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ORDINAL CONDITIONAL FUNCTIONS: A DYNAMIC THEORY OF EPISTEMIC STATES*

1. INTRODUCTION

Many of the philosophically most interesting notions are overtly or covertly epistemological. Overtly epistemological notions are, of course, the concept of belief itself, the concept of subjective probability, and, presumably the most important, the concept of a reason in the sense of a theoretical reason for believing something. Covertly epistemological notions are much more difficult to understand; maybe, they are not epistemological at all. However, a very promising strategy for understanding them is to try to conceive of them as covertly epistemological. One such notion is the concept of objective probability;[1] the concept of explanation is another. A third, very important one is the notion of causation, which has been epistemologically problematic ever since Hume. Finally, there is the notion of truth. Many philosophers believe that there is much to be said for a coherence theory of truth or internal realism; they hold some version of the claim that something for which it is impossible to get a true reason cannot be true, and that truth is therefore covertly epistemological.

Now, if one wants to approach these concepts in a more formal way in order to understand them more clearly and more precisely, the first step will be to try to get a formal grip on epistemology. Here, I am concerned only with this first step.[2] Considering the impressive amount of work in formal epistemology, two general points arise.

The first is very familiar, though it still strikes me as somehow odd; it consists in the fact that formal epistemology, i.e. the formal representation of epistemic states, may be divided into a probabilistic and a deterministic branch (and some things which don't quite fit into the scheme). In a deterministic epistemology, as I call it, one talks about a proposition *being simply believed true or false or neither* by some epistemic subject. The formal machinery established for this works with belief sets, truth in all doxastic alternatives, or similar things well known from epistemic logic.[3] In a probabilistic epistemology, belief is more finely graded and comes in numerical degrees. The formal machinery appropriate to it is, of course, probability theory.

This dichotomy is naturally prepared for on the intuitive level. All the intuitive notions we have for subjective and objective probability fall on the probabilistic side. Plain belief, of course, belongs to the deterministic side. And so does truth; the simplest reason for this is, I think, that an arbitrary, perhaps uncountable conjunction of truths is still a truth – this being a formal property of truth which cannot be modelled probabilistically. However, the dichotomy is not complete on the intuitive level. The concept of a reason is certainly neutral between the two forms of epistemology. The same holds for the concept of explanation, as we have learned from Hempel, and for the concept of causation, as has been stressed by many who take probabilistic causation seriously.

Of course, one would like to get rid of this dichotomy, i.e. to reduce one side of it to the other; and this can only mean reducing deterministic to probabilistic epistemology. However, this is not so easy, as is highlighted by the famous lottery paradox. Indeed, the different behaviour of conjunction in deterministic and probabilistic formalisms seems to entirely exclude such a reduction. Then, we should do the second best, i.e. we should develop both forms of epistemology as far as possible and then look what we can say about their relations.

Now, however, we have to consider the second point, namely that deterministic epistemology is in a much poorer shape than probabilistic epistemology. One important aspect is that probabilistic epistemology is well entrenched in a behavioral theory, i.e. decision theory; and this is hardly counterbalanced by the fact that a deterministic epistemology can be more easily used in a theory of language.[4] What is more important, however, is that the inner functioning of deterministic epistemology is so much poorer. Usual probabilistic conditionalization and the generalized conditionalization of Jeffrey (1965), ch. 11, give a plausible account of rational epistemic changes. Probability theory also provides a good model for the impact of evidence and counter-evidence on our beliefs, for the weighing of reasons and counter-reasons; it provides, in other words, a good explication for relevance, potential or conditional relevance, and irrelevance in the epistemic sense. As far as I can see, deterministic epistemology can, in the present state, not produce equivalent achievements.

That is precisely what this paper is about; I shall try to raise deterministic epistemology to the level of probabilistic theorizing. More specifically, I shall try to give a more satisfying account of rational changes, i.e. of the dynamics of deterministic epistemic states. It is to be expected, and will become evident, that this brings advance also on the other scores mentioned. Moreover, it will turn out that the problems I am concerned with are in fact present and unsolved at the probabilistic side as well; thus the paper will also add something to probabilistic epistemology.

This being my focus, I greatly simplify my business by proceeding from the obsolete view that belief is a strictly propositional attitude, i.e. that the objects of belief are complete propositions as expressed by eternal sentences. I thereby neglect other serious problems with epistemic states such as the de-re/de-dicto distinction, the fact that belief is most likely neither propositional nor sentential, but something midway, and the observation that belief seems to be as heavily indexical as language itself. But there is no agreed formal epistemology for handling these problems, and our dynamic problem is certainly intricate enough; hence, I comply with that old view and its associated method of possible world talk.

Having thus laid out the general setting, I shall proceed in the following way. First of all, I'd like to keep separate the story I have to tell and the comments relating it to existing ideas and conceptions. My reason for this is not the novelty of the story (only one feature is really new, as far as I know); rather, I wish to do so because: I think that the story is simple and self-contained; I do not want anything read into it which is not explicitly written into it; and the danger of misreading is the greater, the sooner one mixes up this story with similar, but not completely congruent stories. Thus, I defer all comparative remarks to the final Section 8. The story I want to tell starts in Section 2 with a presentation of what I take to be the essentials of the received deterministic conception of epistemic states. In Section 3, I shall state a crucial problem and argue that it cannot be adequately treated within that received conception. In Section 4, I shall introduce my proposal for a solution of this problem i.e. the concept of an ordinal conditional function, and in Sections 5 and 6 the theory of ordinal conditional functions is developed up to a point where it may not be too much to say that this theory offers a genuine qualitative counterpart to probability theory.[5] Finally, Section 7 explains why the whole story also has a considerable bearing for probabilistic epistemology.

2. SIMPLE CONDITIONAL FUNCTIONS

Having made things simple by assuming belief to be propositional, we shall work with the common, technically convenient framework of possible worlds. Thus, throughout this paper, W is to denote a non-empty set of possible worlds (or a sample space, in probabilistic terms).[6] A *proposition* then is just any subset of W.

The most straightforward deterministic representation of an epistemic state is, of course, as a set of propositions, namely those propositions believed true in that state. Will any set of propositions do? No. Usually, it is required, as conditions of rationality, that such a set of propositions be consistent and deductively closed. One might object that this requires an unattainable logical perfection rather than a form of rationality. Indeed; but the logical perfection is already assumed by taking belief to be propositional. For, taking belief to be propositional means that, for any two sentences having the same content, i.e. expressing the same proposition, an epistemic subject should recognize them to have the same content. Thus, it means that epistemic subjects have perfect semantic knowledge which embraces perfect logical knowledge. And given that, the conditions of rationality seem perfectly acceptable; any indication that a subject violates these conditions is also evidence that his semantic knowledge is not perfect.[7]

Formally, these conditions amount to this: If $\mathcal{B}$ is a set of propositions, then $\mathcal{B}$ is consistent iff $\cap\mathcal{B} \neq \emptyset$, and $\mathcal{B}$ is deductively closed iff we have $A \in \mathcal{B}$ whenever there is a $\mathcal{B}' \subseteq \mathcal{B}$ with $\cap\mathcal{B}' \subseteq A$.[8] From this, it follows immediately that, for consistent and deductively closed $\mathcal{B}$, $A \in \mathcal{B}$ iff $\cap\mathcal{B} \subseteq A$. Thus, we can represent an epistemic state simply by a single non-empty proposition C, and the set of propositions believed true in that state is $\{A \mid C \subseteq A\}$. We shall call this proposition C the *net content* of that epistemic state.

If we represent epistemic states simply by their net contents, what can we say about their temporal change? To begin with, it is clear that epistemic changes may have many causes: experiences, forgetfulness, wishful thinking, drugs, etc. And it is also clear that from our armchair position we can at best hope to say something about *rational* epistemic changes on the ground of experience, information and the like. So, suppose that the epistemic state of the subject X at time t has the net content C and that the proposition A represents all the information X gets and accepts between t and t'. What then is the net content C' of

X's epistemic state at t', provided X is not subject to arational influences? We have to distinguish two cases here:

First, consider the case where $C \cap A \neq \emptyset$, i.e. where the new information is compatible with the old beliefs of X. In this case, it is reasonable to assume that $C' \subseteq C \cap A$, since the new information, because of its compatibility with C, does not force X to give up any of his old beliefs. And it is also reasonable to assume that $C \cap A \subseteq C'$; otherwise, X would at t' believe some proposition not implied by his old beliefs and the new information, and there is no good reason for doing so. Thus, rational belief change is in this case characterized by $C' = C \cap A$.

The other case to consider is that $C \cap A = \emptyset$, i.e. that the new information contradicts the old beliefs. This is a very common case; we often learn that we were wrong. And usually, it is an undramatic case; the rearrangement of beliefs usually takes place without much difficulty. However, all attempts to spell out objective principles for the rearrangement of beliefs in this case have failed. The only thing that can at present be confidently said about this case is that X arrives at *some* new epistemic state which includes the belief in A (since A was supposed to be accepted information), i.e. that $\emptyset \neq C' \subseteq A$.

We are thus left with an incomplete account of rational belief change. How can we improve upon the situation? Well, I shall not try to say anything more substantial about the last critical case – as so many have tried to do by invoking such things as lawlike sentences, modal categories, similarity, epistemic importance, informational value, etc., which may appear to be antecedently understandable. Rather, the only thing I shall try to do is to turn what appears to be a partially undetermined process on the surface level of the net contents of epistemic states into a completely determined process on some suitable deeper level. Thus, all the notions introduced in the course of my story are only meant to provide a theoretical substructure to this surface level which derives its meaning exclusively from what it says about the surface level (which I indeed assume to be antecedently understandable). In a sense, we shall only go beneath and not beyond what we have already said. I stress this point, because it seems to involve changing the usual tactics towards our question.

So, what can be done along these lines? Since the above observations about epistemic changes hold for any possible information, we can, as a first reasonable step, define a function which collects all the possible changes of the net contents of epistemic states brought about by all possible informations. Such functions are defined in

DEFINITION 1. The function g is a *simple conditional function (SCF)* iff g is a function from the set of all non-empty subsets of W into the set of all subsets of W such that the following conditions hold for all non-empty $A, B \subseteq W$:

(a) $\emptyset \neq g(A) \subseteq A$,

(b) if $g(A) \cap B \neq \emptyset$, then $g(A \cap B) = g(A) \cap B$.

The interpretation of SCFs is clear: If we use an SCF g for describing X at t, it says that, if A is the information X accepts by $t' > t$, $g(A)$ is the net content of X's epistemic state at t; or briefly: X believes at t B conditional on A iff $g(A) \subseteq B$. This includes that the net content of X's epistemic state at t itself is given by $g(W)$, since the tautological information W leaves X's epistemic state unchanged; hence, X believes B at t iff $g(W) \subseteq B$. An SCF thus provides a *response scheme* to all possible informations.

It is also clear that an SCF should have the properties fixed in Def. 1: The exclusion of the empty set from the domain of an SCF reflects the fact that a contradiction is not an acceptable information. Clause (a) says that, whatever information is accepted, the beliefs remain consistent and include the information. And clause (b) is a natural generalization of what we have said about the case where the new information is compatible with the old beliefs: Our above consideration concluded that, in the present terms, $g(B) = g(W) \cap B$, if $g(W) \cap B \neq \emptyset$; and if we take not, as we did, $g(W)$, but rather the state informed by A, i.e. $g(A)$, as the starting point of that consideration, we just get clause (b).[9]

An SCF is, we understand, a response scheme to all possible informations. Now, a natural further step, which has not been made so far, is to assume that the response scheme which holds for a subject X at some time t is already embodied in the epistemic state of X at t. This means, however, that we give up representing epistemic states simply by their net contents. Rather, we now conceive them as more complicated things representable by SCFs. This is an advance; we can now state a rule for the dynamics of belief which is completely determinate: If the SCF g represents the epistemic state of X at t and if A is the information X accepts between t and t', then X believes B at t' iff $g(A) \subseteq B$ (provided X is not subject to arational influences).

Is this the end of the story? No, for a very simple reason which will be introduced in the next section. Before that, let me introduce an intuitively and technically very useful concept which is equivalent to that of an SCF. Here as well as in all later sections, $\alpha, \beta, \gamma, \ldots, \zeta, \ldots$ will always be used to denote ordinal numbers.

DEFINITION 2. The sequence $(E_\alpha)_{\alpha<\zeta}$ is a *well-ordered partition*, a *WOP* (of W) iff we have for all $\alpha, \beta < \zeta$: $E_\alpha \neq \emptyset$, $E_\alpha \cap E_\beta = \emptyset$ for $\alpha \neq \beta$, and $\bigcup_{\alpha<\zeta} E_\alpha = W$.

DEFINITION 3. If $(E_\alpha)_{\alpha<\zeta}$ is a WOP and g an SCF, we say that $(E_\alpha)_{\alpha<\zeta}$ *represents* g iff for each non-empty $A \subseteq W$ $g(A) = E_\beta \cap A$, where $\beta = \min\{\alpha | E_\alpha \cap A \neq \emptyset\}$.

THEOREM 1. *Each SCF is represented by exactly one WOP, and each WOP represents exactly one SCF.*

Proof. Let g be an SCF. Define by transfinite recursion: $E_\beta = g(W \setminus \bigcup_{\alpha<\beta} E_\alpha)$. Let ζ be the smallest α for which $E_\alpha = \emptyset$. It is obvious that $(E_\alpha)_{\alpha<\zeta}$ is a WOP. Does it represent g? Yes, as may be seen thus: Let A be a non-empty subset of W and $\beta = \min\{\alpha | E_\alpha \cap A \neq \emptyset\}$. Then we have with the help of clause (b) of Def. 1: $g(A) = g(W \setminus \bigcup_{\alpha<\beta} E_\alpha) \cap A = g(W \setminus \bigcup_{\alpha<\beta} E_\alpha) \cap A = E_\beta \cap A$.

Conversely, let $(E_\alpha)_{\alpha<\zeta}$ be a WOP. Let the function g be defined for all non-empty $A \subseteq W$ as in Def. 3. It is obvious that g then satisfies clause (a) of Def. 1. Now suppose that $g(A) \cap B \neq \emptyset$. This means that $E_\beta \cap A \cap B \neq \emptyset$, where $\beta = \min\{\alpha | E_\alpha \cap A \neq 0\}$. Hence, we also have $\beta = \min\{\alpha | E_\alpha \cap A \cap B \neq \emptyset\}$. This implies that $g(A \cap B) = g(A) \cap B$. Thus, g also satisfies clause (b) of Def. 1, i.e. is an SCF.

Finally, the uniqueness claims of Theorem 1 again are rather obvious. Q.E.D.

A WOP $(E_\alpha)_{\alpha<\zeta}$ is easily interpretable as an *ordering of disbelief* in possible worlds; E_0 contains the possible worlds not disbelieved at all, E_1 contains the least disbelieved worlds, E_2 the second least disbelieved, and so on.[10] The rule for changing beliefs then takes a very simple form: If you now have the ordering $(E_\alpha)_{\alpha<\zeta}$ of disbelief, then you now believe that the true world is among the not disbelieved worlds, i.e. in E_0; thus, E_0 is the net content of your present state. And if you get information A, then you believe that the true world is among the least disbelieved within that information, i.e. in your new net content $E_\beta \cap A$, where $\beta = \min\{\alpha | E_\alpha \cap A \neq \emptyset\}$. What Theorem 1 shows is that response schemes (SCFs) are equivalent to such orderings of disbelief; so we may, and shall indeed, carry through the following considerations in terms of WOPs.

3. A PROBLEM WITH SIMPLE CONDITIONAL FUNCTIONS

So far, we have arrived at conceiving epistemic states as SCFs or WOPs. But there is a problem; the rule for epistemic change we have stated is simply insufficient. In this rule, the old epistemic state was represented by an SCF, but the ensuing epistemic state was still represented in the former way by its net content. This will not do, of course. Having decided to represent epistemic states by SCFs, we must represent *all* epistemic states we are talking of in this way; that is, we must also represent the ensuing state by some SCF, and we must say which SCF that is. The problem becomes pressing, if we consider several successive epistemic changes. The above rule explains the first of these changes; but after that we are back on the surface level of net contents, where we cannot apply the above rule to account for the further changes.

The problem is obvious and grave; but it has received surprisingly little attention. In fact, the only place I found where the problem is explicitly recognized in this way is in Harper (1976, pp. 95ff.), where he tries to solve its probabilistic counterpart with respect to Popper measures.[11] What can we do about it? Well, let's at least try to solve it within our representation of epistemic states. If this should fail, as it will, we shall at least see more clearly what is missing.

It will be intuitively more transparent in this attempt to work with orderings of disbelief, i.e. WOPs. Thus, let the old epistemic state be represented by the WOP

$$E_0, E_1, E_2, \ldots, E_\zeta$$

(which we suppose only for illustrative reasons to have a last term), and let A be the information to be accepted and $\beta = \min\{\alpha \mid E_\alpha \cap A \neq \emptyset\}$. Some new epistemic state ensues which should also be represented by a WOP. Can we determine this new WOP in a reasonable way?

A first proposal might be this: It seems plausible to assume that, after information A is accepted, all the possible worlds in A are less disbelieved than the worlds in $\bar{A}$ (where $\bar{A}$ is the relative complement $W \setminus A$ of A). Further, it seems reasonable to assume that, by getting information only about A, the ordering of disbelief of the worlds

within A remains unchanged, and likewise for the worlds in $\bar{A}$. Both assumptions already determine uniquely the new ordering of disbelief; it is given by the sequence

$$E_\beta \cap A, \ldots, E_\zeta \cap A, \quad E_0, \ldots, E_{\beta-1},$$
$$E_\beta \cap \bar{A}, \ldots, E_\zeta \cap \bar{A},$$

where – this is important – all empty terms must still be deleted; otherwise, we wouldn't have a WOP. Wasn't that a quick solution? Well, it isn't a good one. Let me point out three shortcomings:

First, according to this proposal, epistemic changes are not reversible; there is no operation of the specified kind which reinstalls the old ordering of disbelief. In fact, there is in general no way at all, even if we know β, to infer from the new WOP what the old one was. The technical reason for this is just the deletion of empty terms, since after they have been deleted, we no longer know where they have been deleted. However, it is certainly desirable to be able to account for the reversibility of epistemic changes.

Secondly, according to this proposal, epistemic changes are not commutative. If A and B are two logically independent propositions, it is easily checked that getting informed first about A and then about B leads to one WOP, getting informed first about B and then about A leads to another WOP, and getting informed at once about $A \cap B$ leads to still another WOP. This is definitely an inadequacy. To be sure, one wouldn't always want epistemic changes to commute. The two pieces of information may somehow conflict, in which case the order in which they are received may matter. But the normal case is certainly that information just accumulates, and in this case the order of information should be irrelevant. However, according to our proposal it is irrelevant only in trivial cases.

Thirdly, the assumption that, after getting informed about A, all worlds in $\bar{A}$ are more disbelieved than all worlds in A seems too strong. Certainly, the first member, i.e. the net content of the new WOP, must be a subset of A; thus, at least some worlds in A must get less disbelieved than the worlds in $\bar{A}$. But it is utterly questionable whether even the most disbelieved world in A should get less disbelieved than even the least disbelieved world in $\bar{A}$; this could be effected at best by the most certain information.

This last consideration suggests a second proposal. Perhaps one should put only the least disbelieved and not all worlds in A at the top of the new WOP which then looks thus:

$$E_\beta \cap A, \quad E_0, \ldots, E_{\beta-1}, \quad E_\beta \backslash A, \quad E_{\beta+1}, \ldots, E_\zeta.$$

Here again, empty terms still have to be deleted ($E_\beta \backslash A$ may be empty). However, that's no good, either. This proposal does not fare better with respect to the reversibility and commutativity of epistemic changes, as may be easily verified. Moreover, we have now gone to the other extreme. The information A is now treated as only minimally reliable; it is given up as soon as only a single consequence of the things believed together with A, i.e. of $E_\beta \cap A$, turns out to be false.

One may try further. But I think that the case already looks hopeless. There is no good solution to our problem within the confines of SCFs or WOPs. Nevertheless, there are two important conclusions to be drawn from these efforts.

One conclusion is this: In the first proposal the information A was accepted maximally firmly; in the second it was accepted minimally firmly. We considered both extremes undesirable. But then no degree of firmness is the right one for all cases. Rather, the natural consequence is that, in order to specify the new epistemic state, we must say not only which information it is that changes the old state; we must also specify with which firmness this information is incorporated into the new state. This consequence is most important; it means that we have so far neglected a parameter which plays a crucial role in epistemic changes. No wonder that we tried in vain.

The other conclusion is this: We discovered that the reversing of epistemic changes was impossible because of the deletion of empty terms. This suggests that we should generalize the concept of a WOP to the effect that such a partition may contain empty terms. This is what we shall do. Technically, this is a small trick which will, however, make all the difference. Note that this has another important consequence. There may then be two such generalized partitions which order the possible worlds in exactly the same way and which thus differ only by having empty terms at different places. These two partitions should be viewed as two different epistemic states; and this implies that not only the ordering of worlds, but also their relative distances in these partitions are relevant. Mathematically, this means that we have to consider not only the order, but also the arithmetical properties of ordinals.

Now we are well prepared. We only have to adhere to these conclusions. The first conclusion will be developed in Section 5, the second right now.

4. ORDINAL CONDITIONAL FUNCTIONS

It is more convenient to formalize such generalized partitions as functions from possible worlds to ordinals. Moreover, we shall explicitly relativize these functions to a given field of propositions. So far, there was no need for this relativization; but now, when things get more technical, it will prove very useful. The same is done in probability theory, where it is important to compare or relate probability measures on different σ-fields. So, let us define:

DEFINITION 4. Let $\mathcal{A}$ be a complete field of propositions over W (i.e. a non-empty set of subsets of W closed under complementation and arbitrary union and intersection). Then we call κ an *$\mathcal{A}$-measurable ordinal conditional function* (*$\mathcal{A}$-OCF*), if and only if κ is a function from W into the class of ordinals such that $\kappa^{-1}(0) \neq \emptyset$ and for all atoms[12] A of $\mathcal{A}$ and all $w,w' \in A$ $\kappa(w) = \kappa(w')$. Moreover, we define for any $A \in \mathcal{A}\backslash\{\emptyset\}$ $\kappa(A) = \min\{\kappa(w) \mid w \in A\}$.[13]

It is obvious that OCFs generalize WOPs and thus SCFs. The measurability condition is also obvious; it demands that an $\mathcal{A}$-OCF does not discriminate possible worlds which are not discriminated in $\mathcal{A}$.

Two simple observations will be permanently used:

THEOREM 2. *Let* κ *be an* $\mathcal{A}$-OCF. *Then we have*

(a) *for each* $A \in \mathcal{A}\backslash\{\emptyset, W\}$, $\kappa(A) = 0$ *or* $\kappa(\bar{A}) = 0$ *or both*,

(b) *for all* $A,B \in \mathcal{A}\backslash\{\emptyset\}$, $\kappa(A \cup B) = \min\{\kappa(A), \kappa(B)\}$.

Intuitively, an OCF is not only an ordering, but a *grading of disbelief* in possible worlds. It is clear how such a grading of disbelief is to be understood as a deterministic epistemic state: In state κ, the true world is always believed to be in $\kappa^{-1}(0)$; thus $\kappa^{-1}(0)$ is the net content of the epistemic state κ, and hence the stipulation that $\kappa^{-1}(0) \neq \emptyset$. A is then *believed* in the state κ iff $\kappa^{-1}(0) \subseteq A$, i.e. $\kappa(\bar{A}) > 0$. (Beware: $\kappa(A) = 0$ only means that A is not believed to be false in state κ; and this leaves open the possibility that also $\kappa(\bar{A}) = 0$, i.e. that A is also not believed true in state κ.)

Relative to an OCF κ, we may also introduce degrees of firmness of belief (and thereby slightly reduce the contorted talk of disbelief). If we, for a moment, also allow for negative ordinals, we may say that A *is believed with firmness* α *relative to* κ iff either $\kappa(A) = 0$ and $\alpha = \kappa(\bar{A})$ or $\kappa(A) > 0$ and $\alpha = -\kappa(A)$. Thus, in state κ one believes or disbelieves A iff, respectively, one believes A with positive or negative firmness; firmness 0 means that one is neutral to A. And we might also say that A *is more plausible than* B iff A is believed with greater firmness than B, i.e. iff $\kappa(\bar{A}) > \kappa(\bar{B})$ or $\kappa(A) < \kappa(B)$.[14]

It is clear that the role of taking the minimum corresponds to the role addition has in probability theory; compare the definition of $\kappa(A)$ with the probabilistic formula $P(A) = \Sigma_{w \subset A} P(\{w\})$. The two sides of the correspondence differ, however, in a very characteristic way. To put it somewhat metaphorically:

In probability theory, epistemically interpreted, possible worlds have a probability *mass*. They compete for their share of the total mass available; and in epistemic changes these shares get redistributed. Thus, this competition may be conceived as a sort of territorial fight where the parties aim at getting as large as possible. A proposition may then be conceived as a team consisting of its members; and each such team is as weighty and fares as well in this competition as the sum of the masses of its members.

In the theory of OCFs, possible worlds have, by contrast, *grades* of disbelief. They compete for grades, 0 being the top grade above an unending sequence of lower grades; and in epistemic changes their grades will get rearranged. Thus, this competition may be conceived as a sort of race where the parties aim at reaching the top. A proposition may again be conceived as a team consisting of its members; but in this race, each such team is just as good as its best members.[15]

Exactly how do the grades get rearranged in epistemic changes? This is the subject of the next section.

5. CONDITIONALIZATION AND GENERALIZED CONDITIONALIZATION

In what follows I shall make use of the somewhat uncommon *left-sided subtraction* of ordinals[16] which is defined in the following way: Let α and β be two ordinals with $\alpha \leq \beta$; then $-\alpha + \beta$ is to be that uniquely determined ordinal ξ, for which $\alpha + \xi = \beta$.[17]

Moreover, we shall throughout use the following auxiliary concept:

DEFINITION 5. Let κ be an $\mathcal{A}$-OCF and $A \in \mathcal{A}\backslash\{\emptyset\}$. Then the *A-part of* κ is to be that function $\kappa(\cdot|A)$[18] defined on A for which for all $w \in A$ $\kappa(w|A) = -\kappa(A) + \kappa(w)$. For $B \in \mathcal{A}$ with $A \cap B \neq \emptyset$ we also define $\kappa(B|A) = \min\{\kappa(w|A)|w \in A \cap B\} = -\kappa(A) + \kappa(A \cap B)$.

Thus, if $\mathcal{A}' = \{A \cap B | B \in \mathcal{A}\}$, $\mathcal{A}'$ is a complete field of subsets of A, and $\kappa(\cdot|A)$ is an $\mathcal{A}'$-OCF. One might say that the A-part of κ is the restriction of κ to A shifted to 0, i.e. in such a way that the minimum taken is 0. It will soon become clear why I have here chosen the same notation as is used in probability theory.

With the aid of this concept we can define the notion central to the dynamics of epistemic states:

DEFINITION 6. Let κ be an $\mathcal{A}$-OCF, $A \in \mathcal{A}\backslash\{\emptyset, W\}$, and α an ordinal. Then $\kappa_{A,\alpha}$ is to be that $\mathcal{A}$-OCF for which

$$\kappa_{A,\alpha}(w) = \begin{cases} \kappa(w|A), & \text{if } w \in A \\ \alpha + \kappa(w|\bar{A}), & \text{if } w \in \bar{A} \end{cases}.$$

We call $\kappa_{A,\alpha}$ the *A,α-conditionalization of* κ.

Thus, the A,α-conditionalization of κ is the union of the A-part of κ and of the $\bar{A}$-part of κ shifted up by α grades. Trivially, we have $\kappa_{A,\alpha}(A) = 0$ and $\kappa_{A,\alpha}(\bar{A}) = \alpha$; hence, A is believed in $\kappa_{A,\alpha}$ with firmness α. By having introduced the parameter α, we have now taken account of the first conclusion of Section 3.

Def. 6 conforms to the intuitive requirement that getting informed only about A does not change the epistemic state restricted to A, or $\bar{A}$, i.e. the *grading* of disbelief within A, or $\bar{A}$. In other words, the A,α-conditionalization of κ leaves the A-part as well as the $\bar{A}$-part of κ unchanged; they are only shifted in relation to one another. Thereby, we have finally also made use of and given meaning to the relative distances of possible worlds in an OCF, as was implied by our second conclusion of Section 3.

The failure of WOPs may now be seen to have a simple mathematical reason: it's just that the set of all WOPs is not closed under all the above shiftings; therefore, no reasonable conditionalization could be defined for them. With the OCFs, this problem disappears; the class of all OCFs *is* closed under all these shiftings.[19]

The A,α-conditionalization of κ should not always be interpreted as the change of κ which results from obtaining the information A with positive firmness. There are two exceptional cases. For the first case, suppose that $\kappa(\bar{A}) = \beta > 0$; thus, A is believed already in κ. Now, if $\alpha = \beta$, there is no change at all; if $\alpha > \beta$, then one has got additional reason for A whereby the belief in A is strengthened; and if $\alpha < \beta$, then one has got some reason against A whereby the belief in A is weakened, though not destroyed. The second case is the A,0-conditionalization of κ. This may best be described as the neutralization of A and $\bar{A}$, since in $\kappa_{A,0}$ neither A nor $\bar{A}$ is believed. In both cases, it would be inappropriate to say that one was informed about A. But the epistemic changes described in them may certainly be found in reality and are thus properly covered by Def. 6.[20,21]

The problems we had with our proposals in Section 3 no longer trouble us. Of course, epistemic changes according to Def. 6 are reversible:

THEOREM 3. *Let κ be an $\mathcal{A}$-OCF and $A \in \mathcal{A}\setminus\{\emptyset,W\}$ such that $\kappa(A) = 0$ and $\kappa(\bar{A}) = \beta$. Then we have $(\kappa_{A,\alpha})_{A,\beta} = (\kappa_{\bar{A},\alpha})_{A,\beta} = \kappa$.*

Moreover, accumulating information commutes. Here, as in the sequel, we shall say that two ordinals α and β *commute* iff $\alpha + \beta = \beta + \alpha$.

THEOREM 4. *Let κ be an $\mathcal{A}$-OCF and $A,B \in \mathcal{A}\setminus\{\emptyset,W\}$ such that $\kappa(A \cap B) = \kappa(A \cap \bar{B}) = \kappa(\bar{A} \cap B) = 0$, and let α and β be two commuting ordinals. Then we have $(\kappa_{A,\alpha})_{B,\beta} = (\kappa_{B,\beta})_{A,\alpha}$.*

Proof. Set $C_1 = A \cap B$, $C_2 = A \cap \bar{B}$, $C_3 = \bar{A} \cap B$, and $C_4 = \bar{A} \cap \bar{B}$, and for $n = 1,\ldots,4$ $\kappa(C_n) = a_n$, $\kappa_{A,\alpha}(C_n) = b_n$, $(\kappa_{A,\alpha})_{B,\beta}(C_n) = c_n$, $\kappa_{B,\beta}(C_n) = d_n$, and $(\kappa_{B,\beta})_{A,\alpha}(C_n) = e_n$. It suffices to show that $c_n = e_n$ for $n = 1,\ldots,4$: We have assumed that $a_1 = a_2 = a_3 = 0$. By Def. 6 we now get:

$$b_1 = 0, \quad b_2 = 0, \quad b_3 = \alpha, \quad b_4 = \alpha + a_4, \quad \text{and}$$
$$d_1 = 0, \quad d_2 = \beta, \quad d_3 = 0, \quad d_4 = \beta + a_4.$$

Again applying Def. 6, we get from this:

$$c_1 = 0, \quad c_2 = \beta, \quad c_3 = \alpha, \quad c_4 = \beta + \alpha + a_4, \quad \text{and}$$
$$e_1 = 0, \quad e_2 = \beta, \quad e_3 = \alpha, \quad e_4 = \alpha + \beta + a_4.$$

Thus $c_n = e_n$ for $n = 1,2,3$, and also $c_4 = e_4$, since α and β commute.

Q.E.D.

The conclusion of Theorem 4 holds also under more general conditions. These, however, are not so illuminating as to justify the clumsy calculations needed.

We may further generalize our topic. As is well known, Jeffrey (1965, ch. 11) made a substantial contribution to the dynamics of probabilistic epistemic states by discovering generalized conditionalization. There, a probability measure P is conditionalized not by some proposition A, but rather by a probability measure Q on some set of propositions. Q represents here some new state of information with respect to these propositions, and the generalized conditionalization of P by Q describes how the total epistemic state P changes because of this new state of information. Nobody seems to have even thought of doing the same for deterministically conceived epistemic states; but here, the parallel extends in quite a natural way:

DEFINITION 7. Let $\mathcal{B}$ be a complete subfield of $\mathcal{A}$, κ an $\mathcal{A}$-OCF, and λ a $\mathcal{B}$-OCF. Then κ_λ is to be that $\mathcal{A}$-OCF for which for all atoms B of $\mathcal{B}$ and all $w \in B$ $\kappa_\lambda(w) = \lambda(B) + \kappa(w|B)$. We call κ_λ the *λ-conditionalization of* κ.

Def. 6 is only a special case of Def. 7:

THEOREM 5. *Let κ be an $\mathcal{A}$-OCF, $A \in \mathcal{A}\setminus\{\emptyset,W\}$, and λ that $\{\emptyset,A,\bar{A},W\}$-measurable OCF for which*

$$\lambda(w) = \begin{cases} 0 & \text{for } w \in A, \\ \alpha & \text{for } w \in \bar{A}. \end{cases}$$

Then, $\kappa_\lambda = \kappa_{A,\alpha}$.

Of course, generalized conditionalization is reversible, too:

THEOREM 6. *Let $\mathcal{B}$, κ, and λ be as in Def. 7; and let κ' be the $\mathcal{B}$-measurable coarsening of κ defined by $\kappa'(w) = \kappa(B)$ for all atoms B of $\mathcal{B}$ and all $w \in B$. Then $(\kappa_\lambda)_{\kappa'} = \kappa$.*

With the aid of Def. 7 we can state our most general rule for rational epistemic change: Let X's epistemic state at time t with respect to the field $\mathcal{A}$ of propositions be represented by the $\mathcal{A}$-OCF κ. Suppose further that the experiences between t and t' directly affect only X's attitude towards propositions in the field $\mathcal{B}$ and cause him to adopt the $\mathcal{B}$-OCF λ as epistemic state with respect to $\mathcal{B}$. Then κ_λ represents X's epistemic state at t' with respect to $\mathcal{A}$ (provided X is not subject to arational influences).

This formulation of the rule brings out a fact which seems by now to be well accepted in epistemology in general. It was realized in probabilistic epistemic modelling with Jeffrey's generalized conditionalization (this was its revolutionary point), but it does not seem to have been clearly recognized in deterministic epistemic modelling: I mean the fact that what is described by rules of epistemic change are never rational inner reactions to outward circumstances or happenings, but always rational adjustments of the overall epistemic state to inner epistemic changes in particular quarters; how these initial epistemic changes come about is in any case a matter to which a rationality assessment cannot be reasonably applied and which therefore falls outside the scope of investigations like this one. This fact is formally mirrored, here as in Jeffrey, by the fact that epistemic states, probability measures or OCFs, are conditionalized by things of their own kind; talking of conditionalization by propositions (or events), albeit technically correct, has been intuitively very misleading.

6. INDEPENDENCE AND CONDITIONAL INDEPENDENCE

Related to conditionalization, there is another important topic in probability theory in particular, but also in epistemology in general: namely dependence and independence. I know of no reasonable definition of independence for deterministic representations of epistemic states. Logical independence will not do, of course, since almost everything is logically independent of almost everything. The best we can do within the domain of SCFs is to say that A is epistemically independent of B relative to the SCF g, if and only if $g(B) \subseteq A$ iff $g(\bar{B}) \subseteq A$ and $g(B) \subseteq \bar{A}$ iff $g(\bar{B}) \subseteq \bar{A}$, i.e. iff acceptance of B, or of $\bar{B}$, does not matter to whether A or $\bar{A}$ or neither is believed. However, this implies, for example, that each A believed true in state g is independent of each B

believed neither true nor false in g; and this is certainly much too much independence.

Not surprisingly, there is no problem with independence with respect to OCFs:

DEFINTION 8. Let κ be an $\mathcal{A}$-OCF and $\mathcal{B}$ and $\mathcal{C}$ two complete subfields of $\mathcal{A}$. Then $\mathcal{C}$ is *independent of* $\mathcal{B}$ *with respect to* κ iff for all atoms B of $\mathcal{B}$ and all atoms C of $\mathcal{C}$ $B \cap C \neq \emptyset$ and $\kappa(B \cap C) = \kappa(B) + \kappa(C)$. $\mathcal{B}$ *and* $\mathcal{C}$ are *independent* (with respect to κ) iff $\mathcal{C}$ is independent of $\mathcal{B}$ and $\mathcal{B}$ is independent of $\mathcal{C}$. Moreover, if $A,B \in \mathcal{A}\setminus\{\emptyset,W\}$, A is *independent of* B (with respect to κ) iff $\{\emptyset,A,\bar{A},W\}$ is independent of $\{\emptyset,B,\bar{B},W\}$, and A *and* B are *independent* (with respect to κ) iff A is independent of B and B is independent of A.

Def. 8 copies probabilistic independence concepts as far as possible. Note that independence with respect to OCFs need not be symmetric, simply because addition of ordinals is not commutative; therefore the distinction between "A is independent of B" and "A and B are independent".

Independence so defined has the properties we would expect.

THEOREM 7. *If* $\mathcal{C}$ *is independent of* $\mathcal{B}$ *with respect to* κ, *then for all* $B \in \mathcal{B}\setminus\{\emptyset\}$ *and all* $C \in \mathcal{C}\setminus\{\emptyset\}$:

$$\kappa(B \cap C) = \kappa(B) + \kappa(C).$$

Proof. Let $B' \in \mathcal{B}\setminus\{\emptyset\}$ *and* $C' \in \mathcal{C}\setminus\{\emptyset\}$. Let further $\mathcal{B}'$ be the set of atoms of $\mathcal{B}$ which are subsets of B' and $\mathcal{C}'$ the set of atoms of $\mathcal{C}$ which are subsets of C'. Hence, $B' = \cup\mathcal{B}'$ and $C' = \cup\mathcal{C}'$, and moreover, $\kappa(B') = \min\{\kappa(B) \mid B \in \mathcal{B}'\}$ and $\kappa(C') = \min\{\kappa(C) \mid C \in \mathcal{C}'\}$. Then we have

$$\begin{aligned}\kappa(B' \cap C') &= \min\{\kappa(B \cap C) \mid B \in \mathcal{B}', C \in \mathcal{C}'\} = \\ &= \min\{\kappa(B) + \kappa(C) \mid B \in \mathcal{B}', C \in \mathcal{C}'\} = \\ &= \min\{\kappa(B) \mid B \in \mathcal{B}'\} + \min\{\kappa(C) \mid C \in \mathcal{C}'\} = \\ &= \kappa(B') + \kappa(C').\end{aligned}$$

Q.E.D.

The converse of Theorem 7 is obviously true. An immediate consequence of Theorem 7 and Def. 5 and 7 is

THEOREM 8. *The following three assertions are equivalent:*

(a) $\mathcal{C}$ *is independent of* $\mathcal{B}$ *with respect to* κ,

(b) *for all* $B \in \mathcal{B}\setminus\{\emptyset\}$ *and* $C \in \mathcal{C}\setminus\{\emptyset\}$ $\kappa(C|B) = \kappa(C)$ *holds true,*

(c) *for each* $\mathcal{B}$*-OCF* λ *and each* $C \in \mathcal{C}\setminus\{\emptyset\}$ $\kappa_\lambda(C) = \kappa(C)$ *holds true.*

Theorem 8 particularly clearly shows the intuitive adequacy of Def. 8.

The parallel to probability theory may be extended further. In probability theory, one also defines independence for families of subfields. This can be done here as well.

DEFINITION 9. Let $(\mathcal{B}_\alpha)_{\alpha<\beta}$ be a sequence of complete subfields of $\mathcal{A}$ and κ an $\mathcal{A}$-OCF. Then $(\mathcal{B}_\alpha)_{\alpha<\beta}$ is called *independent with respect to* κ iff for all atoms B_α of $\mathcal{B}_\alpha$ $(\alpha < \beta)$ $\bigcap_{\alpha<\beta} B_\alpha \neq \emptyset$ and $\kappa(\bigcap_{\alpha<\beta} B_\alpha) = \Sigma_{\alpha<\beta}\kappa(B_\alpha)$.

The connection to Def. 8 is stated in

THEOREM 9. $(\mathcal{B}_\alpha)_{\alpha<\beta}$ *is independent iff for all* $\gamma < \beta$ *the complete field generated by* $\bigcup_{\gamma\leq\alpha<\beta} \mathcal{B}_\alpha$ *is independent of the complete field generated by* $\bigcup_{\alpha<\gamma} \mathcal{B}_\alpha$.

Proof. Define $\mathcal{C}_\gamma$ and $\mathcal{D}_\gamma$ to be, respectively, the complete field generated by $\bigcup_{\alpha<\gamma} \mathcal{B}_\alpha$ and $\bigcup_{\gamma\leq\alpha<\beta} \mathcal{B}_\alpha$. Now suppose first that for all atoms B_α of $\mathcal{B}_\alpha$ $(\alpha < \beta)$ $\kappa(\bigcap_{\alpha<\beta} B_\alpha) = \Sigma_{\alpha<\beta}\,\kappa(B_\alpha)$. This implies that for all atoms B_α of $\mathcal{B}_\alpha$ $(\alpha < \gamma)$ $\kappa(\bigcap_{\alpha<\gamma} B_\alpha) = \Sigma_{\alpha<\gamma}\,\kappa(B_\alpha)$, and similarly, that for all atoms B_α of $\mathcal{B}_\alpha$ $(\gamma \leq \alpha < \beta)$ $\kappa(\bigcap_{\gamma\leq\alpha<\beta} B_\alpha) = \Sigma_{\gamma\leq\alpha<\beta}\,\kappa(B_\alpha)$. Thus, we have $\kappa(\bigcap_{\alpha<\beta} B_\alpha) = \kappa(\bigcap_{\alpha<\gamma} B_\alpha) + \kappa(\bigcap_{\gamma\leq\alpha<\beta} B_\alpha)$, and this means that $\mathcal{D}_\gamma$ is independent of $\mathcal{C}_\gamma$.

Conversely, suppose that for all $\gamma < \beta$ $\mathcal{D}_\gamma$ is independent of $\mathcal{C}_\gamma$. For $\gamma = 1$, this says that for all atoms B_α of $\mathcal{B}_\alpha$ $(\alpha < \beta)$ $\kappa(\bigcap_{\alpha<\beta} B_\alpha) = \kappa(B_0) + \kappa(\bigcap_{1\leq\alpha<\beta} B_\alpha)$. This implies in particular that for all atoms B_α of $\mathcal{B}_\alpha$ $(\alpha = 0,1)$ $\kappa(B_0 \cap B_1) = \kappa(B_0) + \kappa(B_1)$. For $\gamma = 2$, we therefore get that for all atoms B_α of $\mathcal{B}_\alpha$ $(\alpha < \beta)$ $\kappa(\bigcap_{\alpha<\beta} B_\alpha) = \kappa(B_0 \cap B_1) + \kappa(\bigcap_{2\leq\alpha<\beta} B_\alpha) = \kappa(B_0) + \kappa(B_1) + \kappa(\bigcap_{2\leq\alpha<\beta} B_\alpha)$. Continuing this line of reasoning by transfinite induction till β then leads to the desired result. Q.E.D.

An immediate consequence of Theorem 9 is

THEOREM 10. *Let* $(\mathcal{B}_\alpha)_{\alpha<\beta}$ *be independent. Let* $(\Gamma_\gamma)_{\gamma<\delta}$ *be a partition of* $\{\alpha|\alpha < \beta\}$ *such that we have for all* $\gamma,\gamma' < \delta$: *if* $\gamma < \gamma'$, *then* $\alpha < \alpha'$ *for all* $\alpha \in \Gamma_\gamma$ *and* $\alpha' \in \Gamma_{\gamma'}$. *Let finally* $\mathcal{C}_\gamma$ *be the complete field generated by* $\bigcup_{\alpha\in\Gamma_\gamma} B_\alpha$. *Then the sequence* $(\mathcal{C}_\gamma)_{\gamma<\delta}$ *is also independent.*

In probability theory, the corresponding theorem is known as the theorem of the composition of independent fields.

As a last topic, let me take up conditional independence. It is well known that conditional independence is central for a probabilistic theory of causality. Thus, this topic will become important, when one turns to deterministic theories of causality. Here, however, I take it up only for demonstrating the parallel between probability measures and OCFs a bit further.

DEFINITION 10. Let $\mathcal{B}$ and $\mathcal{C}$ be two complete subfields of $\mathcal{A}$, κ an $\mathcal{A}$-OCF, and $A \in \mathcal{A}\setminus\{\emptyset\}$. Then $\mathcal{C}$ is *independent of* $\mathcal{B}$ *conditional on* A (or *given* A) *with respect to* κ iff for all atoms B of $\mathcal{B}$ and all atoms C of $\mathcal{C}$ with $A \cap B \cap C \neq \emptyset$ $\kappa(B \cap C|A) = \kappa(B|A) + \kappa(C|A)$. If $\mathcal{D}$ is another complete subfield of $\mathcal{A}$, then $\mathcal{C}$ is *independent of* $\mathcal{B}$ *conditional on* $\mathcal{D}$ (or *given* $\mathcal{D}$) (with respect to κ) iff for each atom D of $\mathcal{D}$ $\mathcal{C}$ is independent of $\mathcal{B}$ given D. Further phrases may be defined in analogy to Def. 8.

The intuitive interpretation of Def. 10 should be clear and is supported by the fact that Theorems 7 and 8 hold correspondingly for conditional independence. The following theorems are more interesting; the expression "$\mathcal{B} + \mathcal{C}$" used in them is meant to denote the complete field generated by $\mathcal{B} \cup \mathcal{C}$.

THEOREM 11. *Let* $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, *and* $\mathcal{E}$ *be four complete subfields of* $\mathcal{A}$. *Suppose that* $\mathcal{C}$ *is independent of* $\mathcal{B}$ *given* $\mathcal{D} + \mathcal{E}$ *and that* $\mathcal{D}$ *is independent of* $\mathcal{B}$ *given* $\mathcal{E}$. *Then* $\mathcal{C} + \mathcal{D}$ *is independent of* $\mathcal{B}$ *given* $\mathcal{E}$.

Proof. Let B, C, D, and E be variables for atoms of $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$, respectively. The first assumption says that for all B, C, D, and E with $B \cap C \cap D \cap E \neq \emptyset$:

$\kappa(B \cap C|D \cap E) = \kappa(B|D \cap E) + \kappa(C|D \cap E)$, i.e. by Def. 5,
$-\kappa(D|E) + \kappa(B \cap C \cap D|E) =$
$= -\kappa(D|E) + \kappa(B \cap D|E) + (-\kappa(D|E) + \kappa(C \cap D|E))$, i.e.
$\kappa(B \cap C \cap D|E) = \kappa(B \cap D|E) + (-\kappa(D|E) + \kappa(C \cap D|E))$.

The second assumption states that for all B, D, and E with $B \cap D \cap E \neq \emptyset$:

$$\kappa(B \cap D|E) = \kappa(B|E) + \kappa(D|E).$$

The last two equations together yield that for all B, C, D, and E with $B \cap C \cap D \cap E \neq \emptyset$

$$\kappa(B \cap C \cap D|E) = \kappa(B|E) + \kappa(C \cap D|E);$$

and that's what we had to prove. Q.E.D.

In the same way, the result symmetric to Theorem 11 may be proved:

THEOREM 12. *If $\mathcal{B}$ is independent of $\mathcal{C}$ given $\mathcal{D} + \mathcal{E}$ and independent of $\mathcal{D}$ given $\mathcal{E}$, then $\mathcal{B}$ is independent of $\mathcal{C} + \mathcal{D}$ given $\mathcal{E}$.*

Moreover, we have

THEOREM 13. *If $\mathcal{B}$ is independent of $\mathcal{C} + \mathcal{D}$ given $\mathcal{E}$ and independent of $\mathcal{C} + \mathcal{E}$ given $\mathcal{D}$, then $\mathcal{B}$ is also independent of $\mathcal{C} + \mathcal{D} + \mathcal{E}$ given $\mathcal{D} \cap \mathcal{E}$.*

Proof. Let B, C, D, and E be as in the proof of Theorem 11, and let F be a variable for the atoms of $\mathcal{D} \cap \mathcal{E}$ (which, to be sure, is also a complete field). The first premise says that for all B, C, D, E, and F with $B \cap C \cap D \cap E \neq \emptyset$ and $D, E \subseteq F$:

$$\begin{aligned}
&\kappa(C \cap D \cap B|E \cap F) = \\
&= \kappa(C \cap D|E \cap F) + \kappa(B|E \cap F), \text{ i.e. by Def. 5,} \\
&-\kappa(E|F) + \kappa(C \cap D \cap E \cap B|F) = \\
&= -\kappa(E|F) + \kappa(C \cap D \cap E|F) + (-\kappa(E|F) + \kappa(E \cap B|F)), \text{i.e.} \\
(1)\quad &\kappa(C \cap D \cap E \cap B|F) = \\
&= \kappa(C \cap D \cap E|F) + (-\kappa(E|F) + \kappa(E \cap B|F)).
\end{aligned}$$

Likewise, the second premise says that for all B, C, D, E, and F with $B \cap C \cap D \cap E \neq \emptyset$ and $D, E \subseteq F$:

$$\begin{aligned}
&\kappa(C \cap E \cap B|D \cap F) = \\
&= \kappa(C \cap E|D \cap F) + \kappa(B|D \cap F), \text{ i.e. as before}
\end{aligned}$$

$$\begin{aligned}
(2)\quad &\kappa(C \cap D \cap E \cap B|F) = \\
&= \kappa(C \cap D \cap E|F) + (-\kappa(D|F) + \kappa(D \cap B|F)).
\end{aligned}$$

(1) and (2) imply that for all $D, E \subseteq F$

$$-\kappa(E|F) + \kappa(E \cap B|F) = -\kappa(D|F) + \kappa(D \cap B|F).$$

This in turn implies that for all atoms E,E' of $\mathcal{E}$ with $E,E' \subseteq F$

$$(3) \qquad -\kappa(E|F) + \kappa(E \cap B|F) = -\kappa(E'|F) + \kappa(E' \cap B|F).$$

Now, there must be an atom E_0 of $\mathcal{E}$ with $E_0 \subseteq F$ such that $\kappa(E_0|F) = 0$, since $0 = \kappa(F|F) = \min\{\kappa(E|F)|E \subseteq F\}$. Thus, we have, using (3), $\kappa(E_0 \cap B|F) = \min\{\kappa(E \cap B|F)|E \subseteq F\} = \kappa(B|F)$. Using (3) once more, this yields that for all $E \subseteq F$

$$-\kappa(E|F) + \kappa(E \cap B|F) = \kappa(B|F).$$

Substituting this result in (1), we finally get

$$\kappa(C \cap D \cap E \cap B|F) = \kappa(C \cap D \cap E|F) + \kappa(B|F)$$

for all B, C, D, E, and F with $B \cap C \cap D \cap E \neq \emptyset$ and $D, E \subseteq F$.
Q.E.D.

The assertion symmetric to Theorem 13 does not necessarily hold.

These theorems are as analogous to probabilistic theorems as can be.[22] Here I would like to end for the time being. I think there can be no doubt that OCFs are vastly superior to SCFs or WOPs.

7. CONNECTIONS WITH PROBABILITY THEORY

So far, all this has been a story wholly within deterministic epistemology. But there is in fact an exact probabilistic duplicate of our story progressing from net contents to OCFs. The probabilistic counterparts to our net contents are probability measures. With net contents we had the problem that we could say nothing about the new net content resulting from information incompatible with the old net content; this problem induced us to introduce the SCFs. The corresponding problem is that probabilities conditional on propositions having probability ● are not defined in standard probability theory; we can say nothing about the new probability measure resulting from information having probability 0 in the old epistemic state. One solution to this problem, perhaps the most prominent one, consists in introducing Popper measures.[23] These are indeed the probabilistic counterparts to our SCFs; it is known that it is an SCF which, if adapted to the algebraic framework of probability theory (which operates with σ-fields instead of complete fields), represents the 0-1-structure of a Popper measure P, i.e. the relation $\{\langle A,B\rangle \mid P(B|A) = 1\}$.[24] This means, however, that Popper measures are as insufficient for a dynamic theory of epistemic states as SCFs are; this was very clearly pointed out by Harper (1976, pp. 95f). Hence, the probabilistic story calls for continuation, too. It is quite obvious what this should look like; just define probabilistic counterparts to OCFs which would be something like functions from propositions to ordered pairs consisting of an ordinal and a real between 0 and 1. I won't now pursue this in technical detail, since this fusion of probability theory and the theory of OCFs appears to me to be fairly straightforward. But the advantage of such probabilified OCFs over Popper measures is quite clear; it is the same as that of OCFs over SCFs.

One point, however, is still open; we do not yet have an *explanation* of why OCFs behave so much like probability measures (which is, of course, not given by the proposed fusion of probability theory and the theory of OCFs). But this explanation may be made, I think, along the following lines within the framework of nonstandard probability theory:[25] Let P be a nonstandard probability measure for which there is an infinitesimal i such that for each A $P(A)$ is of the same order as i^n for some (nonstandard) natural number n (i.e. $P(A)/i^n$ is finite, but not infinitesimal). Now define $\kappa(B|A) = n$ iff $P(B|A)$ is of the same order as i^n. Then κ is like an OCF within this framework. Indeed, we have thereby defined a homomorphism from a class of nonstandard probability measures onto the class of (nonstandard) OCFs which maps, first, addition of probabilities into taking the minimum of OCF-values and which maps, secondly, multiplication and division of probabilities into addition and subtraction of OCF-values. More specifically, whenever A and B are independent according to P, they are so according to κ; and for the κ so defined, we have $\kappa(B|A) = \kappa(A \cap B) - \kappa(A)$. This would explain why OCFs obey the same laws as probability measures concerning independence and conditionalization.[26]

8. DISCUSSION

I see three points where the foregoing story should be related to the actual state of discussion.

The first point is that all concepts introduced in Section 2 are, of course, absolutely standard. SCFs are better known under the label of (class) selection functions, which play a central role in conditional logic; in this context, a number of slightly different concepts of a selection function have been proposed, and it is well known that nearly every semantics for conditional logic is based on some such concept.[27] Moreover, if $(E_\alpha)_{\alpha<\zeta}$ is a WOP, then the sequence $(\bigcup_{\alpha\leq\beta} E_\alpha)_{\beta<\zeta}$ is a (universal) system of similarity spheres (at one possible world) in the sense of David Lewis. (In general, a system of spheres need not to be well-ordered, of course.) Thus, Theorem 1 can be already found in Lewis (1973, at pp. 58f.) and in other places.

Why, then, did I define the SCFs in the way I did? Well, I have stated my reasons for doing so fully in Section 2. These reasons are debatable, but I have the impression that the slight differences between the various concepts of a selection function are motivated rather by differing opinions about conditional logic than about the dynamics of belief; and I was exclusively concerned with the latter which must not be mixed up with the former. (That was one reason why I deferred this comment.) To be sure, I completely side with Ernest W. Adams, Brian Ellis, Peter Gärdenfors, and others in maintaining that the various uses of the conditional can only be correctly and uniformly understood by relating them to a dynamic theory of epistemic states. But this relation is, I think, not yet sufficiently understood.

To be a bit more specific: If one accepts something like the straight thesis that the sentence "if A, then B" is accepted in (or true relative to) some epistemic state if and only if B is accepted in the revision of that state by A,[28] then one is bound to strain one or other side of this biconditional. A clear case, in my view, is provided by the very common causal conditionals. For, as I in effect argue in my (1983), if the concept of revising epistemic states is only to tell how beliefs change, then, according to this thesis, the conditional "if A, then B" only states something about the evidential relations between A and B, i.e. about A's being a reason for B, and thus does not yet express a causal relation between A and B. But let's not go further into this; my remark should only show why I want to confine myself to the dynamics of epistemic states and to leave aside the complicated relations to conditional logic.

The second point is this: If the central problem stated in Section 3 has been known at least since Harper (1976), what has been done to solve it? Surprisingly, not very much; and one reason for this is, it seems to me, that the issue has been obscured by what I have just complained about, i.e. by not clearly separating the dynamics of belief and conditional logic. In fact, I have found only three ideas which are addressed to this issue or can be so understood; and I shall deal with them, for the sake of brevity, only at a strategic level:

The first idea is that our problem of accounting for iterated belief changes appears to be analogous to the problem of providing a semantics for a language with iterated conditionals. The standard solution to the latter problem is to associate an SCF, a selection function, a system of similarity spheres, or whatever with *every* possible world (so that each conditional sentence has again a set of possible worlds as its truth condition). There is no need now to assess the semantic problem and its solution, though I always had the impression that in iterated intensional constructions the syntactic horse bolts with the semantic rider. The main point is that I don't see how the seeming analogy could be brought to bear; for, how should such a function from possible worlds to SCFs or whatever be interpreted as an epistemic state?

A second related idea is this: Enrich the language in which propositions are expressed by a conditional and thus by conditional sentences and propositions, and then exploit this new structural richness of the epistemic objects for a solution of our problem. This is, very roughly, the strategy applied by Harper (1976, pp. 95ff.). Ellis (1979, pp. 53ff.) and Gärdenfors (1979 and 1981, sect. II and III), seem to endorse it as well. However this strategy is brought to work in detail, it seems to be wrong from the start for two reasons: Our problem with SCFs shows, one should think, that it is the characterization of epistemic states as SCFs and not the structure of the epistemic objects which is too poor; one would expect that a dynamic theory of epistemic states does not force us to make special assumptions about the underlying structure of the epistemic objects. So, this strategy seems to focus on the wrong point (whereas our OCFs conform to this expectation). Moreover, there is the problem of how the conditional is interpreted within this strategy. In order to keep within the spirit of their approach, Harper and the others want to interpret it in terms of the dynamics of belief so far elaborated. This, however, amounts in fact to assuming second or higher order epistemic states which are partially about propositions describing properties of lower order epistemic states. Interesting as this may be, this move is uncalled for; one would expect that the problem with SCFs can be solved strictly at the level of first-order epistemic

states. Thus, this second idea seems to be an unconvincing mixture of the dynamics of belief and conditional logic.

The third and last approach is found in Gärdenfors (1984). The machinery developed there consists of belief sets, which are essentially equivalent to our net contents, and a relation of epistemic importance between sentences or propositions. With this machinery, Gärdenfors is able to describe successive changes of belief sets and thus gives a solution to our problem with SCFs – *provided* that the relation of epistemic importance is kept fixed. But why should it be so? An ordering of disbelief in our sense does essentially the same job as a belief set plus a relation of epistemic importance (though our respective interpretations of the two things do not match precisely); thus, that relation should be viewed as a part of an epistemic state which may change, too. Gärdenfors' approach therefore seems to me to provide only a restricted solution to our problem.

All this considered, there is enough reason to look for a solution to our problem elsewhere, as I have done in Sections 3–5.

The final point in need of a comment is that our OCFs look rather familiar; our degrees of disbelief seem more or less identical with the degrees of potential surprise in Shackle (1969). Indeed, the similarity is amazing, and the more so as Shackle developed his ideas long ago (before there was any conditional logic) and in quite a different scientific department. Since in particular his intuitive explanation of his functions of potential surprise perfectly fit my OCFs, it may be worthwhile to identify the points of difference, although this comparison is bound to be forced and somewhat unfair just because of the very different setting of his work.

According to Shackle (1969, p. 80),[29] a *function y of potential surprise* (an *FPS*) may be defined to be a function from a given field of propositions into the closed interval [0,1] such that for all propositions A and B

(1) $y(\emptyset) = 1$,
(2) either $y(A) = 0$ or $y(\bar{A}) = 0$ or both,
(3) $y(A \cup B) = \min\{y(A), y(B)\}$.

1 is the arbitrarily chosen maximal degree of potential surprise which is taken at least by $\emptyset$, and (2) and (3) are identical with my Theorem 2 (Sect. 4). Thus, there seem to be hardly any differences between FPSs and OCFs; but there are four.

One point is that OCFs satisfy the generalization of (3) to arbitrary unions; but, as expressed in Notes 8 and 13, I do not attach much importance to this. Since in this generalization min is not weakened to inf, it forces the range of OCFs to be well-ordered; and then ordinals are the natural values for OCFs. Thus, the difference with respect to (3) also accounts for the differing ranges of FPSs and OCFs; but we shall see that there is more to the difference in the ranges.

Another difference is about the maximal degree of potential surprise. I also could have introduced a number larger than any ordinal as the OCF-value for $\emptyset$; but this did not look nice, and so I preferred to make qualifications to the effect that $\emptyset$ is not imported into the domain of an OCF. The important point here is that I therefore do not allow any other proposition to take the maximal value. The reason is that, once a proposition were disbelieved to the maximal degree, it would always be disbelieved to the maximal degree, at least according to my rules of belief change; rational belief change could then no longer be treated within my framework. This was something I wanted to avoid. Shackle, by contrast, makes free use of the maximal degree of potential surprise. And Levi (1980, p. 7) explicitly assigns it to each proposition that is incompatible with what he calls a corpus of knowledge, and he therefore has trouble, e.g. in (1983), with specifying rules for changing such corpora of knowledge.

The essential point is that Shackle has no precise and workable account of conditional degrees of potential surprise, of changes of FPSs, etc. This becomes apparent in his handling of conjunctions. In his (1969, pp. 80ff. and 199ff.), he sticks to the postulate that

$$y(A \cap B) = \max\{y(A), y(B|A)\} \tag{4}$$

(where I have adapted the notation and where $y(B|A)$ is in fact undefined). In contrast to this, our Definition 5, which is fundamental for our Sections 5 and 6, is equivalent to

$$\kappa(A \cap B) = \kappa(A) + \kappa(B|A). \tag{5}$$

Shackle has obviously considered accepting something like (5) instead of (4); but he says little about why he finally rejected it. In his (1969, p. 205), he says only that (4) would be simpler and less unrealistic than something like (5).

A final significant difference may be inferred from (1)–(5). Shackle (1969, ch. XV–XVII) clearly intends his FPSs to be measurable on a ratio scale. But it is hard to see precisely how this scale is established

and where it is really used; it seems that we may conceive FPSs as purely ordinal concepts.[30] In any case, FPSs as displayed by (1)–(4) are purely ordinal, as may be seen from the exclusive use of mathematical operations like max and min. But if this is so, FPSs correspond to our WOPs (or the functions definable by WOPs according to Note 13). This would mean that the decisive step towards OCFs is perhaps intended, but not really taken by FPSs.

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NOTES

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[1] Which is clearly conceived as covertly epistemological by Lewis (1980), for instance.

[2] In my (1983), I have proposed a way of progressing from the notion of reason to the notion of cause. I have refrained there from introducing the formal machinery developed in this paper; footnote 18 of that paper marks the point where what I there called selection functions and shall call here simple conditional functions should be replaced by the ordinal conditional functions to be defined – for reasons more fully explained here in Section 3.

[3] I am not happy with the term "deterministic epistemology", but I could not find a better one. It derives from the natural and familiar distinction between deterministic and probabilistic causation which, in my opinion, is closely related to the different forms of epistemology.

[4] Think, e.g., of the disquotation principle saying that if X sincerely and seriously utters "p", then X *believes* that p. This is an important, though not generally true linguistic fact; and it is hard to see what a probabilistic version of it could look like.

[5] This is not to be confused with what is ordinarily called qualitative probability which is a relational, comparative concept.

[6] Where I don't at all oppose construing a possible world as a maximal consistent set of sentences of a given langauge, as a valuation of that language, or the like.

[7] This consideration suggests that the idealization of belief as propositional should be overcome not by seeking for a stricter objective individuation of the objects of belief, but by getting a grip on the subjective imperfections of semantic knowledge.

[8] One might argue about whether $\mathcal{B}'$ should here be assumed to be countable or finite or neither. With my definition of deductive closure I have assumed what has been called the generalized consequence principle; cf. e.g. Pollock (1976, pp. 19f.) or Gärdenfors (1981, p. 308). I do so, because I find this principle convincing given the idealization of perfect semantic knowledge and because it makes things technically much simpler. However, as far as I see, everything I say in this paper could be adapted to a weaker assumption without essential complications.

[9] Some, e.g. Lewis (1973, p. 58), prefer to replace the condition that $g(A) \neq \emptyset$ (which is tantamount to universality) by the condition that $A \subseteq B$ and $g(A) \neq \emptyset$ imply $g(B) \neq \emptyset$. However, that's much of a muchness. The only difference is this: With the alternative definition one can prove that there is a $D \subseteq W$ such that $g(A) = \emptyset$ iff $A \subseteq D$. Now alter g by putting $g(A) = A$ for $A \subseteq D$ and leave it otherwise unchanged (thus, there is no change, if $D = \emptyset$); then g complies with our definition. In both cases, accepting an information $A \subseteq D$ leads to complete epistemic collapse. In our case, only the information is then believed; all induction ceases. In the alternative case, everything is then believed (if this makes sense); induction goes crazy. I find our description of that desperate situation a bit more pleasant; besides, SCFs in our sense are more easily generalized to the ordinal conditional functions introduced in Section 4.

[10] Continuously using these negative terms is a somewhat clumsy and contorted mode of expression. But Isaac Levi has convinced me that this is precisely the intuitively appropriate terminology.

[11] I shall say in Section 7 how Popper measures relate to the present subject.

[12] A is an atom of $\mathcal{A}$ iff $A \neq \emptyset$ and there is no $B \in \mathcal{A}$ with $\emptyset \subset B \subset A$. Complete fields of sets are always atomic.

[13] This latter function for propositions is the more important one. The corresponding notion at the level of WOPs is the function assigning to each proposition A the number $\min\{\alpha | E_\alpha \cap A \neq \emptyset\}$, which we have frequently used, though not explicitly introduced. Note, by the way, that it is our acceptance of the generalized consequence principle (cf. Note 8) which is in the end responsible for the possibility of reducing the propositional function to a function defined for possible worlds.

[14] All this is a sort of exercise in intuitively interpreting OCFs. Degrees of firmness could also have been introduced relative to WOPs; but this would have been misleading, because, relative to WOPs, numbers have a purely ordinal meaning.

[15] Still, it would be inappropriate to say that only the best members count. Imagine a proposition having only one member with a good grade, the rest being very far behind. Then, if this good member fell back very badly, so would the whole proposition. If, however, the rest were not so bad, the top member could fall back without disastrous consequences for the team. In this sense the rest matter, too.

[16] It would be a natural idea to restrict the range of OCFs to the set of natural numbers. In fact, much of the following could thereby be simplified since usual arithmetic is simpler than the arithmetic of ordinals. For the sake of formal generality I do not impose this restriction. But larger ranges may also be intuitively needed. For example, it is tempting to use OCFs with larger ranges to represent the stubbornness with which some beliefs are held in the face of seemingly arbitrarily augmentable counter-evidence.

[17] For details cf. Klaua (1969), p. 173.

[18] This is a short notation for the function assigning to each w in the domain the value $\kappa(w|A)$.

[19] This was pointed out to me by Godehard Link.

[20] It is easy to link Def. 6 with Gärdenfors (1984). Gärdenfors there discusses contractions and minimal changes of what he calls belief sets, where these belief sets are essentially equivalent to our net contents. Keeping in mind that $\kappa^{-1}(0)$ is the net content of state κ, we may define the minimal change of $\kappa^{-1}(0)$ needed to accept A as $\kappa_{A,\alpha}^{-1}(0)$ for some $\alpha > 0$ (this does not depend on which $\alpha > 0$ we choose). And we may define the contraction of $\kappa^{-1}(0)$ with respect to A as $\kappa^{-1}(0)$, if $\kappa(\bar{A}) = 0$, and as $\kappa_{A,0}^{-1}(0)$, if $\kappa(\bar{A}) > 0$. It is then easy to prove that contractions and minimal changes so defined have all the properties (1)–(21) Gärdenfors (1984, pp. 140–142) wants them to have.

[21] A self-comment: In my (1983), I explicated the notion that A is a reason for B relative to SCFs (which I there called selection functions). This has now turned out to be inadequate, but it is easily repaired: *A is a reason for B* in the state κ iff B is believed in κ with greater firmness given A than given $\bar{A}$, i.e. iff $\kappa(\bar{B}|A) > \kappa(\bar{B}|\bar{A})$ or $\kappa(B|A) < \kappa(B|\bar{A})$. The rest of the paper is easily adapted to this new definition. (Instead of "A is a reason for B in a given epistemic state" one may also say that A *means* B in that state. This is, it seems to me, the most basic meaning of meaning on which other (linguistic) concepts of meaning may be built.)

[22] Cf. e.g. Spohn (1980), Theorem 1(d) and (e). – Indeed, I wonder how far the mathematical analogy could be extended. What I have shown is that the probabilistic theory of dependence, independence, and conditionalization can be carried over to OCFs. The Definition 7 of generalized conditionalization suggests that the concept of a *mixture* may also be meaningfully carried over from probability measures to OCFs. This might be worth exploring. One essential point of dissimilarity is that, as far as I see, there is no meaning to a theory of integration within the theory of OCFs.

[23] Cf. e.g. van Fraassen (1976).

[24] Cf. Harper (1976, pp. 87ff.), or my (1986). The dimensionally well-ordered families of probability measures introduced in the latter paper are the counterparts to our WOPs; and these families represent Popper measures just as WOPs represent SCFs according to Theorem 1.

[25] The idea is essentially due to Kurt Weichselberger. I have merged his idea with an idea I found in Skyrms (1983, p. 158).

[26] One may perhaps conclude that I should have carried through the whole business of OCFs within a nonstandard framework form the start. However, I am happier with the standard version presented, and I did not want to burden my theory with nonstandard number models.

[27] Cf. e.g. Nute (1980, ch. 1 and 3).

[28] Gärdenfors (1981, p. 207), e.g., explicitly accepts this thesis.

[29] Cf. also Levi (1980, p. 7).

[30] On p. 188, Shackle (1969) says that the assumption of the cardinality of his tool is "by no means indispensible to its main purpose".

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