Ordinal Bayesian Incentive Compatible Representations of Committees*

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Abstract

We consider the problem of strategic manipulation or decision schemes that provide an adequate representation (in some sense) of the distribution of power within a committee. "Strategic-proof representation" is very restrictive: it implies that the committee contains exactly one minimal winning coalition. So we introduce the weaker concept of "Ordinally Bayesian Incentive Compatible representation" and prove the existence of such representations for weak games under some conditions. Finally, constructing examples, we show first how necessary these conditions are – including the use of change in the voting procedure – and second that we cannot avoid Condorcet's paradox.

1. Introduction

The most commonly used method for solving social choice problems is to design adequate committees. Hence, the importance of studying voting procedures, that properly correspond to such ad hoc committees, has been well recognized. In particular, the theory of strategic voting in committees has received considerable attention recently in the framework of games with complete information (see, Schofield, 1985; Peleg, 1984; Holzman, 1986). However in many cases the voting situation is characterized by the incomplete information that every member has concerning the relevant characteristics of the other members. This informational constraint gives a more complex structure to the problem of strategic voting. We note that Pattanaik [12] already expressed the need for treating voting games with incomplete information.

The purpose of the present investigation is to analyze this problem in the framework of games with incomplete information as formalized by Harsanyi (1967-68) (see also Mertens and Zamir, 1985). The main result will be the construction of voting procedures reflecting quite accurately the distribution of power in committees of a particular type and also avoiding insincere behavior of the individual members of these committees. The basic concept in that respect will be the property of "Bayesian Incentive Compatibility" introduced and investigated in d'Aspremont and Gérard-Varet (1979, 1982) (see also Myerson, 1985).

However, some general remarks concerning the "realistic" features of such procedures have to be made. First voting procedures form a particular class of decision mechanisms since they are based on a restricted number of the voter's characteristics, namely those determining their rankings of the set of alternatives. This is compatible with most observed procedures and will justify the

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introduction of a concept of "Ordinally Bayesian Incentive Compatibility". Second, it is usual to design a committee for some length so that it can eventually decide on several issues. Hence, although the distribution of power remains invariant during that time, the voting procedure may vary according to the number of alternatives in each issue. What we require is that the procedures reflect the given distribution of power for every issue. Third, and this observation will be one of our results, such voting procedures cannot be always deterministic that is, social choice functions. Hence, the decision-making procedures, which we shall concentrate on, will combine voting with chance and our study will rely on previous work about such "Decision Schemes" (see e.g. Gibbard, 1977; Hylland, 1980; Barbera, 1979). Finally, it will be observed that neither the use of Bayesian Incentive Compatibility nor the introduction of Decision Schemes implies that Condorcet's "voting paradox" can be resolved.

The paper is organized as follows. First we define committees as simple games and give some basic properties. Then we consider the problem of associating with each committee a decision scheme which is an adequate representation (in some sense) of the distribution of power within the committee. This ends Section 2. In Section 3 we take up the problem of strategic manipulation of the voting procedure. We begin by showing that the strategy-proof concept normally used in the complete information framework (see Gibbard, 1977) is very strong and essentially restricts us to committees having exactly one minimal winning coalition and to their representation by random dictatorship. Then, in the incomplete information framework, we introduce the new and weaker concept of "Ordinally Bayesian Incentive Compatibility" (OBIC) for decision schemes and show, under some restrictions, the existence of OBIC-representations for weak games. In Section 4, we analyze several examples showing that we cannot avoid these restrictions, including the use of nondegenerate decision schemes, and also that we cannot resolve the "voting paradox". Concluding remarks are given in Section 5.

2. Committees and decision schemes

In this section we introduce a general notion of voting procedure, called a "decision scheme" that allows probabilistic choice. We shall associate such procedures to various kinds of committees, formally defined here as "simple games", which are supposed to describe various distributions of power among some set of voters. More specifically we shall look for decision schemes reflecting correctly the distribution of power of a given committee.

Let us start by recalling some definitions about simple games (committees). A *simple game* G is an ordered pair (N, W), where $N = \{1, \ldots, n\}, n \geq 2$, is the set of *players* (or voters) and W is a set of coalitions (i.e. non-empty subsets of N), whose members are called the *winning coalitions*. We shall most often restrict our attention to simple games (N, W) which are both *monotonic*, i.e.,

$$[S \in W \text{ and } S \subset T] \;\Rightarrow\; T \in W$$

and *proper*, i.e.,

$$S \in W \;\Rightarrow\; N - S \notin W.$$

In a simple game $G = (N, W)$, a *minimal winning coalition* is a winning coalition which contains no other winning coalition. A monotonic simple game is determined by its set of minimal winning coalitions. Also a *veto player* is a player which belongs to every winning coalition. In the sequel we shall use the following definitions.

Definition 2.1 *Let $G = (N, W)$ be a monotonic simple game. G is* **weak** *if the set V of veto players, i.e., $V = \cap\{S \mid S \in W\}$, is nonempty.*

Definition 2.2 *Let $G = (N, W)$ be a monotonic simple game. A permutation π of N is a* **symmetry** *of G if, for every $S \in W$,*

$$\pi(S) = \{\pi(i) \mid i \in S\} \in W.$$

The group of all symmetries of G will be denoted by $SYM(G)$ and if $SYM(G) = S_N$, the group of all permutations of N, then G is said to be *symmetric*.

We remark that a symmetric game (N, W) is completely determined by a pair (n, k), where n is the number of players and k is the size of a minimal winning coalition.

In our approach a simple game is to be viewed as a specification of the distribution of power among a set of voters. However this becomes relevant only if some kind of voting procedure is considered. For that purpose we use the notion of a decision scheme which is a procedure that associates a lottery on the set of alternatives to each ranking of alternatives by the different voters. More precisely, let A be a finite set of m *alternatives*, $m \geq 2$. We assume that no individual is indifferent between any two alternatives, and accordingly denote by L the set of all *linear orders* on A, i.e. the complete, reflexive, transitive and antisymmetric binary relations on A. Clearly, if $R \in L$, we may write $R = (t_1(R), t_2(R), \ldots, t_m(R))$, where $t_k(R)$ is the k-th best alternative according to R. As in Gibbard (1977) let us have:

Definition 2.3 *A* **decision scheme** *(DS) is a function $d : A \times L^N \to [0, 1]$ which satisfies: $\sum_{x \in A} d(x, R^N) = 1$ for all $R^N \in L^N$ (where L^N is the set of all functions from N to L).*

We may then define a certain number of possible requirements on decision schemes.

Definition 2.4 *For $R^N \in L^N$ denote by $PAR(R^N)$ the set of Pareto-optimal alternatives with respect to R^N – i.e. $x \in PAR(R^N)$ if for no $y \in A$, $y \neq x$ and yR^ix for all $i \in N$. A DS d is* **Paretian ex post** *if, for all $R^N \in L^N$,*

$$x \notin PAR(R^N) \Rightarrow d(x, R^N) = 0.$$

Definition 2.5 *Let d be a DS. A permutation π of N is a* **symmetry** *of d if, for all R^N in L^N,*

$$d(x, R^N) = d(x, (R^{\pi(1)}, \ldots, R^{\pi(n)})), \text{ for all } x \in A.$$

The group of all symmetries of d is denoted $SYM(d)$ and, if $SYM(d) = S_N$, then d is said to be **anonymous**.

Definition 2.6 *Let d be a DS. For any σ, a permutation of A, and for $R \in L$, let $\sigma(R)$ denote the linear order such that: for all $x, y \in A$, $\sigma(x)\sigma(R)\sigma(y)$ iff xRy. Then d is said to be* **neutral** *if for every permutation σ of A and every $R^N \in L^N$,*

$$d(x, R^n) = d(\sigma(x), (\sigma(R^1), \ldots, \sigma(R^n))), \text{ for all } x \in A.$$

Definition 2.7 *Let $R^N \in L^N$ and let $x \in A$. We shall denote by R^N_x any profile obtained from R^N by an improvement of the position of x in the following sense: for all $a, b \in A - \{x\}$ and for all $i \in N$:*

$$aR^ib \Leftrightarrow aR^i_xb, \text{ and}$$
$$xR^ib \Rightarrow xR^i_xb.$$

Then a DSd is said to be **monotonic** *if for every $R^N \in L^N$ and every $x \in A$,*

$$d(x, R_x^N) \geq d(x, R^N).$$

Definition 2.8 *A DSd is said to satisfy the* **attainability condition** *if for every $x \in A$ there exists $R^n \in L^N$ such that $d(x, R^N) = 1$. Notice that if d is Paretian ex post then d satisfies the attainability condition.*

Now that we have defined what we mean by a committee and different kinds of decision schemes, we shall associate the two notions. Our objective in this is to "represent" any committee by feasible choice procedures reflecting correctly the distribution of power in the committee. We shall proceed as follows. First to any DSd we shall associate a simple game, say $G(d)$. Then, to represent a simple game G, we shall take the DS'sd such that $G(d) = G$. But there are several ways to define the function $G(\cdot)$. We shall in this section limit ourselves to the following.

Definition 2.9 *Let $m \geq 2$ and $n \geq 2$. Let $d : A \times L^N \to [0, 1]$ be a DS and let $x \in A$. A coalition S is* **winning** *for x (with respect to d) if*

$$[R^N \in L^N \text{ and } t_1(R^i) = x \text{ for all } i \in S] \Rightarrow d(x, R^N) = 1.$$

The set of all winning coalitions for x is denoted by $W^*(d, x)$. The **first simple game associated with** *d is the game $G^*(d) = (N, W^*)$ where*

$$W^* = \cap\{W^*(d, x) \mid x \in A\}.$$

It should be noticed that $G^*(d)$ is monotonic and proper (recall that $m \geq 2$).

Definition 2.10 *Let $G = (N, W)$ be a proper and monotonic simple game. A $DSd : A \times L^N \to [0, 1]$, with $m \geq 2$, is a* **representation of** G **of order** m *if*

$$G^*(d) = G.$$

Finally one may add to the representative character of such a DS with respect to some proper and monotonic simple game G, by requiring that the DS respects the symmetries of G. This leads to the following.

Definition 2.11 *Let a DSd be a representation of a simple game G. Then d is a* **faithful** *representation of G if $SYM(d) = SYM(G)$.*

Now we discuss briefly the existence of "nice" representations, that is Paretian ex post, neutral, monotonic, and faithful representations. First we recall that a social choice correspondence (SCC) is a function $H : L^N \to 2^A$, where 2^A is the set of nonempty subsets of A. If H is an SCC, then we associate with H the DSd_H defined in the following way:

$$d_H(x, R^N) = \begin{cases} 1/|H(R^N)|, & \text{if } x \in H(R^N); \\ 0, & \text{if } x \notin H(R^N), \end{cases}$$

where, here and in the sequel, if B is a finite set then $|B|$ denotes the number of members of B. We remark that: (1) If H is Paretian (respectively neutral, monotonic), then d_H is Paretian ex post (respectively neutral, monotonic); (2) $SYM(d_H) = SYM(H)$, the symmetry group of H; (3) $G^*(d_H) = G^*(H)$ (see Definition 3.1.3 of Peleg, 1984). If we combine the last remark with Example 3.2.20 of Peleg (1984) we obtain the following result.

Remark 2.1 Let $G = (N, W)$ be a nonnull (i.e., $N \in W$), proper, and monotonic simple game. Then G has a Paretian ex post, neutral, monotonic, and faithful representation (by DS's) of every order $m \geq 2$.

Also, we remark that the set of all $DS's$, as well as the set of all Paretian ex post (respectively neutral or monotonic) $DS's$, are convex polytopes in $(Re)^k$ with $k = m(m!)^n$.

Finally, we examine the geometric properties of the set of representations of a given game.

Lemma 2.1 *Let $G = (N, W)$ be a proper and monotonic simple game, and let $|A| = m \geq 2$. Then the set of all representations of G of order m is convex.*

Proof $d : A \times L^N \to [0, 1]$ is a representation if and only if

(a) if $S \in W$, $R^N \in L^N$, and $x = t_1(R^i)$ for all $i \in S$, then $d(x, R^N) = 1$ and

(b) if $S \notin W$ then there exists $R_0^N \in L^N$ and $x_0 \in A$ such that

$$t_1(R_0^i) = x_0 \text{ for all } i \in S \text{ and } d(x_0, R_0^N) < 1.$$

Clearly, (a) and (b) determine a convex set of $DS's$. ■

Example 2.1 Let $A = \{a, b\}$, $N = \{1, 2\}$, $W = \{\{1, 2\}\}$, and $G = (N, W)$. Then $L = \{R_1, R_2\}$, where $R_1 = (a, b)$ and $R_2 = (b, a) \cdot d : A \times L^N \to [0, 1]$ is a representation of G if and only if

(a) $d(a, (R_1, R_1)) = d(b, R_2, R_2)) = 1$;

(b) $d(a, (R_1, R_2)) < 1$ or $d(b, (R_2, R_1)) < 1$;

(c) $d(b, (R_1, R_2)) < 1$ or $d(a, (R_2, R_2)) < 1$.

Thus the set of all representations of G is convex but not closed.

3. Representation theory and incentive compatibility

Given a committee and a voting procedure, strategic voting may happen if some voter has an interest, in some sense, not to vote according to his true preferences. However, the way to define this problem differs greatly according to the type of information the different voters have. In this respect two approaches can be used: complete information and incomplete information. We shall concentrate on the latter approach. However, we first report briefly our results for the first case. Our objective is to point out that in this case only unanimity games up to the addition of dummies (i.e. games with a single minimal winning coalition) have incentive compatible (i.e., strategy-proof) representations.

3.1 Strategy-proof representations

In the complete information case every voter is supposed to know all the relevant characteristics of the others and, in particular, their true preferences. Because representations of simple games have properties which are normally associated with the preferences which the voters announce – which need not be true – the end result of the voting procedure may be distorted. To avoid this manipulation problem we may look for decision schemes which are strategy-proof in the sense of Gibbard (1977). For the sake of completeness we recall the relevant definitions.

Again, let A be a set of m alternatives, $m \geq 2$, and let $N = \{1, \ldots, n\}$, $n \geq 2$, be a set of voters. If $R \in L$ then we denote by $\Phi(R)$ the set of all utility functions for R. Thus, $u : A \to Re$ is in $\Phi(R)$ if and only if for all $x, y \in A$

$$xRy \Leftrightarrow u(x) \geq u(y).$$

Definition 3.1 *Let $d : A \times L^N \to [0,1]$ be a DS, let $R^N \in L^N$, and let $k \in N$. d is* **potentially manipulable** *by k at R^N if there exist $u \in \Phi(R^k)$ and $Q^k \in L$ such that*

$$\sum_{x \in A} u(x) d(x, R^N) < \sum_{x \in A} u(x) d(x, (R^{N-\{k\}}, Q^k)).$$

d is **manipulable** *if there exist $R^N \in L^N$ and $k \in N$ such that d is potentially manipulable by k at R^N. d is strategy-proof if it is not manipulable.*

Definition 3.2 *A DS d is* **dictatorial** *if, for some $i \in N$, $d = \delta_i$ where δ_i is defined by*

$$\delta_i(t_1(R^i), R^N) = 1 \text{ for all } R^N \in L^N.$$

Now we can state our result on strategy-proof representations.

Theorem 3.1 *Let $G = (N, W)$ be a nonnull (i.e., $N \in W$), proper, and monotonic simple game. Then G has a strategy-proof representation of every order $m \geq 3$ if and only if W contains exactly one minimal winning coalition.*

Proof *Sufficiency.* Let A be a finite set of m alternatives, $m \geq 2$, and let S be the minimal winning coalition of G. Choose real numbers $a_i > 0$, $i \in S$, such that $\sum_{i \in S} a_i = 1$, and let $d = \sum_{i \in S} a_i \delta_i$ (i.e., "random dictatorship"). Then d is strategy-proof. We claim that $G^*(d) = G$. Indeed, let $x \in A$. Then a coalition $T \in W^*(d, x)$ if and only if $T \supset S$. Thus $W^*(d) = \{T \mid T \supset S\} = W$.
Necessity. Let A be a finite set of m alternatives, $m \geq 3$, and let $d : A \times L^N \to [0,1]$ be a strategy-proof representation of G. Now, $N \in W$ and $W = W^*(d)$. Thus, $N \in W^*(d)$. Therefore, d satisfies the attainability condition (see Definition 2.8). Hence, by Corollary 1* of Hylland (1980), there exist non-negative numbers $a_1, \ldots, a_n$, such that $\sum_{i \in N} a_i = 1$ and $\sum_{i \in N} a_i \delta_i$. Let $S = \{i \mid a_i > 0\}$. Then

$$W^*(d) = \{T \mid T \supset S\} = W$$

and S is the only minimal winning coalition in W. ■

We conclude from the proof of Theorem 3.3 that a strategy-proof representation of a nonnull, monotonic, and proper simple game must be a random dictatorship (provided that $m \geq 3$).

Corollary 3.1 *Let $G = (n, n)$ and let A be a finite set of m alternatives, $m \geq 2$. Then G has a strategy-proof, faithful, Paretian ex post, neutral, and monotonic representation.*

Proof Let $d = \sum_{i \in N} \delta_i / n$. Then d has all the desired properties. ■

3.2 Ordinally Bayesian incentive compatible $DS's$

The second approach to the problem of strategic voting, which we shall concentrate on in this work, is to introduce a framework of incomplete information. Indeed, in many voting contexts, it cannot be assumed that every player does know completely all the relevant characteristics of the others. It can only be assumed that every player has beliefs concerning these characteristics. To model this incomplete information (as in Harsanyi, 1967-68), we introduce, for each player $i \in N$, a set T_i of possible "types" as well as probability distributions representing his conditional beliefs for every $t_i \in T_i$. Since for the design of voting procedures we are only interested by the linear orders of the voters on the set of alternatives, we assume that every T^i can be decomposed as a product $\widetilde{T}^i \times L$: an element $t_i = (\tau^i, R^i) \in T_i$ contains thus the linear order R^i of voter i and all the other relevant features of i, denoted τ^i. To summarize we introduce the notion of information structure:

Definition 3.3 *An* **information-structure** *(IS) is a* $(2n+1)$*-tuple*

$$I = (A; T^1, \ldots, T^n; p^1, \ldots, p^n)$$

where

(i) A *is a finite set of* m *alternatives,* $m \geq 2$*;*

(ii) T^i *is a* **finite** *set of possible types for player* $i \in N$*, of the form* $T^i = \widetilde{T}^i \times L$*;*

(iii) For every $i \in N$ *and* $t^i \in T^i, p^i(\cdot \mid t^i)$ *is a probability distribution over* $T^{-i} = \bigtimes_{j \neq i} T^j$.

Now, given an information structure I and, for every voter $i \in N$, a utility function $u^i : A \times T^i \to Re$ such that

$$u^i(\cdot; t^i) \in \Phi(R^i) \text{ for all } t^i = (\tau^i, R^i) \in T^i,$$

one may associate to every DS $d : A \times L^N \to [0,1]$ a game with incomplete information in the sense of Harsanyi (1967-68) on the $4n$-tuple,

$$\Gamma = (L, \ldots, L; T^1, \ldots, T^n; p^1, \ldots, p^n; u^1, \ldots, u^n).$$

In this game every player $i \in N$ chooses to announce a preference order in L as a function of his own type. Hence his set of strategies is the set of decision rules $S^i = \{s \mid s : T^i \to L\}$. Accordingly the payoff function of every player i of type t^i is computed as follows:

$$\sum_{t^{-i} \in T^{-i}} p^i(t^{-i} \mid t^i) \sum_{x \in A} u^i(x; t^i) d(x, (s^1(t^1), \ldots, s^n(t^n))),$$

for all $(s^1, \ldots, s^n) \in S^1 \times \ldots \times S^n$. We shall use the notation:

$$\widetilde{U}^i = \{u^i \mid u^i : A \times T^i \to Re \text{ such that } u^i(\cdot; t^i) \in \Phi(R^i) \text{ whenever } t^i = (\tau^i, R^i) \in T^i\}$$

and

$$\widetilde{U}^N = \bigtimes_{i \in N} \widetilde{U}^i.$$

For the game Γ one may define the following noncooperative equilibrium concept.

Definition 3.4 *Let I be an IS, $u^N \in \widetilde{U}^N$, and d be a DS. A* **(Bayesian) equilibrium** *of the game Γ is an n-tuple $(s^1_*, \ldots, s^n_*) \in S^1 \times \ldots \times S^n$ such that: for all $i \in N$, $t^i \in T^i$ and $R \in L$,*

$$\sum_{t^{-i} \in T^{-i}} p^i(t^{-i} \mid t^i) \sum_{x \in A} u^i(x; t^i) d(x, (s^{-i}_*(t^{-i}), R))$$
$$\leq \sum_{t^{-i} \in T^{-i}} p^i(t^{-i} \mid t^i) \sum_{x \in A} u^i(x; t^i) d(x, (s^1_*(t^1), \ldots, s^n_*(t^n))).$$

Among possible strategies for a player $i \in N$ is the one which consists in announcing his true preference order.

Definition 3.5 *In a game Γ associated with I, some IS, with $(u^1, \ldots, u^n) \in \widetilde{U}^N$ and with some DS d, the* **truth-telling strategy** *of player i is the strategy $\hat{s}^i \in S^i$ such that: $\hat{s}^i(t^i) = R$ whenever $t^i = (\tau^i, R) \in T^i$.*

Finally this allows to define the two generalized concepts of strategy proofness that we shall use in the present incomplete information framework.

Definition 3.6 *Let I be an IS and $u^N \in \widetilde{U}^N$. A DS $d : A \times L^N \to [0,1]$ is* **Bayesian Incentive Compatible** *(BIC) if the truth-telling strategy n-tuple $(\hat{s}^1, \ldots, \hat{s}^n)$ is a (Bayesian) equilibrium of the corresponding game with incomplete information Γ, i.e., for all $i \in N$, $t^i \in T^i$ and $R \in L$*

$$\sum_{t^{-i} \in T^{-i}} p^i(t^{-i} \mid t^i) \sum_{x \in A} u^i(x; t^i) d(x, (\hat{s}^{-i}(t^{-i}), R))$$
$$\leq \sum_{t^{-i} \in T^{-i}} p^i(t^{-i} \mid t^i) \sum_{x \in A} u^i(x; t^i) d(x, (\hat{s}^1(t^1), \ldots, \hat{s}^n(t^n))).$$

Definition 3.7 *Let I be an IS. A DS $d : A \times L^N \to [0,1]$ is* **Ordinally Bayesian Incentive Compatible** *(OBIC) if it is BIC for every $u^N \in \widetilde{U}^N$.*

We see that the set of all BIC $DS's$ is characterized by a finite system of linear inequalities. So it is a convex polytope Also if a DS is strategy-proof then it is BIC for any IS (with the same set of alternatives) and any $u^N \in \widetilde{U}^N$ and hence OBIC (see d'Aspremont and Gérard-Varet, 1979, Theorem 4). In particular dictatorial $DS's$ are BIC (see Dasgupta et al., 1979). Moreover, we have

Theorem 3.2 *Let A be a given set of alternatives. Then a DS $d : A \times L^N \to [0,1]$ is strategy-proof if and only if it is OBIC for any $IS = (A; T^1, \ldots, T^n; p^1, \ldots, p^n)$.*

Proof It remains to show that d is strategy-proof if it is OBIC for every IS. Suppose there is some $R^N_0 \in L^N, i \in N, v \in \Phi(R^i_0)$ and $Q^i_0 \in L$ such that

$$\sum_{x \in A} v(x) d(x, R^N_0) < \sum_{x \in A} v(x) d(x, (R^{N-\{i\}}_0, Q^i_0)),$$

i.e., d is manipulable. Then defining an IS $I = (A, T^1, \ldots, T^n; p^1, \ldots, p^n)$ where $T^j = L$ for all j and $p^i(R^{N-\{i\}}_0 \mid R^i_0) = 1$ we get that

$$p^i(R^{N-\{i\}}_0 \mid R^i_0) \sum_{x \in A} u^i(x; R^i_0) d(x; R^N_0)$$
$$< p^i(R^{N-\{i\}}_0 \mid R^i_0) \sum_{x \in A} u^i(x; R^i_0) d(x; (R^{N-\{i\}}_0, Q^i_0))$$

for $u^i \in \widetilde{U}^i$ such that $u^i(\cdot; R_0^i) = v(\cdot)$. Hence d is not OBIC with respect to I. ■

Theorem 3.3 *Let $I = (A; T^1, \ldots, T^n; p^1, \ldots, p^n)$ be an IS. A DSd is OBIC if and only if for every $i \in N$, every $t^i = (\tau^i, R) \in T^i$ and every $Q \in L$,*

$$\begin{aligned}&\sum_{j=0}^{k-1} \sum_{t^{-i} \in T^{-i}} p^i(t^{-i} \mid t^i) d(t_{m-j}(R), (\hat{s}^{-i}(t^{-i}), R)) \\ \leq &\sum_{j=0}^{k-1} \sum_{t^{-i} \in T^{-i}} p^i(t^{-i} \mid t^i) d(t_{m-j}(R), (\hat{s}^{-i}(t^{-i}), Q)), k = 1, \ldots, m.\end{aligned} \tag{1}$$

The set of OBIC $DS's$ with respect to I is a convex polytope.

Before going into the proof of this theorem notice that the inequalities (1) have the following simple interpretation. For every $i \in N$, every $t^i = (\tau^i, R) \in T^i$, and every "bottom set" B in the order R, the probability of B is made smallest (by i) by telling the truth. Also, we need a lemma.

Lemma 3.1 *Let $R \in L$ and let p and q be probability distributions on A. Then*

$$\sum_{a \in A} p(a)u(a) \geq \sum_{a \in A} q(a)u(a)$$

for all $u \in \Phi(R)$ if and only if

$$\sum_{j=0}^{k-1} p(t_{m-j}(R)) \leq \sum_{j=0}^{k-1} q(t_{m-j}(R)) \textit{ for } k = 1, \ldots, m.$$

Lemma 3.1 is a well-known result on stochastic dominance (see e.g., Hanoch and Levy, 1969).
Proof of Theorem 3.2 $d : A \times L^N \to [0, 1]$ is OBIC with respect to I if and only if for every $i \in N$ and every $t^i \in T^i$

$$\begin{aligned}&\sum_{x \in A} u^i(x; t^i) \sum_{t^{-i} \in T^{-i}} p^i(t^{-i} \mid t^i) d(x, \hat{s}^N(t^N)) \\ \geq &\sum_{x \in A} u^i(x; t^i) \sum_{t^{-i} \in T^{-i}} p^i(t^{-i} \mid t^i) d(x, (\hat{s}^{-i}(t^{-i}), Q))\end{aligned}$$

for all $u^i \in \Phi(\hat{s}^i(t^i))$ and all $Q \in L$. So by Lemma 3.1 d is OBIC if and only if for every $i \in N$, every $t^i = (\tau^i, R) \in T^i$ and every $Q \in L$ the system (1) holds. Thus the set of all OBIC $DS's$ with respect to I is characterized by a finite system of linear inequalities and the result follows. ■

3.3 Existence of OBIC representations for weak games

We shall now introduce our main result which is about OBIC representations (with additional properties). It consists of proving the existence of such representations of simple games with at least one vetoer, having "free beliefs" in the following sense (see d'Aspremont and Gérard-Varet, 1982).

Definition 3.8 *A player $i \in N$ has* **free beliefs** *with respect to some IS $I = (A; T^1, \dots, T^n; p^1, \dots, p^n)$. If*

$$p^i(\cdot \mid t^i) = p^i(\cdot \mid \bar{t}^i) \text{ for all } t^i, \bar{t}^i \in T^i.$$

So consider an $IS\ I = (A; T^1, \dots, T^n; p^1, \dots, p^n)$ and a weak game $G = (N, W)$. The proof will go by construction of a specific OBIC representation of G. For that purpose we need the following definition.

Definition 3.9 *Let $R^N \in L^n$; for $x, y \in A, x \neq y$, and for $S \in W$, x* **dominates** *y (with respect to R^N) via S, written x Dom $(R^n, S)y$ if xR^iy for all $i \in S$; x* **dominates** *y (with respect to R^N), written x Dom $(R^N)y$, if there exists $T \in W$ such that x Dom $(R^N, T)y$; also, an alternative $c \in A$ is called a* **Condorcet alternative** *(with respect to R^N), if c Dom $(R^N)x$ for all $x \in A - \{c\}$. Lastly, a $DS\ d : A \times L^N \to [0,1]$ such that $d(c, R^N) = 1$ whenever c is a Condorcet alternative with respect to $R^N \in L^N$, is said to satisfy the* **Condorcet condition**.

Now, for some vetoer i in the game G, for some $e \in A$ and $\alpha_e : A - \{e\} \to [0,1]$ we define a $DS\ d = d(\cdot, \cdot \mid i, e, \alpha)$ by the following rules:

1. if $t_1(R^i) \neq e$ then $d(t_1(R^i), R^N) = 1$;
2. if $t_1(R^i) = e$ and e Dom $(R^N)t_2(R^i)$, then $d(e, R^N) = 1$;
3. in all other cases $d(t_2(R^i), R^N) = \alpha(t_2(R^i))$, and $d(e, R^N) = 1 - \alpha(t_2(R^i))$.

Remark 3.1 $d(\cdot, \cdot \mid i, e, \alpha)$ is Paretian ex post and monotonic (see Definitions 2.4 and 2.7).

Claim 3.1 For all $R^N \in L^N$ and for all $k \neq i$, $d = d(\cdot, \cdot \mid i, e, \alpha)$ is not potentially manipulable by k at R^N (see Definition 3.1).

Proof Let $k \in N - \{i\}$ and let $R^N \in L^N$. Denote $x = t_1(R^i)$ and $y = t_2(R^i)$. For all $Q_1, Q_2 \in L$, if $[xQ_1y \Leftrightarrow xQ_2y]$ then $d(x, (R^{N-\{k\}}, Q_1)) = d(x, (R^{N-\{k\}}, Q_2))$. Since d is monotonic, it is not potentially manipulable by k at R^N. ∎

Claim 3.2 If there exists $x \neq e$ such that $\alpha(x) > 0$ then $d(\cdot, \cdot \mid i, e, \alpha)$ is a representation of G (see Definition 2.10).

Proof Let $S \in W$ and let $x \in A$. If $R^N \in L^N$ and $t_1(R^h) = x$ for all $h \in S$ then $t_1(R^i) = x$. (Indeed, $S \supset V$; see Definition 2.1. Also, x Dom$(R^N, S)t_2(R^i)$. Hence, $d(x, R^N) = 1$. Thus, $S \in W^*$ (see Definition 2.9).

Suppose now that $S \notin W$. If $i \notin S$ then, clearly, $S \notin W^*$. Thus, we may assume that $i \in S$. Let $x \in A - \{e\}$ satisfy $\alpha(x) > 0$, and let $R^N \in L^N$ satisfy, $t_1(R^h) = e$ for all $h \in S$, $t_2(R^i) = x$ and $t_1(R^h) = x$ for all $h \notin S$. Since $S \notin W$, $d(e, R^N) = 1 - \alpha(x) < 1$. Thus, $S \notin W^*(d, e)$ (see Definition 2.9). Hence, $S \notin W^*$. ∎

Claim 3.3 Assume that i has free beliefs. Then there exists $\alpha : A - \{e\} \to [0,1]$ such that (i) $\alpha(x) > 0$ for some x; and (ii) $d(\cdot, \cdot \mid i, e, \alpha)$ is OBIC (with respect to I).

Proof Let $t^i = (\tau^i, R)$. If $t_1(R) \neq e$ then, clearly, R is a best reply to $\hat{s}^{-i}$, when i's type is t^i. If $t_1(R) = e$, $Q \in L$ and $t_1(Q) \neq e$, then R is at least as good as Q (with respect to every $u^i(\cdot; t^i) \in \Phi(R)$). So, let $Q \in L$ and $t_1(Q) = e$.

Now, R is at least as good a reply as Q (with respect to every $u^i(\cdot;t^i) \in \Phi(R)$), if and only if

$$\begin{aligned} &\sum_{t^{-i}\in T^{-i}} p^i(t^{-i} \mid t^i) d(t_2(R), (\hat{s}^{-i}(t^{-i}), R)) \\ \leq &\sum_{t^{-i}\in T^{-i}} p^i(t^{-i} \mid t^i) d(t_2(Q), (\hat{s}^{-i}(t^{-i}), Q)). \end{aligned} \tag{2}$$

We shall now compute the sum: $\sum_{t^{-i}\in T^{-i}} p^i(t^{-i} \mid t^i) d(\cdot, (\hat{s}^{-i}(t^{-i}), \cdot))$.

A coalition T is blocking if $N - T \notin W$. Let

$$B = \{S \mid i \notin S \text{ and } S \text{ is blocking}\}$$

and, for $x \neq e$,

$$E(x) = \{t^{-i} = (\tau^k, R^k)_{k\neq i} \in T^{-i} \mid \{k \mid xR^k e\} \in B\}.$$

Then, for every $Q \in L$ such that $t_1(Q) = e$,

$$\sum_{t^{-i}\in T^{-i}} p^i(t^{-i} \mid t^i) d(t_2(Q), (\hat{s}^{-i}(t^{-i}), Q)) = p^i(E(t_2(Q)) \mid t^i)\alpha(t_2(Q)).$$

Thus (2) is equivalent to

$$p^i(E(t_2(R)) \mid t^i)\alpha(t_2(R)) \leq p^i(E(t_2(Q)) \mid t^i)\alpha(t_2(Q)). \tag{3}$$

Thus, our problem is to find $\alpha : A - \{e\} \to [0, 1]$ such that *(i)* $\alpha(x) > 0$ for some x; and *(ii)* (3) is satisfied for every $t^i \in T^i$ and every $Q \in L$ which satisfies $t_1(Q) = e$. Now, by assumption, i has free beliefs. Thus, *(ii)* is equivalent to:

$$p^i(E(x))\alpha(x) = p^i(E(y))\alpha(y) \text{ for all } x, y \in A - \{e\}. \tag{4}$$

Let $x_1, \ldots, x_{m-1}$ be an ordering of $A - \{e\}$ such that

$$p^i(E(x_{t+1})) \geq p^i(E(x_t)),\ t = 1, \ldots, m-2.$$

If $p^i(E(x_1)) = 0$, let $\alpha(x_1) = 1$ and $\alpha(x_t) = 0$ for $t = 2, \ldots, m-1$. Then α solves our problem. Finally, if $p^i E(x_1)) > 0$ let $\alpha(x_1) = 1$ and

$$\alpha(x_t) = p^i(E(x_1))/p^i(E(x_t)),\ t = 2, \ldots, m-1.$$

Again, α solves our problem. We conclude from Claim 3.1 that $d(\cdot, \cdot \mid i, e, \alpha)$ is OBIC (with respect to I). ■

We can now summarize all the results we have just obtained by the following.

Theorem 3.4 *Let $G = (N, W)$ be a weak game and let $I = (A; T^1, \ldots, T^n; p^1, \ldots, p^n)$ be an IS. Assume that there exists $i \in V$ with free beliefs. Then there exists a DS $d : A \times L^N \to [0, 1]$ with the following properties.*

(i) d is Paretian ex post and monotonic;

(ii) d is a representation of G which satisfies the Condorcet condition;

(iii) d is OBIC with respect to I.

Proof By Remark 3.1 we know that (i) holds for $d = d(\cdot, \cdot \mid i, e, \alpha)$ as specified by (1)–(3) above. Also (ii) holds by Claim 3.2 and, using Claim 3.1 and Claim 3.3, we get (iii). ■

We see, that, by construction, the DS used in this existence proof has no symmetry property. Some individual i and some alternative e are both privileged. However, by the convexity of the set of all Paretian ex post and monotonic $DS's$ that satisfy the Condorcet condition and are OBIC-representations of the game G, we can introduce some symmetry as follows.

Corollary 3.2 *Let $G = (N, W)$ be a weak game and let $I = (A; T^1, \ldots, T^n; p^1, \ldots, p^n)$ be an IS such that $V^0 = \{i \in V \mid i \text{ has free beliefs}\} \neq \emptyset$. Also, for every $i \in V^0$ and $e \in A$, define first the DS $d(\cdot, \cdot \mid i, e, \alpha_e)$ – following the rules (1)–(3) and the proof of Claim 3.3 – and*

$$d^i(\cdot, \cdot) = \sum_{e \in A} \frac{1}{m} d(\cdot, \cdot \mid i, e, \alpha_e),\ m = |A|,$$
$$d^0(\cdot, \cdot) = \sum_{i \in V^0} \frac{1}{v^0} d^i(\cdot, \cdot), \qquad v^0 = |V^0|.$$

Then the $DS's$ $d^i(\cdot, \cdot)$ and $d^0(\cdot, \cdot)$ are Paretian ex post, monotonic and OBIC representations of G which satisfy the Condorcet condition.

The beliefs of the players may not be symmetric with respect to the alternatives. Hence, the $DS's$ d^i and d^0 may *not* be neutral (see Definition 2.6). Also, if $|V| \geq 2$ then d^i is *not* a faithful representation (of G) by its construction. Moreover, if symmetric players (in G) have different beliefs, then there may exist no OBIC and faithful representations.

Although symmetry properties are difficult to obtain in general, we may show that we have obtained a nice representation of G in still another aspect. Consider a DS d. We have given (see Definition 2.9) the "first" simple game associated with d. We may now associate with d two other games.

Definition 3.10 *Let $d : A \times L^N \to [0, 1]$ be a DS, and let $x \in A$. A coalition S is α-**effective** for x (with respect to d), if there exists $Q^S \in L^S$ such that for all $R^{N-S} \in L^{N-S} d(x, (Q^S, R^{N-S})) = 1$. The set of all α-effective coalitions for x is denoted by $W_\alpha(d, x)$. The* **second simple game associated with** *d is the game $G_\alpha(d) = (N, W_\alpha)$, where*

$$W_\alpha = \cap\{W_\alpha(d, x) \mid x \in A\}.$$

Definition 3.11 *Let $d : A \times L^N \to [0, 1]$ be a DS, and let $x \in A$. A coalition S is β-**effective** for x (with respect to d), if for every $R^{N-S} \in L^{N-S}$ there exists $Q^S \in L^S$ such that $d(x, (Q^S, R^{N-S})) = 1$. The set of all β-effective coalitions for x is denoted by $W_\beta(d, x)$. The* **third simple game associated with** *d is the game $G_\beta(d) = (N, W_\beta)$, where*

$$W_\beta = \cap\{W_\beta(d, x) \mid x \in A\}.$$

We notice that, for every $DS\, d$,

$$W^*(d,x) \subset W_\alpha(d,x) \subset W_\beta(d,x), \text{ for all } x \in A,$$

and that both $G_\alpha(d)$ and $G_\beta(d)$ are monotonic; in addition $G_\alpha(d)$ is proper ($G_\beta(d)$ might not be, as shown by Example 3.1.13 of Peleg, 1984). Also d satisfies the attainability condition (see Definition 2.8) if and only if $N \in W_\beta$.

Now the additional property we want for decision schemes is the following.

Definition 3.12 *A $DS\, d : A \times L^N \to [0,1]$ is* **tight** *if*

$$W^*(d,x) = W_\beta(d,x), \text{ for all } x \in A.$$

This allows us to prove:

Corollary 3.3 *A $DS\, d^i$, as defined in Corollary 3.2, is tight whenever p^i is positive.*

Proof Indeed, we shall prove a stronger result, namely $W = W_\beta(d^i, e)$ for every $e \in A$. Since d^i is a representation of G, the weak game given in Corollary 3.2,

$$W \subset W^*(d^i, e) \subset W_\beta(d^i, e), \text{ for every } e \in A.$$

Thus, we have only to prove the reverse inclusion. So, let $e \in A$ and $S \notin W$. If $i \notin S$ then, obviously, $S \notin W_\beta(d^i, e)$. Thus, let $i \in S$. Since p^i is positive, $\alpha_e(x) > 0$ for all $x \in A - \{e\}$ (see the proof of 3.3). Let $Q^{N-S} \in L^{N-S}$ satisfy $t_m(Q^k) = e$ for all $k \in N - S$. Then, by our construction

$$d(e, (R^S, Q^{N-S}) \mid i, e, \alpha_e) < 1 \text{ for every } R^S \in L^S.$$

Hence, $d^i(e, (R^S, Q^{N-S})) < 1$ for every $R^S \in L^S$. Thus, $S \notin W_\beta(d^i, e)$. ■

4. Examples

In this section we provide an example which shows that the conditions of Theorem 3.4 are virtually necessary. This example is based on a particular class of beliefs and, proves, essentially, that Theorem 3.4 is sharp. Then, using again the same particular class of beliefs, we give a second example showing that Paretian ex post OBIC representations of some games require, when they are "core-selections", the introduction of non-deterministic decision schemes (i.e., social choice functions cannot do). Finally, we observe that Bayesian "approximations" of the "voting paradox" have no BIC representations. Thus, there is no general existence proof for such representations. We start with the following definition and lemma.

Definition 4.1 *Let $I = (A; T^1, \ldots, T^n; p^1, \ldots, p^n)$ be an IS and let $i \in N$. i has* **comprehensive beliefs** *if the following conditions are satisfied:*

(i) $T^i = L^{N-\{i\}} \times L$;

(ii) for all $(R_*^{N-\{i\}}, R_*^i) \in T^i : p^i(t^{-i} \mid R_*^{N-\{i\}}, R_*^i)) > 0$ only if $t^j = (\tau^j, R_*^j)$ for all $j \neq i$.

Lemma 4.1 *Let $I = (A; T^1, \ldots, T^n; p^1, \ldots, p^n)$ be an IS and let $d : A \times L^N \to [0,1]$ be OBIC (with respect to I). If k has comprehensive beliefs, then d is not manipulable by k.*

Proof Assume, on the contrary, that there exists $R_*^N \in L^N$ such that d is potentially manipulable by k at R_*^N. Hence, there exists $v \in \Phi(R_*^k)$ and $Q^k \in L$ such that

$$\sum_{x \in A} v(x) d(x, R_*^N) < \sum_{x \in A} v(x) d(x, (R_*^{N-\{k\}}, Q^k)).$$

Let $u^k(x; (R_*^{N-\{k\}}, R_*^k)) = v(x)$ for all $x \in A$. Then

$$\begin{aligned}
&\sum_{t^{-k} \in T^{-k}} p^k(t^{-k} \mid R_*^{N-\{k\}}, R_*^k)) \sum_{x \in A} u^k(x; (R_*^{N-\{k\}}, R_*^k)) d(x, \hat{s}^N(t^N)) \\
=&\sum_{x \in A} v(x) d(x, R_*^N) < \sum_{x \in A} v(x) d(x, (R_*^{N-\{k\}}, Q^k)) \\
=& \sum_{t^{-k} \in T^{-k}} p^k(t^{-k} \mid (R_*^{N-\{k\}}, R_*^k)) \sum_{x \in A} u^k(x; (R_*^{N-\{k\}}, R_*^k)) d(x, (\hat{s}^{-k}(t^{-k}), Q^k)).
\end{aligned}$$

Thus, d is not OBIC with respect to I, and the desired contradiction has been obtained. ■

We now prove, by means of an example, the following claim. Assume – to exclude the possibility of strategy-proof representation – that G is not a unanimity game (up to the addition of dummies). Further assume that only the beliefs of one player, player i, may be restricted. Then, G may have no OBIC representation, unless i is a vetoer.

Example 4.1 Let $G = (N, W)$ be a proper and monotonic simple game, let $i \in N - V$, and let $I = (A; T^1, \ldots, T^n; p^1, \ldots, p^n)$ be an IS. Assume that (i) G has at least two minimal winning coalitions; (ii) $|A| \geq 3$, and (iii) every $k \in N - \{i\}$ has comprehensive beliefs. Then G has no OBIC representation.

Assume, on the contrary, that $d : A \times L^N \to [0, 1]$ is an OBIC representation of G. By Lemma 4.1 d is not manipulable by every $k \in N - \{i\}$. Now choose $Q^i \in L$ and consider the $DS\, d^* : A \times L^{N-\{i\}} \to [0, 1]$ given by

$$d^*(x, R^{N-\{i\}}) = d(x, (R^{N-\{i\}}, Q^i)).$$

d^* is strategy-proof. Because i is not a vetoer, $N - \{i\} \in W$. Hence, d satisfies the attainability condition (see Definition 2.8). Thus, $d^* = \sum_{j \neq i} a_j \delta_j$, where $a_j \geq 0$, $j \in N - \{i\}$, and $\sum_{j \neq i} a_j = 1$ (see Definition 3.2 and Corollary 1* of Hylland, 1980). Let $a_k > 0$. If $k \notin V$ then $N - \{k\} \in W$. Now choose $R^{N-\{i\}} \in L^{N-\{i\}}$ such that $t_1(R^h) = t_1(Q^i)$ if $h \neq k$, and $t_1(R^k) = t_m(Q^i)$ (where $|A| = m$). Then

$$1 > 1 - a_k = d^*(t_1(Q^i), R^{N-\{i\}}) = d(t_1(Q^i), (R^{N-\{i\}}, Q^i)).$$

Thus, d is not a representation of G in this case. Hence, we may assume $\{k \mid a_k > 0\} \subset V$. This implies that V is winning with respect to d^*. Because we may assume that this is true for every $Q^i \in L$, V is winning with respect to d. Now, V is not in W (because G has more than one minimal winning coalition). Hence d is not a representation of G.

Now we shall show, again by means of an example, that randomization is essential for the validity of Theorem 3.4. More precisely, we shall demonstrate the nonexistence of OBIC social choice functions (i.e. deterministic $DS's$) under the assumptions of that theorem. Thus, the use of (nondegenerate) $DS's$ is essential for the theory of voting with incomplete information. This remark may serve as a justification for introducing and investigating $DS's$, that was very much needed for a long time.

We now turn to our second example. First, we need a definition and a lemma.

Definition 4.2 *A DS $d : A \times L^N \to [0,1]$ is a* **social choice function** *(SCF), if, for every $R^N \in L^N$ and every $x \in A$, $d(x, R^N) \in \{0,1\}$.*

Lemma 4.2 *Let $G = (N, W)$ be a proper, and monotonic game, let $n \geq 2$, let $i \in N$, and let $I = (A; T^1, \ldots, T^n; p^1, \ldots, p^n)$ be an IS. Assume that (i) $V \subset \{i\}$, and (ii) every $k \in N - \{i\}$ has comprehensive beliefs. Now let an SCF $d : A \times L^N \to \{0,1\}$ be an OBIC representation of G. Then, for every $Q^i \in L$ the SCF $d^* : A \times L^{N-\{i\}} \to \{0,1\}$ given by*

$$d^*(x, R^{N-\{i\}}) = d(x, (R^{N-\{i\}}, Q^i)) \text{ for all } x \in A,$$

is duple, that is the range of d^* contains at most two alternatives (x is in the range of d^* if $d^*(x, R^{N-\{i\}}) = 1$ for some $R^{N-\{i\}} \in L^{N-\{i\}}$).

Proof Let $Q^i \in L$. By Lemma 4.1 d^* is nonmanipulable. Hence, by the Gibbard-Satterthwaite Theorem d^* is dictatorial or duple (see Theorem 2.5.5 in Peleg, 1984). Assume, on the contrary, that d^* is dictatorial. Let A^* be the range of d^*. Then there exists $k \in N - \{i\}$ such that for every $R^{N-\{i\}} \in L^{N-\{i\}}$ the following is true: $d^*(x, R^{N-\{i\}}) = 1$ if $x \in A^*$ and xR^ky for every $y \in A^*$. Clearly, we may assume $|A^*| \geq 3$. Now, $k \notin V$. Hence, $N - \{k\} \in W$. Choose $R^{N-\{i\}} \in L^{N-\{i\}}$ such that $t_1(R^h) = t_1(Q^i)$ if $h \neq k$, and $t_1(R^k) \in A^* - \{t_1(Q^i)\}$. Then $d^*(t_1(R^k), R^{N-\{i\}}) = 1 = d(t_1(R^k), (R^{N-\{i\}}, Q^i))$, contradicting the assumption that d is a representation of G. ■

Corollary 4.1 *Under the assumptions of Lemma 4.2 the following claim is true. Let A^* be the range of d^* and let L_* be the set of all linear orderings of A^*. Then there exists an SCF $\tilde{d} : A^* \times L_*^{N-\{i\}} \to \{0,1\}$ such that, for every $R^{N-\{i\}} \in L^{N-\{i\}}$ and every $x \in A$, $d^*(x, R^{N-\{i\}}) = \tilde{d}(x, R^{N-\{i\}} \mid A^*)$, where $R^{N-\{i\}} \mid A^*$ is the restriction of $R^{N-\{i\}}$ to A^*.*

Corollary 4.1 follows from the fact that d^* is nonmanipulable (see e.g., Gibbard, 1977).

Corollary 4.2 *Let $d : A \times L^N \to \{0,1\}$ be an OBIC representation of G as in Lemma 4.2. Assume further that d is a core-selection, that is, for every $R^N \in L^N$ if $d(x, R^N) > 0$ then x is not dominated with respect to R^N (see Definition 3.9). Then the range of d^* contained in $\{t_1(Q^i), t_2(Q^i)\}$.*

The proof of Corollary 4.2 is straightforward.

Example 4.2 Let $G = (\{1,2,3\}, \{\{1,2\}, \{1,3\}, \{1,2,3\}\})$, $A = \{a,b,c\}$, and $I = (A, T^1, T^2, T^3; p^1, p^2, p^3)$, and let $R_1 = (a,b,c)$, $R_2 = (a,c,b)$, $R_3 = (b,a,c)$, $R_4 = (b,c,a)$, $R_5 = (c,a,b)$, and $R_6 = (c,b,a)$. Assume that (i) 2 and 3 have comprehensive beliefs; (ii) $T^1 = L$, ad (iii) 1 has free beliefs that satisfy $p^1(R_1, R_5) = x_1$, $p^1(R_3, R_6) = x_2$, $p^1(R_2, R_3) = x_3$, $p^1(R_4, R_5) = x_4$, $p^1(R_1, R_4) = x_5$, $p^1(R_2, R_6) = x_6$, where $p^1(R_k, R_1) = \sum_{(\tau^2,\tau^3)} p^1(\tau^2, R_\ell), (\tau^3, R_\ell), 0 < x_1 < x_2 < x_3 < x_4 < x_5 < x_6$ and $\sum_{i=1}^6 x_i = 1$. Then there exists no SCF $d : A \times L^N \to \{0,1\}$ with the following properties: (a) d is an OBIC representation of G, and (b) d is a core-selection.

Indeed, assume on the contrary, that $d : A \times L^N \to \{0,1\}$ satisfies (a) and (b). Let $Q^1 \in L$, let $T = \{2,3\}$, and let $A^* = \{t_1(Q^1), t_2(Q^1)\}$. By the foregoing discussion there exists an SCF $\tilde{d} : A^* \times L_*^T \to \{0,1\}$ such that $d(x,(R^T,Q^1)) = \tilde{d}(x, R^T \mid A^*)$ for every $x \in A^*$ and every $R^T \in L^T$ (here L_* is the set of linear orderings of A^*). We denote

$$\alpha(t_2(Q^1), Q^1) = \tilde{d}(t_2(Q^1), ((t_2(Q^1), t_1(Q^1)), (t_2(Q^1), t_1(Q^1)))). \tag{5}$$

Also, we observe that if $R^T \mid A^* \neq ((t_2(Q^1), t_1(Q^1)), (t_2(Q^1), t_1(Q^1)))$ then $d(t_2(Q^1), (R^T, Q^1)) = 0$ because d is a core-selection. Now d is OBIC. Hence,

$$\begin{aligned}\sum_{R^T \in L^T} p^1(R^T) d(t_2(Q^1), (R^T, Q^1)) \le &\sum_{R^T \in L^T} p^1(R^T) d(t_3(Q^1), (R^T, Q^*)) \\ &+ \sum_{R^T \in L^T} p^1(R^T) d(t_2(Q^1), (R^T, Q^*))\end{aligned} \tag{6}$$

for every $Q^* \in L$ (notice that $d(t_3(Q^1), (R^T, Q^1)) = 0$ for every $R^T \in L^T$). Now choose $Q^1 = R^1$ and $Q^* = R_2$. (6) yields

$$\begin{aligned}&\sum_{R^T \in L^T} p^1(R^T) d(b, (R^T, R_1)) = x_2 \alpha(b, R_1) \\ \le &\sum_{R^T \in L^T} p^1(R^T) d(c, (R^T, R_2)) = x_4 \alpha(c, R_2).\end{aligned}$$

Thus, $x_2\alpha(b,R_1) \le x_4\alpha(c,R_2)$. If we interchange R_2 and R_1 in the foregoing inequalities we obtain $x_4\alpha(c,R_2) \le x_2\alpha(b,R_1)$. Hence $x_2\alpha(b,R_1) = x_4\alpha(c,R_2)$. Because $\alpha(b,R_1), \alpha(c,R_2) \in \{0,1\}$, and $0 < x_2 < x_4$, we have $\alpha(b,R_1) = \alpha(c,R_2) = 0$. Similarly, one proves that $\alpha(t_2(Q^1), Q^1) = 0$ for every $Q^1 \in L$. Hence d must satisfy $d(t_1(R^1), R^N) = 1$ for every $R^N \in L^N$, that is $d = \delta_1$. Because δ_1 is not a representation of G, the desired contradiction has been obtained.

We now remark that the $DS\, d(\cdot,\cdot \mid i,e,\alpha)$ of Theorem 3.4 is a core-selection. We conclude this section with an example which shows that voting problems which are "close" to the "voting paradox" have no BIC representations.

Example 4.3 Let $G = (N, W)$ be a proper and monotonic simple game. Assume that there exists a partition $\{S_1, S_2, S_3\}$ of N with the following property. For each $1 \le j \le 3$ there exists $i_j \in S_j$ such that $\{i_j\} \cup S_t \in W$, where $1 \le t \le 3$ and $t \equiv j + 2(3)$. Now consider the $IS\, I = (A; T^1, \ldots, T^n; p^1, \ldots, p^n)$, where $A = \{a,b,c\}$, $T^i = L$, $i = 1, \ldots, n$, and p^i, $i \in N$, are defined as follows. Let $R_1 = (a,b,c)$, $R_2 = (c,a,b)$, and $R_3 = (b,c,a)$. For $i \in S_j$, $j = 1,2,3$, let $p_i(R_j) = 1 - \varepsilon$, where $1 > \varepsilon > 0$, and $p_i(R) = \varepsilon/5$ for $R \in L - \{R_j\}$. We define, for $i \in N$,

$$p^i(R^{N-\{i\}} \mid R^i) = \prod_{k \neq i} p_k(R^k) \ \text{ for all } R^{N-\{i\}} \in L^{N-\{i\}} \text{ and } R^i \in L.$$

Thus, all the players have free beliefs. Now, we consider utility functions $u^i : A \times L \to Re$, $i \in N$, that satisfy for some $\delta > 0$:

$$\begin{array}{lllll} u^i(a;R_1) = 1+\delta, & u^i(b;R_1) = 1, & \text{and} & u^i(c;R_1) = 0, & \text{if} \quad i \in S_1; \\ u^i(c;R_2) = 1+\delta, & u^i(a;R_2) = 1, & \text{and} & u^i(b;R_2) = 0, & \text{if} \quad i \in S_2; \\ u^i(b;R_3) = 1+\delta, & u^i(c;R_3) = 1, & \text{and} & u^i(a;R_3) = 0, & \text{if} \quad i \in S_3. \end{array}$$

We assume that $(1-\varepsilon)^{n-1} > (1+\delta)(2/3 + 2^{n-1}\varepsilon)$.

Let $d : A \times L^N \to [0,1]$ be a representation of G. We claim that d is not BIC (with respect to I and $u^i, i \in N$). Let R_*^N be given by $R_*^i = R_j$ if $i \in S_j, j = 1,2,3$. Without loss of generality we may assume that $d(c, R_*^N) \geq 1/3$. Let $h \in S_1$ such that $\{h\} \cup S_3 \in W$. We compute

$$\begin{aligned}
&\sum_{R^{N-\{h\}}} \prod_{k \neq h} p_k(R^k) \sum_{x \in A} u^h(x; R_*^h) d(x, (R^{N-\{h\}}, R_*^h)) \\
\leq &(1-\varepsilon)^{n-1}(1+\delta)\frac{2}{3} + (1-(1-\varepsilon)^{n-1})(1+\delta) \\
\leq &\frac{2}{3}(1+\delta) + (1+\delta)\sum_{i=1}^{n-1} \varepsilon^i (-1)^{i+1} \begin{pmatrix} n-1 \\ i \end{pmatrix} \leq (1+\delta)\left(\frac{2}{3} + \varepsilon 2^{n-1}\right).
\end{aligned}$$

Now let $\hat{R} = (b, a, c)$. Then $d(b, (R_*^{N-\{h\}}, \hat{R})) = 1$ and

$$\sum_{R^{N-\{h\}}} \prod_{k \neq h} p_k(R^k) \sum_{x \in A} u^k(x; R_*^h) d(x, (R^{N-\{h\}}, \hat{R})) \geq (1-\varepsilon)^{n-1}.$$

Thus, d is not BIC (because $(1-\varepsilon)^{n-1} > (1+\delta)(2/3 + \varepsilon 2^{n-1})$).

We conclude the example by describing an infinite family of weighted majority games

$$G_n = [n-1; n-2, \overbrace{1, \ldots, 1}^{n-1}],\ n = 3, 4, \ldots$$

that possess partitions with the foregoing properties. Indeed, the minimal winning coalitions in G_n are $\{1, i\}$, $i = 2, \ldots, n$, and $\{2, \ldots, n\}$. Hence the partition $\{\{1\}, \{3, \ldots, n\}, \{2\}\}$ has all the required properties.

For $n = 3$, $G_3 = (3, 2)$ is the 3-person simple majority game, and our Bayesian voting problem is "close" to the "voting paradox". (The "voting paradox" itself is obtained when $n = 3$ and $\varepsilon = 0$.)

5. Concluding remarks

We now summarize our results. Let $I = (A; T^1, \ldots, T^n; p^1, \ldots, p^n)$ be an IS. Then the class of OBIC $DS's$ (with respect to I) is a (non-empty) polytope. It certainly includes all strategy-proof $DS's$. Only when a power distribution among coalitions, that is a simple game, $G = (N, W)$ is specified, the selection of an OBIC DS representing it becomes a meaningful and important problem in the theory of voting. The selection of an efficient (OBIC) DS without reference to a power structure, like in, e.g. Myerson (1985), is, in our model, both (mathematically) trivial and uninteresting from the point of view of applications (because it may simply result, for instance, in a choice of a dictatorial DS). It is not surprising, therefore, that our results are highly dependent on the simple game G. If G has only one minimal winning coalition, a strategy-proof representation may be obtained. If W contains more than one minimal winning coalition, then we prove the existence of an OBIC representation under the assumption that there is a vetoer with free beliefs.

Also, we show that if the beliefs of all players except one are unrestricted, then, in order for an OBIC representation to exist the player with constrained beliefs must be a vetoer (see Example 4.3). This shows that our main existence result, Theorem 3.4, is essentially sharp (in the case of one player with restricted beliefs). Of course, one might obtain additional existence results by imposing restrictions on the beliefs of *groups* of players. However, an analysis of a generalized version of the "voting paradox" shows that there exist (infinitely many) simple games that do not possess OBIC representations even when *all* the players have free (and consistent) beliefs. Thus, there is no hope for a general existence theorem.

Finally, we examine another aspect of the proof of the main result, Theorem 3.4. We show that *it may be impossible to obtain an OBIC representation* (under the assumptions of that theorem) *by means of a deterministic DS* (i.e., a social choice function). This last result may serve as a justification for the study, in social choice theory, of procedures combining voting with chance.

We conclude with the following two remarks.

1. It is possible to define in our model a notion of ("interim") efficiency in the sense of Myerson [11]. Indeed, let $I = (A; T^1, \dots, T^n; p^1, \dots, p^n)$ be an IS and let d and d^* be OBIC $DS's$ (with respect to I). Then d^* dominates d if for every $i \in N$ and for every $t^i \in T^i$

$$\sum_{t^{-i} \in T^{-i}} p^i(t^{-i} \mid t^i) \sum_{x \in A} u^i(x; t^i) d^*(x, \hat{s}(t^N)) \\ \geq \sum_{t^{-i} \in T^{-i}} p^i(t^{-i} \mid t^i) \sum_{x \in A} u^i(x; t^i) d(x, \hat{s}(t^N))$$

for every $u^i(\cdot; t^i) \in \Phi(\hat{s}(t^i))$, with at least one strict inequality. Now, an OBIC d is (interim) *efficient* if it is undominated.

The notion of ex post efficiency that we have used (see Definition 2.4) is very different in spirit: it does not combine efficiency with incentive compatibility, it is an "unconstrained" (with respect to incentives) efficiency. Hence, in general, this notion is not comparable with interim-efficiency. Moreover, since the set of OBIC representations of a committee is not closed in most cases, we were not able to prove the existence of interim-efficient OBIC representations.

2. One might think that the approach used here is unduly intricate and that we could have used the "revelation principle" (see Myerson, 1985) to look more straightforwardly for OBIC representations of simple games. However this other approach could not lead to our results. First, applying the revelation principle means looking for mechanisms that are functions of all the characteristics of the players and this, we have argued, goes against real-life observations of voting procedures. In particular we should have worked with more general mechanisms (called "direct") than decision schemes. But, second and more importantly, by looking "directly" for such mechanisms as being incentive compatible, we would then be unable to derive from them decision schemes (as mechanisms based on less information) and preserve at the same time the power structure. We would not obtain "representations". However the precise comparison of what can be obtained by the application of our requirements to what can be obtained by the application of the "revelation principle" is difficult and should be a topic for future research.

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