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Ordering Unicyclic Graphs with Large Average Eccentricities

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Abstract. The eccentricity of a vertex u in a connected graph G is the maximum distance from u to other vertices of G. The n-vertex unicyclic graphs with the i-th largest average eccentricity are determined for all i up to $\left\lfloor \frac{n}{2} \right\rfloor - 1$ with $n \geq 6$.

1. Introduction

We consider only simple graphs. Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$. For $u, v \in V(G)$, the distance between u and v in G, denoted by $d_G(u, v)$, is the length (number of edges) of a shortest path connecting u and v in G. For $u \in V(G)$, the eccentricity of u in G, denoted by $e_G(u)$, is the maximum distance from u to other vertices of G. The average eccentricity of G is

$$avec(G) = \frac{1}{n} \sum_{u \in V(G)} e_G(u),$$

where $n = |V(G)|$. The average eccentricity has been used as a molecular descriptor (topological index) for various structure-property models, see [3]. It is named the eccentric mean by Buckley and Harray [1]. Dankelmann et al. [2] found an upper bound for the average eccentricity in terms of number of vertices and minimum degree. We established in [4] various properties for the average eccentricity, including lower and upper bounds and especially, the ordering of trees (connected graphs of no cycles) by average eccentricity.

In this paper, we determine the n-vertex unicyclic graphs (connected graphs of a unique cycle) with the i-th largest average eccentricity for all i up to $\lfloor \frac{n}{2} \rfloor - 1$ with $n \geq 6$.

2. Preliminaries

Let P_n be the path on n vertices. For $n \geq 4$ and $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$, let T_n^i be the tree formed by attaching a pendent vertex v_{n-1} to a vertex v_i of the path $P_{n-1} = v_0 v_1 \ldots v_{n-2}$.

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Lemma 2.1. [4] *Among all the n-vertex trees, T_n^i for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor - 1$ is the unique graph with the $(i+1)$th-largest average eccentricity, where*

$$avec(T_n^i) = \begin{cases} \dfrac{3(n-1)^2 + 2n - 3 - 4i}{4n}, & \text{if } n \text{ is even} \\ \dfrac{3(n-1)^2 + 2n - 2 - 4i}{4n}, & \text{if } n \text{ is odd.} \end{cases}$$

For $2 \leq d \leq n-1$, let $T^{(n,d)} = \left\{T_{n,d}^a : 1 \leq a \leq \lfloor \frac{n+1-d}{2} \rfloor\right\}$, where $T_{n,d}^a$ is the n-vertex tree obtained by attaching a and $n+1-a-d$ pendent vertices respectively to the two end vertices of the path P_{d-1}.

Lemma 2.2. [4] *Let G be an n-vertex trees with diameter d, where $2 \leq d \leq n-1$. Then*

$$avec(G) \leq g_n(d)$$

with equality if and only if $G \in T^{(n,d)}$, where

$$g_n(d) = \frac{1}{n}\left[\left\lfloor \frac{3(d+1)^2 - 2(d+1)}{4} \right\rfloor + (n-d-1)d\right].$$

Also, $g_n(d)$ is increasing for $2 \leq d \leq n-1$.

Let $\mathbb{U}_{n,3}$ be the set of n-vertex unicyclic graphs with cycle length 3, and $\mathbb{U}_{n,\geq 4}$ be the set of n-vertex unicyclic graphs with cycle length at least 4.

3. Result

If G is a connected graph such that $G-e$ is also connected for $e \in E(G)$, then $e_G(u) \leq e_{G-e}(u)$ for $u \in V(G)$ and thus $avec(G) \leq avec(G-e)$.

For $0 \leq i \leq \lfloor \frac{n-3}{2} \rfloor$, let $P_{n,3}(i)$ be the n-vertex unicyclic graph formed by attaching two pendent paths with i and $n-3-i$ vertices respectively, to the two vertices of a triangle.

Lemma 3.1. *Among the graphs in $\mathbb{U}_{n,3}$ with $n \geq 6$, $P_{n,3}(i)$ with $0 \leq i \leq \lfloor \frac{n-4}{2} \rfloor$ is the unique graph with the $(i+1)$th-largest average eccentricity, where*

$$avec(P_{n,3}(i)) = \begin{cases} \dfrac{3n^2 - 4n - 4i - 4}{4n}, & \text{if } n \text{ is even} \\ \dfrac{3n^2 - 4n - 4i - 3}{4n}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Obviously, $P_{n,3}(i)$ with $0 \leq i \leq \lfloor \frac{n-4}{2} \rfloor$ may be obtained from T_n^{i+1} by adding an edge $v_i v_{n-1}$. From this fact we may easily find that vertices in $P_{n,3}(i)$ and T_n^{i+1} with the same labeling have equal eccentricity, and thus $avec(P_{n,3}(i)) = avec(T_n^{i+1})$ for $0 \leq i \leq \lfloor \frac{n-4}{2} \rfloor$. By Lemma 2.1,

$$avec(P_{n,3}(i)) = \begin{cases} \dfrac{3n^2 - 4n - 4i - 4}{4n}, & \text{if } n \text{ is even} \\ \dfrac{3n^2 - 4n - 4i - 3}{4n}, & \text{if } n \text{ is odd} \end{cases}$$

for $0 \leq i \leq \lfloor \frac{n-4}{2} \rfloor$, and $avec(T_n^j) > avec(T_n^{j+1})$ for $0 \leq j \leq \lfloor \frac{n-4}{2} \rfloor$. Then

$$avec(P_{n,3}(j)) > avec(P_{n,3}(j+1)) \text{ for } 0 \leq j \leq \left\lfloor \frac{n-6}{2} \right\rfloor. \tag{1}$$

Suppose that $G \in \mathbb{U}_{n,3} \setminus \bigcup_{j=0}^{\lfloor \frac{n-4}{2} \rfloor} \{P_{n,3}(j)\}$. We will show that

$$avec(G) < avec\left(P_{n,3}\left(\left\lfloor \frac{n-4}{2} \right\rfloor\right)\right).$$

Let d be the diameter of G. Note that $2 \le d \le n-2$.

Case 1. $d = n-2$. Then $G = P_{n,3}(\frac{n-3}{2})$, with odd n. We have $avec\left(P_{n,3}\left(\frac{n-3}{2}\right)\right) - avec\left(T_n^{\frac{n-3}{2}}\right) = \frac{1}{n}\left(\frac{n-1}{2} - \frac{n+1}{2}\right) < 0$, and thus

$$avec\left(P_{n,3}\left(\frac{n-3}{2}\right)\right) < avec\left(T_n^{\frac{n-3}{2}}\right) = avec\left(P_{n,3}\left(\left\lfloor \frac{n-4}{2} \right\rfloor\right)\right).$$

Case 2. $d = n-3$. Then G is an n-vertex unicyclic graph formed by attaching one pendent vertex to $P_{n-1,3}(i)$ with $0 \le i \le \lfloor \frac{n-4}{2} \rfloor$. It is easy to find that there exists an edge e on the cycle of G such that the diameter of $G-e$ is equal to $n-3$. By Lemmas 2.1 and 2.2, $avec(G) \le avec(G-e) \le g_n(n-3) \le \frac{3(n-1)^2-13}{4n} < avec\left(T_n^{\lfloor \frac{n-2}{2} \rfloor}\right) = avec\left(P_{n,3}\left(\lfloor \frac{n-4}{2} \rfloor\right)\right)$.

Case 3. $2 \le d \le n-4$. There is an edge e' on the cycle of G such that the diameter of $G-e$ is at most $n-3$. By Lemmas 2.1 and 2.2, $avec(G) \le avec(G-e') \le g_n(n-3) < avec\left(T_n^{\lfloor \frac{n-2}{2} \rfloor}\right) = avec\left(P_{n,3}\left(\lfloor \frac{n-4}{2} \rfloor\right)\right)$.

Combining Cases 1–3, we have $avec(G) < avec\left(P_{n,3}\left(\left\lfloor \frac{n-4}{2} \right\rfloor\right)\right)$ for $G \in \mathbb{U}_{n,3} \setminus \bigcup_{j=0}^{\lfloor \frac{n-4}{2} \rfloor} \{P_{n,3}(j)\}$, from which, together with (1), we have the result. $\square$

For $0 \le i \le \lfloor \frac{n-4}{2} \rfloor$, let $P_{n,4}(i)$ be the n-vertex unicyclic graph formed by attaching two pendent paths with i and $n-4-i$ vertices respectively, to the two non-adjacent vertices of a quadrangle.

Lemma 3.2. *Among the graphs in $\mathbb{U}_{n,\ge 4}$ with $n \ge 6$, $P_{n,4}(i)$ with $0 \le i \le \lfloor \frac{n-6}{2} \rfloor$ is the unique graph with the $(i+1)$th-largest average eccentricity, where*

$$avec(P_{n,4}(i)) = \begin{cases} \dfrac{3n^2-4n-4i-8}{4n}, & \textit{if } n \textit{ is even} \\ \dfrac{3n^2-4n-4i-7}{4n}, & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. Obviously, $P_{n,4}(i)$ with $0 \le i \le \lfloor \frac{n-6}{2} \rfloor$ may be obtained from T_n^{i+2} by adding an edge $v_i v_{n-1}$. From this fact we may easily find that vertices in $P_{n,4}(i)$ and T_n^{i+2} with the same labeling have equal eccentricity for $0 \le i \le \lfloor \frac{n-6}{2} \rfloor$. Thus by Lemma 2.1, we have

$$avec(P_{n,4}(i)) = avec(T_n^{i+2}) = \begin{cases} \dfrac{3n^2-4n-4i-8}{4n}, & \text{if } n \text{ is even} \\ \dfrac{3n^2-4n-4i-7}{4n}, & \text{if } n \text{ is odd} \end{cases}$$

for $0 \le i \le \lfloor \frac{n-6}{2} \rfloor$, and $avec(T_n^j) > avec(T_n^{j+1})$ for $0 \le i \le \lfloor \frac{n-4}{2} \rfloor$. Then

$$avec(P_{n,4}(j)) > avec(P_{n,4}(j+1)) \text{ for } 0 \le j \le \left\lfloor \frac{n-8}{2} \right\rfloor. \tag{2}$$

Suppose $G \in \mathbb{U}_{n,\ge 4} \setminus \bigcup_{j=0}^{\lfloor \frac{n-6}{2} \rfloor} \{P_{n,4}(j)\}$. We will show that

$$avec(G) < avec\left(P_{n,4}\left(\left\lfloor \frac{n-6}{2} \right\rfloor\right)\right).$$

Let m be the cycle length of G, and d the diameter of G.

Case 1. $m = 4$ and $d = n-2$. Then $G = P_{n,4}\left(\left\lfloor \frac{n-4}{2} \right\rfloor\right)$. We have $avec\left(P_{n,4}\left(\left\lfloor \frac{n-4}{2} \right\rfloor\right)\right) - avec\left(T_n^{\lfloor \frac{n-2}{2} \rfloor}\right) = \frac{1}{n}\left(\left\lfloor \frac{n-2}{2} \right\rfloor - \left\lceil \frac{n}{2} \right\rceil\right) < 0$, and thus

$$avec\left(P_{n,4}\left(\left\lfloor \frac{n-4}{2} \right\rfloor\right)\right) < avec\left(T_n^{\lfloor \frac{n-2}{2} \rfloor}\right) = avec\left(P_{n,4}\left(\left\lfloor \frac{n-6}{2} \right\rfloor\right)\right).$$

Case 2. $m = 4$ and $2 \le d \le n-3$, or $m \ge 5$. Note that $2 \le d \le n - \lceil \frac{m}{2} \rceil$. It is easy to find there exists an edge e on the cycle of G such that the diameter of $G - e$ is at most $n-3$. By Lemmas 2.1 and 2.2 $avec(G) \le avec(G-e) \le g_n(n-3) < avec\left(T_n^{\lfloor \frac{n-2}{2} \rfloor}\right) = avec\left(P_{n,4}\left(\lfloor \frac{n-6}{2} \rfloor\right)\right)$. Thus $avec(G) < avec\left(P_{n,4}\left(\lfloor \frac{n-6}{2} \rfloor\right)\right)$.

Combining Cases 1 and 2, we have $avec(G) < avec\left(P_{n,4}\left(\left\lfloor \frac{n-6}{2} \right\rfloor\right)\right)$ for $G \in \mathbb{U}_{n;\ge 4} \setminus \bigcup_{j=0}^{\lfloor \frac{n-6}{2} \rfloor} \{P_{n,4}(j)\}$, from which, together with (2), we have the result. $\square$

Note that $avec(P_{n,3}(i)) = avec(P_{n,4}(i-1))$ for $1 \le i \le \lfloor \frac{n-4}{2} \rfloor$. Combining Lemmas 3.1 and 3.2, we can have the following main result.

Theorem 3.3. *Among the n-vertex unicyclic graphs with $n \ge 6$, $P_{n,3}(0)$ is the unique graph with the first largest average eccentricity, equal to $\frac{3n^2-4n-4}{4n}$ for even n and $\frac{3n^2-4n-3}{4n}$ for odd n, $P_{n,3}(i)$ and $P_{n,4}(i-1)$ are the unique graphs with the $(i+1)$th-largest average eccentricity, equal to $\frac{3n^2-4n-4i-4}{4n}$ for even n and $\frac{3n^2-4n-4i-3}{4n}$ for odd n where $1 \le i \le \lfloor \frac{n}{2} \rfloor - 2$.*

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