_i}\right)^{\beta_i} \right\} \left\{ 1 - h^{-1}(u_j)\left(1 - (F(x))^{\alpha_j}\right)^{\beta_j} \right\} \right].$$
(3.2)

Considering the fact that, $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{V}_n$ and $(\boldsymbol{\beta}, \boldsymbol{h(p)}) \in \mathcal{U}_n$, for $i \leq j$, let us consider $\alpha_i \geq \alpha_j$, $\beta_i \leq \beta_j$ and $u_i \leq u_j$. So, for all $x \geq 0$ it can be written that $(1 - (F(x))^{\alpha_i})^{\beta_i} \geq (1 - (F(x))^{\alpha_j})^{\beta_j}$.

Again, as $h(u)$ is decreasing and convex in $u$, $u_i \leq u_j$ gives $\frac{dh^{-1}(u_i)}{du_i} \leq \frac{dh^{-1}(u_j)}{du_j}$, which yields

$$-\left(1 - (F(x))^{\alpha_i}\right)^{\beta_i} \frac{dh^{-1}(u_i)}{du_i} \geq -\left(1 - (F(x))^{\alpha_j}\right)^{\beta_j} \frac{dh^{-1}(u_j)}{du_j}. \tag{3.3}$$

Again, as $u_i \leq u_j$ and $h(u)$ is decreasing in $u$, it is easy to show that $h^{-1}(u_i) \geq h^{-1}(u_j)$, which in turn implies that

$$1 - h^{-1}(u_i)\left(1 - (F(x))^{\alpha_i}\right)^{\beta_i} \leq 1 - h^{-1}(u_j)\left(1 - (F(x))^{\alpha_j}\right)^{\beta_j}. \tag{3.4}$$

Substituting the results (**??**) and (**??**) in (**??**), we get $\frac{\partial \Psi}{\partial u_i} - \frac{\partial \Psi}{\partial u_j} \geq 0$. Thus by Lemma **??** it can be proved that $\Psi$ is Schur-concave in $\boldsymbol{u}$.

Again, for $i \leq j$, if $\alpha_i \leq \alpha_j$, $\beta_i \geq \beta_j$ and $u_i \geq u_j$ are taken, then proceeding in the same line and using Lemma **??**, it can be proved that $\Psi$ is Schur-concave in $\boldsymbol{u}$.

Thus the result is proved by Lemma **??**. $\qquad\square$


**Theorem 3.2** *For $i = 1, 2, \ldots, n$, let $T_i$ and $W_i$ be two sets of mutually independent random variables with $T_i \sim Kw\text{-}G(\alpha, \beta_i, F)$ and $W_i \sim Kw\text{-}G(\alpha, \delta_i, F)$. Further, suppose that $I_i$ be a set of independent Bernoulli rvs, independent of $T_i$'s ($W_i$'s) with $E(I_i) = p_i, i = 1, 2, ..., n$. If $h : [0,1] \to \Re_+$ is a differentiable, and decreasing function, then $\boldsymbol{\beta} \overset{w}{\succeq} \boldsymbol{\delta}$ implies $X_{n:n} \geq_{st} Y_{n:n}$ if $(\boldsymbol{\beta}, \boldsymbol{h(p)}), (\boldsymbol{\delta}, \boldsymbol{h(p)}) \in \mathcal{U}_n$.*

**Proof:** From (**??**), let us assume that

$$\Psi_1(\boldsymbol{\beta}, \boldsymbol{h(p)}) = \prod_{i=1}^{n} \left[1 - h^{-1}(u_i)\left(1 - (F(x))^{\alpha}\right)^{\beta_i}\right],$$

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where $h(p_i) = u_i$. Differentiating $\Psi_1$ partially, with respect to $\beta_i$ we get

$$\frac{\partial \Psi_1}{\partial \beta_i} = \left[ -h^{-1}(u_i)\left(1 - (F(x))^\alpha\right)^{\beta_k} \log\left(1 - (F(x))^\alpha\right) \right] \prod_{k \neq i=1}^{n} \left[ 1 - h^{-1}(u_k)\left(1 - (F(x))^\alpha\right)^{\beta_k} \right] \geq 0.$$

So, $\Psi_1$ is increasing in each $\beta_i$. Again, it can be shown that

$$\frac{\partial \Psi_1}{\partial \beta_i} - \frac{\partial \Psi_1}{\partial \beta_j} \stackrel{sign}{=} \left[ \frac{h^{-1}(u_j)\left(1 - (F(x))^\alpha\right)^{\beta_j} \log\left(1 - (F(x))^\alpha\right)}{1 - h^{-1}(u_j)\left(1 - (F(x))^\alpha\right)^{\beta_j}} - \frac{h^{-1}(u_i)\left(1 - (F(x))^\alpha\right)^{\beta_i} \log\left(1 - (F(x))^\alpha\right)}{1 - h^{-1}(u_i)\left(1 - (F(x))^\alpha\right)^{\beta_i}} \right].$$
$$(3.5)$$

Now,

$$\frac{\partial}{\partial \beta} \left( \frac{\left(1 - (F(x))^\alpha\right)^\beta}{1 - h^{-1}(u)\left(1 - (F(x))^\alpha\right)^\beta} \right) \stackrel{sign}{=} \left(1 - (F(x))^\alpha\right)^\beta \log\left(1 - (F(x))^\alpha\right) \leq 0,$$

implying that $\frac{(1-(F(x))^\alpha)^\beta}{1-h^{-1}(u)(1-(F(x))^\alpha)^\beta}$ is decreasing in $\beta$. Again, as $h(u)$ is decreasing in $u$, then

$$\frac{\partial}{\partial u} \left( \frac{h^{-1}(u)}{1 - h^{-1}(u)\left(1 - (F(x))^\alpha\right)^\beta} \right) = \frac{\partial h^{-1}(u)}{\partial u} \leq 0,$$

giving that $\frac{h^{-1}(u)}{1-h^{-1}(u)(1-(F(x))^\alpha)^\beta}$ is decreasing in $u$. Thus, as $(\boldsymbol{\beta}, \boldsymbol{h(p)}) \in \mathcal{U}_n$, for $i \leq j$ taking $\beta_\leq \beta_j$ and $u_i \leq u_j$ and noticing the fact that $h^{-1}(u)$ is decreasing in $u$, it can be written that

$$\frac{h^{-1}(u_j)\left(1 - (F(x))^\alpha\right)^{\beta_j}}{1 - h^{-1}(u_j)\left(1 - (F(x))^\alpha\right)^{\beta_j}} \leq \frac{h^{-1}(u_j)\left(1 - (F(x))^\alpha\right)^{\beta_i}}{1 - h^{-1}(u_j)\left(1 - (F(x))^\alpha\right)^{\beta_i}} \leq \frac{h^{-1}(u_i)\left(1 - (F(x))^\alpha\right)^{\beta_i}}{1 - h^{-1}(u_i)\left(1 - (F(x))^\alpha\right)^{\beta_i}},$$

which implies

$$\frac{h^{-1}(u_j)\left(1 - (F(x))^\alpha\right)^{\beta_j} \log\left(1 - (F(x))^\alpha\right)}{1 - h^{-1}(u_j)\left(1 - (F(x))^\alpha\right)^{\beta_j}} \geq \frac{h^{-1}(u_i)\left(1 - (F(x))^\alpha\right)^{\beta_i} \log\left(1 - (F(x))^\alpha\right)}{1 - h^{-1}(u_i)\left(1 - (F(x))^\alpha\right)^{\beta_i}}.$$

Hence, from (**??**), we get $\frac{\partial \Psi_1}{\partial \beta_i} - \frac{\partial \Psi_1}{\partial \beta_j} \geq 0$. Thus by Lemma **??** it can be proved that $\Psi$ is Schur-concave in $\boldsymbol{u}$. Again, for $i \leq j$, if $\beta_i \geq \beta_j$ and $u_i \geq u_j$ are taken, then also using Lemma **??**, it can be proved that $\Psi$ is Schur-concave in $\boldsymbol{u}$.

Thus the result is proved by Lemma **??**. $\qquad\square$

Now the question arises— what will happen if the matrix of the shape $(\boldsymbol{\beta})$ and the shock parameters of one sample majorizes the other when the other shape parameter $(\alpha)$ remains constant? The theorem given below answers that the majorized matrix of the parameters leads to better performance of the sample. Combining Theorems **??** and **??**, the following theorem can be obtained.

**Theorem 3.3** *For $i = 1, 2, \ldots, n$, let $T_i$ and $W_i$ be two sets of mutually independent random variables with $T_i \sim Kw-G(\alpha, \beta_i, F)$ and $W_i \sim Kw-G(\alpha, \delta_i, F)$. Further, suppose that $I_i$ $(I_i^*)$ be a set of independent Bernoulli rvs, independent of $X_i$'s $(Y_i$'s) with $E(I_i) = p_i$ $(E(I_i^*) = p_i^*)$, $i = 1, 2, \ldots, n$ and $h : [0, 1] \rightarrow \Re_+$ is a differentiable, decreasing and convex function. If $(\boldsymbol{\beta}, \boldsymbol{h(p)}) \in \mathcal{U}_n$, and $(\boldsymbol{\delta}, \boldsymbol{h(p^*)}) \in \mathcal{U}_n$, then*

$$\begin{bmatrix} \boldsymbol{h(p)} \\ \boldsymbol{\beta} \end{bmatrix} >^w \begin{bmatrix} \boldsymbol{h(p^*)} \\ \boldsymbol{\delta} \end{bmatrix} \quad \text{implies } X_{n:n} \geq_{st} Y_{n:n}.$$

The Counterexample, given below shows that the conditions $(\boldsymbol{\beta}, \boldsymbol{h(p)}) \in \mathcal{U}_n$, and $(\boldsymbol{\delta}, \boldsymbol{h(p^*)}) \in \mathcal{U}_n$ are necessary conditions for the result of Theorem **??** to hold.

**Counterexample 3.1** *For fixed $\alpha = 2.0$ and $h(p) = -\log(p)$, let $\boldsymbol{\beta} = (3, 2, 1)$, $\boldsymbol{\delta} = (2.5, 2.5, 1)$, $\boldsymbol{h(p)} = (1, 2, 3)$ and $\boldsymbol{h(p^*)} = (1, 2.5, 2.5)$. So, $(\boldsymbol{\beta}, \boldsymbol{h(p)}) \in \mathcal{V}_n$, and $(\boldsymbol{\delta}, \boldsymbol{h(p^*)}) \in \mathcal{V}_n$ and*

$$\begin{bmatrix} \boldsymbol{h(p)} \\ \boldsymbol{\beta} \end{bmatrix} >^w \begin{bmatrix} \boldsymbol{h(p^*)} \\ \boldsymbol{\delta} \end{bmatrix}.$$

*But, Figure **??** shows there exists no stochastic ordering between $X_{n:n}$ and $Y_{n:n}$.*
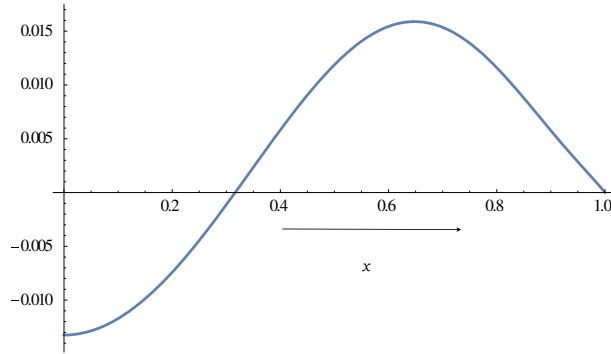


Figure 1: Graph of $F_{3:3}(x) - G_{3:3}(x)$

The next result can be easily concluded from the previous theorem.

**Theorem 3.4** *For $i = 1, 2, \ldots, n$, let $T_i$ and $W_i$ be two sets of mutually independent random variables with $T_i \sim Kw-G(\alpha, \beta_i, F)$ and $W_i \sim Kw-G(\alpha, \delta_i, F)$. Further, suppose that $I_i$ $(I_i^*)$ be a set of independent Bernoulli rvs, independent of $X_i$'s $(Y_i$'s) with $E(I_i) =$*

$p_i$ $(E(I_i^*) = p_i^*)$, $i = 1, 2, ..., n$ and $h : [0,1] \to \Re_+$ is a differentiable, decreasing and convex function. If $(\boldsymbol{\beta}, \boldsymbol{h}(\boldsymbol{p})) \in \mathcal{U}_n$, and $(\boldsymbol{\delta}, \boldsymbol{h}(\boldsymbol{p^*})) \in \mathcal{U}_n$, then

$$\begin{bmatrix} \boldsymbol{h}(\boldsymbol{p}) \\ \boldsymbol{\beta} \end{bmatrix} >> \begin{bmatrix} \boldsymbol{h}(\boldsymbol{p^*}) \\ \boldsymbol{\delta} \end{bmatrix} \text{ implies } X_{n:n} \geq_{st} Y_{n:n}.$$

Although Theorem **??** shows that there exists stochastic ordering between $X_{n:n}$ and $Y_{n:n}$ when $(\boldsymbol{\beta}, \boldsymbol{h}(\boldsymbol{p}))$ and $(\boldsymbol{\delta}, \boldsymbol{h}(\boldsymbol{p^*}))$ are ordered in the sense of row weakly majorization, the next counterexample shows that no such ordering exists between $X_{n:n}$ and $Y_{n:n}$ when $(\boldsymbol{\alpha}, \boldsymbol{h}(\boldsymbol{p}))$ and $(\boldsymbol{\alpha^*}, \boldsymbol{h}(\boldsymbol{p^*}))$ are ordered in the same sense, keeping $\beta$ as fixed.
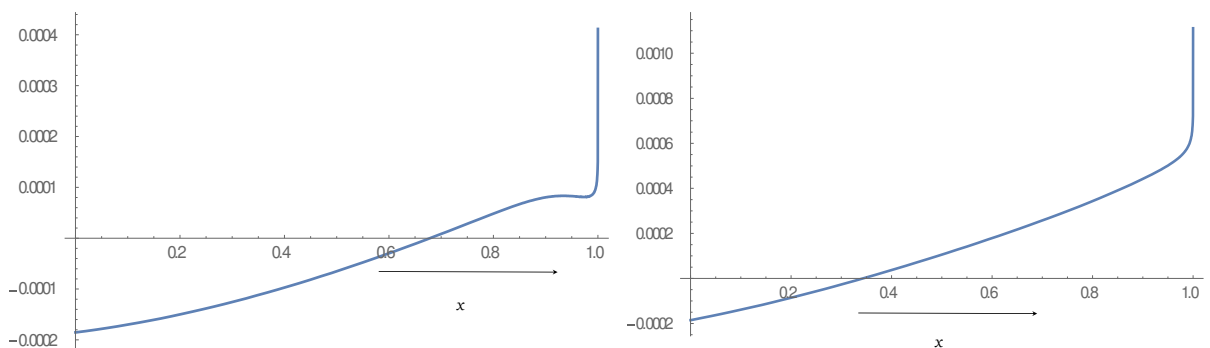
**Counterexample 3.2** *For $\beta = 0.01$ and $h(p) = -\log(p)$, let $\boldsymbol{\alpha} = (1, 2, 30)$, $\boldsymbol{\alpha^*} = (3, 5, 25)$, $\boldsymbol{h}(\boldsymbol{p}) = (0.3, 0.2, 0.1)$ and $\boldsymbol{h}(\boldsymbol{p^*}) = (0.25, 0.25, 0.1)$. So, clearly $(\boldsymbol{\alpha}, \boldsymbol{h}(\boldsymbol{p})) \in \mathcal{V}_n$, $(\boldsymbol{\alpha^*}, \boldsymbol{h}(\boldsymbol{p^*})) \in \mathcal{V}_n$ and*

$$\begin{bmatrix} \boldsymbol{h}(\boldsymbol{p}) \\ \boldsymbol{\alpha} \end{bmatrix} >^w \begin{bmatrix} \boldsymbol{h}(\boldsymbol{p^*}) \\ \boldsymbol{\alpha^*} \end{bmatrix}.$$

*But, Figure **??**(a) shows that there exists no stochastic ordering between $X_{3:3}$ and $Y_{3:3}$. Again for the same $\beta$ and $h(p)$, if $\boldsymbol{\alpha} = (1, 2, 30)$, $\boldsymbol{\alpha^*} = (3, 5, 25)$, $\boldsymbol{h}(\boldsymbol{p}) = (0.1, 0.2, 0.3)$ and $\boldsymbol{h}(\boldsymbol{p^*}) = (0.1, 0.25, 0.25)$ are taken then clearly $(\boldsymbol{\alpha}, \boldsymbol{h}(\boldsymbol{p})) \in \mathcal{U}_n$, $(\boldsymbol{\alpha^*}, \boldsymbol{h}(\boldsymbol{p^*})) \in \mathcal{U}_n$ and*

$$\begin{bmatrix} \boldsymbol{h}(\boldsymbol{p}) \\ \boldsymbol{\alpha} \end{bmatrix} >^w \begin{bmatrix} \boldsymbol{h}(\boldsymbol{p^*}) \\ \boldsymbol{\alpha^*} \end{bmatrix}.$$

*But, Figure **??**(b) shows that there exists no stochastic ordering between $X_{3:3}$ and $Y_{3:3}$.*



(a) For $(\boldsymbol{\alpha}, \boldsymbol{h}(\boldsymbol{p})), (\boldsymbol{\alpha^*}, \boldsymbol{h}(\boldsymbol{p^*})) \in \mathcal{V}_n$      (b) For $(\boldsymbol{\alpha}, \boldsymbol{h}(\boldsymbol{p})), (\boldsymbol{\alpha^*}, \boldsymbol{h}(\boldsymbol{p^*})) \in \mathcal{U}_n$

Figure 2: Graph of $F_{3:3}(x) - G_{3:3}(x)$

Next, we compare the largest order statistics of two heterogeneous samples from $Kw$-G random variables with different homogenous parent cdfs. Let $X_1$ and $X_2$ be two random variables with continuous distribution functions $F_1(\cdot)$ and $F_2(\cdot)$ and density functions $f_1(\cdot)$ and $f_2(\cdot)$ respectively. Also suppose that $T_i \sim Kw\text{-G}(\alpha_i, \beta_i, F_1)$ and $W_i \sim Kw\text{-G}(\gamma_i, \delta_i, F_2)$ $(i = 1, 2, \ldots, n)$ be two sets of $n$ independent random variables. Therefore, for all $x \geq 0$

$$F_{X_{n:n}}(x) = \prod_{i=1}^{n} \left[ 1 - p_i \left( 1 - (F_1(x))^{\alpha_i} \right)^{\beta_i} \right], \tag{3.6}$$

and

$$G_{Y_{n:n}}(t) = \prod_{i=1}^{n} \left[ 1 - p_i^* \left( 1 - (F_2(x))^{\gamma_i} \right)^{\delta_i} \right].$$

The next theorems also show that under certain conditions on parameters, usual stochastic ordering between $X_1$ and $X_2$ implies the same between $X_{n:n}$ and $Y_{n:n}$.

**Theorem 3.5** *For $i = 1, 2, \ldots, n$, let $T_i$ and $W_i$ be two sets of mutually independent random variables with $T_i \sim Kw-G\left(\alpha_i, \beta_i, F_1\right)$ and $W_i \sim Kw-G\left(\alpha_i, \beta_i, F_2\right)$. Further, suppose that $I_i$ $(I_i^*)$ be a set of independent Bernoulli rvs, independent of $T_i$'s $(W_i$'s) with $E(I_i) = p_i$ $(E(I_i^*) = p_i^*), i = 1, 2, ..., n$. If $h : [0, 1] \to \Re_+$ is a differentiable, decreasing, and strictly convex function, then $X_1 \geq_{st} X_2$ and $\boldsymbol{h(p)} \overset{w}{\succeq} \boldsymbol{h(p^*)}$ implies $X_{n:n} \geq_{st} Y_{n:n}$ for $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{V}_n$ and $(\boldsymbol{\beta}, \boldsymbol{h(p)}), (\boldsymbol{\beta}, \boldsymbol{h(p^*)}) \in \mathcal{U}_n$.*

**Proof:** Let us consider another random variable $Z_i$ such that $Z_i \sim Kw\text{-G}(\alpha_i, \beta_i, F_1)$. As $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathcal{V}_n$ and $(\boldsymbol{\beta}, \boldsymbol{h(p)}), (\boldsymbol{\beta}, \boldsymbol{h(p^*)}) \in \mathcal{U}_n$, from Theorem **??**, it can be shown that $\boldsymbol{h(p)} \overset{w}{\succeq} \boldsymbol{h(p^*)}$ implies $X_{n:n} \geq_{st} Z_{n:n}$. Thus, by the definition of usual stochastic ordering, it can be written that

$$\prod_{i=1}^{n} \left[ 1 - h^{-1}(u_i) \left( 1 - (F_1(x))^{\alpha_i} \right)^{\beta_i} \right] \leq \prod_{i=1}^{n} \left[ 1 - h^{-1}(u_i^*) \left( 1 - (F_1(x))^{\alpha_i} \right)^{\beta_i} \right]. \tag{3.7}$$

Now, $X_1 \geq_{st} X_2$ implies $F_1(x) \leq F_2(x) \, \forall x$, which yields $\left( 1 - (F_1(x))^{\alpha_i} \right)^{\beta_i} \geq \left( 1 - (F_2(x))^{\alpha_i} \right)^{\beta_i}$, which in turn implies that

$$1 - h^{-1}(u_i^*) \left( 1 - (F_1(x))^{\alpha_i} \right)^{\beta_i} \leq 1 - h^{-1}(u_i^*) \left( 1 - (F_2(x))^{\alpha_i} \right)^{\beta_i}. \tag{3.8}$$

So from (**??**) and (**??**) it can be written that

$$\prod_{i=1}^{n} \left[ 1 - h^{-1}(u_i) \left( 1 - (F_1(x))^{\alpha_i} \right)^{\beta_i} \right] \leq \prod_{i=1}^{n} \left[ 1 - h^{-1}(u_i^*) \left( 1 - (F_2(x))^{\alpha_i} \right)^{\beta_i} \right].$$

This proves the result. □

**Theorem 3.6** *For $i = 1, 2, \ldots, n$, let $T_i$ and $W_i$ be two sets of mutually independent random variables with $T_i \sim Kw-G(\alpha, \beta_i, F_1)$ and $W_i \sim Kw-G(\alpha, \delta_i, F_2)$. Further, suppose that $I_i$ be a set of independent Bernoulli rvs, independent of $T_i$'s ($W_i$'s) with $E(I_i) = p_i, i = 1, 2, ..., n$. If $h : [0, 1] \rightarrow \Re_+$ is a differentiable, and decreasing function, then $X_1 \geq_{st} X_2$, and $\boldsymbol{\beta} \overset{w}{\succeq} \boldsymbol{\delta}$ implies $X_{n:n} \geq_{st} Y_{n:n}$ if $(\boldsymbol{\beta}, \boldsymbol{h(p)}), (\boldsymbol{\delta}, \boldsymbol{h(p)}) \in \mathcal{U}_n$.*

**Proof:** Considering $Z_i \sim Kw\text{-}G(\alpha, \delta_i, F_1)$, and using the same logic as in Theorem **??** the result can be proved with the help of Theorem **??**. □

Combining Theorems **??** and **??**, the following theorem on row weakly majorization is obtained.

**Theorem 3.7** *For $i = 1, 2, \ldots, n$, let $T_i$ and $W_i$ be two sets of mutually independent random variables with $T_i \sim Kw-G(\alpha, \beta_i, F_1)$ and $W_i \sim Kw-G(\alpha, \delta_i, F_2)$. Further, suppose that $I_i$ ($I_i^*$) be a set of independent Bernoulli rvs, independent of $X_i$'s ($Y_i$'s) with $E(I_i) = p_i$ ($E(I_i^*) = p_i^*$), $i = 1, 2, ..., n$ and $h : [0, 1] \rightarrow \Re_+$ is a differentiable, decreasing and convex function. If $(\boldsymbol{\beta}, \boldsymbol{h(p)}) \in \mathcal{U}_n$, and $(\boldsymbol{\delta}, \boldsymbol{h(p^*)}) \in \mathcal{U}_n$, then $X_1 \geq_{st} X_2$, and*

$$\begin{bmatrix} \boldsymbol{h(p)} \\ \boldsymbol{\beta} \end{bmatrix} >^w \begin{bmatrix} \boldsymbol{h(p^*)} \\ \boldsymbol{\delta} \end{bmatrix} \text{ implies } X_{n:n} \geq_{st} Y_{n:n}.$$

The next result is an immediate consequence of the previous theorem.

**Theorem 3.8** *For $i = 1, 2, \ldots, n$, let $T_i$ and $W_i$ be two sets of mutually independent random variables with $T_i \sim Kw-G(\alpha, \beta_i, F_1)$ and $W_i \sim Kw-G(\alpha, \delta_i, F_2)$. Further, suppose that $I_i$ ($I_i^*$) be a set of independent Bernoulli rvs, independent of $X_i$'s ($Y_i$'s) with $E(I_i) = p_i$ ($E(I_i^*) = p_i^*$), $i = 1, 2, ..., n$ and $h : [0, 1] \rightarrow \Re_+$ is a differentiable, decreasing and convex function. If $(\boldsymbol{\beta}, \boldsymbol{h(p)}) \in \mathcal{U}_n$, and $(\boldsymbol{\delta}, \boldsymbol{h(p^*)}) \in \mathcal{U}_n$, then $X_1 \geq_{st} X_2$, and*

$$\begin{bmatrix} \boldsymbol{h(p)} \\ \boldsymbol{\beta} \end{bmatrix} >> \begin{bmatrix} \boldsymbol{h(p^*)} \\ \boldsymbol{\delta} \end{bmatrix} \text{ implies } X_{n:n} \geq_{st} Y_{n:n}.$$

# 4 Application

The implications of the results as derived in the previous section are explained here with the help of one application. The application is provided considering Weibull as the parent cdf of the $Kw$-G distribution. Cordeiro *et al.* (2011) have shown that the $Kw$-W distribution fits lifetime data better than some known existing lifetime distributions such

as generalized Weibull, generalied exponential and Weibull distributions. The data, as taken from Meeker and Escobar (1998) is described as the times of failure and running times for a sample of devices from a field-tracking study of a larger system. At a certain point in time, thirty units were installed in normal service conditions. Two causes of failure were observed for each unit that failed: the failure caused by an accumulation of randomly occurring damage from power-line voltage spikes during electric storms and the failure caused by normal product wear. The $Kw$-W distribution is found to perform best in terms of Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Consistent Akaike Information Criterion (CAIC) along with Kolmogorov-Smirnov (K-S) distance. The following application of $Kw$-W distribution as a lifetime model explains the implication of the results.

Suppose there are two parallel systems consisting of $n$ independent and heterogeneous components each. We are interested to compare the performance of the systems stochastically when each component of the systems experiences a random shock which may cause the component(s) to fail. Assume further that the lifetimes of the components are distributed as $Kw$-W with cdf of the $i$th component as given by

$$G(x) = 1 - [1 - \{1 - exp(-(\lambda x)^\gamma)\}^{\alpha_i}]^{\beta_i}, \quad x > 0, \ \alpha_i, \beta_i, \lambda, \gamma > 0 \ \forall \ i = 1, 2, ..., n. \quad (4.1)$$

Following equation (**??**), let $T_i$ and $W_i$ ($i = 1, 2, ..., n$) be the lifetime of the $i$th component of the systems 1 and 2 respectively with $T_i \sim Kw\text{-W}(\alpha, \beta_i, \lambda, \gamma)$ and $W_i \sim Kw\text{-}$ W$(\alpha, \delta_i, \theta, \gamma)$. Now, each component of the systems 1 and 2 receives a shock at random which may cause the component to fail with probability $p_i$ and $p_i^*$ respectively. If $I_i$ denote independent Bernoulli rvs, independent of the $T_i$s, with $E(I_i) = p_i$, the rv $X_i = I_i T_i$ corresponds to the lifetime of the $i$th component of the system 1 under shock. Theorem **??** guarantees that for parallel systems of components having independent $Kw$-Weibull distributed lifetimes with one of the shape parameter ($\boldsymbol{\alpha}$) vector common, the other majorized shape and shock parameter vectors lead to a larger systems lifetime in usual stochastic ordering for $\left(\frac{\lambda}{\theta}\right)^\gamma \leq 1$. In other words, for fixed parameter vector $\boldsymbol{\alpha}$, if the shape and shock parameter vectors $(\boldsymbol{\beta}, \boldsymbol{h}(\boldsymbol{p}))$ of system 1 are more dispersed than the same $(\boldsymbol{\delta}, \boldsymbol{h}(\boldsymbol{p^*}))$, of system 2 (although average is the same for both the vectors), then the lifetime of system 1 will be larger than that of system 2 keeping the other conditions same as mentioned earlier.

# 5   Concluding Remarks

Parallel systems (maximum order statistic) being one of the building blocks of many complex coherent systems are always required to be compared stochastically in the

context of reliability optimization and life testing experiments. Such comparisons are generally carried out with the assumption that the components of the system fail with certainty. In practice, the components may experience random shocks which eventually doesn't guarantee its failure. This paper compares the maximum order statistics of two independent and heterogeneous samples from $Kw$-G distribution with associated random shock. It is proved that for two samples with common shape parameter vector ($\boldsymbol{\alpha}$), the majorized matrix of the shape ($\boldsymbol{\beta}$) and shock parameters ($\boldsymbol{h(p)}$) leads to better system reliability. It is also shown through counter examples that no such results exist when the matrix of shape ($\boldsymbol{\alpha}$) and shock parameters ($\boldsymbol{h(p)}$) of one system majorizes the same of the other for fixed $\boldsymbol{\beta}$. The results of this paper are applicable to a wide variety of distributions generated from $Kw$ distribution through the cdf $F$ as discussed in the Introduction, viz. $Kw$-N, $Kw$-W, $Kw$-Ga, $Kw$-Gu etc. Results related to ordering properties of maximum order statistics of two independent and heterogeneous samples from $Kw$-G distribution can be easily derived from the current results by taking $p_1 = p_2 = ... = p_n$.

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