

## Ordering two-qubit states with concurrence and negativity

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We study the ordering of two-qubit states with respect to the degree of bipartite entanglement using the Wootters concurrence—a measure of the entanglement of formation—and the negativity—a measure of the entanglement cost under the positive-partial-transpose-preserving operations. For two-qubit pure states, the negativity is the same as the concurrence. However, we demonstrate analytically on simple examples of various mixtures of Bell and separable states that the entanglement measures can impose different orderings on the states. We show which states, in general, give the maximally different predictions (i) when one of the states has the concurrence greater but the negativity smaller than those for the other state and (ii) when the states are entangled to the same degree according to one of the measures, but differently according to the other.

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The entropy of entanglement is essentially the unique measure of entanglement for pure states of bipartite systems [1]. By contrast, a generalization of the entropy of entanglement to describe mixed states is by no means unique (even for two qubits), leading to various entanglement measures (for a review see [2]), including entanglement of formation, distillable entanglement [3,4], and relative entropy of entanglement [5]. The quantification of mixed-state entanglement is still in early stages with many open questions.

Eisert and Plenio [6] raised an intriguing problem of ordering the density operators with respect to the amount of entanglement. Specifically, certain two entanglement measures  $E'$  and  $E''$  are defined to give the same state ordering if the condition [6]

$$E'(\rho_1) < E'(\rho_2) \Leftrightarrow E''(\rho_1) < E''(\rho_2) \quad (1)$$

is satisfied for any density operators  $\rho_1$  and  $\rho_2$ . No counterexample to Eq. (1) can be found by comparing pure states only or pure and Werner states [6]. However, the standard entanglement measures do not give the same ordering in the sets of two-qubit mixed states, as first observed by applying Monte Carlo simulations by Eisert and Plenio [6] and then investigated by others in Refs. [7–12]. Counterexamples to Eq. (1) can also be constructed for  $d$ -level qudit pure states if  $d \geq 3$  as shown by Życzkowski and Bengtsson [9]. Virmani and Plenio [8] proved that all good asymptotic entanglement measures, which reduce to the entropy of entanglement for pure states, are either equivalent or do not have the same state ordering. The property that ordering of some states depends on the applied measures of entanglement “in itself is a very surprising conclusion” [8] but is physically reasonable as these incomparable states cannot be transformed to each other with unit efficiency by any local operations and classical communication (LOCC).

In this paper, we present an analysis of different orderings of two-qubit states induced by concurrence and negativity. The ordering of two-qubit states by the same measures has already been studied by Eisert and Plenio [6], but by using only a numerical simulation (of  $10^4$  pairs of entangled states  $\{\rho_1, \rho_2\}$ ). For three qubits, analytical counterexamples to Eq. (1) are known even for pure states [9,11].

First, we briefly describe the entanglement measures important for our comparison.

The entanglement of formation  $E_F$  of a mixed state  $\rho$ , according to Bennett *et al.* [3,4], is the minimized average entanglement of any ensemble of pure states  $|\psi_i\rangle$  realizing  $\rho$ :

$$E_F(\rho) = \inf \sum_i p_i E(|\psi_i\rangle\langle\psi_i|), \quad (2)$$

where the infimum is taken over all pure-state decompositions  $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$  and  $E(|\psi_i\rangle\langle\psi_i|)$  is the entropy of entanglement easily determined by the von Neumann entropy. In a special case of two qubits, Wootters [14] proved that the entanglement of formation of a state  $\rho$  is given by a simple formula

$$E_F(\rho) = H\left(\frac{1}{2}[1 + \sqrt{1 - C^2(\rho)}]\right), \quad (3)$$

where  $H(x) = -x \log_2 x - (1-x) \log_2 (1-x)$  is the binary entropy with the argument related to the Wootters concurrence defined by

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \quad (4)$$

where the  $\lambda_i$ 's are (in nonincreasing order) the square roots of the eigenvalues of  $\rho(\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y)$ ;  $\sigma_y$  is the Pauli spin matrix and complex conjugation is denoted by an asterisk. Both  $E_F(\rho)$  and  $C(\rho)$  range from 0 for a separable state to 1 for a maximally entangled state.

We also consider another entanglement measure referred to as the negativity, which can be considered a quantitative version of the Peres-Horodecki criterion [15]. The negativity for a two-qubit state  $\rho$  can be defined as [16,6,17]

$$N(\rho) = \max\{0, -2\mu_{\min}\}, \quad (5)$$

where  $\mu_{\min}$  is the minimal eigenvalue of the partial transpose of  $\rho$ . Similarly to the concurrence, the negativity, given by Eq. (5), ranges from 0 for a separable state to 1 for a maximally entangled state. As shown by Vidal and Werner [17], the negativity is an entanglement monotone (including convexity) and thus can be considered a useful measure of entanglement. The logarithmic negativity defined by [17]

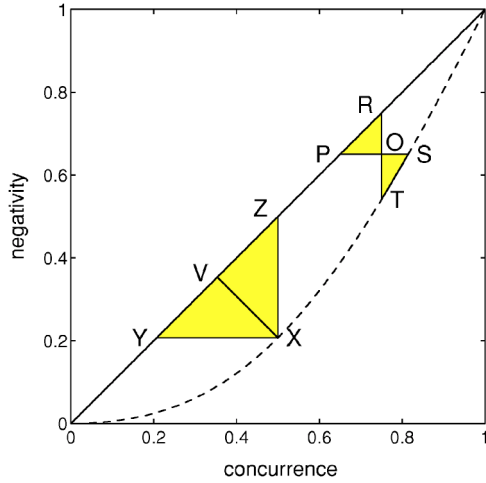


FIG. 1. Negativity versus concurrence. Interpretation of the marked regions is given in the text.

$$E_N(\rho) = \log_2[N(\rho) + 1] \quad (6)$$

measures the entanglement cost of a quantum state  $\rho$  for the exact preparation of any finite number of copies of the state under quantum operations preserving the positivity of the partial transpose (PPT) as proposed by Audenaert *et al.* [18] and proved by Ishizaka [19] for any two-qubit states. Moreover, the logarithmic negativity determines upper bounds on the teleportation capacity and the entanglement of distillation [17].

In the following, we will analyze two-qubit states violating the condition (1) induced by the concurrence (the entanglement of formation) and negativity (the PPT-entanglement cost) only. And thus by referring to Eq. (1) we always mean  $E' = C$  and  $E'' = N$ .

For an arbitrary two-qubit pure state, given by

$$|\Psi\rangle = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle, \quad (7)$$

where  $c_{ij}$  are the normalized complex amplitudes, the concurrence and negativity are the same and simply given by

$$N(|\Psi\rangle) = C(|\Psi\rangle) = 2|c_{00}c_{11} - c_{01}c_{10}|. \quad (8)$$

Nevertheless,  $N(\rho)$  and  $C(\rho)$  can differ for a mixed state  $\rho$ . In general, as shown by Verstraete *et al.* [13], the negativity  $N(\rho)$  of a two-qubit state  $\rho$  can never exceed its concurrence  $C(\rho)$  and

$$N(\rho) \geq \sqrt{[1 - C(\rho)]^2 + C^2(\rho)} - [1 - C(\rho)], \quad (9)$$

as presented in Fig. 1. The states corresponding to these lower and upper bounds have the minimal and maximal negativity for a fixed concurrence. The class of the maximal negativity states can be characterized by the condition that the eigenvector corresponding to the negative eigenvalue of the partial transpose of  $\rho$  is a Bell state [13]. Apart from pure states (7), the class of the maximal negativity states includes the Bell diagonal states [13,14] with the celebrated Werner states defined by [20]

$$\rho_W(p) = p|\psi_B\rangle\langle\psi_B| + \frac{1-p}{4}I \otimes I, \quad (10)$$

where the parameter  $p \in \langle 0, 1 \rangle$ ,  $I$  is the identity operator of a single qubit, and  $|\psi_B\rangle$  is the singlet state:

$$|\psi_B\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle). \quad (11)$$

The negativity and concurrence of  $\rho_W(p)$  are equal to each other for any value of  $p$  as given by

$$N(\rho_W(p)) = C(\rho_W(p)) = \max\left\{0, \frac{3p-1}{2}\right\}. \quad (12)$$

Since both pure and Werner two-qubit states are the maximal negativity states, it is clear why they do not violate condition (1), which is another explanation of the Eisert-Plenio result [6].

The structure of the class of the minimum negativity states was given by Verstraete *et al.* [13] as a solution of the Lagrange constrained problem for the manifold of states with constant concurrence. They found that the minimum negativity states have two vanishing eigenvalues and the other two corresponding to eigenvectors which are a Bell state and separable state orthogonal to it. As an example of the minimum negativity state, we analyze the state [2]

$$\rho_H(p) = p|\psi_B\rangle\langle\psi_B| + (1-p)|00\rangle\langle 00|, \quad (13)$$

where  $0 \leq p \leq 1$  and  $|\psi_B\rangle$  is the Bell state given by Eq. (11). The concurrence and negativity of  $\rho_H$  are given by

$$C(\rho_H) = p,$$

$$N(\rho_H) = \sqrt{(1-p)^2 + p^2} - (1-p), \quad (14)$$

respectively, being equal to each other for  $p=0$  and  $p=1$  only. The state, given by Eq. (13), is a mixture of a maximally entangled state and a separable state orthogonal to it as required by the condition of Verstraete *et al.* for the minimum negativity states [13]. Thus, by replacing  $|00\rangle$  in Eq. (13) by another separable state orthogonal to  $|\psi_B\rangle$  [e.g., by  $|11\rangle$  or  $(|00\rangle + |01\rangle + |10\rangle + |11\rangle)/2$ ], other minimum negativity states satisfying Eq. (14) can be obtained.

By analyzing Fig. 1 for a given state  $\rho_1$  corresponding to some point  $(C(\rho_1), N(\rho_1))$ , it is easy to identify all other states  $\rho_2$ , corresponding to points  $(C(\rho_2), N(\rho_2))$ , which lead to a violation of Eq. (1). E.g., the state  $\rho_1$  described by point  $O(X)$  and the other states  $\rho_2$  corresponding to an arbitrary point in regions  $OPR$  and  $OST$  ( $XYZ$ ) violate condition (1). Clearly, maximal violation of Eq. (1) holds if one of the states (say,  $\rho_1$ ) is the maximum negativity state and the other ( $\rho_2$ ) is the minimum negativity state. To analyze the degree of violation of Eq. (1), we will calculate  $\Delta C\{\rho_1, \rho_2\} \equiv C(\rho_1) - C(\rho_2)$ ,  $\Delta N\{\rho_1, \rho_2\} \equiv N(\rho_1) - N(\rho_2)$ , and

$$\delta\{\rho_1, \rho_2\} \equiv -\min(0, \Delta C\{\rho_1, \rho_2\} \Delta N\{\rho_1, \rho_2\}). \quad (15)$$

In Fig. 2, the function  $\delta(\rho_1, \rho_2)$  is plotted versus all possible values of  $C(\rho_1)$  of the maximum negativity states and  $C(\rho_2)$

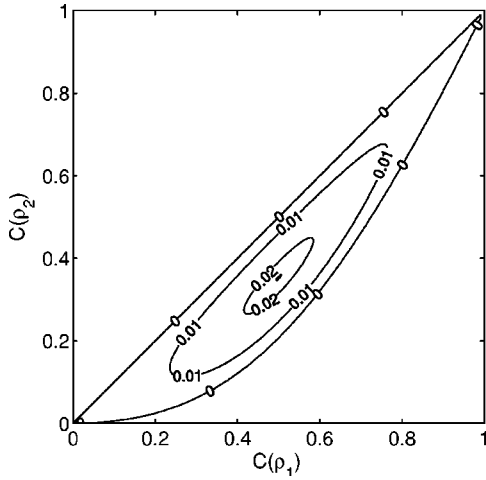


FIG. 2. Contour plot of  $\delta(\rho_1, \rho_2)$  vs  $C(\rho_1)$  and  $C(\rho_2)$  for the maximum negativity states  $\rho_1$  and the minimum negativity states  $\rho_2$ .

of the minimum negativity states. A closer look at Eq. (9) leads us to the conclusions

$$\max_{\substack{\rho_1, \rho_2 \\ N(\rho_1)=N(\rho_2)}} |\Delta C\{\rho_1, \rho_2\}| = \Delta C\{\rho_X, \rho_Y\} = 1 - \frac{\sqrt{2}}{2}, \quad (16)$$

$$\max_{\substack{\rho_1, \rho_2 \\ C(\rho_1)=C(\rho_2)}} |\Delta N\{\rho_1, \rho_2\}| = \Delta N\{\rho_Z, \rho_X\} = 1 - \frac{\sqrt{2}}{2}, \quad (17)$$

$$\max_{\rho_1, \rho_2} \delta\{\rho_1, \rho_2\} = \delta\{\rho_V, \rho_X\} = \frac{\kappa^2}{2}, \quad (18)$$

where  $\kappa = (\sqrt{2}-1)/2$ ,  $\rho_J$  ( $J=V, Y, Z$ ) are the maximum negativity states and  $\rho_X$  is the minimum negativity state having the following concurrences and negativities:  $C(\rho_V) = N(\rho_V) = \sqrt{2}/4$ ,  $C(\rho_Z) = N(\rho_Z) = C(\rho_X) = 1/2$ , and  $C(\rho_Y) = N(\rho_Y) = N(\rho_X) = \kappa$ , as depicted by the corresponding points in Fig. 1. As an explicit example of states maximally violating Eq. (1), one can choose  $\rho_X = \rho_H(1/2)$ , given by Eq. (13), and the following Werner states (10):  $\rho_Y = \rho_W(\sqrt{2}/3)$ ,  $\rho_Z = \rho_W(2/3)$ , and

$\rho_V = \rho_W(1/3 + \sqrt{2}/6)$ . Alternatively, instead of the Werner states, one can take the following pure states (7):

$$|\Psi(p)\rangle = \sqrt{p}|01\rangle + \sqrt{1-p}|10\rangle, \quad (19)$$

which implies that  $\rho_Y$  can be given by  $|\Psi(p)\rangle\langle\Psi(p)|$  for  $p = 1/2 \pm \sqrt{1+2\sqrt{2}}/4$ ,  $\rho_Z$  for  $p = 1/2 \pm \sqrt{3}/4$ , and  $\rho_V$  for  $p = 1/2 \pm \sqrt{14}/8$ .

Let us also consider the two-qubit states

$$\bar{\rho}(p, q) = p|\psi_B\rangle\langle\psi_B| + (1-p)|\psi_q\rangle\langle\psi_q|, \quad (20)$$

being a mixture of the Bell state, given by Eq. (11), and the separable state

$$|\psi_q\rangle = \sqrt{1-q}|00\rangle + \sqrt{q}|01\rangle, \quad (21)$$

where the parameters  $p, q \in (0, 1)$ . The negativity of  $\bar{\rho}(p, q)$  depends on both  $p$  and  $q$  according to

$$N(\bar{\rho}(p, q)) = \sqrt{1-2p(1-p)(1-q)} - (1-p), \quad (22)$$

while the concurrence, given by

$$C(\bar{\rho}(p, q)) = p, \quad (23)$$

is clearly independent of  $q$ . In a special case for  $q=0$ , Eq. (20) goes over into Eq. (13) describing the minimum negativity state, while for  $q=1$ , Eq. (20) describes the maximum negativity state as  $N(p, 1) = C(p, 1)$ .

In the following, we will analyze three classes of the states given by Eq. (20).

(i) The first class is formed by those states with the same negativity—say,  $N'$ . From Eq. (22), one finds that the states  $\rho' = \bar{\rho}(p, q')$ , given by Eq. (20) for

$$q' = \frac{N'[N' + 2(1-p)] - p^2}{2p(1-p)}, \quad (24)$$

have the  $p$ -dependent concurrence,  $C(\rho') = p$ , but constant negativity,  $N(\rho') = N' = \text{const}$ , for all  $p \in \langle N', \sqrt{2N'(N'+1)} - N' \rangle$ . This result is confirmed graphically in Fig. 3(a) for a few choices of  $N'$ . In particular, for  $N' = \kappa$ , one gets the  $p$ -parametrized ( $\kappa \leq p \leq 1/2$ ) states

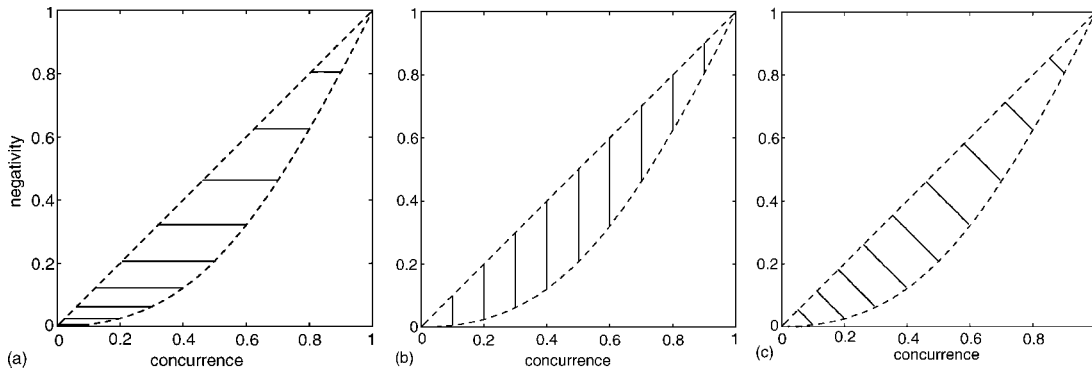


FIG. 3. Negativity vs concurrence for the three classes of states (20) for the parameter  $p = 0.1, 0.2, \dots, 0.9$ : (a)  $\rho'$  where  $N' = N(\rho_H)$ , (b)  $\rho''$ , and (c)  $\rho'''$  where  $\rho = \rho_H$ .

$$\rho_{XY}(p) \equiv \bar{\rho}\left(p, \frac{(1-2p)(2\sqrt{2}+2p-1)}{8p(1-p)}\right), \quad (25)$$

which are visualized by the points on the  $XY$  line in Fig. 1. The states  $\rho_{XY}(p)$  for  $p=\kappa$  (corresponding to a maximum negativity state) and  $p=1/2$  (minimum negativity state) have maximally different concurrences and the same negativity:

$$\Delta C\{\rho_{XY}(1/2), \rho_{XY}(\kappa)\} = 1 - \frac{\sqrt{2}}{2}. \quad (26)$$

(ii) To the second class belong those  $\rho'' = \bar{\rho}(p'', q)$  having the same concurrence. This condition is easily fulfilled by fixing  $p=p''$  in Eq. (20); then, the concurrence  $C'' \equiv C(p'', q) = p'' = \text{const}$  for all values of  $q$ , while the negativity  $N(p'', q)$  ranges from  $\sqrt{(1-p'')^2 + (p'')^2} - (1-p'')$  to  $p''$  as shown in Fig. 3(b) for a few choices of  $p''$ . In particular, for  $p''=1/2$ , we get the  $q$ -parametrized ( $q \in \langle 0, 1 \rangle$ ) states

$$\rho_{XZ}(q) \equiv \bar{\rho}(1/2, q) = \frac{1}{2}(|\psi_B\rangle\langle\psi_B| + |\psi_q\rangle\langle\psi_q|), \quad (27)$$

which are described by the points on the  $XZ$  line in Fig. 1. The states  $\rho_{XZ}(q)$  for  $q=0$  (corresponding to the minimum negativity state) and  $q=1$  (maximum negativity state) have maximally different negativities for the same concurrence:

$$\Delta N\{\rho_{XZ}(1), \rho_{XZ}(0)\} = 1 - \frac{\sqrt{2}}{2}. \quad (28)$$

(iii) Finally, we analyze such states  $\rho''' = \bar{\rho}(p, q''')$  of the form (20) for which predictions concerning negativity and concurrence are exactly opposite to those for a given state  $\rho$ , i.e.,

$$\Delta C\{\rho, \rho'''\} = -\Delta N\{\rho, \rho'''\}. \quad (29)$$

This condition is fulfilled if the parameter  $q'''$  is given by

$$q''' = 1 + \frac{[N(\rho) + C(\rho) + 1 - 2p]^2 - 1}{2p(1-p)} \quad (30)$$

for  $\frac{1}{2}[C(\rho) + N(\rho)] \leq p \leq C(\rho) < 1$  as shown in Fig. 3(c). In particular, if  $N(\rho) = \kappa$  and  $C(\rho) = 1/2$ , one arrives at the  $p$ -parametrized ( $\sqrt{2}/4 \leq p \leq 1/2$ ) states

$$\rho_{XV}(p) \equiv \bar{\rho}\left(p, \frac{(1-2p)(2\sqrt{2}+1-2p)}{4p(1-p)}\right), \quad (31)$$

exhibiting  $N(\rho_{XV}(p)) = \sqrt{2}/2 - p$  and, as usual,  $C(\rho_{XV}(p)) = p$ , which are described by the points on the  $XV$  line in Fig. 1. Predictions of the concurrence and negativity for the states  $\rho_{XV}(p)$  with  $p=1/2$  (corresponding to the minimum negativity state) and  $p=\sqrt{2}/4$  (maximum negativity state) are maximally different, as given by

$$\delta\left\{\rho_{XV}\left(\frac{1}{2}\right), \rho_{XV}\left(\frac{\sqrt{2}}{4}\right)\right\} = \frac{\kappa^2}{2}, \quad (32)$$

which is the upper bound determined by Eq. (18).

We analyzed ordering of density matrices of two qubits with respect to the bipartite entanglement quantified by the Wootters concurrence  $C$ , a measure of the entanglement of formation, and by the negativity  $N$ , a measure of the PPT-entanglement cost. We have presented simple two-qubit states  $\rho_1$  and  $\rho_2$  (where one of them can be pure) having the entanglement measures different in such a way that (i)  $N(\rho_1) = N(\rho_2)$  but  $C(\rho_1) \neq C(\rho_2)$ , (ii)  $C(\rho_1) = C(\rho_2)$  but  $N(\rho_1) \neq N(\rho_2)$ , or (iii) the concurrence  $C(\rho_1)$  is smaller than  $C(\rho_2)$  but the negativity  $N(\rho_1)$  is greater than  $N(\rho_2)$ . Using the bounds of Verstraete *et al.* [13], we have also found analytically to what degree the concurrence and negativity can give different orderings of two-qubit states.

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